H. KIMURA
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ON F-DATA OF AUTOMORPHISM GROUPS OF COMPACT RIEMANN SURFACES
—THE CASE OF A_5—

BY HIDEYUKI KIMURA

Introduction.

Let X be a compact Riemann surface of genus $g(\geqq 2)$. We denote by $\mathrm{Aut}(X)$ the group of all conformal automorphisms on X. We take a basis of the space of abelian differentials of the first kind on X. We consider the canonical representation $\rho : \mathrm{Aut}(X) \to GL(g, \boldsymbol{C})$, for the basis. We denote by $\rho(AG; X)$ and $\rho(\sigma; X)$ the images of a subgroup $AG \subset \mathrm{Aut}(X)$ and an element $\sigma \in \mathrm{Aut}(X)$ by ρ, respectively. In the previous paper [2], for the $G(\cong D_8, Q_8) \subset GL(g, \boldsymbol{C})$, satisfying the CY- and RH-conditions, we have investigated surjective homomorphisms $\varphi : \Gamma(G) \to G$ to determine whether G arises from a compact Riemann surface of genus g. But there exists $G(\cong A_5) \subset GL(g, \boldsymbol{C})$, satisfying the CY- and RH-conditions, such that we can not determine that G arises from a compact Riemann surface of genus g by the same method. Therefore we introduce the collection of nonnegative integers which consists of information about characters of ρ and fixed points of AG. We shall call this F-data. In this paper, we study F-data of A_5 and determine what F-data of A_5 arises from a compact Riemann surface of genus g.

Notation. We denote by $\boldsymbol{Z}$, $\boldsymbol{C}$ and $\boldsymbol{Z}_{\geqq 0}$ the ring of rational integers, the complex number field and the set of nonnegative integers, respectively. For a finite set S we denote by $\#S$ the cardinality of S. For an element σ of a finite group we denote by $\#\sigma$ the order of σ. We denote by g an integer $(\geqq 2)$.

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§ 1. Preliminaries.

In this section we give preliminary results. We use the same notation and terminology as introduced in [3]. Throughout this section we denote by G a finite group.

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DEFINITION. We say that $G \subset GL(g, \boldsymbol{C})$ arises from a compact Riemann surface of genus g, if there exist a compact Riemann surface X of genus g and a subgroup $AG \subset \mathrm{Aut}(X)$ such that $\rho(AG; X)$ is $GL(g, \boldsymbol{C})$-conjugate to G.

1.1 We give a necessary and sufficient condition for an element of prime order of $GL(g, \boldsymbol{C})$ to arise from a compact Riemann surface of genus g, see [4] and [7].

THEOREM ([4]). *Let A be an element of prime order n of $GL(g, \boldsymbol{C})$. Then the following two conditions are equivalent:*

(1) *There is a compact Riemann surface X of genus g and an automorphism σ of X such that $\rho(\sigma; X)$ is conjugate to A.*

(2) *There are $s(\geqq 0)$ integers $\nu_1, \cdots, \nu_s$ which are prime to n such that* $\mathrm{Tr}\, A = 1 + \sum_{i=1}^{s} \frac{\zeta^{\nu_i}}{1-\zeta^{\nu_i}}$, *where* $\zeta = \zeta_n = \exp \frac{2\pi\sqrt{-1}}{n}$.

1.2 We define the CY- and RH-conditions, see [3] and [5].

DEFINITION. We say that a finite group $G \subset GL(g, \boldsymbol{C})$ satisfies the CY-condition if every element of $CY(G)$ arises from a compact Riemann surface of genus g.

Remark. It is known that A_5 has only elements of prime orders, i.e., 2, 3 and 5. The above theorem, mentioned in 1.1, suffices to check that $G(\cong A_5)$ satisfies the CY-condition.

DEFINITION. We say that G satisfies the RH-condition if G satisfies the E-condition and $l(H: G)$ is a nonnegative integer for any $H \in CY(G)$.

1.3 Now, we introduce the EX-condition, which is a necessary condition for G to arise from a compact Riemann surface. We explain a criterion whether G, satisfying the EX-condition, arises from a compact Riemann surface or not, see [6].

DEFINITION. Assume that $G \subset GL(g, \boldsymbol{C})$ satisfies the RH-condition. We say that G satisfies the EX-condition if there exists a surjective homomorphism $\varphi: \Gamma(G) \to G$ with $\#\varphi(\gamma_j) = m_j$ $(j=1, \cdots, r)$.

If $G \subset GL(g, \boldsymbol{C})$ satisfies the EX-condition, there exist a compact Riemann surface X of genus g and an injective homomorphism $G \to \mathrm{Aut}(X)$. Then for any element σ $(\#\sigma = m > 1)$ of G and $u \in \boldsymbol{Z}$ $((u, m)=1)$ we have (cf. [6])

$$\#\{P \in X \mid \zeta_P(\sigma) = \zeta_m^u\} = \sum_{m \mid m_j} \frac{1}{m_j} \#\{\alpha \in G \mid \sigma = \alpha\, \varphi(\gamma_j)^{u m_j / m} \alpha^{-1}\}.$$

By the Eichler trace formula, we have

$$\operatorname{Tr}\rho(\sigma\,;X)=1+\sum_{(u,m)=1}\sum_{m\mid m_j}\frac{1}{m_j}\#\{\alpha\in G\mid\sigma=\alpha\varphi(\gamma_j)^{um_j/m}\alpha^{-1}\}\frac{\zeta_m^u}{1-\zeta_m^u}.$$

If there exists a surjective homomorphism $\varphi:\Gamma(G)\to G$ such

$$\operatorname{Tr}\sigma=\operatorname{Tr}\rho(\sigma\,;X)\quad\text{for every}\quad\sigma\in G,$$

then we see that G arises from the compact Riemann surface X.

1.4 We denote by A_5 the alternating group of degree 5, i.e., the group which consists of all the even permutations of 5 letters. The character table of A_5 is as follows:

	(1)	(12)(34)	(123)	(12345)	(13524)
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	5	1	-1	0	0
χ_4	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_5	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

§2. A necessary and sufficient condition for CY- and RH-conditions.

2.1. PROPOSITION. *Let G be a finite subgroup of $GL(g, \boldsymbol{C})$ being isomorphic to A_5, χ_G be the character of the natural representation $G\to GL(g, \boldsymbol{C})$. Let $n_1\chi_1+\cdots+n_5\chi_5$, $n_i\in\boldsymbol{Z}_{\geq 0}$ be the decomposition into irreducible characters of χ_G. Then G satisfies the CY- and RH-conditions if and only if n_i's satisfy the following relations:*

(1) $1\geqq n_1+n_3-2n_4$

(2) $1\geqq n_1+n_2-n_3$

(3) $1\geqq n_1-n_2+n_4$

(4) $n_4=n_5$.

Remark. If G satisfies the CY- and RH-conditions, then we have

$$\text{(0)}\quad g=n_1+4n_2+5n_3+6n_4,$$

which means the degree of character χ_G.

Proof. We prove the *if*-part. We fix an isomorphism $\iota: A_5\to G$ and denote by A, B and C the images of (23)(45), (142) and (12345) via ι, respectively. We remark that, by Property 6 (I-1), two of A, B and C generate G. First we show that G satisfies the CY-condition. To see this, it is sufficient to show that A, B and C satisfy the condition (2) in Theorem 1.1.

The case of A.
Put

$$s:=2-2\chi_G(A)=2-2(n_1+n_3-n_4-n_5).$$

Then we see that s is a nonnegative integer by (1) and (4). If we put

$$\nu_1=\cdots=\nu_s=1,$$

then we have

$$\operatorname{Tr} A=1+s\frac{-1}{1-(-1)}.$$

Thus A arises from a compact Riemann surface of genus g by Theorem 1.1.

The case of B.
Put

$$s:=2-2\chi_G(B)=2-2(n_1+n_2-n_3).$$

Then we see that $s/2$ is a nonnegative integer by (2). If we put

$$\nu_1=\cdots=\nu_{s/2}=1,\ \nu_{(s/2)+1}=\cdots=\nu_s=2,$$

then we have

$$\operatorname{Tr} B=1+\frac{s}{2}\left(\frac{\omega}{1-\omega}+\frac{\omega^2}{1-\omega^2}\right),\quad \text{where } \omega=\zeta_3.$$

Thus B arises from a compact Riemann surface of genus g by Theorem 1.1.

The case of C.
Put

$$s:=2-2\chi_G(C)=2-2\left(n_1-n_2+\frac{1+\sqrt{5}}{2}n_4+\frac{1-\sqrt{5}}{2}n_5\right).$$

Then we see that $s/2$ is a nonnegative integer by (3) and (4). We take $p,q\in\boldsymbol{Z}_{\geqq 0}$ with $p+q=s/2$. If we put

$$\nu_1=\cdots=\nu_p=1,\ \nu_{p+1}=\cdots=\nu_{2p}=4,$$

$$\nu_{2p+1}=\cdots=\nu_{2p+q}=2,\ \nu_{2p+q+1}=\cdots=\nu_s=3,$$

then we have

$$\operatorname{Tr} C=1+p\left(\frac{\zeta}{1-\zeta}+\frac{\zeta^4}{1-\zeta^4}\right)+q\left(\frac{\zeta^2}{1-\zeta^2}+\frac{\zeta^3}{1-\zeta^3}\right),$$

where $\zeta=\zeta_5$.

Thus C arises from a compact Riemann surface of genus g by Theorem 1.1.

This means that G satisfies the CY-condition. It is easy to see that G satisfies the RH-condition. In fact we have

$$l(\langle A\rangle : G)=1-(n_1+n_3-n_4-n_5)$$

$$l(\langle B\rangle : G)=1-(n_1+n_2-n_3)$$

$$l(\langle C\rangle : G)=1-\left(n_1-n_2+\frac{1+\sqrt{5}}{2}n_4+\frac{1-\sqrt{5}}{2}n_5\right),$$

which are nonnegative integers by (1), $\cdots$, (4).

The *only-if*-part follows immediately from the fact:

$$\chi_G(C)=\chi_G(C^2) \quad \text{implies } n_4=n_5 .$$

Therefore we obtain our proposition.

Remark. In the case of B, since B is G-conjugate to B^2, we have

$$\#\{i \mid \nu_i=1\}=\#\{i \mid \nu_i=2\} .$$

In the case of C, since C(resp. C^2) is G-conjugate to C^4(resp. C^3), we have

$$\#\{i \mid \nu_i=1\}=\#\{i \mid \nu_i=4\} \quad (\text{resp. } \#\{i \mid \nu_i=2\}=\#\{i \mid \nu_i=3\}).$$

2.2 We introduce an F-data of A_5.

DEFINITION. We say that a collection of nonnegative integers $(n_1, \cdots, n_5; p, q)$, $p\geqq q$, is an F-data of A_5 if there exists a group $G(\cong A_5)\subset GL(g, \boldsymbol{C})$ satisfying the CY- and RH-conditions and that

$$\chi_G=n_1\chi_1+\cdots+n_5\chi_5 ,$$

$$1-\chi_G(C)=p+q \qquad \text{for every} \quad C(\in G) \text{ of order } 5.$$

Instead of $G(\cong A_5)\subset GL(g, \boldsymbol{C})$ which satisfies the CY- and RH-conditions, we consider an F-data $(n_1, \cdots, n_5; p, q)$ of A_5.

DEFINITION. Let $(n_1, \cdots, n_5; p, q)$ be an F-data of A_5. Define g by (0). We say that an F-data $(n_1, \cdots, n_5; p, q)$ of A_5 arises from a compact Riemann surface of genus g if there exist a compact Riemann surface X of genus g, a subgroup $AG(\cong A_5)\subset \mathrm{Aut}(X)$ and an element $C(\in AG)$ of order 5 such that

$$\mathrm{Tr}\,\rho(\circ\,; X)|_{AG}=n_1\chi_1+\cdots+n_5\chi_5=\chi_G$$

and

$$\left.\begin{aligned}\#\{P\in X \mid \zeta_P(C)=\zeta\}&=\#\{P\in X \mid \zeta_P(C)=\zeta^4\}=p ,\\ \#\{P\in X \mid \zeta_P(C)=\zeta^2\}&=\#\{P\in X \mid \zeta_P(C)=\zeta^3\}=q .\end{aligned}\right\}\cdots(*)$$

§ 3. Characterization of automorphism groups.

Hereafter, for simplicity, we put $l(\circ)=l(\circ : G)=l(\langle\circ\rangle : G)$.

3.1. THEOREM. *The notation being as in Proposition 2.1, let* $(n_1, \cdots, n_5; p, q)$ *be an F-data of* A_5. *If* $(n_1, \cdots, n_5; p, q)$ *does not arise from a compact Riemann surface of genus g, then*

$$(n_1, \cdots, n_5; p, q)=(0, 1, 0, 0, 0; 2, 0) \quad (g=4),$$

$$=(0, 2, 1, 0, 0; 2, 1) \quad (g=13)$$

or

$$=(1, 1, 1, 1, 1; 0, 0) \quad (g=16).$$

Remark. The F-data (0, 1, 0, 0, 0; 1, 1) (resp. (0, 2, 1, 0, 0; 3, 0)) arises from a compact Riemann surface of genus 4 (resp. 13) but not (0, 1, 0, 0, 0; 2, 0) (resp. (0, 2, 1, 0, 0; 2, 1)).

Before proving the theorem, we give some properties of A_5.

Property 1.

For every element h of A_5, there exist elements $a, b(\in A_5)$ such that $h=[a, b]$, where $[a, b]=aba^{-1}b^{-1}$.

Proof. It is sufficient to consider representatives of conjugacy classes of A_5, since we have the relation:

$$g^{-1}[x, y]g=[g^{-1}xg, g^{-1}yg]$$

for $g\in A_5$.

(1) order 2.

Put

$$a=(234), \quad b=(134).$$

Then we have

$$[a, b]=(12)(34).$$

(2) order 3.

Put

$$a=(13)(45), \quad b=(23)(35).$$

Then we have

$$[a, b]=(123).$$

(3) order 5.

Put

$$a_1=(25)(34), \quad b_1=(13)(45),$$

$$a_2=(25)(34), \quad b_2=(15)(24).$$

Then we have

$$[a_1, b_1]=(12345), \quad [a_2, b_2]=(13524).$$

Property 2.

For every $\theta \in A_5$, θ is A_5-conjugate to θ^{-1}.

Proof. In the case $\#\theta=2$, 3, it is easily verified from the character table of A_5. In the case $\#\theta=5$, it is verified from the following relation:

$$(12345)=(25)(34)(12345)^{-1}(25)(34).$$

Property 3.
Let ε and ε' be elements of A_5 of order 5. If ε is A_5-conjugate to ε', then the order of $\varepsilon\varepsilon'$ is not 2.

Property 4.
Let ε and ε' be distinct elements of A_5 of order 5. If ε is A_5-conjugate to ε' and $\varepsilon\varepsilon'$ is of order 5, then $\varepsilon\varepsilon'$ is A_5-conjugate to ε.

To prove Properties 3 and 4, we use a result from character theory (c. f. [G]):

THEOREM. *Denote the conjugacy classes of finite group G by K_i and let y_i be an element of K_i, $1 \leqq i \leqq r$. Then if λ_{ijk} is the number of times of given element of K_k can be expressed as an ordered product of an element of K_i and an element of K_j, we have*

$$\lambda_{ijk}=\frac{\#K_i \cdot \#K_j}{\#G} \sum_{m=1}^{r} \frac{\chi_m(y_i)\chi_m(y_j)\overline{\chi_m(y_k)}}{\chi_m(1)}$$

for $1 \leqq i, j, k \leqq r$.

Proof of Property 3. We apply the above theorem. We take a conjugacy class of order 5 as K_i and the conjugacy class of order 2 as K_k. Put $K_j=K_i$. Then we have $\lambda_{ijk}=0$. This means that there are no elements $\varepsilon, \varepsilon' \in K_i$ such that $\varepsilon\varepsilon' \in K_k$.

Proof of Property 4. We take a conjugacy class of order 5 as K_i and the other conjugacy class of order 5 as K_k. Put $K_j=K_i$. Then we have $\lambda_{ijk}=1$. This means than $x=y=z^3$ is the unique solution of the equation $z=x \cdot y (z \in K_k, x, y \in K_i)$. This completes the proof.

Property 5.
Let θ be an element of A_5 and N a positive integer. Then there are N elements $\theta_1, \cdots, \theta_N$ ($\in A_5$, not necessarily distinct) being A_5-conjugate to θ such that $\theta=\theta_1 \cdots \theta_N$.

Proof. This follows from the relations $(12)(34)=(13)(24)\cdot(14)(23)$, $(14352)=(12345)\cdot(13425)$, $(14325)=(13524)\cdot(12354)$. We prove only the case $\#\theta=2$. Case $N\equiv 0 \pmod 2$. The above relation means that there exist elements θ', θ'' of order 2 such that $\theta=\theta' \times \theta''$. Then we may take $\theta_1=\theta'$, $\theta_2=\cdots=\theta_N=\theta''$. Case $N \equiv 1 \pmod 2$. We may take $\theta_1=\cdots=\theta_N=\theta$.

Property 6.

We have the following presentations (I-1), $\cdots$, (VI-1) for A_5.

(I-1) $A_5=\langle\gamma, \delta, \varepsilon\,;\,\gamma^2=\delta^3=\varepsilon^5=\gamma\delta\varepsilon=1\rangle$.

(for example, $\gamma=(23)(45)$, $\delta=(142)$, $\varepsilon=(12345)$).

In the following, we write only relations and mean that ε_i's are A_5-conjugate to each other and ε_i's are not A_5-conjugate to η_i's.

(I-2) $(\gamma_1)^2=(\gamma_2)^2=(\gamma_3)^2=\varepsilon^5=\gamma_1\gamma_2\gamma_3\varepsilon=1$,

(for example, $\gamma_1=(23)(45)$, $\gamma_2=(12)(35)$, $\gamma_3=(14)(35)$, $\varepsilon=(12345)$).

(I-3) $\gamma^2=(\delta_1)^3=(\delta_2)^3=(\delta_3)^3=\gamma\delta_1\delta_2\delta_3=1$,

(for example, $\gamma=(23)(45)$, $\delta_1=(142)$, $\delta_2=(123)$, $\delta_3=(345)$).

(I-4) $\gamma^2=(\varepsilon_1)^5=(\varepsilon_2)^5=(\varepsilon_3)^5=\gamma\varepsilon_1\varepsilon_2\varepsilon_3=1$,

(for example, $\gamma=(23)(45)$, $\varepsilon_1=\varepsilon_3=(12345)$, $\varepsilon_2=(13254)$).

(I-5) $(\gamma_1)^2=(\gamma_2)^2=(\gamma_3)^2=(\gamma_4)^2=(\gamma_5)^2=\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5=1$,

(for example, $\gamma_1=(23)(45)$, $\gamma_2=(12)(35)$, $\gamma_3=(14)(35)$, $\gamma_4=(15)(24)$, $\gamma_5=(14)(23)$).

(I-6) $(\gamma_1)^2=(\gamma_2)^2=(\gamma_3)^2=\delta^3=\gamma_1\gamma_2\gamma_3\delta=1$,

(for example, $\gamma_1=(15)(24)$, $\gamma_2=(14)(23)$, $\gamma_3=(23)(45)$, $\delta=(142)$).

(I-7) $(\gamma_1)^2=(\gamma_2)^2=(\varepsilon_1)^5=(\varepsilon_2)^5=\gamma_1\gamma_2\varepsilon_1\varepsilon_2=1$,

(for example, $\gamma_1=\gamma_2=(23)(45)$, $\varepsilon_1=(15432)$, $\varepsilon_2=(12345)$).

(I-8) $(\gamma_1)^2=(\gamma_2)^2=(\gamma_3)^2=\varepsilon^5=\eta^5=\gamma_1\gamma_2\gamma_3\varepsilon\eta=1$,

(for example, $\gamma_1=\gamma_3=(14)(35)$, $\gamma_2=(12)(35)$, $\varepsilon=(12345)$, $\eta=(15234)$).

(I-9) $\gamma^2=\delta^3=(\varepsilon_1)^5=(\varepsilon_2)^5=\gamma\delta\varepsilon_1\varepsilon_2=1$,

(for example, $\gamma=(23)(45)$, $\delta=(142)$, $\varepsilon_1=(14352)$, $\varepsilon_2=(15243)$).

(I-10) $(\gamma_1)^2=(\gamma_2)^2=\delta^3=\varepsilon^5=\gamma_1\gamma_2\delta\varepsilon=1$,

(for example, $\gamma_1=(24)(35)$, $\gamma_2=(25)(34)$, $\delta=(142)$, $\varepsilon=(12345)$).

(I-11) $\gamma^2=(\delta_1)^3=(\delta_2)^3=\varepsilon^5=\gamma\delta_1\delta_2\varepsilon=1$,

(for example, $\gamma=(23)(45)$, $\delta_1=\delta_2=(124)$, $\varepsilon=(12345)$).

(II-1) $\gamma^2=\varepsilon^5=\eta^5=\gamma\varepsilon\eta=1$,

(for example, $\gamma=(13)(25)$, $\varepsilon=(12345)$, $\eta=(12435)$).

(II-2) $(\gamma_1)^2=(\gamma_2)^2=\varepsilon^5=\eta^5=\gamma_1\gamma_2\varepsilon\eta=1$,

(for example, $\gamma_1=(12)(35)$, $\gamma_2=(15)(23)$, $\varepsilon=(12345)$, $\eta=(12435)$).

(II-3) $\gamma^2=\delta^3=\varepsilon^5=\eta^5=\gamma\delta\varepsilon\eta=1$,

(for example, $\gamma=(13)(45)$, $\delta=(245)$, $\varepsilon=(12345)$, $\eta=(12435)$).

(III-1) $(\delta_1)^3=(\delta_2)^3=\varepsilon^5=\delta_1\delta_2\varepsilon=1$,

(for example, $\delta_1=(354)$, $\delta_2=(132)$, $\varepsilon=(12345)$).

(III-2) $(\gamma_1)^2=(\gamma_2)^2=(\delta_1)^3=(\delta_2)^3=\gamma_1\gamma_2\delta_1\delta_2=1$,

(for example, $\gamma_1=(15)(24)$, $\gamma_2=(14)(23)$, $\delta_1=(354)$, $\delta_2=(132)$).

(III-3) $(\delta_1)^3=(\delta_2)^3=(\delta_3)^3=(\delta_4)^3=\delta_1\delta_2\delta_3\delta_4=1$,

(for example, $\delta_1=(354)$, $\delta_2=(132)$, $\delta_3=(123)$, $\delta_4=(345)$).

(IV-1) $\delta^3=(\varepsilon_1)^5=(\varepsilon_2)^5=\delta\varepsilon_1\varepsilon_2=1$,

(for example, $\delta=(134)$, $\varepsilon_1=(12345)$, $\varepsilon_2=(13542)$).

(V-1) $\delta^3=\varepsilon^5=\eta^5=\delta\varepsilon\eta=1$,

(for example, $\delta=(145)$, $\varepsilon=(12345)$, $\eta=(14532)$).

(V-2) $(\varepsilon_1)^5=(\varepsilon_2)^5=(\varepsilon_3)^5=\eta^5=\varepsilon_1\varepsilon_2\varepsilon_3\eta=1$,

(for example, $\varepsilon_1=(13542)$, $\varepsilon_2=(15243)$, $\varepsilon_3=(12345)$, $\eta=(14532)$).

(V-3) $(\delta_1)^3=(\delta_2)^3=\varepsilon^5=\eta^5=\delta_1\delta_2\varepsilon\eta=1$,

(for example, $\delta_1=\delta_2=(154)$, $\varepsilon=(12345)$, $\eta=(14532)$).

(V-4) $\varepsilon^5=\eta^5=(\varepsilon\varepsilon^{-1}\eta\eta^{-1})=1$,

(for example, $\varepsilon=(15432)$, $\eta=(14532)$).

(VI-1) $(\varepsilon_1)^5=(\varepsilon_2)^5=(\varepsilon_3)^5=\varepsilon_1\varepsilon_2\varepsilon_3=1$,

(for example, $\varepsilon_1=(12345)$, $\varepsilon_2=(12534)$, $\varepsilon_3=(12453)$).

Proof of Property 6. It is sufficient to verify that $\gamma, \delta, \varepsilon, \cdots$ generate A_5. Here we remark the following FACT.

FACT. (c.f. [ATLAS]) Maximal subgroups of A_5 are A_4, D_{10} and S_3. For example, we consider the case (I-1). If $\langle\gamma, \delta, \varepsilon\rangle$ is a proper subgroup of A_5, since it has an element of order 5, $D_{10}\supset\langle\gamma, \delta, \varepsilon\rangle$. However, D_{10} does not contain an element of order 3. This means $A_5=\langle\gamma, \delta, \varepsilon\rangle$. The remaining cases are proved by the similar method.

Proof of Theorem. Let the notation be as in Proposition 2.1. Recall that

$$RH(G)=[n_1, 60; \overbrace{2, \cdots, 2}^{l(A)}, \overbrace{3, \cdots, 3}^{l(B)}, \overbrace{5, \cdots, 5}^{l(C)}]$$

and put

$$\Gamma(G)=\langle\alpha_1, \beta_1, \cdots, \alpha_{n_1}, \beta_{n_1}, \gamma_1, \cdots, \gamma_{l(A)}, \delta_1, \cdots, \delta_{l(B)}, \varepsilon_1, \cdots, \varepsilon_p, \eta_1, \cdots, \eta_q;$$

$$\prod_{j=1}^{l(A)}\gamma_j \prod_{k=1}^{l(B)}\delta_k \prod_{l=1}^{p}\varepsilon_l \prod_{m=1}^{q}\eta_m \prod_{i=1}^{n_1}[\alpha_i, \beta_i]=1$$

$$\gamma_1^2=\cdots=\gamma_{l(A)}^2=\delta_1^3=\cdots=\delta_{l(B)}^3=\varepsilon_1^5=\cdots=\varepsilon_p^5=\eta_1^5=\cdots=\eta_q^5=1\rangle.$$

We study whether G in Definition 2.2 satisfies the EX-condition or not. Using φ's, we verify whether $(n_1, \cdots, n_5; p, q)$ satisfy condition (*). We divide our proof into three cases according as $n_1 \geqq 2$, $n_1=1$ or $n_1=0$.

We assume that $n_1 \geqq 2$. We define $\varphi: \Gamma(G) \to G$ as follows: $\gamma_1, \cdots, \gamma_{l(A)} \to A$, $\delta_1, \cdots, \delta_{l(B)} \to B$, $\varepsilon_1, \cdots, \varepsilon_p \to C$, $\eta_1, \cdots, \eta_q \to C^2$, $\alpha_1 \to A$, $\beta_1 \to B$, $\alpha_2 \to U$, $\beta_2 \to V$, $\alpha_3, \beta_3, \cdots, \alpha_{n_1}, \beta_{n_1} \to 1$, where we choose U and V so that $\varphi(\prod \gamma_j \prod \delta_k \prod \varepsilon_l \prod \eta_m \prod [\alpha_\iota, \beta_\iota])=1$ holds. By virtue of Property 1, we can find them. Recall that A and B are the images of (23)(45) and (142), respectively. By Property 6 (I-1) we see that φ is surjective. To verify $\mathrm{Tr}\,\rho(\circ; X)=\chi_G(\circ)$, it is sufficient to check only for $\sigma=A$, B and C.

$$\mathrm{Tr}\,\rho(A; X)=1+2l(A)\frac{-1}{1+1}=1-l(A)=\chi_G(A).$$

$$\mathrm{Tr}\,\rho(B; X)=1+l(B)\left(\frac{\omega}{1-\omega}+\frac{\omega^2}{1-\omega^2}\right)=1-l(B)=\chi_G(B),$$

where $\omega=\zeta_3$.

$$\begin{aligned}\mathrm{Tr}\,\rho(C; X)&=1+p\left(\frac{\zeta}{1-\zeta}+\frac{\zeta^4}{1-\zeta^4}\right)+q\left(\frac{\zeta^2}{1-\zeta^2}+\frac{\zeta^3}{1-\zeta^3}\right)\\ &=1-l(C)=\chi_G(C),\end{aligned}$$

where $\zeta=\zeta_5$.

Thus, in this case, we see that $(n_1, \cdots, n_5; p, q)$ arises from a compact Riemann surface of genus g. In the following cases, we define only those φ's which have the desired property.

Next, we assume that $n_1=1$.

(i) The case $l(A)>0$ & $l(B)>0$.
We define $\varphi: \Gamma(G) \to G$ as follows: $\gamma_1, \cdots, \gamma_{l(A)} \to A$, $\delta_1, \cdots, \delta_{l(B)} \to B$, $\varepsilon_1, \cdots, \varepsilon_p \to C$, $\eta_1, \cdots, \eta_q \to C^2$, $\alpha_1 \to U$, $\beta_1 \to V$, where we choose U and V so that $\varphi(\prod \gamma_j \prod \delta_k \prod \varepsilon_l \prod \eta_m \prod [\alpha_\iota, \beta_\iota])=1$ holds. By virtue of Property 1, we can find them.

(ii) The case $l(A)=l(B)=0$.
By the assumption, we have $l(C)=n_4 \geqq 1$.

(ii-1) $l(C)=n_4=1$.
Recall that we fix isomorphism $\iota: A_5 \to G$. We define $\varphi: \Gamma(G) \to G$ as follows: $\varepsilon_1 \to \iota((12453))$, $\alpha_1 \to \iota((354))$, $\beta_1 \to \iota((13)(25))$.

(ii-2) $l(C)=n_4 \geqq 2$.

(ii-2-a) $p \geqq 2$.
We define $\varphi: \Gamma(G) \to G$ as follows: $\varepsilon_1 \to \iota((12345))$, $\varepsilon_2, \cdots, \varepsilon_p \to \iota((12534))$, $\eta_1, \cdots, \eta_q \to \iota((12534)^2)$, $\alpha_1 \to U$, $\beta_1 \to V$, where we choose U and V so that $\varphi(\prod \gamma_j \prod \delta_k \prod \varepsilon_l \prod \eta_m \prod [\alpha_\iota, \beta_\iota])=1$ holds.

(ii-2-b) $p=q=1$.
We define $\varphi: \Gamma(G)\to G$ as follows: $\varepsilon_1\to\iota((12345))$, $\eta_1\to\iota((12435))$, $\alpha_1\to U$, $\beta_1\to V$, where we choose U and V so that $\varphi(\Pi\gamma_j\Pi\delta_k\Pi\varepsilon_l\Pi\eta_m\Pi[\alpha_\iota, \beta_\iota])=1$ holds.

(iii) The case $l(A)=0$ & $l(B)>0$.
(iii-1) $l(C)>0$.
We define $\varphi: \Gamma(G)\to G$ as follows: $\delta_1, \cdots, \delta_{l(B)}\to B$, $\varepsilon_1, \cdots, \varepsilon_p\to C$, $\eta_1, \cdots, \eta_q\to C^2$, $\alpha_1\to U$, $\beta_1\to V$, where we choose U and V so that $\varphi(\Pi\gamma_j\Pi\delta_k\Pi\varepsilon_l\Pi\eta_m\Pi[\alpha_\iota, \beta_\iota])=1$ holds.

(iii-2) $l(C)=0$.
By the assumption, we have $l(B)=n_4\geqq 1$.
Considering Property 5, we can reduce this case to $\delta_1\to\iota((134))$, $\alpha_1\to\iota((12345))$, $\beta_1\to\iota((12435))$.

(iv) The case $l(A)>0$ & $l(B)=0$.
(iv-1) $l(C)>0$.
We define $\varphi: \Gamma(G)\to G$ as follows: $\gamma_1, \cdots, \gamma_{l(A)}\to A$, $\varepsilon_1, \cdots, \varepsilon_p\to C$, $\eta_1, \cdots, \eta_q\to C^2$, $\alpha_1\to U$, $\beta_1\to V$, where we choose U and V so that $\varphi(\Pi\gamma_j\Pi\delta_k\Pi\varepsilon_l\Pi\eta_m\Pi[\alpha_\iota, \beta_\iota])=1$ holds.

(iv-2) $l(C)=0$.
(iv-2-a) $l(A)\geqq 2$.
Considering Property 4, we can reduce this case to $\gamma_1\to\iota((25)(34))$, $\gamma_2\to\iota((14)(25))$, $\alpha_1\to\iota((12345))$, $\beta_1\to\iota((12435))$.

(iv-2-b) $l(A)=1$. (i.e. $n_1=n_2=n_3=n_4=n_5=1$, $p=q=0$.)
In this case, there is no φ having the desired properties. To see this, it is sufficient to show that there are no elements $\alpha, \beta(\in A_5)$ such that $A_5=\langle\alpha, \beta\mid \#[\alpha, \beta]=2\rangle$.

The case $\#\alpha=5$.
Since α is A_5-conjugate to $\beta\alpha^{-1}\beta^{-1}$, by Property 3, $\#[\alpha, \beta]\neq 2$.

The case $\#\alpha=\#\beta=3$.
By the abstract definition of A_4, i.e., $A_4=\langle S, T\mid S^3=T^3=(ST)^2=1\rangle$, we have $\#(\alpha\beta)\neq 2$, $\#(\alpha^{-1}\beta)\neq 2$, (of course $\#(\alpha\beta)\neq 1$, $\#(\alpha^{-1}\beta)\neq 1$). Suppose that $\#(\alpha\beta)=\#(\alpha^{-1}\beta)=3$. Then $\langle\alpha, \beta\rangle$ must be contained in $(3, 3\mid 3, 3)$ which is a group of order 27, see [1]. This is absurd.
Therefore we have $\#(\alpha\beta)=5$ or $\#(\alpha^{-1}\beta)=5$. Assume $\#(\alpha\beta)=5$. Since $\alpha\beta$ is A_5-conjugate to $\alpha^{-1}\beta^{-1}$, by Property 3, $\#[\alpha, \beta]\neq 2$. Next assume $\#(\alpha^{-1}\beta)=5$. Since $\alpha^{-1}\beta$ is A_5-conjugate to $\alpha\beta^{-1}$, by Property 3, $\#[\alpha^{-1}, \beta]=\#(\alpha^{-1}[\alpha, \beta]\alpha)=\#[\alpha, \beta]\neq 2$.

The case $\#\alpha=3$ & $\#\beta=2$.
By the abstract definitions of A_4 and S_3, i.e., $A_4=\langle S, T\mid S^3=T^2=(ST)^3=1\rangle$, $S_3=\langle S, T\mid S^3=T^2=(ST)^2=1\rangle$, we have $\#(\alpha\beta)\neq 3$, $\#(\alpha\beta)\neq 2$. Therefore $\#(\alpha\beta)=5$.

Since $\alpha\beta$ is A_5-conjugate to $\alpha^{-1}\beta^{-1}$, by Property 3, $\#[\alpha, \beta]\neq 2$.

The case $\#\alpha=\#\beta=2$.
Assume $\#[\alpha, \beta]=2$. Then we have

$$[\alpha, \beta]^2=1 \longleftrightarrow \alpha\beta=\beta\alpha \longleftrightarrow [\alpha, \beta]=1.$$

This is contradiction.
Thus we see that the F-data (1, 1, 1, 1, 1; 0, 0) does not arise from a compact Riemann surface of genus 16.

Finally, we assume that $n_1=0$. Then

$$l(A)=1-n_3+2n_4, \quad l(B)=1-n_2+n_3, \quad l(C)=1+n_2-n_4.$$

By simple calculation we see that the triple $(l(A), l(B), l(C))$ does not coincide any one of following:

(0, 0, 0), (1, 0, 0), (0, 1, 0), (2, 0, 0), (1, 1, 0), (0, 2, 0), (3, 0, 0),

(2, 1, 0), (1, 2, 0), (0, 3, 0), (4, 0, 0),

(0, 0, 1), (1, 0, 1), (0, 1, 1), (2, 0, 1),

(0, 0, 2), (1, 1, 1).

In the following, instead of defining φ, we give the relation (of A_5) which guarantees the extistence of φ.

(i) The case $l(A)=0$ & $l(B)=0$ & $l(C)\geqq 3$.
(i-1) $p\geqq 3$, $q=0$.
(i-2) $p\geqq 3$, $q=1$.
(i-3) $p\geqq 2$, $q\geqq 2$,
(i-4) $p=2$, $q=1$.
Considering Property 5, we can reduce (i-1), (i-2) and (i-3) to Property 6 (VI-1), (V-2) and (V-4), respectively. In the case of (i-4), by Property 4, the F-data (0, 2, 1, 0, 0; 2, 1) does not arise from a compact Riemann surface of genus 13.

(ii) The case $l(A)\geqq 1$ & $l(B)\geqq 1$ & $l(C)\geqq 2$.
(ii-1) $p\geqq 2$, $q=0$.
(ii-2) $p\geqq 1$, $q\geqq 1$.
Considering Property 5, we can reduce (ii-1) and (ii-2) to Property 6 (I-9) and (II-3), respectively.

(iii) The case $l(A)\geqq 2$ & $l(B)\geqq 1$ & $l(C)=1$.
Considering Property 5, we can reduce this case to Property 6 (I-10).

(iv) The case $l(A)=1$ & $l(B)\geqq 2$ & $l(C)=1$.
Considering Property 5, we can reduce this case to Property 6 (I-11).

(v) The case $l(A)\geqq 3$ & $l(B)=0$ & $l(C)\geqq 1$.
(v-1) $p\geqq 1$, $q=0$.
(v-2) $p\geqq 1$, $q\geqq 1$.
Considering Property 5, we can reduce (v-1) and (v-2) to Property 6 (I-2) and (I-8), respectively.

(vi) The case $l(A)=0$ & $l(B)\geqq 2$ & $l(C)\geqq 1$.
(vi-1) $p\geqq 1$, $q=0$.
(vi-2) $p\geqq 1$, $q\geqq 1$.
Considering Property 5, we can reduce (vi-1) and (vi-2) to Property 6 (III-1) and (V-3), respectively.

(vii) The case $l(A)=1$ & $l(B)=0$ & $l(C)\geqq 2$.
(vii-1) $p\geqq 3$, $q=0$.
(vii-2) $p\geqq 1$, $q\geqq 1$.
(vii-3) $p=2$, $q=0$.
Considering Property 5, we can reduce (vii-1) and (vii-2) to Property 6 (I-4) and (II-1), respectively. In the case of (vii-3), by Property 3, the F-data (0, 1, 0, 0, 0; 2, 0) does not arise from a compact Riemann surface of genus 4.

(viii) The case $l(A)=2$ & $l(B)=0$ & $l(C)\geqq 2$.
(viii-1) $p\geqq 2$, $q=0$.
(viii-2) $p\geqq 1$, $q\geqq 1$.
Considering Property 5, we can reduce (viii-1) and (viii-2) to Property 6 (I-7) and (II-2), respectively.

(ix) The case $l(A)=0$ & $l(B)=1$ & $l(C)\geqq 2$.
(ix-1) $p\geqq 2$, $q=0$.
(ix-2) $p\geqq 1$, $q\geqq 1$.
Considering Property 5, we can reduce (ix-1) and (ix-2) to Property 6 (IV-1) and (V-1), respectively.

(x) The case $l(A)\geqq 2$ & $l(B)\geqq 2$ & $l(C)=0$.
Considering Property 5, we can reduce this case to Property 6 (III-2).

(xi) The case $l(A)\geqq 3$ & $l(B)=1$ & $l(C)=0$.
Considering Property 5, we can reduce this case of Property 6 (I-6).

(xii) The case $l(A)\geqq 5$ & $l(B)=0$ & $l(C)=0$.
Considering Property 5, we can reduce this case to Property 6 (I-5).

(xiii) The case $l(A)=1$ & $l(B)\geqq 3$ & $l(C)=0$.
Considering Property 5, we can reduce this case to Property 6 (I-3).

(xiv) The case $l(A)=0$ & $l(B)\geqq 4$ & $l(C)=0$.
Considering Property 5, we can reduce this case to Property 6 (III-3).
This completes the proof.

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TOKYO INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATHEMATICS
MEGURO-KU, TOKYO, JAPAN
AND
FELLOWSHIPS OF THE JAPAN SOCIETY
FOR THE PROMOTION OF SCIENCE
FOR JAPANESE JUNIOR SCIENTISTS