EXISTENCE THEORY OF OPTIMAL CONTROLS FOR DISTRIBUTED PARAMETER SYSTEMS

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Abstract

A unified existence theory of optimal controls for general semilinear evolutionary distributed parameter systems is established under the framework of mild (or weak) solutions for evolution equations. The theory applies to optimal control problems with the state equations being parabolic, hyperbolic partial differential equations and ordinary retarded differential equations. The approach also applies to problems governed by elliptic partial differential equations as well as variational inequalities.

§ 1. Introduction.

In [2], an existence theory for the optimal controls of semilinear abstract evolutionary distributed parameter systems was established under the framework of strong solutions for evolution equations (see the references cited therein also). It was asked whether one can establish a similar theory under the framwork of mild (or weak) solutions for evolution equations. In [27], the author started to investigate such a problem for time optimal control problem of semilinear distributed parameter systems. The purpose of this paper is to continue the investigation and establish a unified existence theory of optimal controls for evolutionary semilinear distributed parameter systems under the framework of mild solutions for evolution equations. Also, the same approach applies to elliptic variational inequalities.

The essence of existence theory of optimal controls is to find conditions (as weak as possible), under which some minimizing sequence of admissible pairs is convergent in the set of all admissible pairs (then the existence of an optimal pair follows from the sort of lower semi-continuity of the cost functional). This convergence usually follows from three conditions: 1) the relative com-

pactness of trajectories in some function spaces; 2) the existence of mea-
surable selections for some multifunctions; and 3) the convexity condition
on some relevant set-valued functions. In finite dimensional case, 1) and 2)
follow from some (very weak) conditions ([5, 14]). Thus, essentially, the only
condition imposed is the convexity condition, which seems to be indispensible in
general. In the case that the control domain depending on the state, the con-
vexity condition is given in terms of the so-called Cesari property (or the pro-
perty (Q) [5, 10, 14]). In infinite dimensional case, to obtain a similar theory,
the convexity condition still has to be imposed, of course. Again, in the case
that the control domain depending on the state, such a convexity condition is
imposed in terms of the Cesari property. On the other hand, some relevant
measurable selection theorems for multifunctions valued in general (separable)
metric spaces (including infinite dimensional separable Banach spaces) are still
available ([16, 26]). Thus, the only thing left is the relative compactness of the
trajectories in some functions spaces. This issue should be treated very carefully
due to the infinite dimensionality nature of the space. It is very important to
know that in using the Cesari property, we need the strong convergence of the
trajectories. To archieve this, we have to impose sort of compactness conditions,
one way or another. It is important that these conditions should be satisfied
by the problems governed by usual concrete distributed parameter systems, say
elliptic, parabolic, hyperbolic partial differential equations and retarded ordinary
differential equations, etc. It turns out that our approach not only covers the
systems governed by mentioned equations, but also extends to some variational
inequalities.

   We should note that the problem studied in [6] basically is of elliptic partial
differential system type because the highest order spatial derivative is solved
explicitly. Also, the solution to the state equation was understood as a strong
solution. It seems to us that there may be certain overlaps of our results with
those in [12]. But, for the evolution problems, we use the semigroup and mild
solution approach which is different from those in [12]; and for stationary
problem, we discuss variational inequalities which was not mentioned in [12].
Recently, in [19, 21], also the strong solution was adopted in studying the
similar problem for nonlinear evolution systems. For some other relevant results,
we refer the readers to [1, 4, 10, 11, 13, 15, 17, 18, 20, 24, 26].

§ 2. Evolution Systems with Compact Semigroups.

   In this section, we establish the existence of optimal controls for semilinear
evolutionary distributed parameter systems in which the semigroup involved is
compact. This is the case for systems governed by semilinear parabolic partial
differential equations with proper boudary conditions or by ratarded ordinary
differential equations.

§ 2.1. Statement of the Control Problem.

Let us start with the following assumptions.

(P1) X is a reflexive Banach space, U is a Polish space ([8]) with metric d, and $T>0$ is a constant.

(P2) Operator A: $\mathcal{D}(A) \subset X \to X$ generates a compact semigroup e^{At} on X.

(P3) Mapping f: $[0, T] \times X \times U \to X$ is Borel measurable in (t, x, u) and continuous in (x, u) for almost all $t \in [0, T]$; Mapping f^0: $[0, T] \times X \times U \to \mathbf{R}$ is Borel measurable in (t, x, u), lower semicontinuous in (x, u) and there exists a constant $K \geq 0$, such that

$$(2.1) \qquad f^0(t, x, u) \geq -K, \qquad \forall (t, x, u) \in [0, T] \times X \times U.$$

(P4) Multifunction Γ: $[0, T] \times X \to 2^U$ is pseuedo-continuous (see Appendix for definitions and relevant results about multifunctions).

(P5) $Q \subset X$ is closed and $\Omega \subset X \times X$ is bounded and weakly closed.

Next, we let

$$\mathcal{U}[0, T] = \{u : [0, T] \to U \mid u(\cdot) \text{ is measurable}\}.$$

Any element in $\mathcal{U}[0, T]$ is called a control (on $[0, T]$). The evolution system we are interested in is the following

$$(2.2) \qquad \dot{x}(t) = Ax(t) + f(t, x(t), u(t)), \qquad t \in [0, T].$$

As usual, a (mild) solution $x(\cdot)$ of (2.2) is defined as a solution of the following integral equation:

$$(2.3) \qquad x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} f(\tau, x(\tau), u(\tau)) d\tau, \qquad t \in [0, T].$$

Any solution $x(\cdot)$ of (2.3) is referred as a trajectory of the evolution system corresponding to initial state $x(0)$ and control $u(\cdot)$.

DEFINITION 2.1. A pair $(x(\cdot), u(\cdot))$ is said to be semi-admissible if (2.3) is satisfied and

$$(2.4) \qquad u(t) \in \Gamma(t, x(t)), \qquad \text{a.e.} \quad t \in [0, T].$$

The pair is said to be admissible if it is semi-admissible and

$$(2.5) \qquad (x(0), x(T)) \in \Omega,$$

$$(2.6) \qquad x(t) \in Q \qquad t \in [0, T],$$

$$(2.7) \qquad f^0(\cdot, x(\cdot), u(\cdot)) \in L^1(0, T).$$

In the case $(x(\cdot), u(\cdot))$ is semi-admissible (admissible, resp.), we refer $x(\cdot), u(\cdot)$ and $(x(\cdot), u(\cdot))$ semi-admissible (admissible) trajectory, control and pair, respectively. Hereafter, we let

$$\mathcal{A}_s = \{(x(\cdot), u(\cdot)) \in C([0, T]; X) \times \mathcal{U}[0, T] | (x(\cdot), u(\cdot)) \text{ is semi-admissible}\},$$

$$\mathcal{X}_s = \{x(\cdot) \in C([0, T]; X) | \exists u(\cdot) \in \mathcal{U}[0, T], \text{ such that } (x(\cdot), u(\cdot)) \in \mathcal{A}_s\},$$

$$\mathcal{U}_s = \{u(\cdot)) \in \mathcal{U}[0, T] | \exists x(\cdot) \in C([0, T]; X), \text{ such that } (x(\cdot), u(\cdot)) \in \mathcal{A}_s\}.$$

We define $\mathcal{A}_{ad}$, $\mathcal{X}_{ad}$ and $\mathcal{U}_{ad}$ similar to $\mathcal{A}_s$, $\mathcal{X}_s$ and $\mathcal{U}_s$ replacing semi-admissibility by admissibility. Next, we introduce the cost functional to be of the following Lagrange form:

$$(2.8) \qquad J(x(\cdot), u(\cdot)) = \int_0^T f^0(t, x(t), u(t)) dt, \qquad \forall (x(\cdot), u(\cdot)) \in \mathcal{A}_{ad}.$$

Then, our optimal control problem can be stated as follows.

PROBLEM P.   Find $(\bar{x}(\cdot), \bar{u}(\cdot)) \in \mathcal{A}_{ad}$, such that

$$(2.9) \qquad J(\bar{x}(\cdot), \bar{u}(\cdot)) = \min_{(x(\cdot), u(\cdot)) \in \mathcal{A}_{ad}} J(x(\cdot), u(\cdot)).$$

If such a pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ exists, we call $\bar{x}(\cdot)$, $\bar{u}(\cdot)$ and $(\bar{x}(\cdot), \bar{u}(\cdot))$ an optimal trajectory, control and pair, respectively.

For simplicity, in this paper, we only consider the fixed duration problem. Our approach also applies to the case with non-fixed duration problem (see [27] for time optimal control problem). We should note that unlike in the finite dimensional case, the non-fixed duration problem can not be simply transformed to a fixed duration one (as in [5]). Also, our approach applies to problems with Mayer or Bolza type cost functionals.

§ 2.2. Cesari Property.
Let us recall the following.

DEFINITION 2.2.   ([5, 14]) Let Y be a Banach space and Z be a metric space. Let $\Lambda: Z \to 2^Y$ be a multifunction. We say Λ possesses the Cesari property at $z_0 \in Z$, if

$$(2.10) \qquad \bigcap_{\delta > 0} \overline{co} \Lambda(\mathcal{N}_\delta(z_0)) = \Lambda(z_0),$$

where $\overline{co}D$ stands for the closed convex hull of the set D and $\mathcal{N}_\delta(z_0)$ is the δ-open neighborhood of the point z_0. If Λ has the Cesari property at every point $z \in Z$, we simply say that Λ has the Cesari property on Z.

PROPOSITION 2.3.   *Let Y be a Banach space and Z be a metric space. Let $\Lambda: Z \to 2^Y$ be upper semicontinuous, convex and closed valued. Then, Λ has the Cesari property on Z.*

Proof. Let $z_0 \in Z$ be fixed. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$(2.11) \qquad \Lambda(\mathcal{N}_\delta(z_0)) \subset \mathcal{N}_\varepsilon(\Lambda(z_0)).$$

Since $\Lambda(z_0)$ is convex, so is $\mathfrak{N}_\varepsilon(\Lambda(z_0))$. Thus,

$$(2.12) \qquad \overline{co}\,\Lambda(\mathfrak{N}_\delta(z_0)) \subset \overline{\mathfrak{N}_\varepsilon(\Lambda(z_0))}.$$

then, it follows that

$$(2.13) \qquad \bigcap_{\delta>0} \overline{co}\,\Lambda(\mathfrak{N}_\delta(z_0)) \subset \bigcap_{\varepsilon>0} \overline{\mathfrak{N}_\varepsilon(\Lambda(z_0))} = \overline{\Lambda(z_0)} = \Lambda(z_0),$$

which proves our proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now, let us introduce the following set

$$(2.14) \qquad \mathcal{E}(t,\,x)=\{(y^0,\,y)\in \boldsymbol{R}\times X \mid y^0 \geqq f^0(t,\,x,\,u),\ y=f(t,\,x,\,u),\ u\in \Gamma(t,\,x)\},$$
$$\forall(t,\,x)\in[0,\,T]\times X.$$

In the proof of the existence of optimal pairs for Problem P, the following hypothese will play a cruicial role.

(P6) For almost all $t\in[0,\,T]$, the map $\mathcal{E}(t,\,\cdot): X\to 2^{\boldsymbol{R}\times X}$ has the Cesari property on Q.

Following result gives a sufficient condition ensuring (P6).

PROPOSITION 2.4. *Let the following hold ·*

(P3′) *For almost all $t\in[0,\,T]$, the map $f(t,\,\cdot,\,\cdot)$ is continuous uniformly in $u\in U$ and $f^0(t,\,\cdot,\,\cdot)$ is lower semicontinuous uniformly in $u\in U$, i.e., for any given $x\in X$ and any $\varepsilon>0$, there exists a $\sigma=\sigma(x)>0$, such that whenever $x'\in \mathfrak{N}_\sigma(x)$ and $d(u,\,u')<\sigma$, one has*

$$(2.15) \qquad \begin{cases} |f(t,\,x',\,u')-f(t,\,x,\,u)|<\varepsilon, \\ f^0(t,\,x',\,u')>f^0(t,\,x,\,u)-\varepsilon. \end{cases}$$

(P4′) *For almost all $t\in[0,\,T]$, the map $\Gamma(t,\,\cdot): X\to 2^U$ is upper semicontinuous on X.*

Then, for almost all $t\in[0,\,T]$, $\mathcal{E}(t,\,\cdot)$ has the Cesari property on X if and only if for almost all $t\in[0,\,T]$, the set $\mathcal{E}(t,\,x)$ is convex and closed.

Remark 2.5. The above proposition looks very similar to the result of [5, pp. 72-74] for finite dimensional situation. However, here, we are in infinite dimensional space and we have not assumed any compactness! In some sense, our result can be regarded as an improvement of the one just mentioned. We will see that the proof given below is different from that given in [5].

Proof of Proposition 2.4. Let $t\in[0,\,T]$ be given such that (P3′) and (P4′) hold at this t and let $x\in X$ be fixed. By (P3′), for any $\varepsilon>0$, there exists a $\sigma=\sigma(x)>0$, such that whenever $x'\in \mathfrak{N}_\sigma(x)$ and $d(u,\,u')<\sigma$, (2.15) holds. Next, by (P4′), we can find $0<\delta=\delta(x)<\sigma$, such that

(2.16) $\Gamma(t, \mathfrak{N}_\delta(x))\subset\mathfrak{N}_\sigma(\Gamma(t, x))$.

Then, for any $(y_0^\delta, y^\delta)\in\mathcal{E}(t, \mathfrak{N}_\delta(x))$, there exist an $x^\delta\in\mathfrak{N}_\delta(x)$ and a $u^\delta\in\Gamma(t, x^\delta)$ $\subset\mathfrak{N}_\sigma(\Gamma(t, x))$, such that

(2.17) $\begin{cases} y_0^\delta\geqq f^0(t, x^\delta, u^\delta), \\ y^\delta=f(t, x^\delta, u^\delta). \end{cases}$

Then, it is clear that there exists a $u\in\Gamma(t, x)$ with $d(u^\delta, u)<\sigma$. Thus, note the fact that $\delta\leqq\sigma$, we see from (2.15) and (2.17) that

(2.18) $\begin{cases} y_0^\delta\geqq f^0(t, x^\delta, u^\delta)>f^0(t, x, u)-\varepsilon, \\ |y^\delta-f(t, x, u)|=|f(t, x^\delta, u^\delta)-f(t, x, u)|<\varepsilon. \end{cases}$

This means

(2.19) $(y_0^\delta, y^\delta)\in\mathfrak{N}_\varepsilon(\mathcal{E}(t, x))$.

Hence, the set $\mathcal{E}(t, \cdot)$ is upper semicontinuous. Then our conclusion follows from Proposition 2.3. $\square$

In the above result, we see that (P3′) and (P4′) are not very restrictive. Thus, (P6) essentially says that $\mathcal{E}(t, x)$ is convex and closed. Following result gives a sufficient condition for $\mathcal{E}(t, x)$ being convex.

PROPOSITION 2.6. *Let $(t, x)\in[0, T]\times X$ be fixed. Let $f(t, x, \Gamma(t, x))$ be convex and there exists a convex function $\varphi(\cdot; t, x): X\rightarrow R$, such that*

(2.20) $f^0(t, x, u)=\varphi(f(t, x, u); t, x), \qquad \forall u\in\Gamma(t, x)$.

Then, $\mathcal{E}(t, x)$ is convex.

Proof. Let $(y_i^0, y_i)\in\mathcal{E}(t, x)$, $i=1, 2$, and $\lambda\in(0, 1)$. We can find $u_1, u_2\in$ $\Gamma(t, x)$, such that

(2.21) $\begin{cases} y_i^0\geqq f^0(t, x, u_i), \\ y_i=f(t, x, u_i), \end{cases} \quad i=1, 2$.

By the convexity of $f(t, x, \Gamma(t, x))$, there exists a $u_3\in\Gamma(t, x)$, such that

$\lambda y_1+(1-\lambda)y_2\equiv\lambda f(t, x, u_1)+(1-\lambda)f(t, x, u_2)=f(t, x, u_3)$.

By (2.20), we have

$\lambda y_1^0+(1-\lambda)y_2^0\geqq\lambda f^0(t, x, u_1)+(1-\lambda)f^0(t, x, u_2)$

$=\lambda\varphi(f(t, x, u_1); t, x)+(1-\lambda)\varphi(f(t, x, u_2); t, x)$

$$\geqq \varphi(\lambda f(t, x, u_1) + (1-\lambda)f(t, x, u_2); t, x)$$

$$= \varphi(f(t, x, u_3); t, x)$$

$$= f^0(t, x, u_3).$$

Hence, $\mathcal{E}(t, x)$ is convex. □

It is easy to check that if $f(t, x, u)$ is linear in u, $f^0(t, x, u)$ is convex in u and $\Gamma(t, x)$ is convex, then $\mathcal{E}(t, x)$ is convex. The above result gives some other situations which guaranteeing the convexity of $\mathcal{E}(t, x)$. By some simple observation, we can see that in order the set $\mathcal{E}(t, x)$ to be convex, it is necessary that $f(t, x, \Gamma(t, x))$ is convex and the functions $f(t, x, \cdot)$, $f^0(t, x, \cdot)$ and the set $\Gamma(t, x)$ should be compatible in some sense. Condition (2.20) is one of such compatibility conditions.

§ 2.3. Properties of Compact Semigroups.

In this subsection, we present a few results for compact C_0-semigroups. First of all, we have

LEMMA 2.7. ([22]) *Let* e^{At} *be a compact* C_0-*semigroup on Banach space* X. *Then,* $t \mapsto e^{At}$ *is continuous in the operator norm in* $(0, \infty)$.

Followig result will play an important role in sequel.

LEMMA 2.8. *Let* e^{At} *be a compact* C_0-*semigroup on some Banach space* X. *Let* $p > 1$ *and define*

$$(2.22) \qquad \mathcal{S}(g(\cdot)) = \int_0^{\cdot} e^{A(\cdot - \tau)} g(\tau) d\tau, \qquad \forall g(\cdot) \in L^p(0, T; X).$$

Then, $\mathcal{S}: L^p(0, T; X) \to C([0, T]; X)$ *is compact.*

It seems to us that such a result should be known. But we could not find an exact reference. For reader's convenience, we present a proof in the following.

Proof of Lemma 2.6. Let $g_k(\cdot) \in L^p(0, T; X)$ with

$$(2.23) \qquad \|g_k(\cdot)\|_{L^p(0, T; X)} \leqq 1, \qquad \forall k \geqq 1.$$

We need to prove that $\{\mathcal{S}(g_k(\cdot))\}_{k \geq 1}$ is relatively compact in $C([0, T]; X)$. To this end, we first prove that for each $t \in [0, T]$, the set $\{\mathcal{S}(g_k(\cdot))(t)\}_{k \geq 1}$ is relatively compact in X. In fact, the case $t = 0$ is trivial. We let $t \in (0, T]$. Then, for any $\varepsilon > 0$, there exists a $\delta \in (0, t]$, such that

$$(2.24) \qquad \left| \int_{t-\delta}^t e^{A(t-\tau)} g_k(\tau) d\tau \right| < \frac{\varepsilon}{2}, \qquad \forall k \geqq 1.$$

Next, we let

$$y_k \equiv \int_0^{t-\delta} e^{A(t-\delta-\tau)} g_k(\tau) d\tau, \qquad \forall k \geq 1.$$

Then, we see that the set $\{y_k\}_{k \geq 1}$ is bounded in X. Thus by the compactness of $e^{A\delta}$, we can find a finite set $\{z_i, 1 \leq i \leq m\}$ in X, such that

(2.25) $$\{e^{A\delta} y_k\}_{k \geq 1} \subset \bigcup_{i=1}^m \mathcal{N}_{\varepsilon/2}(z_i).$$

Then, we have

(2.26) $$\{\mathcal{S}(g_k(\cdot))(t)\}_{k \geq 1} \subset \bigcup_{i=1}^m \mathcal{N}_\varepsilon(z_i).$$

Hence, for each $t \in [0, T]$, $\{\mathcal{S}(g_k(\cdot))(t)\}_{k \geq 1}$ is relatively compact in X. Next, we show that $\{\mathcal{S}(g_k(\cdot))\}_{k \geq 1}$ is equicontinuous on $[0, T]$. In fact, for $t' > t > 0$ and $0 < \delta \leq t$, we have

$$\mathcal{S}(g_k(\cdot))(t') - \mathcal{S}(g_k(\cdot))(t) = \int_t^{t'} e^{A(t'-\tau)} g_k(\tau) d\tau$$

$$+ \int_0^{t-\delta} (e^{A(t'-\tau)} - e^{A(t-\tau)}) g_k(\tau) d\tau$$

$$+ \int_{t-\delta}^t (e^{A(t'-\tau)} - e^{A(t-\tau)}) g_k(\tau) d\tau$$

$$\equiv I_1 + I_2 + I_3.$$

By some direct estimation, we can find some constant C, independent of k, such that

$$|I_1| \leq C |t'-t|^{p/p-1},$$

$$|I_2| \leq C \left(\int_\delta^t \|e^{A(t'-t+\tau)} - e^{A\tau}\|^q d\tau \right)^{1/q},$$

$$|I_3| \leq C \delta^{1/q},$$

with $1/p + 1/q = 1$. Thus, by Lemma 2.7, we obtain the equicontinuity of the set $\{\mathcal{S}(g_k(\cdot))\}$ on $[0, T]$. Then by Arzela-Ascoli Theorem ([28]), we obtain the compactness of the operator $\mathcal{S}$.   □

   COROLLARY 2.9.   *Let e^{At} be a compact C_0-semigroup on some Banach space X and $p > 1$. Let $g_k(\cdot) \in L^p(0, T; X)$ satisfy*

(2.27) $$g_k(\cdot) \overset{w}{\longrightarrow} g(\cdot), \quad in \quad L^p(0, T; X).$$

Then,

(2.28) $$\lim_{k \to \infty} \sup_{0 \leq t \leq T} \left\| \int_0^t e^{A(t-\tau)} (g_k(\tau) - g(\tau)) d\tau \right\| = 0.$$

§ 2.4. Existence of Semi-admissible Pairs.

In this subsetion, we study the existence of semi-admissible pairs defined as in Definition 2.1. To this end, we first introduce the following:

(P3″) The map $f\colon [0, T] \times X \times U \to X$ is Borel measurable in (t, x, u), continuous in (x, u) and for some constant $L > 0$

$$(2.29) \quad \begin{cases} |f(t, x, u) - f(t, \hat{x}, u)| \leq L|x - \hat{x}|, & \forall t \in [0, \infty),\ x,\ \hat{x} \in X,\ u \in U, \\ |f(t, x, u)| \leq L(1 + |x|), & \forall t \in [0, \infty),\ x \in X,\ u \in U. \end{cases}$$

(P6′) For almost all $t \in [0, \infty)$, the set $f(t, x, \Gamma(t, x))$ satisfies the following:

$$(2.30) \quad \bigcap_{\delta > 0} \overline{co}\, f(t, \mathfrak{N}_\delta(x), \Gamma(\mathfrak{N}_\delta(t, x))) = f(t, x, \Gamma(t, x)).$$

(P6″) Let $\Lambda(t, x) = f(t, x, \Gamma(t, x))$. For almost all $t \in [0, T]$, the map $\Lambda(t, \cdot)\colon X \to 2^X$ has the Cesari property.

It is clear that (P6′) implies (P6″). In fact, one has

$$(2.31) \quad \Lambda(t, \mathfrak{N}_\delta(x)) \subseteq f(t, \mathfrak{N}_\delta(x), \Gamma(\mathfrak{N}_\delta(t, x))).$$

Furthermore, we have the following

PROPOSITION 2.10. *Let $\Gamma\colon [0, T] \times X \to 2^U$ be upper semicontinuous and closed set valued (see Appendix), $f(t, x, u)$ be uniformly continuous in (x, u) for any $t \in [0, T]$. Then, the following are equivalent*
(i) *(P6′) holds.*
(ii) *(P6″) holds.*
(iii) *$f(t, x, \Gamma(t, x))$ is closed and convex.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are immediate.
(iii) $\Rightarrow$ (i). By the uniform continuity of $f(t, x, u)$ in (x, u), for any $\varepsilon > 0$, there exists a $\sigma > 0$, such that

$$(2.32) \quad f(t, \mathfrak{N}_\sigma(x), \mathfrak{N}_\sigma(\Gamma(t, x))) \subset \mathfrak{N}_\varepsilon(f(t, x, \Gamma(t, x))).$$

On the other hand, by the upper semicontinuity of Γ, we can find a $\delta \in (0, \sigma]$, such that

$$(2.33) \quad \Gamma(\mathfrak{N}_\delta(t, x)) \subset \mathfrak{N}_\sigma(\Gamma(t, x)).$$

Thus, (i) follows from the convexity and closedness of $f(t, x, \Gamma(t, x))$. $\quad \square$

Our main result of this subsection is the following:

THEOREM 2.11. *Let (P1), (P2), (P3′), (P4) and (P6′) hold. Then, for any $y \in X$,*

$$(2.34) \quad \mathcal{A}_s([0, T]\,;\ y) \equiv \{(x(\cdot),\ u(\cdot)) \in \mathcal{A}_s \mid x(0) = y\} \neq \varnothing.$$

Moreover, $\mathscr{X}_s([0, T]; y) \equiv \{x(\cdot) \in \mathscr{X}_s \mid x(0) = y\}$ *is compact in* $C([0, T]; X)$.

Proof. For any $k \geq 1$, Let

$$t_j = \frac{j}{k} T, \qquad 0 \leq j \leq k-1.$$

We set

$$(2.35) \qquad u_k(t) = \sum_{j=0}^{k-1} u^j \chi_{[t_j, t_{j+1})}(t), \qquad t \in [0, T].$$

Here, u^j's are constructed as follows: First, we take

$$u^0 \in \Gamma(0, y).$$

By (2.29), we know that there exists a unique $x_k(\cdot)$ satisfies

$$x_k(t) = e^{At}y + \int_0^t e^{A(t-\tau)} f(\tau, x_k(\tau), u^0)d\tau, \qquad \forall t \in [t_0, t_1].$$

Then, we take

$$u^1 \in \Gamma(t_1, x_k(t_1)).$$

We can continue the above procedure to obtain $x_k(\cdot)$ on $[t_1, t_2]$, etc. By induction, we end up with the following:

$$(2.36) \qquad \begin{cases} x_k(t) = e^{At}y + \int_0^t e^{A(t-\tau)} f(\tau, x_k(\tau), u_k(\tau))d\tau, \qquad \forall t \in [0, T], \\ u_k(t) \in \Gamma(t_j, x_k(t_j)), \qquad t \in [t_j, t_{j+1}), \qquad 0 \leq j \leq k-1. \end{cases}$$

By Gronwall's inequality and (2.29), we see that for some absolute constant $C > 0$.

$$(2.37) \qquad \begin{cases} |x_k(t)| \leq C, \qquad \forall t \in [0, T], \\ |f(t, x_k(t), u_k(t))| \leq C, \qquad \text{a.e. } t \in [0, T], \end{cases} \qquad k \geq 1.$$

By Lemma 2.8, we know that $\{x_k(\cdot)\}_{k \geq 1}$ is relatively compact in $C([0, T]; X)$. Then we may assume

$$(2.38) \qquad x_k(\cdot) \xrightarrow{s} \bar{x}(\cdot), \qquad \text{in } C([0, T]; X),$$

for some $\bar{x}(\cdot) \in C([0, T]; X)$. Also, we may let

$$(2.39) \qquad f(\cdot, x_k(\cdot), u_k(\cdot)) \xrightarrow{*} \bar{f}(\cdot), \qquad \text{in } L^\infty(0, T; X),$$

for some $\bar{f}(\cdot) \in L^\infty(0, T; X)$. By the compactness of the operator $\mathcal{S}$, we have

$$(2.40) \qquad \bar{x}(t) = e^{At}y + \int_0^t e^{A(t-\tau)} \bar{f}(\tau)d\tau, \qquad t \in [0, T].$$

By (2.38), for any $\delta > 0$, there exists a k_0, such that

(2.41) $\qquad\qquad x_k(t) \in \mathfrak{N}_\delta(\bar{x}(t)), \qquad \forall t \in [0, T], \; k \geq k_0.$

On the other hand, by the definition of $u_k(\cdot)$, for k large, one has

(2.42) $\quad u_k(t) \in \Gamma(t_j, x_k(t_j)) \subset \Gamma(\mathfrak{N}_\delta(t, \bar{x}(t))), \qquad \forall t \in [t_j, t_{j+1}), \; 0 \leq j \leq k-1.$

Next, by (2.39) and Mazur Theorem, we may let $\alpha_{ij} \geq 0$, $\sum_{i \geq 1} \alpha_{ij} = 1$, such that for some $p < \infty$,

(2.43) $\quad \phi_j(\cdot) \equiv \sum \alpha_{ij} f(\cdot, x_{i+j}(\cdot), u_{i+j}(\cdot)) \xrightarrow{s} \bar{f}(\cdot), \qquad \text{in} \quad L^p(0, T, X).$

Then, we can assume

(2.44) $\qquad\qquad \phi_j(t) \xrightarrow{s} \bar{f}(t), \qquad \text{in } X, \quad \text{a.e. } t \in [0, T].$

On the other hand, by (2.41) and (2.42), we see that for j large, one has

(2.45) $\qquad \phi_j(t) \in co\, f(t, \mathfrak{N}_\delta(\bar{x}(t)), \Gamma(\mathfrak{N}_\delta(t, \bar{x}(t)))), \qquad \text{a.e. } t \in [0, T].$

Thus, for any $\delta > 0$, we have

(2.46) $\qquad \bar{f}(t) \in \overline{co}\, f(t, \mathfrak{N}_\delta(\bar{x}(t)), \Gamma(\mathfrak{N}_\delta(t, \bar{x}(t)))), \qquad \text{a.e. } t \in [0, T].$

By (P6′), we get

(2.47) $\qquad\qquad \bar{f}(t) \in f(t, \bar{x}(t), \Gamma(t, \bar{x}(t))), \qquad \text{a.e. } t \in [0, T].$

Then, by Theorems A.3 and A.4 of the Appendix, we know that there exists a $\bar{u}(\cdot) \in \mathcal{U}[0, T]$, such that

(2.48) $\qquad \begin{cases} \bar{u}(t) \in \Gamma(t, \bar{x}(t)), & \text{a.e. } t \in [0, T], \\ \bar{f}(t) = f(t, \bar{x}(t), \bar{u}(t)), & \text{a.e. } t \in [0, T]. \end{cases}$

Combining (2.40) and (2.48), we see that

$$(\bar{x}(\cdot), \bar{u}(\cdot)) \in \mathcal{A}_s([0, T]; y).$$

Thus, (2.34) follows. Finally, let $\{x^k(\cdot)\}_{k \geq 1} \subset \mathcal{X}_s([0, T]; y)$ with

$$\|x^k(\cdot)\|_{C([0,T];X)} \leq C, \qquad \forall k \geq 1.$$

Then, by the above proof, we see that $\{x^k(\cdot)\}_{k \geq 1}$ is relatively compact in $C([0, T]; X)$. Moreover, if for some subsequence (still denoted by itself), one has

$$x^k(\cdot) \xrightarrow{s} \hat{x}(\cdot), \qquad \text{in } C([0, T]; X),$$

then, by (P6′), we must have $\hat{x}(\cdot) \in \mathcal{X}_s([0, T]; y)$. Thus, $\mathcal{X}_s([0, T]; y)$ is compact in $C([0, T]; X)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We should note that in the case $\Gamma(t, x) \equiv \Gamma(t)$, $\forall x \in X$, Theorem 2.11 holds under much simpler conditions and the proof becomes much simpler.

§ 2.5. Existence of Optimal Pairs.

In this subsetion, we present the existence of optimal pairs for Problem (P).

THEOREM 2.12. *Let* (P1)-(P6) *hold. Let there exist a sequence of minimizing sequence* $\{x_k(\cdot), u_k(\cdot)\} \in \mathcal{A}_{ad}$, *such that for some* $p > 1$ *and* $C > 0$,

$$(2.49) \qquad \|f(., x_k(\cdot), u_k(\cdot))\|_{L^p(0, T; X)} \leq C, \qquad \forall k \geq 1.$$

Then, Problem (P) *admits at least one optimal pair.*

Proof. Without loss of generality, we may assume

$$(2.50) \qquad f(\cdot, x_k(\cdot), u_k(\cdot)) \xrightarrow{w} \bar{f}(\cdot), \qquad in \quad L^p(0, T; X),$$

for some $\bar{f}(\cdot) \in L^p(0, T; X)$. Then, by Corollary 2.9, we obtain

$$(2.51) \qquad \int_0^t e^{A(t-\tau)} f(\tau, x_k(\tau), u_k(\tau)) d\tau \xrightarrow{s} \int_0^t e^{A(t-\tau)} \bar{f}(\tau) d\tau,$$

$$\text{uniformly in } t \in [0, T].$$

Also, by the boundedness of Ω, refelxivity of X and the compactness of $e^{At}(t > 0)$, we may assume

$$(2.52) \qquad \begin{cases} x_k(0) \xrightarrow{w} x_0, \\ e^{At} x_k(0) \xrightarrow{s} e^{At} x_0, \qquad \forall t \in (0, T]. \end{cases}$$

Let

$$(2.53) \qquad \bar{x}(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} \bar{f}(\tau) d\tau, \qquad t \in [0, T].$$

Then, by the above, we have

$$(2.54) \qquad \lim_{k \to \infty} \|x_k(t) - \bar{x}(t)\| = 0, \qquad t \in (0, T].$$

Then, by (2.50) and Mazur Theorem, we may let $\alpha_{ij} \geq 0$, $\sum_{i \geq 1} \alpha_{ij} = 1$, such that for some $p < \infty$,

$$(2.55) \qquad \phi_j(\cdot) \equiv \sum_{i \geq 1} \alpha_{ij} f(\cdot, x_{i+j}(\cdot), u_{i+j}(\cdot)) \xrightarrow{s} \bar{f}(\cdot), \qquad in \quad L^p(0, T; X).$$

We set

$$(2.56) \qquad \phi_j^0(\cdot) \equiv \sum_{i \geq 1} \alpha_{ij} f^0(\cdot, x_{i+j}(\cdot), u_{i+j}(\cdot)),$$

and set (note (2.1))

(2.57) $$\bar{f}^0(t)=\varliminf_{j\to\infty}\phi_j^0(t)\geqq -K, \quad \text{a.e.} \quad t\in[0, T].$$

Similar to [5], we have

(2.58) $$(\bar{f}^0(t), \bar{f}(t))\in\mathscr{E}(t, \bar{x}(t)), \quad \text{a.e.} \quad t\in[0, T],$$

and (by Fatou's Lemma)

(2.59) $$\int_0^T \bar{f}^0(t)dt\leqq\varliminf_{k\to\infty} J(x_k(\cdot), u_k(\cdot))=\inf_{(x(\cdot),\ u(\cdot))\in\mathscr{A}_{ad}} J(x(\cdot), u(\cdot)).$$

Then, by Theorems A.3 and A.4 of the Appendix, we obtain $\bar{u}(\cdot)$, a measurable selection of $\Gamma(\cdot, \bar{x}(\cdot))$, such that

(2.60) $$\begin{cases} \bar{f}^0(t)\geqq f^0(t, \bar{x}(t), \bar{u}(t)), \\ \bar{f}(t)=f(t, \bar{x}(t), \bar{u}(t)), \end{cases} \quad \text{a.e.} \quad t\in[0, T].$$

Thus, noting (P5), we can easily see that $(\bar{x}(\cdot), \bar{u}(\cdot))\in\mathscr{A}_{ad}$, and

(2.61) $$J(\bar{x}(\cdot), \bar{u}(\cdot))\leqq\int_0^T \bar{f}^0(t)dt\leqq\inf_{(x(\cdot),\ u(\cdot))\in\mathscr{A}_{ad}} J(x(\cdot), u(\cdot)).$$

This means that $(\bar{x}(\cdot), \bar{u}(\cdot))$ is an optimal pair. $\square$

We should note that (2.49) is a very weak condition. This condition holds if (P3″) holds.

Remark 2.13. For the case $\Gamma(t, x)\equiv\Gamma(t), \forall x\in X$, we can relax the Cesari condition somehow (similar to [5]).

Remark 2.14. If the semigroup e^{At} is compact and analytic, then, the map f can be more general. For example, in the case that A is the Laplacian in some bounded domain in $\boldsymbol{R}^n$ with suitable boundary conditions, then, the non-linear term f is allowed to contain the first order spatial derivatives of the state. Of course, the assumptions ensuring the above results should be changed properly.

It is very easy to present concrete examples covered by the above theory. We prefer not to give these details here.

§ 3. Second Order Evolution Systems.

In this section, we will discuss the optimal control problems with the systems governed by following evolution equation:

(3.1) $$\ddot{x}(t)+Ax(t)=f(t, x(t), u(t)), \quad t\in[0, T],$$

with some symmetric operator A. The motivation is the controlled wave or beam equations. It is known that we may transfer the above into a first order

evolution system. In fact, by setting (formally)

$$z=\begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \qquad \mathcal{A}=\begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \qquad F(t,\,x,\,u)=\begin{pmatrix} 0 \\ f(t,\,x,\,u) \end{pmatrix},$$

we have (formally again)

$$(3.2) \qquad\qquad \dot{z}(t)=\mathcal{A}z(t)+F(t,\,z(t),\,u(t)).$$

Then, it is standard that $\mathcal{A}$ generates a C_0-group (*instead of a C_0-semigroup!*) $e^{\mathcal{A}t}$ in the underline space. Thus, $e^{\mathcal{A}t}$ can not be compact! Then, the theory established in the previous section does not apply to the situation we have here. We note that in [2], the above type second order evolution controlled systems was not discussed.

§ 3.1. Some Preliminary results on Second Order Evolution Systems.

Let us start with some basic assumptions.

(W1) V and H are separable Hilbert spaces with norms $\|\cdot\|$ and $|\cdot|$, respectively, and with duals V' and $H'\equiv H$, respectively. The embedding $V\hookrightarrow H$ is dense and compact. The duality pairing between V and V' is $\langle\cdot,\,\cdot\rangle$ and the inner product of H is $(\cdot,\,\cdot)$. The norm of V' is denoted by $\|\cdot\|_*$. U is a Polish space ([8]) and T is a positive constant.

(W2) $A\in\mathcal{L}(V,\,V')$ is symmetric and coercive, i.e.,

$$(3.3) \qquad\qquad \langle Ax,\,y\rangle=\langle x,\,Ay\rangle, \qquad \forall x,\,y\in V,$$

and for some $\alpha>0$,

$$(3.4) \qquad\qquad \langle Ax,\,x\rangle\geqq\alpha\|x\|^2, \qquad \forall x\in V.$$

From (W1), we know that $V\hookrightarrow H=H'\hookrightarrow V'$. The space H is usually referred as the pivot space. Now, for $g(\cdot)\in L^2(0,\,T;\,V')$ and $(x_0,\,x_1)\in V\times H$, we consider the following evolution equation

$$(3.5) \qquad\qquad \ddot{x}(t)+Ax(t)=g(t), \qquad \text{a.e.}\;\; t\in[0,\,T],\;\text{in}\;V',$$

with the initial conditions

$$(3.6) \qquad\qquad x(0)=x_0, \qquad \text{in}\;H,$$

$$(3.7) \qquad\qquad \dot{x}(0)=x_1, \qquad \text{in}\;V'.$$

We note that (3.6) and (3.7) stand for the following, respectively:

$$\lim_{t\to 0}|x(t)-x_0|=0, \qquad \lim_{t\to 0}\|\dot{x}(t)-x_1\|_*=0.$$

Before introducing the definition of a solution to (3.5)–(3.7), we first give the following result.

PROPOSITION 3.1. *The following are equivalent·*

(i) *Function* $x(\cdot)\in L^2(0, T ; V)\cap W^{1,2}([0, T]; H)\cap W^{2,2}([0, T]; V')$ *satisfies* (3.5)-(3.7).

(ii) *Function* $x(\cdot)\in L^2(0, T ; V)\cap W^{1,2}([0, T]; H)$ *satisfies* (3.6) *and*

$$(3.8) \quad \int_0^T \langle g(t), v(t)\rangle dt = -(x_1, v(0)) - \int_0^T (\dot{x}(t), \dot{v}(t))dt + \int_0^T \langle Ax(t), v(t)\rangle dt,$$

$$\forall v(\cdot)\in L^2(0, T ; V)\cap W^{1,2}([0, T]; H), \quad v(T)=0 \text{ in } H.$$

(iii) *Function* $x(\cdot)\in L^2(0, T ; V)\cap W^{1,2}([0, T]; H)$ *satisfies* (3.6)-(3.7) *and for any* $v\in V$, $(\dot{x}(\cdot), v)$ *is absolutely continuous in* $[0, T]$ *and*

$$(3.9) \quad \frac{d}{dt}(\dot{x}(t), v)+\langle Ax(t), v\rangle = \langle g(t), v\rangle, \quad \text{a.e. } t\in[0, T].$$

The proof is straightforward. Then, we may give the following.

DEFINITION 3.2. A function $x(\cdot)\in L^2(0, T ; V)\cap W^{1,2}([0, T]; H)$ is called a solution of (3.5)-(3.7) if one of (i)-(iii) in Proposition 3.1 holds.

By usual Galerkin type method, we can prove the following result.

PROPOSITION 3.3. *Let A satisfy* (W2) *and* $(x_0, x_1)\in V\times H$, $g(\cdot)\in L^1(0, T ; H)$. *Then,* (3.5)-(3.7) *admits a unique solution*

$$(3.10) \quad \begin{aligned} x(\cdot)\in L^\infty(0, T ; V)&\cap W^{1,\infty}([0, T]; H)\cap W^{2,2}([0, T]; V') \\ &\hookrightarrow C([0, T]; H)\cap C^1([0, T]; V'). \end{aligned}$$

Moreover, it holds that

$$(3.11) \quad |\dot{x}(t)|^2+\langle Ax(t), x(t)\rangle \leq \left\{[|x_1|^2+\langle Ax_0, x_0\rangle]^{1/2}+\int_0^t |g(\tau)|d\tau\right\}^2, \quad t\in[0, T].$$

Next, we introduce another kind of solution, the so-called mild solution. To this end, we let

$$(3.12) \quad \tilde{\mathcal{A}}=\begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}: V\times V' \longrightarrow V'\times V',$$

and let

$$(3.13) \quad \begin{cases} \mathscr{D}(\mathcal{A})=\{(x, y)\in V\times H|\ \tilde{\mathcal{A}}(x, y)\in V\times H\}=\{x\in V|\ Ax\in H\}\times V, \\ \mathcal{A}=\tilde{\mathcal{A}}|_{\mathscr{D}(\mathcal{A})}: \mathscr{D}(\mathcal{A})\subset V\times H \longrightarrow V\times H. \end{cases}$$

Following result is standard ([22]).

PROPOSITION 3.4. *The operator $\mathcal{A}$ generates a C_0-group on the space $V\times H$.* If we let (formally)

$$z=\begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \qquad z_0=\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \qquad h=\begin{pmatrix} 0 \\ g \end{pmatrix},$$

then, (3.5) *reads*

(3.14)
$$\begin{cases} \dot{z}(t)=\mathcal{A}z(t)+h(t), \\ z(0)=z_0. \end{cases}$$

Since $e^{\mathcal{A}t}$ is a C_0-group on $Z\equiv V\times H$, we may study (3.14) directly without looking at (3.5). As usual, a function $z(\cdot)\in C([0, T]; Z)$ is called a mild solution of (3.14) if it satisfies the following integral equation:

(3.15)
$$z(t)=e^{\mathcal{A}t}z_0+\int_0^t e^{\mathcal{A}(t-\tau)}h(\tau)d\tau, \qquad t\in[0, T].$$

On the other hand, adopted from [3], we introduce the notion of weak solution of (3.14) as follows:

DEFINITION 3.5. A function $z(\cdot)\in C([0, T]; Z)$ is called a weak solution of (3.14), if for any $z^*\in\mathcal{D}(\mathcal{A}^*)$, the map $\langle z(\cdot), z^*\rangle$ is absolutely continuous on $[0, T]$, and

(3.16)
$$\begin{cases} \dfrac{d}{dt}\langle z(t), z^*\rangle=\langle z(t), \mathcal{A}^*z^*\rangle+\langle h(t), z^*\rangle, \qquad \text{a.e.} \quad t\in[0, T], \\ \langle z(t), z^*\rangle|_{t=0}=\langle z_0, z^*\rangle. \end{cases}$$

It is important that we have the following equivalent result which is a consequence of the result of [3]. In the following lemma, $\mathcal{A}$ and Z need not to be the same as above.

LEMMA 3.6. *Suppose $\mathcal{A}$ generates a C_0-semigroup on some Banach space Z and $h(\cdot)\in L^1(0, T; Z)$. Then, $z(\cdot)\in C([0, T]; Z)$ is a mild solution of (3.14) (i.e., (3.15) holds) if and only if $z(\cdot)$ is a weak solution of (3.14) (in the sense of Definition 3.5).*

From this result, we end up with the following theorem.

THEOREM 3.7. *Let (W2) hold and $g(\cdot)\in L^1(0, T; H)$. Then,*
 (i) *If $x(\cdot)$ is a solution of (3.5)-(3.7) in the sense of Definition 3.2, then, $\begin{pmatrix} x(\cdot) \\ \dot{x}(\cdot) \end{pmatrix}$ is a mild solution of (3.14).*
 (ii) *If $\begin{pmatrix} x(\cdot) \\ y(\cdot) \end{pmatrix}$ is a mild solution of (3.14), then $y(\cdot)=\dot{x}(\cdot)$ and $x(\cdot)$ is a solution of (3.5)-(3.7) in the sense of Definition 3.2.*

Proof. First of all, we can easily check that

$$(3.17) \quad \begin{cases} \mathscr{D}(\mathcal{A}^*) = H \times V, \\ \mathcal{A}^* \begin{pmatrix} \varphi \\ \phi \end{pmatrix} = \begin{pmatrix} -A\phi \\ \varphi \end{pmatrix} \equiv \begin{pmatrix} 0 & -A \\ I & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \phi \end{pmatrix}, \quad \forall \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \in \mathscr{D}(\mathcal{A}^*). \end{cases}$$

(i) Let $x(\cdot)$ be a solution of (3.5)-(3.7) in the sense of Definition 3.2. Then, by setting $y(\cdot) = \dot{x}(\cdot)$, we have

$$(3.18) \quad \frac{d}{dt}(y(t), \phi) = \langle -Ax(t), \phi \rangle + (g(t), \phi), \quad \text{a.e.} \quad t \in [0, T], \quad \forall \phi \in V.$$

It is easy to see that (since $x(\cdot) \in W^{1,2}([0, T]; H)$)

$$(3.19) \quad \frac{d}{dt}(x(t), \varphi) = (y(t), \varphi), \quad \text{a.e.} \quad t \in [0, T], \quad \forall \varphi \in H.$$

Thus, we have

$$(3.20) \quad \begin{aligned} \frac{d}{dt} \left\langle \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \right\rangle &= (y(t), \varphi) + \langle -Ax(t), \phi \rangle + (g(t), \phi) \\ &= \left\langle \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \mathcal{A}^* \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 0 \\ g(t) \end{pmatrix}, \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \right\rangle, \\ &\quad \forall \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \in H \times V \equiv \mathscr{D}(\mathcal{A}^*). \end{aligned}$$

Thus, $\begin{pmatrix} x(\cdot) \\ y(\cdot) \end{pmatrix}$ is a weak solution of (3.14). By Lemma 3.6, it is a mild solution of (3.14).

(ii) Let $\begin{pmatrix} x(\cdot) \\ y(\cdot) \end{pmatrix}$ be a mild solution of (3.14). Then, it is a weak solution of (3.14) by Lemma 3.6. That means (3.18) and (3.19) hold. Then, it is ready to show that $x(\cdot) \in W^{1,2}([0, T]; H)$ and

$$y(\cdot) = \dot{x}(\cdot).$$

While, (3.19) is exactly the same as (3.9) and thus our conclusion follows. □

Hereafter, we refer $x(\cdot)$ as a mild solution of (3.5)-(3.7) if $\begin{pmatrix} x(\cdot) \\ \dot{x}(\cdot) \end{pmatrix}$ is a mild solution of (3.14). We have seen that the solutions defined in Definition 3.2 are mild solutions. We will simply refer them the solutions of (3.5)-(3.7).

§3.2. Optimal Control Problem, Existence of Optimal Pairs.

In this subsection, we first state our optimal control problem. Then, we will establish the existence of optimal pairs for our problem. First of all, we let Assumptions (W3), (W4) and (W6) be the same as (P3), (P4) and (P6), with X being relpaced by H and we also assume

(W5) The set $Q \subset H \times V'$ is closed and $\Omega \subset V \times V \times H \times H$ is bounded and

weakly closed.

Our controlled evolution system is given by (3.1). For any given $x_0 \in V$, $x_1 \in H$ and $u(\cdot) \in \mathcal{U}[0, T]$, we may talk about the mild (or weak) solution of

$$(3.21) \qquad \begin{cases} \ddot{x}(t) + A x(t) = f(t, x(t), u(t)), & \text{a.e. } t \in [0, T], \\ x(0) = x_0, \\ \dot{x}(0) = x_1. \end{cases}$$

More precisely, we have the following.

DEFINITION 3.8. A function $x(\cdot) \in L^2(0, T; V) \cap W^{1,2}([0, T]; H)$ is called a mild solution of (3.21) (corresponding to $x_0 \in V$, $x_1 \in H$ and $u(\cdot) \in \mathcal{U}[0, T]$) if $f(\cdot, x(\cdot), u(\cdot)) \in L^2(0, T; H)$ and $x(\cdot)$ is a mild solution of (3.5) with $g(\cdot)$ being $f(\cdot, x(\cdot), u(\cdot))$.

DEFINITION 3.9. A pair $(x(\cdot), u(\cdot)) \in (L^2(0, T; V) \cap W^{1,2}([0, T]; H)) \times \mathcal{U}[0, T]$ is said to be semi-admissible if $x(\cdot)$ is a mild solution of (3.21) corresponding to $u(\cdot)$ and

$$(3.22) \qquad u(t) \in \Gamma(t, x(t)), \qquad \text{a.e. } t \in [0, T].$$

Moreover, if the following also hold:

$$(3.23) \qquad (x(0), x(T), \dot{x}(0), \dot{x}(T)) \in \Omega \subset V \times V \times H \times H,$$

$$(3.24) \qquad (x(t), \dot{x}(t)) \in Q \subset H \times V', \qquad \text{a.e. } t \in [0, T],$$

$$(3.25) \qquad f^0(\cdot, x(\cdot), u(\cdot)) \in L^1(0, T),$$

then, we call $(x(\cdot), u(\cdot))$ an admissible pair.

We let

$$\mathcal{A}_s = \{(x(\cdot), u(\cdot)) \in C([0, T]; H) \times \mathcal{U}[0, T] \mid (x(\cdot), u(\cdot)) \text{ is semi-admissible}\},$$

and

$$\mathcal{A}_{ad} = \{(x(\cdot), u(\cdot)) \in C([0, T]; H) \times \mathcal{U}[0, T] \mid (x(\cdot), u(\cdot)) \text{ is admissible}\}.$$

Next, we introduce the cost functional.

$$(3.26) \qquad J(x(\cdot), u(\cdot)) = \int_0^T f^0(t, x(t), u(t)) dt, \qquad \forall (x(\cdot), u(\cdot)) \in \mathcal{A}_{ad}.$$

Our optimal control problem can be stated as follows:

PROBLEM W. Find $(\bar{x}(\cdot), \bar{u}(\cdot)) \in \mathcal{A}_{ad}$, such that

$$(3.27) \qquad J(\bar{x}(\cdot), \bar{u}(\cdot)) = \inf_{(x(\cdot), u(\cdot)) \in \mathcal{A}_{ad}} J(x(\cdot), u(\cdot)).$$

If such a pair exists, we refer $\bar{x}(\cdot)$, $\bar{u}(\cdot)$ and $(\bar{x}(\cdot), \bar{u}(\cdot))$ an optimal trajectory, control and pair, respectively.

Our main result of this scetion is the following theorem.

THEOREM 3.10. *Let* (W1)-(W6) *hold. Let there exist a minimizing sequence* $\{x_k(\cdot), u_k(\cdot))\} \subset \mathcal{A}_{ad}$, *such that for some constant* $C > 0$,

$$(3.28) \qquad \int_0^T |f(t, x_k(t), u_k(t))|^2 dt \leqq C, \qquad \forall k \geqq 1.$$

Then, Problem W admits at least one optimal pair.

Proof. For each $k \geqq 1$, from (3.4), (3.11) and the boundedness of Ω, we have that

$$|\dot{x}_k(t)|^2 + \alpha \|x_k(t)\|^2 \leqq \{[\,|\dot{x}_k(0)|^2$$

$$(3.29) \qquad + \langle Ax_k(0), x_k(0) \rangle]^{1/2} + \int_0^t |f(t, x_k(t), u_k(t))| \, d\tau\}^2 \leqq C,$$

$$\forall t \in [0, T], \quad k \geqq 1,$$

for some constant C. Thus, we may assume that

$$(3.30) \qquad \begin{cases} \dot{x}_k(\cdot) \xrightarrow{\;*\;} \dot{\bar{x}}(\cdot), & \text{in} \quad L^\infty(0, T; H), \\ x_k(\cdot) \xrightarrow{\;*\;} \bar{x}(\cdot), & \text{in} \quad L^\infty(0, T; V). \end{cases}$$

Due to the compactness of the embedding $V \hookrightarrow H$, we know that the embedding

$$L^2(0, T; V) \cap W^{1,2}([0, T]; H) \hookrightarrow L^2(0, T; H),$$

is compact ([23]). Thus, we may let

$$(3.31) \qquad x_k(\cdot) \xrightarrow{\;s\;} \bar{x}(\cdot), \qquad \text{in} \quad L^2(0, T; H),$$

$$(3.32) \qquad x_k(t) \xrightarrow{\;s\;} \bar{x}(t), \qquad \text{in} \quad H, \qquad \text{a.e.} \quad t \in [0, T].$$

On the other hand, by the boundedness of Ω, we may assume

$$(3.33) \qquad \begin{cases} x_k(0) \xrightarrow{\;w\;} \bar{x}_0, & \text{in} \; V, \\ \dot{x}_k(0) \xrightarrow{\;w\;} \bar{x}_1, & \text{in} \; H. \end{cases}$$

Also, from (3.28), we may let

$$(3.34) \qquad f(\cdot, x_k(\cdot), u_k(\cdot)) \xrightarrow{\;w\;} \bar{f}(\cdot), \qquad \text{in} \quad L^2(0, T; H).$$

Then, by Definition 3.2, for any $v(\cdot) \in L^2(0, T; V) \cap W^{1,2}([0, T]; H)$ with $v(T) = 0$, we have

$$\text{(3.35)} \quad \int_0^T \langle f(t,\ x_k(t),\ u_k(t)),\ v(t)\rangle dt = -(\dot{x}_k(0),\ v(0))$$
$$-\int_0^T (\dot{x}_k(t),\ \dot{v}(t))dt + \int_0^T \langle x_k(t),\ Av(t)\rangle dt.$$

Let $k \to \infty$, we obtain

$$\text{(3.36)} \quad \int_0^T \langle \bar{f}(t),\ v(t)\rangle dt = -(x_1,\ v(0))$$
$$-\int_0^T (\bar{x}(t),\ \dot{v}(t))dt + \int_0^T \langle \bar{x}(t),\ Av(t)\rangle dt.$$

From (3.32), we can find $\sigma \in (0,\ T)$, such that

$$\text{(3.37)} \quad x_k(\sigma) \xrightarrow{\ s\ } \bar{x}(\sigma).$$

Then, for any $h \in H$, we have

$$(x_0,\ h) = \lim_{k \to \infty} (x_k(0),\ h)$$
$$= \lim_{k \to \infty} \left\{ (x_k(\sigma),\ h) - \int_0^\sigma (\dot{x}_k(\tau),\ h)d\tau \right\}$$
$$= (\bar{x}(\sigma),\ h) - \int_0^\sigma (\dot{\bar{x}}(\tau),\ h)d\tau$$
$$= (\bar{x}(0),\ h).$$

That is

$$\text{(3.38)} \quad \bar{x}(0) = x_0, \quad \text{in } H.$$

By Definition 3.2, we know that

$$\text{(3.39)} \quad \begin{cases} \ddot{\bar{x}}(t) + A\bar{x}(t) = \bar{f}(t), & \text{a.e. } t \in [0,\ T], \quad \text{in } V', \\ x(0) = x_0, & \text{in } H, \\ \dot{x}(0) = x_1, & \text{in } V'. \end{cases}$$

Then, by a similar method used in section 2, we can prove that for some $\bar{f}^0(\cdot) \in L^1(0,\ T)$,

$$\text{(3.40)} \quad (\bar{f}^0(t),\ \bar{f}(t)) \in \overline{co}\mathcal{E}(t,\ \bar{x}(t)), \quad \text{a.e. } t \in [0,\ T],$$

and

$$\text{(3.41)} \quad \int_0^T \bar{f}^0(t)dt \leq \inf_{\mathcal{A}_{ad}} J(x(\cdot),\ u(\cdot)).$$

Then, by Theorems A3 and A4 of Appendix, we can find some $\bar{u}(\cdot) \in \mathcal{U}[0,\ T]$, such that

$$(3.42) \quad \begin{cases} \bar{f}^0(t) \geqq f^0(t,\ \bar{x}(t),\ \bar{u}(t)), \\ \bar{f}(t) = f(t,\ \bar{x}(t),\ \bar{u}(t)), \qquad \text{a.e.} \quad t \in [0,\ T]. \\ \bar{u}(t) \in \Gamma(t,\ \bar{x}(t)), \end{cases}$$

That is $(\bar{x}(\cdot),\ \bar{u}(\cdot)) \in \mathcal{A}_{ad}$. Hence, $(\bar{x}(\cdot),\ \bar{u}(\cdot))$ is an optimal pair. $\square$

We see that the key point is the strong convergence of the trajectories, which is guaranteed by the suitable compactness conditions. We also note that one can obtain the existence of semi-admissible pairs as we presented in section 2. Since the idea is the same, we omit the details here.

§ 4. Elliptic Variational Inequalities.

In this section, we are going to discuss the existence of optimal controls for elliptic variational inequalities. Let us start with some assumptions.

(E1) Ω is a bounded region in $\mathbf{R}^n$ with a smooth boundary $\partial\Omega$, U is a metric space.

(E2) β is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$, A is a second order uniformly elliptic differential operator of divergince form. More precisely,

$$(4.1) \qquad Ay(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x) \frac{\partial y(x)}{\partial x_i} \right), \qquad \forall y(\cdot) \in H^1(\Omega),$$

with $a_{ij}(\cdot) \in C^1(\bar{\Omega})$ and for some $\alpha_0 > 0$,

$$(4.2) \qquad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqq \alpha_0 |\xi|^2, \qquad \forall \xi \in \mathbf{R}^n, \quad x \in \Omega.$$

(E3) $f: \Omega \times \mathbf{R} \times U \to \mathbf{R}$ is Borel measurable in $(x,\ y,\ u) \in \Omega \times \mathbf{R} \times \mathbf{R}$ and continuous in $(y,\ u) \in \mathbf{R} \times U$ for almost all $x \in \Omega$, $f^0: \Omega \times \mathbf{R} \times U \to \mathbf{R}$ is Borel measurable in $(x,\ y,\ u) \in \Omega \times \mathbf{R} \times \mathbf{R}$, lower semicontinuous in $(y,\ u) \in \mathbf{R} \times U$ for almost all $x \in \Omega$ and there exists a constant $K \geqq 0$, such that

$$(4.3) \qquad f^0(x,\ y,\ u) \geqq -K, \qquad (x,\ y,\ u) \in \Omega \times \mathbf{R} \times U.$$

(E4) $\Gamma: \Omega \times \mathbf{R} \to 2^U$ is pseudo-continuous.

(E5) $Q \subset \mathbf{R}$ is closed.

Our controlled system is the following:

$$(4.4) \qquad \begin{cases} Ay(x) + \beta(y(x)) \ni f(x,\ y(x),\ u(x)), \qquad \text{a.e.} \quad x \in \Omega, \\ y|_{\partial\Omega} = 0. \end{cases}$$

Let us state the following result which will help us to understand (4.4).

PROPOSITION 4.1. *Let* (E1), (E2) *and the following* (E3') *hold*

(E3') $f: \Omega \times \mathbf{R} \times U \to \mathbf{R}$ *is measurable in* $(x,\ y,\ u) \in \Omega \times \mathbf{R} \times U$ *and there exists*

a constant $L>0$, such that

(4.5) $|f(x, 0, u)| \leq L, \qquad \forall (x, u) \in \Omega \times U,$

(4.6)
$$0 \geq [f(x, y, u) - f(x, \hat{y}, u)](y - \hat{y}) \geq -L|y - \hat{y}|^2,$$
$$\forall (x, u) \in \Omega \times U, \qquad y, \hat{y} \in \mathbf{R}.$$

Then, for any $u(\cdot) \in \mathcal{U} \equiv \{u(\cdot): \Omega \to U \,|\, u(\cdot) \text{ measurable}\}$ and any $1 \leq p < \infty$, there exists a unique solution $y(\cdot) \equiv y(\cdot\,; u(\cdot)) \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ to (4.4). Moreover,

(4.7) $|y(\cdot\,; u(\cdot))|_{W^{2,p}(\Omega)} \leq C_p, \qquad \forall u(\cdot) \in \mathcal{U}.$

The proof can be carried out by using the method of [7] (see [9] also). Actually, from [7], we know that (E3′) can be more general if U is a Banach space.

In our problem, the control and the state constraints are given by

(4.8) $u(x) \in \Gamma(x, y(x)), \qquad \text{a.e.} \quad x \in \Omega,$

(4.9) $y(x) \in Q, \qquad \text{a.e.} \quad x \in \Omega.$

We let $p > n/2$. A pair $(y(\cdot), u(\cdot)) \in (W^{2,p}(\Omega) \cap H_0^1(\Omega)) \times \mathcal{U}$ is said to be semi-admissible if (4.4) and (4.8) are satisfied and it is said to be admissible if (4.4), (4.8)-(4.9) hold and $f^0(\cdot, y(\cdot), u(\cdot)) \in L^1(\Omega)$. We use the notation $\mathcal{A}_s$, $\mathcal{A}_{ad}$, $\mathcal{X}_s$, $\mathcal{X}_{ad}$, $\mathcal{U}_s$ and $\mathcal{U}_{ad}$ similar to those in §2.1. Next, for any $(y(\cdot), u(\cdot)) \in \mathcal{A}_{ad}$, we define the cost functional to be the following:

(4.10) $J(y(\cdot), u(\cdot)) = \int_{\Omega} f^0(x, y(x), u(x)) dx.$

Then, our optimal control problem can be stated as follows:

PROBLEM E.   Find $(\bar{y}(\cdot), \bar{u}(\cdot)) \in \mathcal{A}_{ad}$, such that

(4.11) $J(\bar{y}(\cdot), \bar{u}(\cdot)) = \inf_{\mathcal{A}_{ad}} J(y(\cdot), u(\cdot)).$

If a pair $(\bar{y}(\cdot), \bar{u}(\cdot)) \in \mathcal{A}_{ad}$ exists satisfying (4.11), we call $(\bar{y}(\cdot), \bar{u}(\cdot))$, $\bar{y}(\cdot)$ and $\bar{u}(\cdot)$ an optimal pair, state and control, respectively.

The main purpose of this section is to establish the existence of an optimal pair of Problem E. To this end, let us first introduce the following set:

(4.12)
$$\mathcal{E}(x, y) = \{(z^0, z) \in \mathbf{R}^2 \,|\, z^0 \geq f^0(x, y, u), \; z = f(x, y, u), \; u \in \Gamma(x, y)\},$$
$$\forall (x, y) \in \Omega \times \mathbf{R}.$$

We further introduce the following assumption (compare (P6) and (W6))
   (E6)   For almost all $x \in \Omega$, the map $y \mapsto \mathcal{E}(x, y)$ has the Cesari property on Q.
Now, we are ready to state and prove the main result of this section.

THEOREM 4.2. *Let (E1)-(E6) hold. Let $\{y_k(\cdot),\ u_k(\cdot))\}\subset\mathcal{A}_{ad}$ be a minimizing sequence of Problem E, such that for some absolute constant C,*

(4.13) $$|f(\cdot,\ y_k(\cdot),\ u_k(\cdot))|_{L^{\infty}(\Omega)}\leq C,\qquad \forall k\geq 1.$$

Then Problem E aadmits at teast one optimal pair.

Proof. From the idea of [7], we know that for $p>n/2$,

(4.14) $$|y_k(\cdot)|_{W^{2,p}(\Omega)}\leq C_p,\qquad \forall k\geq 1.$$

Thus, we may let

$$y_k(\cdot)\longrightarrow \bar{y}(\cdot),\qquad \text{weaky in}\ \ W^{2,p}(\Omega),$$

$$\text{strongly in}\ \ C^{\alpha}(\bar{\Omega}),$$

for some $\alpha\in(0, 1)$ and some $\bar{y}(\cdot)\in W^{2,p}(\Omega)\cap H_0^1(\Omega)$. On the other hand, we may let

(4.16) $$f(\cdot,\ y_k(\cdot),\ u_k(\cdot))\longrightarrow \bar{f}(\cdot),\qquad \text{weakly in}\ \ L^p(\Omega),$$

for some $\bar{f}(\cdot)\in L^{\infty}(\Omega)$. Then, by Mazur Theorem, we can find $\alpha_{ij}\geq 0$, $\sum_{i\geq 1}\alpha_{ij}=1$, such that

(4.17) $$\phi_j\equiv \sum_{i\geq 1}\alpha_{ij}f(\cdot,\ y_{i+j}(\cdot),\ u_{i+j}(\cdot))\longrightarrow \bar{f}(\cdot),\qquad \text{strongly in}\ \ L^p(\Omega).$$

We set

(4.18) $$\phi_j^0(\cdot)=\sum_{i\geq 1}\alpha_{ij}f^0(\cdot,\ y_{ij}(\cdot),\ u_{i+j}(\cdot)),$$

and let

(4.19) $$\bar{f}^0(x)=\varliminf_{k\to\infty} f^0(x,\ y_k(x),\ u_k(x))\geq -K,\qquad \text{a.e.}\ \ x\in\Omega.$$

By (4.15), we see that for any $\varepsilon>0$, there exists a j_0, such that for $j\geq j_0$,

(4.20) $$(\phi_j^0(x),\ \phi_i(x))\in co\mathcal{E}(x,\ \mathfrak{N}_{\varepsilon}(\bar{y}(x))),\qquad \text{a.e.}\ \ x\in\Omega.$$

Thus,

(4.21) $$(\bar{f}^0(x),\ \bar{f}(x))\in \overline{co}\mathcal{E}(x,\ \mathfrak{N}_{\varepsilon}(\bar{y}(x))),\qquad \text{a.e.}\ \ x\in\Omega.$$

Then, by (E6), we see that

(4.22) $$(\bar{f}^0(x),\ \bar{f}(x))\in \mathcal{E}(x,\ \bar{y}(x)),\qquad \text{a.e.}\ \ x\in\Omega.$$

Hence, similar to the previous sections, we can find $\bar{u}(\cdot)\in \mathcal{U}$, such that

(4.23)
$$\begin{cases} \bar{f}^0(x) \geqq f^0(x,\,\bar{y}(x),\,\bar{u}(x)), \\ \bar{f}(x) = f(x,\,\bar{y}(x),\,\bar{u}(x)), \qquad \text{a.e.} \quad x \in \Omega. \\ \bar{u}(x) \in \Gamma(x,\,\bar{y}(x)), \end{cases}$$

On the other hand, from the admissibility of $(y_k(\cdot),\,u_k(\cdot))$, the convergence (4.15) and (4.16) and (4.23), we have (see [4])

(4.24)
$$\begin{cases} A\bar{y}(x) + \beta(\bar{y}(x)) \ni f(x,\,\bar{y}(x),\,\bar{u}(x)), \qquad \text{a.e.} \quad x \in \Omega, \\ \bar{y}|_{\partial\Omega} = 0. \end{cases}$$

By (E5), we have

(4.25)
$$\bar{y}(x) \in Q, \qquad \text{a.e.} \quad x \in \Omega.$$

Hence, $(\bar{y}(\cdot),\,\bar{u}(\cdot)) \in \mathcal{A}_{ad}$. By Fatou's Lemma, we see that

(4.26)
$$\begin{aligned} J(\bar{y}(\cdot),\,\bar{u}(\cdot)) &= \int_\Omega f^0(x,\,\bar{y}(x),\,\bar{u}(x))dx \\ &\leqq \varliminf_{k \to \infty} J(y_k(\cdot),\,u_k(\cdot)) \\ &= \inf_{\mathcal{A}_{ad}} J(y(\cdot),\,u(\cdot)). \end{aligned}$$

Thus, $(\bar{y}(\cdot),\,\bar{u}(\cdot))$ is an optimal pair.   □

Remark 4.3. In the above the operator A is not necessarily of second order. Also, we see that if $\beta = 0$, the (4.4) is reduced to a semilinear elliptic partial differential system with the leading operator being of divergence form and coercive. Thus, our theory covers the existence of optimal controls for the mentioned elliptic systems.

Remark 4.4. From the above, we see that the existence of an optimal pair follows essentially from the precompactness of the minimizing sequence in suitable spaces and the sort of convexity conditions. The multivaluedness of term $Ay + \beta(y)$ does not bring any difficulty. However, if this multivalued operator also depends on the control variable, then, the situation becomes technically difficult.

Remark 4.5. It seems to us that our approach is also possible to discuss the existence of optimal controls for evolutionary variational inequalities.

Appendix. Multivalued Mappings.

In this appendix, we recall some results about multivalued mappings. First, we recall the following definition.

DEFINITION A.1. Let T and Z be metric spaces, $\Lambda: T \to 2^Z \equiv \{$nonempty

subsets of Z}.

(i) Λ is said to be continuous (with respect to the Hausdorff metric ρ_H) at $t \in T$, if

(A.1) $$\lim_{\bar{t} \to t} \rho_H(\Lambda(\bar{t}),\, \Lambda(t)) = 0,$$

where

$$\rho_H(\Lambda_1,\, \Lambda_2) = \frac{1}{2} \max\{ \sup_{\lambda_1 \in \Lambda_1} d(\lambda_1,\, \Lambda_2) + \sup_{\lambda_2 \in \Lambda_2} d(\Lambda_1,\, \lambda_2)\}.$$

(ii) Λ is said to be lower semi-continuous at $t \in T$, if for any open set V, whenever

(A.2) $$\Lambda(t) \cap V \neq \varnothing,$$

there exists a $\delta > 0$, such that

(A.3) $$\Lambda(s) \cap V \neq \varnothing, \qquad \forall s \in \mathfrak{N}_{\delta}(t).$$

(iii) Λ is said to be upper semi-continuous at $t \in T$, if for any $\varepsilon > 0$, there exists a $\delta > 0$, such that

(A.4) $$\Lambda(\mathfrak{N}_{\delta}(t)) \subset \mathfrak{N}_{\varepsilon}(\Lambda(t)).$$

(iv) Λ is said to be pseudo-continuous at $t \in T$, if

(A.5) $$\bigcap_{\varepsilon > 0} \overline{\Lambda(\mathfrak{N}_{\varepsilon}(t))} = \Lambda(t).$$

(v) If T is a domain in $\boldsymbol{R}^n$, then, Λ is said to be measurable, if for any closed subset $F \subset Z$, the set

$$\Lambda^{-1} \equiv \{t \in T \mid \Lambda(t) \cap F \neq \varnothing\}$$

is (Lebesgue) measurable.

(vi) Λ is said to be closed and/or convex set valued on T if for all $t \in T$, $\Lambda(t)$ is closed and/or convex.

If in (i)–(iv), the mentioned properties hold for all $t \in T$, we simply say Λ is continuous, upper semi-continuous and psuedo-continuous, respectively.

PROPOSITION A.2. *Let Z be a complete metric space and $\Lambda: T \to Z$ closed set valued. Then,*
   (i) *If T is a metric space, then, the following implication chains hold*

$$\Lambda \text{ is continuous} \Rightarrow \Lambda \text{ is lower semicontinuous}$$

$$\Longleftrightarrow \forall \text{ open set } F \subset Z,\ \Lambda^{-1} \text{ is open in } T;$$

$$\Lambda \text{ is continuous} \Rightarrow \Lambda \text{ is upper semicontinuous}$$

$$\Longleftrightarrow \forall \text{ closed set } F \subset Z,\ \Lambda^{-1} \text{ is closed in } T$$

$\Rightarrow \Lambda$ is *psuedo-continuous*

$\Longleftrightarrow G(\Lambda)\equiv$ *the graph of Λ is closed in $T\times X$.*

(ii) *If T is a metric space and Z is a compact metric space.   Then*

Λ *is continuous $\Longleftrightarrow \Lambda$ is lower and upper semicontinuous;*

Λ *is upper continuous $\Longleftrightarrow \Lambda$ is psuedo-continuous.*

(iii) *If T is a domain in $\mathbf{R}^n$, then,*

Λ *is psuedo-continuous $\Rightarrow \Lambda$ is measurable.*

In [5], the pseudo-continuity and the upper semicontinuity were referred as upper semicontinuity and upper semicontinuity with respect to the inclusion (u. s. c. i.), respectively. Our notion is adopted from [28] for lower, upper semicontinuity and a modification of that given in [2] for pseudo-continuity.

THEOREM A.3. ([16, 25]) *Let T be a locally compact metric space, U be a Polish space, X be a complete metric space, $\Lambda: T\to 2^U$ be measurable taking closed subset values, $f: T\times U\to X$ be measurable in t, locally uniformly continuous in x and $\bar{f}: T\to X$ be measurable with*

(A.4) $\bar{f}(t)\in f(t,\ \Lambda(t)),$ a.e. $t\in T$.

Then, there exists a measurable function $u: T\to U$, such that

(A.5) $\begin{cases} u(t)\in \Lambda(t), \\ \bar{f}(t)=f(t,\ u(t)), \end{cases}$ a.e. $t\in T$.

THEOREM A.4. *Let T be a domain in $\mathbf{R}^n$, X be a metric space, U be a Polish space, $\Gamma: T\times X\to 2^U$ be pseudo-continuous and $\bar{x}: T\to X$ be measurable. Then, $\Gamma(\cdot,\ \bar{x}(\cdot)): T\to 2^U$ is measurable.*

The proof follows easily from the definition and the criteria of measurability for multifunctions given in [16].

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