T.H. KIM
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NONLINEAR ERGODIC THEOREMS OF ALMOST-ORBITS OF NON-LIPSCHITZIAN SEMIGROUPS

TAE HWA KIM

Abstract

In this paper, we shall establish the weak convergence and nonlinear ergodic theorems for reversible semigroups of weakly asymptotically non-expansive type in Banach spaces.

1. Introduction

Let G be a semitopological semigroup, i.e., G is a semigroup with a Hausdorff topology such that for each $a\in G$ the mappings $s\mapsto a\cdot s$ and $s\to s\cdot a$ from G to G are continuous. G is called right reversible if any two closed left ideals of G have nonvoid intersection. In this case, $(G, \succcurlyeq)$ is a directed system when the binary relation "$\succcurlyeq$" on G is defined by

$$t\succcurlyeq s \quad \text{if and only if} \quad \{s\}\cup\overline{Gs}\supseteq\{t\}\cup\overline{Gt}, \quad s, t\in G.$$

Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups (see [7, p. 335]). Left reversibility of G is defined similarly. G is called reversible if it is both left and right reversible.

Let G be a semitopological semigroup with a binary relation "$\succcurlyeq$" which directs G. Let C be a nonempty closed convex subset of a real Banach space E and let a family $\mathfrak{J}=\{S(t): t\in G\}$ be a (continuous) representation of G as continuous mappings on C into C, i.e., $S(ts)x=S(t)S(s)x$ for all $t, s\in G$ and $x\in C$, and for every $x\in C$, the mapping $t\mapsto S(t)x$ from G into C is continuous. A representation $\mathfrak{J}=\{S(t): t\in G\}$ of G on C is called reversible [resp., right (left) reversible] if G is reversible [resp., right (left) reversible]. In this paper, we also consider a non-Lipschitzian semigroup of mappings: a representation $\mathfrak{J}=\{S(t): t\in G\}$ of G on C is said to be a semigroup of weakly asymptotically

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nonexpansive type (simply, w.a.n.t.) on C if, for each $x\in C$ and each bounded subset D of C,

$$\limsup_{t}\sup_{y\in D}(\|S(t)x-S(t)y\|-\|x-y\|)\leqq 0.$$

Immediately, we can see that the semigroups of w.a.n.t. include all semigroups of nonexpansive mappings with directed systems. In particular, if $\mathfrak{J}=\{S(t): t\in G\}$ is a Lipschitzian representation of G with an additional condition, i.e., $\limsup_{t} k_t\leqq 1$ (see [15]), then it is obviously of w.a.n.t. In cases where $G=N$, $S=S(1)$, we have $S(n)=S^n$ for each $n\in N$, where N denotes the set of natural numbers. Then, when the semigroup $\mathfrak{J}=\{S(n): n\in N\}$ is of w.a.n.t., $S: C\to C$ is simply said to be a mapping of w.a.n.t. For a mapping $S: C\to C$ of a.n.t., see Kirk [9]. And we say that a function $u: G\to C$ is an almost-orbit of $\mathfrak{J}=\{S(t): t\in G\}$ (see [13], [15]) if G is right reversible and

$$\lim_{t}(\sup_{s\in G}\|u(st)-S(s)u(t)\|)=0.$$

In [15], Takahashi-Zhang established the weak convergence of an almost-orbit of a noncommutative Lipschitzian semigroup in a Banach space. In [11], Lau-Takahashi also proved the nonlinear ergodic theorems for a noncommutative nonexpansive semigroup in the space. In this paper, we first shall establish a fixed point theorem for an almost-orbit $\{u(t): t\in G\}$ of the reversible semigroup $\mathfrak{J}=\{S(t): t\in G\}$ of w.a.n.t. in a uniformly convex Banach space, which generalizes the commutative version due to Kiang-Tan [8], and also prove the equivalent conditions of weak convergence of the almost-orbit $\{u(t): t\in G\}$ of $\mathfrak{J}=\{S(t): t\in G\}$, which extend the results due to Miyadera [12] and Emmanule [3, Theorem 2]. Next, we shall carry over the weak convergence and nonlinear ergodic theorems due to Lau-Takahashi [11], Takahashi-Zhang [15] to those for the right reversible semigroup $\mathfrak{J}=\{S(t): t\in G$ of w.a.n.t. Our proofs employ the methods of [8], [11], [12], and [15].

2. Fixed Point Theorem

Let C be a nonempty closed convex subset of a real Banach space E and let G be a semitopological semigroup with a binary relation "$\succcurlyeq$" which directs G.

DEFINITION 2.1. A family $\mathfrak{J}=\{S(t): t\in G\}$ of continuous mappings from C into itself is said to be *a (continuous) representation of G on C* if $\mathfrak{J}$ satisfies the following:

(a) $S(ts)x=S(t)S(s)x$ for all $t, s\in G$ and $x\in C$;

(b) for every $x\in C$, the mapping $t\mapsto S(t)x$ from G into C is continuous.

DEFINITION 2.2. Let $\mathfrak{J}=\{S(t): t\in G\}$ be a representation of G on C. $\mathfrak{J}$ is said to be a *semigroup of weakly asymptotically nonexpansive type* (simply, *w. a. n. t.*) on C if, for each $x \in C$ and each bounded subset D of C,

$$\inf_{s \in G} \sup_{t \geq s} \sup_{y \in D} (\|S(t)x - S(t)y\| - \|x - y\|) \leqq 0. \tag{2.1}$$

Immediately, we can see that the semigroups of w. a. n. t. indude all semigroups of nonexpansive mappings with directed systems. In particular, if $\mathfrak{J} = \{S(t) : t \in G\}$ is a Lipschitzian representation of G with an additional condition, i. e., $\limsup_t k_t \leqq 1$ (see [15]), then it is obviously of w. a. n. t. In cases where $G = N$, $S = S(1)$, we have $S(n) = S^n$ for each $n \in N$, where N denotes the set of natural numbers. Then, when the semigroup $\mathfrak{J} = \{S(n) : n \in N\}$ is of w. a. n. t., $S : C \to C$ is simply said to be a mapping of w. a. n. t. For a mapping $S : C \to C$ of a. n. t., see Kirk [9].

DEFINITION 2.3. Let G be right reversible and let $\mathfrak{J} = \{S(t) : t \in G\}$ be as in Definition 2.1. A function $u : G \to C$ is called an *almost-orbit* of $\mathfrak{J} = \{S(t) : t \in G\}$ (see [13], [15]) if

$$\lim_t (\sup_{s \in G} \|u(st) - S(s)u(t)\|) = 0. \tag{2.2}$$

Now let $\{x_\alpha\}$ be a bounded net in C. Then we define

$$r(x, \{x_\alpha\}) = \limsup_\alpha \|x - x_\alpha\|, \tag{2.3}$$

$$r = r(C, \{x_\alpha\}) = \inf\{r(x, \{x_\alpha\}) : x \in C\}, \tag{2.4}$$

and

$$A(C, \{x_\alpha\}) = \{z \in C : r(z, \{x_\alpha\}) = r(C, \{x_\alpha\})\}. \tag{2.5}$$

Then, any element of $A(C, \{x_\alpha\})$ is called to be an *asymptotic center* of the net $\{x_\alpha\}$ in C. It is well known that if C is weakly compact and convex, then $A(C, \{x_\alpha\}) \neq \emptyset$ and if E is uniformly convex, then $A(C, \{x_\alpha\})$ is a singleton set and it will be simply denoted by the unique element; see [4].

From now on, unless other specified, let G, E, C, and $\mathfrak{J} = \{S(t) : t \in G\}$ be as in Definition 2.2 and $F(\mathfrak{J})$ denote the set of all common fixed points of $\mathfrak{J} = \{S(t) : t \in G\}$ in C, i. e., $F(\mathfrak{J}) = \{x \in C : S(t)x = x \text{ for all } t \in G\}$. We begin with the following:

LEMMA 2.4. *Let G be right reversible and let E be uniformly convex. If $\{u(t) : t \in G\}$ is a bounded almost-orbit of $\mathfrak{J} = \{S(t) : t \in G\}$ and $r = r(C, \{u(t)\}) = 0$, then $\lim_t S(s)u(t) = c \in F(\mathfrak{J})$ for each $s \in G$, where $c = A(C, \{u(t)\})$.*

Proof. Since $r = r(c, \{u(t)\}) = 0$, clearly $c = \lim_t u(t) = \lim_t u(st)$ for each $s \in G$. Let $s, t \in G$. Then we obtain

$$\begin{aligned}\|c - S(s)u(t)\| &\leqq \|c - u(st)\| + \|u(st) - S(s)u(t)\| \\ &\leqq \|c - u(st)\| + \phi(t),\end{aligned}$$

where $\phi(t)=\sup_{s\in G}\|u(st)-S(s)u(t)\|$. Since $\{u(t): t\in G\}$ is an almost-orbit of $\mathfrak{J}=\{S(t): t\in G\}$, $\lim_t \phi(t)=0$, and now taking $\limsup_t$ in both sides, we obtain $\lim_t S(s)u(t)=c$, and hence $S(s)c=S(s)(\lim_t u(t))=\lim_t S(s)u(t)=c$, i.e., $c\in F(\mathfrak{J})$. This completes the proof.

We say that $\mathfrak{J}=\{S(t): t\in G\}$ is *proximately nonexpansive* (see [8] if for every $x\in C$, and every $\beta>0$, there exists $t_0\in G$ such that $\|S(t)x-S(t)y\|\leqq(1+\beta)\|x-y\|$ for all $t\geqslant t_0$, and for any $y\in C$. If $\mathfrak{J}=\{S(t): t\in G\}$ is proximately nonexpansive, then it is of w.a.n.t. Indeed, let $x\in C$ and D a bounded subset of C. Then, for given $\varepsilon>0$, take $\beta=\frac{\varepsilon}{M}$, where $M=\sup_{y\in D}\|x-y\|$.

In order to measure the degree of strict convexity (rotundity) of E, we define its *modulus of convexity* δ: $[0, 2]\to[0, 1]$ by

$$\delta(\varepsilon)=\inf\left\{1-\frac{1}{2}\|x+y\| : \|x\|\leqq 1,\ \|y\|\leqq 1, \text{ and } \|x-y\|\geqq\varepsilon\right\}. \tag{2.6}$$

The *characteristic of convexity* ε_0 of E is also defined by

$$\varepsilon_0=\varepsilon_0(E)=\sup\{\varepsilon : \delta(\varepsilon)=0\}. \tag{2.7}$$

It is well-known (see [4]) that the modulus of convexity δ satisfies the following properties:

$$\text{(2.8)}\quad \begin{cases} \text{(a)} & \delta \text{ is increasing on } [0, 2], \text{ and moreover strictly increasing on } [\varepsilon_0, 2]; \\ \text{(b)} & \delta \text{ is continuous on } [0, 2) \text{ (but not necessarily at } \varepsilon=2); \\ \text{(c)} & \delta(2)=1 \text{ if and only if } E \text{ is strictly convex}; \\ \text{(d)} & \delta(0)=0 \text{ and } \lim_{\varepsilon\to 2-}\delta(\varepsilon)=1-\frac{1}{2}\varepsilon_0; \\ \text{(e)} & \|a-x\|\leqq r,\ \|a-y\|\leqq r \text{ and } \|x-y\|\geqq\varepsilon \\ & \Longrightarrow \left\|a-\frac{1}{2}(x+y)\right\|\leqq r(1-\delta(\varepsilon/r)). \end{cases}$$

A Banach space E is said to be *uniformly convex* if $\delta(\varepsilon)>0$ for all positive ε; equivalently $\varepsilon_0=0$. Obviously, any uniformly convex space is both strictly convex and reflexive. By properties above, we can see that if E is uniformly convex, then δ is strictly increasing and continuous on $[0, 2]$ (see also [1]).

Now we can present a fixed point thorem for an almost-orbit $\{u(t): t\in G\}$ of the reversible semigroup $\mathfrak{J}=\{S(t): t\in G)$ of w.a.n.t. in a uniformly convex Banach space, which generalizes the commutative version due to Kiang-Tan [8; Theorem 1].

THEOREM 2.5. *Let G be reversible and let E be uniformly convex. If an almost-orbit $\{u(t): t\in G\}$ of $\mathfrak{J}=\{S(t): t\in G\}$ is bounded, then $A(C, \{u(t)\})=c\in F(\mathfrak{J})$.*

Proof. If $r=r(C, \{u(t)\})=0$, the result follows from Lemma 2.4. Assume $r>0$ and we set

$$d(c)=\limsup_{s}\|c-S(s)c\|.$$

Then, $d(c)<\infty$. In fact, since $\mathfrak{J}=\{S(t): t\in G\}$ is of w.a.n.t., for $c\in C$ and $D=\{u(t): t\in G\}$,

$$\limsup_{t}\sup_{y\in D}(\|S(t)c-S(t)y\|-\|c-y\|)\leqq 0.$$

For $s, t\in G$, we have

$$\begin{aligned}\|c-S(t)c\| &\leqq \|c-u(ts)\|+\|u(ts)-S(t(u(s)\|+\|S(t)u(s)-S(t)c\| \\ &\leqq \|c-u(ts)\|+\phi(s)+\|S(t)c-S(t)u(s)\| \\ &\leqq 2(\|c\|+M)+\phi(s)-\sup_{y\in D}(\|S(t)c-S(t)y\|-\|c-y\|),\end{aligned}$$

where $M=\sup_{t\in G}\|u(t)\|<\infty$ and $\phi(s)=\sup_{t\in G}\|u(ts)-S(t)u(s)\|$. Since $\lim_{s}\phi(s)=0$, taking $\limsup_{t}$ at first and next $\limsup_{s}$ in both sides, we obtain

$$d(c)\leqq 2(\|c\|+M)<\infty.$$

If $d(c)=0$, then, since G is right reversible, we obtain $S(t)c=\lim_{s} S(t)S(s)c= \lim_{s} S(ts)c=\lim_{s} S(s)c=c$ for each $t\in G$, i.e., $c\in F(\mathfrak{J})$. Hence, it suffices to show that $d(c)=0$. If not, let ε be such that $0<\varepsilon<d(c)$. Then, for each $\alpha\in G$, there is $s_\alpha\in G$ such that $s_\alpha\geqslant\alpha$ and

$$\|c-S(s_\alpha)c\|>\varepsilon.$$

Since E is uniformly convex, choose $d>0$ so small that

$$(r+d)\left[1-\delta\left(\frac{\varepsilon}{r+d}\right)\right]<r,$$

where δ is the modulus of convexity of E. For this $d>0$, by (2.1) with $c\in C$ and $D=\{u(s): s\in G\}$, (2.2), and (2.3), there exists $\alpha\in G$ such that

$$\|S(t)c-S(t)y\|\leqq\|c-y\|+\frac{d}{3},$$

$$\sup_{s\in G}\|u(st)-S(s)u(t)\|<\frac{d}{3},$$

and

$$\|c-u(t)\|<r+\frac{d}{3},$$

for all $t\geqslant\alpha$ and all $y\in D$. Then, for this α, we can also choose $s_\alpha\in G$ with $s_\alpha\geqslant\alpha$ and

$$\|c-S(s_\alpha)c\|>\varepsilon .$$

Then, for all $t \geqslant \alpha$, we have

$$\begin{aligned}&\|u(s_\alpha t)-S(s_\alpha)c\|\\ &\leqq\|u(s_\alpha t)-S(s_\alpha)u(t)\|+\|S(s_\alpha)u(t)-S(s_\alpha)c\|\\ &<\frac{d}{3}+\|u(t)-c\|+\frac{d}{3}<r+d .\end{aligned}$$

Let $b_0=s_\alpha\alpha$ and $t \geqslant b_0$. Then, since G is left reversible, we obtain

$$\sup_{t\geq b_0}\|u(t)-S(s_\alpha)c\|=\sup_{t\geq\alpha}\|u(s_\alpha t)-S(s_\alpha)c\|<r+d .$$

Clearly, we obtain

$$\sup_{t\geq b_0}\|c-u(t)\|<r+d .$$

By the definition of δ (see (e) of (2.6)), it follows that

$$\sup_{t\geq b_0}\left\|u(t)-\frac{1}{2}(c+S(s_\alpha)c)\right\|\leqq(r+\sigma)\left[1-\delta\left(\frac{\varepsilon}{r+\sigma}\right)\right].$$

Setting $z=\frac{1}{2}(c+S(s_\alpha)c)\in C$, it implies that

$$\begin{aligned}r&\leqq r(z,\ \{u(t)\})\leqq\sup_{t\geq b_0}\|u(t)-z\|\\ &\leqq(r+\sigma)\left[1-\delta\left(\frac{\varepsilon}{r+\sigma}\right)\right]<r ,\end{aligned}$$

which gives a contradiction. The proof is completed.

3. Weak Convergence

In this section, we present the weak convergence of almost-orbits $\{u(t) \mid t\in G\}$ of the reversible semigroup $\mathfrak{J}=\{S(t): t\in G\}$ of w.a.n.t. For a function $u: G\to C$, let $\omega_w(u)$ denote the set of all weak limits of subnets of the net $\{u(t): t\in G\}$ and we set

(3.1) $$E(u)=\{y\in C: \lim\|u(t)-y\|\text{exists}\} .$$

Then, the following lemma is crucial for our arguement:

LEMMA 3.1. *Let G be right reversible and let $\{u(t): t\in G\}$, $\{v(t): t\in G\}$ almost-orbits of $\mathfrak{J}=\{S(t): t\in G\}$. Then the limit of $\|u(t)-v(t)\|$ exists. In particular, $F(\mathfrak{J})\subseteqq E(u)$.*

Proof. Since $\mathfrak{J}=\{S(t): t\in G\}$ is of w.a.n.t., it follows that, for each

$s \in G$ and $D=\{v(s)\}$,

$$\lim_t \sup(\|S(t)u(s)-S(t)v(s)\|-\|u(s)-v(s)\|) \leqq 0,$$

Therefore, for each $s, t \in G$,

$$\begin{aligned}\|u(ts)-v(ts)\| &\leqq \|u(ts)-S(t)u(s)\|+\|S(t)u(s)-S(t)v(s)\|+\|S(t)v(s)\|-v(ts)\| \\ &\leqq \phi(s)+(\|S(t)u(s)-S(t)v(s)\|-\|u(s)-v(s)\|)+\|u(s)-v(s)\|+\psi(s),\end{aligned}$$

where $\phi(s)=\sup_t\|u(ts)-S(t)u(s)\|$, $\psi(s)=\sup_t\|v(ts)-S(t)v(s)\|$. At first, taking $\lim_t\sup$ at both sides, we get

$$\lim_t \sup\|u(t)-v(t)\| \leqq \lim_t \sup\|u(ts)-v(ts)\| \leqq \phi(s)+\psi(s)+\|u(s)-v(s)\|.$$

Since $\{u(t): t \in G\}$, $\{v(t): t \in G\}$ are almost-orbits of $\mathfrak{J}=\{S(t): t \in G\}$, $\lim_s \phi(s)=0$ and $\lim_s \psi(s)=0$, and now taking $\lim_s \inf$ in both sides, $\lim_t\|u(t)-v(t)\|$ exists. Now let $z \in F(\mathfrak{J})$ and put $v(t) \equiv z$ for all $t \in G$. Then $\{v(t): t \in G\}$ is an almost-orbit and hence the limit of $\|u(t)-z\|$ exists, i.e., $z \in E(u)$.

When $\{x_\alpha\}$ is a net in a Banach space E and $x \in E$, $x_\alpha \to x (x_\alpha \rightharpoonup x)$ means the strong (weak) convergence to x of the net $\{x_\alpha\}$, respectively.

Recall that a Banach space E is said to satisfy *Opial's condition* if, for any net $\{x_\alpha\}$ in E with $x_\alpha \rightharpoonup x \in E$,

$$\lim_\alpha \sup\|x_\alpha-x\| < \lim_\alpha \sup\|x_\alpha-y\|, \quad \forall y(\neq x) \in E. \tag{3.2}$$

(see [10, Lemma 2.1]). For any sequence in E, see [14, Lemma 1]. For more details, see also [3] and [5]. We are now ready to prove the equivalent conditions of weak convergence for the almost-orbit $\{u(t): t \in G\}$ of the reversible semigroup $\mathfrak{J}=\{S(t): t \in G\}$ of w.a.n.t., which extends the results due to Miyadera [12] and Emmanuele [3, Theorem 2].

THEOREM 3.2. *Let G be reversible and let E be uniformly convex with Opial's condition. Let $u=\{u(t): i \in G\}$ be an almost-orbit of $\mathfrak{J}=\{S(t): t \in G\}$. Then the following conditions are equivalent*

(i) *w-$\lim_t u(t)$ exists;*

(ii) *$F(\mathfrak{J}) \neq \emptyset$ and $\omega_w(u) \subseteqq F(\mathfrak{J})$;*

(iii) *$E(u) \neq \emptyset$ and $\omega_w(u) \subseteqq E(u)$.*

Proof. Since E is uniformly convex and $\{u(t): t \in G\}$ is bounded in C by (i), by Theorem 2.5., there exists a unique asymptotic center $A(C, \{u(t)\})=c \in F(\mathfrak{J})$. Then, Opial's condition also gives that $c=w$-$\lim_t u(t)$. Hence we have $\omega_w(u)=\{c\} \subseteqq F(\mathfrak{J})$. Thus (i) implies (ii). By Lemma 3.1., clearly (ii) implies (iii). Also to show that (iii) implies (i), apply the method of Theorem 1 due to Emmanuele [3].

LEMMA 3.3. *Let G be right reversible and let E be uniformly convex. Let $u=\{u(t): t\in G\}$ be an almost-orbit of $\mathfrak{J}=\{S(t): t\in G\}$. Suppose $F(\mathfrak{J})\neq\emptyset$ and let $y\in F(\mathfrak{J})$ and $0<\alpha\leqq\beta<1$. Then, for any $\varepsilon>0$, there is $t_0\in G$ such that*

$$\|S(t)(\lambda u(s)+(1-\lambda)y)-(\lambda S(t)u(s)+(1-\lambda)y\|<\varepsilon$$

for all t, $s\geqslant t_0$ and $\lambda\in[\alpha, \beta]$.

Proof. Since $F(\mathfrak{J})\neq\emptyset$, we may assume that $\{u(t): t\in G\}$ is bounded, and hence $D=\{\lambda u(t)+(1-\lambda)y: t\in G, 0\leqq\lambda\leqq 1\}$ is bounded. Let $\varepsilon>0$ and let $r=\lim_t\|u(t)-y\|$ by Lemma 3.1. If $r=0$, since $\mathfrak{J}=\{S(t): t\in G\}$ is of w.a.n.t. on C, for $y\in C$ and D, there exists $t_0\in G$ such that

$$\|y-S(t)z\|<\|y-z\|+\frac{\varepsilon}{4}$$

and

$$\|u(t)-y\|<\frac{\varepsilon}{4} \quad \text{for } t\geqslant t_0 \text{ and } z\in D.$$

Hence, for s, $t\geqslant t_0$ and $0\leqq\lambda\leqq 1$, we have

$$\begin{aligned}&\|S(t)(\lambda u(s)+(1-\lambda)y)-(\lambda S(t)u(s)+(1-\lambda)y)\|\\ &\leqq\|S(t)(\lambda u(s)+(1-\lambda)y)-y\|+\lambda\|S(t)u(s)-y\|\\ &\leqq 2\Big(\lambda\|u(s)-y\|+\frac{\varepsilon}{4}\Big)<\varepsilon.\end{aligned}$$

Now, let $r>0$. Then we can choose $d>0$ so small that

$$(r+d)\Big[1-c\delta\Big(\frac{\varepsilon}{r+d}\Big)\Big]=r_0<r.$$

where δ is the modulus of convexity of E and $c=\min\{2\lambda(1-\lambda): \alpha\leqq\lambda\leqq\beta\}$. For $k>0$ with $k<\min\{d/2, (r-r_0)/2\}$, as the above, there exists $t_0\in G$ such that

$$r-a<\|u(t)-y\|<r+k,$$

$$\|y-S(t)z\|<\|y-z\|+\frac{c}{4}d, \quad \text{for } t\geqslant t_0,\ z\in D,$$

and

$$\|u(ts)-S(t)u(s)\|<a, \quad \text{for } s\geqslant t_0,\ t\in G.$$

Suppose that

$$\|S(t)(\lambda u(s)+(1-\lambda)y)-(\lambda S(t)u(s)+(1-\lambda)y)\|\geqq\varepsilon$$

for some s, $t\geqslant t_0$ and $\lambda\in[\alpha, \beta]$. Put $z=\lambda u(s)+(1-\lambda)y$, $u=(1-\lambda)(S(t)z-y)$ and $v=\lambda(S(t)u(s)-S(t)z)$. Then, it follows that

$$\|u\|\leqq(1-\lambda)\Big(\|y-z\|+\frac{c}{4}d\Big)$$

$$=(1-\lambda)\Big(\lambda\|u(s)-y\|+\frac{c}{4}d\Big)$$
$$<(1-\lambda)\Big(\lambda\Big(r+\frac{d}{2}\Big)+\frac{c}{4}d\Big)$$
$$<\lambda(1-\lambda)(r+d)$$

and

$$\|v\|<\lambda(1-\lambda)(r+d).$$

We also have that

$$\|u-v\|=\|S(t)z-(\lambda S(t)u(s)+(1-\lambda)y)\|\geqq\varepsilon$$

and $\lambda u+(1-\lambda)v=\lambda(1-\lambda)(S(t)u(s)-y)$. By lemma in [6], we have

$$\begin{aligned}\lambda(1-\lambda)\|S(t)u(s)-y\|&=\|\lambda u+(1-\lambda)v\|\\&\leqq\lambda(1-\lambda)(r+d)\Big[1-2\lambda(1-\lambda)\delta\Big(\frac{\varepsilon}{r+d}\Big)\Big]\\&\leqq\lambda(1-\lambda)r_0,\end{aligned}$$

and hence $\|S(t)u(s)-y\|\leqq r_0$. Thus it follows that

$$\begin{aligned}\|u(ts)-y\|&\leqq\|u(ts)-S(t)u(s)\|+\|S(t)u(s)-y\|\\&<a+r_0<r-a,\end{aligned}$$

which gives a contradiction and the proof is complete.

For $x, y\in E$, we denote by $[x, y]$ the set $\{\lambda x+(1-\lambda)y: 0\leqq\lambda\leqq 1\}$. For $D\subset E$, $\overline{co}D$ denotes the closed convex hull of D. The following lemma was proved by Lau-Takahashi [11, Lemma 3].

LEMMA 3.4. *Let E be uniformly convex with a Fréchet differentiable norm and let $\{x_\alpha\}$ be a bounded net in C. Let $z\in\bigcap_\beta\overline{co}\{x_\alpha: \alpha\succcurlyeq\beta\}$, $y\in C$ and $\{y_\alpha\}$ a net of elements in C with $y_\alpha\in[y, x_\alpha]$ and*

$$\|y_\alpha-z\|=\min\{\|u-z\|: u\in[y, x_\alpha]\}.$$

If $y_\alpha\to y$, then $y=z$.

By using Lemma 3.3 and Lemma 3.4, we obtain the similar result as Theorem 2 in [15] for an almost-orbit $\{u(t): t\in G\}$ of the right reversible semigroup $\mathfrak{F}=\{S(t): t\in G\}$ of w.a.n.t. on C in a uniformly convex Banach space E with a Fréchet differentiable norm.

THEOREM 3.5. *Let G be right reversible and let E be uniformly convex with a Fréchet differentiable norm. Suppose that $u=\{u(t): t\in G\}$ is an almost-orbit of $\mathfrak{F}=\{S(t): t\in G\}$ and $F(\mathfrak{F})\neq\emptyset$. Then the set $\bigcap_{s\in G}\overline{co}\{u(t): t\succcurlyeq s\}\cap F(\mathfrak{F})$ consists of at most one point.*

Proof. Since $F(\mathfrak{J})\neq\emptyset$, we may assume that $\{u(t): t\in G\}$ is bounded. Let $W(u)=\bigcap_{s\in G}\overline{co}\{u(t): t\succcurlyeq s\}$. Suppose that $x, y\in W(u)\cap F(\mathfrak{J})$ and $x\neq y$. Put $z=(x+y)/2$ and $r=\lim_s\|u(s)-y\|$ by Lemma 3.1. Since $z\in W(u)$, we have $\|z-y\|\leqq r$. For each $s\in G$, choose $z(s)\in[u(s), z]$ such that

$$\|z(s)-y\|=\min\{\|v-y\| : v\in[u(s), z]\}.$$

By the definition of $z(s)$, we have $\|z(s)-y\|\leqq\left\|\frac{z(s)+z}{2}-y\right\|\leqq\|z-y\|$ for all $s\in G$. Therefore, if $\liminf_s\|z(s)-y\|=\|z-y\|$, then $\{z(s)\}$ converges strongly to z. Otherwise, there exists some $\varepsilon>0$ and $s_\alpha\in G$ such that $s_\alpha\succcurlyeq\alpha$ and $\|z(s_\alpha)-z\|>\varepsilon$, for every $\alpha\in G$. Then, by the definition of δ (see (e) of (2.6)), we have

$$\left\|\frac{1}{2}(z(s_\alpha)+z)-y\right\|\leqq\|z-y\|\cdot\left[1-\delta\left(\frac{\varepsilon}{\|z-y\|}\right)\right]$$

for every α. It follows from the definition of $z(s_\alpha)$ and the uniform convexity of E that

$$\liminf_s\|z(s)-y\|\leqq\limsup_\alpha\|z(s_\alpha)-y\|$$

$$\leqq\|z-y\|\cdot[1-\delta(\varepsilon/\|z-y\|)]<\|z-y\|,$$

which contradicts the assumption. So, $\lim_s z(s)=z$. Therefore, by Lemma 3.4, we obtain $z=y$ and this contradicts $x\neq y$. To complete the proof, we suppose that

$$\liminf_s\|z(s)-y\|<\|z-y\|.$$

Then, for every $\alpha\in G$, there exist $c>0$ and $t_\alpha\in G$ with $t_\alpha\succcurlyeq\alpha$ such that

$$\|z(t_\alpha)-y\|+c<\|z-y\|$$

and there exists $\alpha_0\in G$ such that

$$r<\|u(\alpha)-y\|+\frac{c}{2},$$

for every $\alpha\succcurlyeq\alpha_0$. Put $z(t_\alpha)=a_\alpha u(t_\alpha)+(1-a_\alpha)z$ for every α. Then there is $\beta>0$ and $\gamma<1$ such that $\beta\leqq a_\alpha\leqq\gamma$ for every $\alpha\succcurlyeq\alpha_0$. In fact, if there exists a_α such that $(1-a_\alpha)M<\frac{c}{2}$, where $M\geqq\sup_{t\in G}\|u(t)-z\|$ and $M>c$, then,

$$|\,\|z(t_\alpha)-y\|-\|u(t_\alpha)-y\|\,|\leqq\|z(t_\alpha)-u(t_\alpha)\|$$

$$=(1-a_\alpha)\|u(t_\alpha)-z\|<\frac{c}{2}$$

and hence $r \leqq \|u(t_\alpha)-y\|+\frac{c}{2}<\|z(t_\alpha)-y\|+c<\|z-y\| \leqq r$. This is a contradiction. If there also exists a_α such that $a_\alpha M<c$, then

$$\big|\,\|z(t_\alpha)-y\|-\|y-z\|\,\big| \leqq \|z(t_\alpha)-z\|$$
$$=a_\alpha\|u(t_\alpha)-z\|<c$$

and hence $\|z-y\|<\|z(t_\alpha)-y\|+c<\|z-y\|$. This is a contradiction. By (2.1) with $y \in C$ and $D=\{\alpha u(t)+(1-\alpha)z : t \in G,\ 0 \leqq \alpha \leqq 1\}$, (2.2), and Lemma 3.3, there exists $s_0 \in G$ with $s_0 \succcurlyeq \alpha_0$ such that

$$\|y-S(s)v\|<\frac{c}{3}+\|y-v\|,$$

$$\|u(st)-S(s)u(t)\|<\frac{c}{3},$$

and

$$\|S(s)(\lambda u(t)+(1-\lambda)z)-(\lambda S(s)u(t)+(1-\lambda)z)\|<\frac{c}{3},$$

for all $s, t \succcurlyeq s_0$, $v \in D$ and $\lambda \in [\beta, \gamma]$. Therefore, for $s \succcurlyeq s_0$, since $t_{s_0} \succcurlyeq s_0$, it follows that

$$\|z(st_{s_0})-y\| \leqq \|a_{s_0}u(st_{s_0})+(1-a_{s_0})z-y\|$$
$$\leqq a_{s_0}\|u(st_{s_0})-S(s)u(t_{s_0})\|+\|S(s)z(t_{s_0})-(a_{s_0}S(s)u(t_{s_0})+(1-a_{s_0})z)\|$$
$$+\|S(s)z(t_{s_0})-y\|<\|z(t_{s_0})-y\|+c<\|z-y\|.$$

Let $\beta_0=s_0t_{s_0}$ and $s \succcurlyeq \beta_0$. Since G is right reversible, we have

$$\sup_{s \geq \beta_0}\|z(s)-y\|=\sup_{s \geq s_0}\|z(st_{s_0})-y\|<\|z-y\|.$$

Thus, we have $z(s) \neq z$ for all $s \succcurlyeq \beta_0$. Now let $s \succcurlyeq \beta_0$ and $u_k=k(z-z(s))+z(s)$ for all $k \geqq 1$. Then $\|u_k-y\| \geqq \|z-y\|$ for all $k \geqq 1$ and hence, by Theorem 2.5 of [2], we have

$$\langle z-u_k,\ J(y-z)\rangle=\langle (1-k)(z-z(s),\ J(y-z)\rangle \geqq 0$$

for all $k \geqq 1$, where J is the duality mapping of E and $\langle x, x^*\rangle$ denotes the value of $x^* \in X^*$ at $x \in E$. Then it follows that $\langle z-z(s), J(y-z)\rangle \leqq 0$ for all $s \succcurlyeq \beta_0$. Then, since $z(s) \in [u(s), z]$, this easily implies that $\langle z-u(s), J(y-z)\rangle \leqq 0$ for all $s \succcurlyeq \beta_0$. Immediately, we obtain $\langle z-w, J(y-z)\rangle \leqq 0$ for all $w \in \overline{co}\{u(s): s \succcurlyeq \beta_0\}$. Put $w=x=z+(z-y)$, then $z=y$. This contradicts $x \neq y$. The proof is completed.

As a direct consequence, we present the following weak convergence of an almost-orbit $\{u(t): t \in G\}$.

THEOREM 3.6. *Let G be right reversible and let E be uniformly convex with a Fréchet differentiable norm. Suppose that $u=\{u(t): t\in G\}$ is an almost-orbit of $\mathfrak{J}=\{S(t): t\in G\}$ and $F(\mathfrak{J})\neq\emptyset$. If $\omega_w(u)\subseteq F(\mathfrak{J})$, then the not $\{u(t): t\in G\}$ converges weakly to an element of $F(\mathfrak{J})$.*

Proof. Be similar to Theorem 3 of [15].

4. Ergodic theorems

We now study in this section the existence of a "ergodic" retraction of C onto the common fixed point set $F(\mathfrak{J})$ of $\mathfrak{J}=\{S(t): t\in G\}$ in C. We begin with the following observation:

THEOREM 4.1. *Let G be right reversible and let E be uniformly convex. Then, the set $F(\mathfrak{J})$ (possibly empty) is closed and convex.*

Proof. By continuity of elements of $\mathfrak{J}$, obviously $F(\mathfrak{J})$ is closed. To prove the convexity of $F(\mathfrak{J})$, it suffices to show that, for $x, y\in F(\mathfrak{J})$ with $x\neq y$, $z=\frac{1}{2}(x+y)\in F(\mathfrak{J})$. If $\lim_t S(t)z=z$, since G is right reversible, we have $S(s)z=\lim_t S(st)z=\lim_t S(t)z=z$ for each $s\in G$ and so $z\in F(\mathfrak{J})$. Hence it suffices to show that $\lim_t S(t)z=z$. If not, there exists $\varepsilon>0$ such that for any $\alpha\in G$, there is $t_\alpha\in G$ with $t_\alpha\succcurlyeq\alpha$ and

$$4\|S(t_\alpha)z-z\|\geqq\varepsilon .$$

Since E is uniformly convex, choose $d>0$ so small

$$(R+d)\Big(1-\delta\Big(\frac{\varepsilon}{R+d}\Big)\Big)<R,$$

where $R=\|x-y\|>0$ and δ is the modulus of convexity of E.

For this $d>0$, since $\mathfrak{J}$ is of w.a.n.t. with $z\in C$ and $D=\{x, y\}$, there is $\alpha_0\in G$ such that, for all $t\succcurlyeq\alpha_0$,

$$\|S(t)z-w\|=\|S(t)z-S(t)w\|\leqq\|z-w\|+\frac{d}{2} \quad \text{for all} \quad w\in D .$$

Thus, $2\|S(t)z-x\|, 2\|S(t)z-y\|\leqq R+d$ for all $t\succcurlyeq\alpha_0$. Put $u=2(S(t_{\alpha_0})z-x)$, $v=2(y-S(t_{\alpha_0})z)$. Then, $\|u\|, \|v\|\leqq R+d$ and $\|u-v\|=4\|S((t_{\alpha_0})z-z)\geqq\varepsilon$. So, we have

$$R=\Big\|\frac{u+v}{2}\Big\|\leqq(R+d)\Big(1-\delta\Big(\frac{\varepsilon}{R+d}\Big)\Big)<R,$$

which gives a contradiction. This completes the proof.

As a direct consequence, we get the following:

COROLLARY 4.2. *Let E be uniformly convex. If a mapping $T: C \to C$ is of weakly asymptotically nonexpansive type, then the fixed point set $F(T)$ of T is in fact closed and convex.*

For each $x \in C$, we define $u(t) = S(t)x$ $(t \in G)$. Then $\{u(t): t \in G\}$ is obviously an almost-orbit of $\mathfrak{J} = \{S(t): t \in G\}$. As a direct consequence, we can prove the following result which generalizes Theorem 8 in [11]. We employs the method of the proof in [11].

THEOREM 4.3. *Let G be right reversible and let E be uniformly convex with a Fréchet differentiable norm. Let $\mathfrak{J} = \{S(t): t \in G\}$ be of w.a.n.t on C. The following are equivalent:*

(i) $\bigcap_{s \in G} \overline{co}\{S(t)x : t \geqslant s\} \cap F(\mathfrak{J}) \neq \emptyset$ *for each* $x \in C$;

(ii) *there exists a retraction P of C onto $F(\mathfrak{J})$ such the $PS(t) = S(t)P = P$ for every $t \in G$ and $Px \in \overline{co}\{S(t)x : t \in G\}$ for every $x \in C$.*

Proof. By Theorem 3.5, for each $x \in C$, $\bigcap_{s \in G} \overline{co}\{S(t)x : t \geqslant s] \cap F(\mathfrak{J})$ contains exactly one point Px. Then, applying the same method of [11, Theorem 8], (i) implies (ii). The converse implication is easy.

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DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL FISHERIES UNIVERSITY OF PUSAN
PUSAN 608-737, KOREA