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ASYMPTOTIC BEHAVIOR OF ALMOST-ORBITS OF NONEXPANSIVE SEMIGROUPS WITHOUT CONVEXITY

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Abstract

We first prove a resul on the asymptotic behavior of almost-orbits of nonexpansive semigroups without convexity in a Hilbert space. This is a generalization of results of Rodé [7] and Takahashi [10]. Further we prove a fixed point theorem for Lipschitzian semigroups without convexity. This is a generalization of results of Lau [3], Takahashi [8], [10] and Ishihara [2].

1. Introduction. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$ and let C be a nonempty subset of H. A mapping $T: C \to C$ is said to be Lipschitzian if there exists a nonnegative number k such that

$$\|Tx-Ty\| \leqq k\|x-y\| \qquad \text{for every} \quad x, y \in C$$

and nonexpansive in the case of $k=1$. Let S be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that for each $s \in S$, the mappings $t \to t\cdot s$ and $t \to s\cdot t$ of S into itself are continuous. Then a family $\mathcal{S}=\{T_s : s \in S\}$ of mappings of C into itself is called a Lipschitzian semigroup on C if it satisfies the following:

(1) $T_{st}x = T_sT_tx$ for all $s, t \in S$ and $x \in C$;

(2) for each $x \in C$, the mapping $s \to T_sx$ is continuous on S;

(3) for each $s \in S$, T_s is a Lipschitzian mapping of C into itself with Lipschitz constant k_s. A Lipschitzian semigroup $\mathcal{S}=\{T_t : t \in S\}$ on C is said to be nonexpansive if $k_s=1$ for every $s \in S$. Recently, Takahashi [10] proved a nonlinear ergodic theorem and a fixed point theorem for nonexpansive semigroups without convexity in a Hilbert space. On the other hand, Miyadera-Kobayasi [4] introduced the notion of an almost-orbit of nonexpansive semigroups and established the weak and strong almost convergence of such an almost-orbit; see also [1], [11], [12].

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In this paper, we first prove a result on the asymptotic behavior of almost-orbits of nonexpansive semigroups without convexity in a Hilbert space. This is a generalization of results of Rodé [7] and Takahashi [10]. Further we prove a fixed point theorem for Lipschitzian semigroups without convexity. This is a generalization of results of Lau [3], Takahashi [8], [10] and Ishihara [2].

2. Asymptotic behavior of almost-orbits. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm and let X be a subspace of $B(S)$ containing constants. Then an element μ of X^* (the dual of X) is a mean on X if and only if

$$\inf_{s\in S} f(s)\leqq\mu(f)\leqq\sup_{s\in S} f(s) \qquad \text{for all} \quad f\in X.$$

Let μ be a mean on X and $f\in X$. Then, according to time and circumstances, we use $\mu_t(f(t))$ instead of $\mu(f)$. For each $s\in S$ and $f\in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ given by

$$(l_s f)(t)=f(st) \quad \text{and} \quad (r_s f)(t)=f(ts) \qquad \text{for all} \quad t\in S.$$

Let X be a subspace of $B(S)$ containing constants and invariant under l_s, $s\in S$ $(r_s,\ s\in S)$. Then a mean μ on X is said to be left invariant (right invariant) if

$$\mu(f)=\mu(l_s f)\ (\mu(f)=\mu(r_s f)) \qquad \text{for all} \quad f\in X \text{ and } s\in S.$$

An invariant mean is a left and right invariant mean. We know following [9]: If X is a left invariant subspace of $B(S)$ containing constants and μ is a left invariant mean on X, then for any $f\in X$,

$$\sup_s \inf_t f(st)\leqq\mu(f)\leqq\inf_s \sup_t f(st).$$

Similarly, if X is a right invariant subspace of $B(S)$ containing constants and μ is a right invariant mean on X, then for any $f\in X$,

$$\sup_s \inf_t f(ts)\leqq\mu(f)\leqq\inf_s \sup_t f(ts).$$

We denote by $C(S)$ the Banach space of all bounded continuous real-valued functions on S with supremum norm.

A continuous function $u: S\to C$ is said to be an almost-orbit of $\mathcal{S}=\{T_t:\ t\in S\}$ if

$$\inf_w \sup_{t,s}\|u(swt)-T_s u(wt)\|=0.$$

If an almost-orbit $u: S\to C$ of $\mathcal{S}=\{T_t:\ t\in S\}$ is bounded and μ is a mean on $C(S)$, then there exists a unique element x_μ of H such that

$$\mu_t(u(t),\ y)=(x_\mu,\ y) \qquad \text{for all} \quad y\in H.$$

In fact, for each $y\in H$,

$$|\mu_t(u(t), y)| \leqq \|\mu\| \sup_t |(u(t), y)| \leqq \sup_t \|u(t)\| \|y\|$$

and hence $\mu_t(u(t), \cdot)$ is a bounded linear functional on H. So, we obtain the desired result by the Riesz representation theorem.

Let $\{\mu_\alpha : \alpha \in A\}$ be a net of means on $C(S)$. Then $\{\mu_\alpha : \alpha \in A\}$ is said to be asymptotically invariant if for each $f \in C(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(l_s f) \longrightarrow 0 \quad \text{and} \quad \mu_\alpha(f) - \mu_\alpha(r_s f) \longrightarrow 0 .$$

THEOREM 1. *Let H be a real Hilbert space and let C be a nonempty subset of H. Suppose that S is a semitopological semigroup such that $C(S)$ has an invariant mean. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C. If an almost-orbit $u: S \to C$ of $\mathcal{S} = \{T_t : t \in S\}$ is bounded and $\bigcap_{s \in S} \overline{co}\{u(st) : t \in S\} \subset C$, then the set $F(\mathcal{S})$ of all common fixed points of T_t, $t \in S$ is nonempty. Moreover, if $\{\mu_\alpha : \alpha \in A\}$ is an asymptotically invariant net of means on $C(S)$, then there exists an element x_0 of $F(\mathcal{S})$ such that x_{μ_α} converges weakly to x_0, where x_{μ_α} is an element of H such that $(\mu_\alpha)_t(u(t), y) = (x_{\mu_\alpha}, y)$ for all $y \in H$.*

Proof. Let μ be an invariant mean on $C(S)$. Then, there exists an element x_μ of H such that $\mu_t(u(t), y) = (x_\mu, y)$ for all $y \in H$. We show $x_\mu \in \bigcap_{s \in S} \overline{co}\{u(st) : t \in S\}$. If not, we have $x_\mu \notin \overline{co}\{u(s_0 t) : t \in S\}$ for some $s_0 \in S$. By the separation theorem, there exists an element y_0 of H such that

$$(x_\mu, y_0) < \inf_{z \in \overline{co}\{u(s_0 t) : t \in S\}} (z, y_0) .$$

So, we have

$$\begin{aligned}(x_\mu, y_0) &< \inf_{z \in \overline{co}\{u(s_0 t) : t \in S\}} (z, y_0) \\ &\leqq \inf_{t \in S} (u(s_0 t), y_0) \\ &\leqq \mu_t(u(s_0 t), y_0) \\ &= \mu_t(u(t), y_0) = (x_\mu, y_0) .\end{aligned}$$

This is a contradiction. Therefore we have $x_\mu \in \bigcap_{s \in S} \overline{co}\{u(st) : t \in S\}$, and hence $x_\mu \in C$.

Since u is continuous and $\{u(t) : t \in S\}$ is bounded, the real-valued function $t \to \|u(t) - y\|^2$ is in $C(S)$ for each $y \in H$. Let $r = \inf_{y \in H} \mu_t \|u(t) - y\|^2$ and $M = \{z \in H : \mu_t \|u(t) - z\|^2 = r\}$. Since for each $y \in H$ and $t \in S$,

$$\|x_\mu - y\|^2 = \|u(t) - y\|^2 - \|u(t) - x_\mu\|^2 - 2(u(t) - x_\mu, x_\mu - y),$$

we have

$$0 \leqq \|x_\mu - y\|^2$$

$$=\mu_t\|u(t)-y\|^2-\mu_t\|u(t)-x_\mu\|^2-2\mu_t(u(t)-x_\mu,\ x_\mu-y)$$

$$=\mu_t\|u(t)-y\|^2-\mu_t\|u(t)-x_\mu\|^2.$$

Hence, for an element y of H, it follows that $x_\mu\neq y$ if and only if $\mu_t\|u(t)-x_\mu\|^2<\mu_t\|u(t)-y\|^2$. This implies that the set M consists a single point x_μ.

We prove that $x_\mu\in F(\mathcal{S})$. We first show that for every $s\in S$ and $y\in H$,

$$\mu_t\|u(st)-y\|^2=\mu_t\|T_su(t)-y\|^2.$$

Since $\{u(t):\ t\in S\}$ is bounded, there exists a positive number M_1 such that $\|u(t)\|\leqq M_1$ for any $t\in S$. Fix $t_0\in S$. Then we have

$$\|T_su(t)-T_su(t_0)\|\leqq\|u(t)-u(t_0)\|\leqq 2M_1$$

and hence

$$\|T_su(t)\|\leqq\|T_su(t_0)\|+2M_1 \qquad \text{for all} \quad t\in S.$$

So, there exists a positive number M_2 such that $\|T_su(t)\|\leqq M_2$ for all $t\in S$. Therefore, we have

$$\begin{aligned}
&|\mu_t\|u(st)-y\|^2-\mu_t\|T_su(t)-y\|^2|\\
&=|\mu_t(\|u(st)-y\|^2-\|T_su(t)-y\|^2)|\\
&\leqq\mu_t((\|u(st)-y\|+\|T_su(t)-y\|)|\,\|u(st)-y\|-\|T_su(t)-y\|\,|)\\
&\leqq(2\|y\|+M_1+M_2)\mu_t\|u(st)-T_su(t)\|\\
&\leqq(2\|y\|+M_1+M_2)\inf_w\sup_t\|u(swt)-T_su(wt)\|=0.
\end{aligned}$$

This implies $\mu_t\|u(st)-y\|^2=\mu_t\|T_su(t)-y\|^2$ for every $s\in S$ and $y\in H$. Using this, we have $x_\mu\in F(\mathcal{S})$. In fact,

$$\begin{aligned}
\mu_t\|u(t)-T_sx_\mu\|^2&=\mu_t\|u(st)-T_sx_\mu\|^2\\
&=\mu_t\|T_su(t)-T_sx_\mu\|^2\\
&\leqq\mu_t\|u(t)-x_\mu\|^2=r,
\end{aligned}$$

and hence $T_sx_\mu\in M$. Since $M=\{x_\mu\}$, we have $T_sx_\mu=x_\mu$ for all $s\in S$. Next, we prove that x_μ is independent of any invariant mean μ on $C(S)$. We know that $\mu_t\|u(t)-z\|^2\leqq\inf_s\sup_t\|u(ts)-z\|^2$ for all $z\in H$. On the other hand, fix $z\in F(\mathcal{S})$ and set $M_3=\sup\|u(t)-z\|$. Then, for any $\varepsilon>0$ there exists an a in S such that

$$\sup_{s,t}\|u(tas)-T_tu(as)\|<\varepsilon.$$

Since for each $s\in S$,

$$\inf_w\sup_t\|u(tw)-z\|^2\leqq\sup_t\|u(tas)-z\|^2$$

$$\leqq \sup_t \|u(tas)-T_t u(as)\|^2 + \sup_t \|T_t u(as)-z\|^2$$
$$+2 \sup_t \|u(tas)-T_t u(as)\| \|T_t u(as)-z\|$$
$$\leqq \varepsilon^2 + \sup_t \|T_t u(as)-T_t z\|^2 + 2\varepsilon \sup_t \|T_t u(as)-T_t z\|$$
$$\leqq \varepsilon^2 + \|u(as)-z\|^2 + 2\varepsilon M_3,$$

we have

$$\inf_w \sup_t \|u(tw)-z\|^2 \leqq \mu_s \|u(as)-z\|^2 + \varepsilon^2 + 2\varepsilon M_3$$
$$= \mu_s \|u(s)-z\|^2 + \varepsilon^2 + 2\varepsilon M_3 .$$

This implies $\inf_w \sup_t \|u(tw)-z\|^2 \leqq \mu_s \|u(s)-z\|^2$ for all $z \in F(\mathcal{S})$. Then,

$$\mu_t \|u(t)-z\|^2 = \inf_s \sup_t \|u(ts)-z\|^2 \qquad \text{for all} \quad z \in F(\mathcal{S}),$$

and hence x_μ is independent of μ. So, we denote the element x_μ by x_0.

Finally, for an asymptotically invariant net $\{\mu_\alpha : \alpha \in A\}$ of means on $C(S)$ we show that w-$\lim x_{\mu_\alpha} = x_0$, where x_{μ_α} is an element of H such that $(\mu_\alpha)_t(u(t), y) = (x_{\mu_\alpha}, y)$ for all $y \in H$. Since $\|\mu_\alpha\| = 1$, $\{\mu_\alpha : \alpha \in A\}$ has a cluster point μ in the sense of w^*-topology. Such a μ is an invariant mean. In fact, for each $\varepsilon > 0$, $f \in C(S)$ and $s \in S$, there exists $\alpha_0 \in A$ such that

$$|\mu_\alpha(f) - \mu_\alpha(l_s f)| \leqq \frac{\varepsilon}{3} \qquad \text{for all} \quad \alpha \geqq \alpha_0 .$$

Since μ is a cluster point of the net $\{\mu_\alpha : \alpha \in A\}$, we can choose $\alpha_1 (\geqq \alpha_0)$ such that

$$|\mu_{\alpha_1}(f) - \mu(f)| \leqq \frac{\varepsilon}{3} \quad \text{and} \quad |\mu_{\alpha_1}(l_s f) - \mu(l_s f)| \leqq \frac{\varepsilon}{3} .$$

Hence, we have

$$|\mu(f) - \mu(l_s f)| \leqq |\mu(f) - \mu_{\alpha_1}(f)| + |\mu_{\alpha_1}(f) - \mu_{\alpha_1}(l_s f)|$$
$$+ |\mu_{\alpha_1}(l_s f) - \mu(l_s f)|$$
$$\leqq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu(f) = \mu(l_s f)$ for every $f \in C(S)$ and $s \in S$. This implies that μ is left invariant. Similarly, μ is right invariant. Since

$$\|x_{\mu_\alpha}\| = \sup_{\|y\| \leqq 1} |(x_{\mu_\alpha}, y)| = \sup_{\|y\| \leqq 1} |(\mu_\alpha)_t(u(t), y)|$$
$$\leqq \sup_t \|u(t)\| ,$$

we get $\{x_{\mu_\alpha} : \alpha \in A\}$ is bounded by virtue of the boundedness of $\{u(t) : t \in S\}$.

Hence we can choose a subnet $\{x_{\mu_{\alpha_\beta}}\}$ of the net $\{x_{\mu_\alpha}: \alpha\in A\}$ which converges weakly to some z in H. If λ is a cluster point of the net $\{\mu_{\alpha_\beta}\}$, then λ is a cluster point of the net $\{\mu_\alpha\}$ and hence λ is an invariant mean. Hence we obtain $z=x_\lambda=x_0$, which implies that x_{μ_α} converges weakly to $x_0\in F(\mathcal{S})$.

Q.E.D.

3. Fixed point theorem. Let X be a subspace of $B(S)$ containing constants. Then, according to Mizoguchi and Takahashi [5], a real-valued function μ on X is called a submean on X if it satisfies the following conditions:

(1) $\mu(f+g)\leqq\mu(f)+\mu(g)$ for every $f, g\in X$;
(2) $\mu(\alpha f)=\alpha\mu(f)$ for every f and $\alpha\geqq 0$;
(3) for $f, g\in X$, $f\leqq g$ implies $\mu(f)\leqq\mu(g)$;
(4) $\mu(c)=c$ for every constant c.

For a submean μ on X, we also use $\mu_t(f(t))$ instead of $\mu(f)$.

LEMMA [5]. *Let S be a semitopological semigroup, let X be a subspace of $B(S)$ containing constants and let μ be a submean on X. Let $\{x_t: t\in S\}$ be a bounded subset of a Hilbert space H and let D be a closed convex subset of H. Suppose that for each $x\in D$, the real-valued function f on S defined by*

$$f(t)=\|x_t-x\|^2 \quad \text{for all} \quad t\in S$$

belongs to X. Then, setting $g(x)=\mu_t\|x_t-x\|^2$ for all $x\in D$ and $r=\inf_{x\in D} g(x)$, there exists a unique element $z\in D$ such that $g(z)=r$ and $r+\|z-x\|^2\leqq g(x)$ for every $x\in D$.

Let X be a subspace of $B(S)$ containing constants and invariant under l_s, $s\in S$. Then a submean μ on X is said to be left invariant if $\mu(f)=\mu(l_s f)$ for all $s\in S$ and $f\in X$.

THEOREM 2. *Let C be a nonempty subspace of a Hilbert space H and let S be a semitopological semigroup. Suppose that X is a subspace of $B(S)$ containing constants and invariant under l_s, $s\in S$ and that there exists a left invariant submean μ on X. Let $\mathcal{S}=\{T_t: t\in S\}$ be a Lipschitzian semigroup on C with Lipschitz constants k_s, $s\in S$, and let u be an almost-orbit of $\mathcal{S}=\{T_t: t\in S\}$ such that $\{u(t): t\in S\}$ is bounded and $\bigcap_{s\in S}\overline{co}\{u(st): t\in S\}\subset C$. If for each $v\in H$, the real-valued function f on S defined by*

$$f(t)=\|u(t)-v\|^2 \quad \text{for all} \quad t\in S$$

and the function h on S defined by

$$h(t)=k_t^2 \quad \text{for all} \quad t\in S$$

belong to X and $\mu_t(k_t^2)\leqq 1$, then there exists an element $z\in C$ such that $T_s z=z$ for all $s\in S$.

Proof. Define a real-valued function g on H by

$$g(y)=\mu_t\|u(t)-y\|^2 \quad \text{for all} \quad y\in H.$$

Then, setting $r=\inf_{y\in H} g(y)$, by Lemma, there exists a unique element $z\in H$ such that $g(z)=r$ and $r+\|z-y\|^2\leqq g(y)$ for every $y\in H$. For each $s\in S$, let Q_s be the metric projection of H onto $\overline{co}\{u(st): t\in S\}$. Then, by Phelps [6], Q_s is nonexpansive and for each $t\in S$,

$$\|u(st)-Q_s z\|^2=\|Q_s u(st)-Q_s z\|^2\leqq\|u(st)-z\|^2.$$

So, we have

$$\begin{aligned}\mu_t\|u(t)-Q_s z\|^2&=\mu_t\|u(st)-Q_s z\|^2\leqq\mu_t\|u(st)-z\|^2\\ &=\mu_t\|u(t)-z\|^2,\end{aligned}$$

and thus $Q_s z=z$. This implies that $z\in\overline{co}\{u(st): t\in S\}$ for any $s\in S$, and hence $z\in\bigcap_{s\in S}\overline{co}\{u(st): t\in S\}$. We prove that $T_s z=z$ for all $s\in S$. Before proving it, we show $\mu_t\|u(st)-T_s z\|^2\leqq k_s^2\mu_t\|u(t)-z\|^2$ for every $s\in S$. Setting $k=\sup_{t\in S} k_t$ and $M=\sup_{t\in S}\|u(t)-z\|$, then for $\varepsilon>0$ there exists an $a\in S$ such that

$$\sup_{t,s}\|u(sat)-T_s u(at)\|<\varepsilon.$$

Since for each t, $s\in S$,

$$\begin{aligned}\|u(sat)-T_s z\|^2&\leqq\|u(sat)-T_s u(at)\|^2+\|T_s u(at)-T_s z\|^2\\ &\quad+2\|u(sat)-T_s u(at)\|\,\|T_s u(at)-T_s z\|\\ &\leqq\varepsilon^2+k_s^2\|u(at)-z\|^2+2\varepsilon kM,\end{aligned}$$

we obtain

$$\begin{aligned}\mu_t\|u(st)-T_s z\|^2&=\mu_t\|u(sat)-T_s z\|^2\\ &\leqq\varepsilon^2+2\varepsilon kM+k_s^2\mu_t\|u(at)-z\|^2\\ &=\varepsilon^2+2\varepsilon kM+k_s^2\mu_t\|u(t)-z\|^2,\end{aligned}$$

and hence

$$\mu_t\|u(st)-T_s z\|^2\leqq k_s^2\mu_t\|u(t)-z\|^2 \quad \text{for all} \quad s\in S.$$

Since by Lemma

$$\|z-y\|^2\leqq\mu_t\|u(t)-y\|^2-\mu_t\|u(t)-z\|^2 \quad \text{for all} \quad y\in H,$$

we have for each $s\in S$

$$\begin{aligned}\|z-T_s z\|^2&\leqq\mu_t\|u(t)-T_s z\|^2-\mu_t\|u(t)-z\|^2\\ &=\mu_t\|u(st)-T_s z\|^2-\mu_t\|u(t)-z\|^2\end{aligned}$$

$$\leqq (k_s^2-1)\mu_t\|u(t)-z\|^2,$$

and hence

$$\mu_s\|z-T_sz\|^2 \leqq (\mu_s(k_s^2)-1)\mu_t\|u(t)-z\|^2 \leqq 0.$$

This implies $\mu_s\|z-T_sz\|^2=0$. Since, for every a, $s\in S$,

$$\|z-T_az\|^2 \leqq 2\|z-T_sz\|^2+2\|T_sz-T_az\|^2,$$

we have

$$\begin{aligned}\|z-T_az\|^2 &\leqq 2\mu_s\|z-T_sz\|^2+2\mu_s\|T_sz-T_az\|^2\\ &=2\mu_s\|T_{as}z-T_az\|^2\\ &\leqq 2k_a^2\mu_s\|T_sz-z\|^2=0.\end{aligned}$$

This implies $T_sz=z$ for all $s\in S$. Q.E.D.

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