

1-TYPE SUBMANIFOLDS OF THE COMPLEX PROJECTIVE SPACE

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§ 0. Introduction.

The standard way to define the complex projective space CP^m is by means of Hopf fibration $\pi : S^{2m+1} \rightarrow CP^m$ where CP^m is obtained as a quotient space of $S^{2m+1} \subset C^{m+1}$ under the natural action of the group S^1 of complex numbers of norm 1. It is not completely trivial to find an isometric embedding of CP^m into a Euclidean space, the most natural one (the first standard embedding) defined as follows. For $p \in CP^m$ pick $z \in \pi^{-1}(p) \subset S^{2m+1}$ and let $\phi(p) = z\bar{z}^t$ (z is regarded as a column vector in C^{m+1}). Then ϕ is a well defined map that embeds CP^m into the set $H(m+1)$ of Hermitian matrices of degree $m+1$, the latter being a Euclidean space of dimension $N = (m+1)^2$. Now if $x : M^n \rightarrow CP^m$ is an isometric immersion of a compact n -dimensional Riemannian manifold into the complex projective space, than we also have the associated immersion $\tilde{x} = \phi \circ x : M^n \rightarrow H(m+1) = E^N$.

On the other hand, there is a notion of finite type immersion $f : M^n \rightarrow E^N$ whereby a compact Riemannian manifold M^n is said to be of k -type (via f) if f is globally decomposable as $f = f_0 + f_1 + \dots + f_k$, where $f_0 = \text{const}$ (could be 0) and f_i 's are vector eigenfunctions of Laplacian from $k+1$ different eigenspaces [3]. For example, a well known result of Takahashi [15] characterizes compact 1-type submanifolds as minimal in a hypersphere of E^N . Studying finite type immersions is difficult in general, but there have been several results on low type submanifolds of CP^m . In particular, A. Ros has the following classification of CR -minimal submanifolds of CP^m which are of 1-type via the first standard embedding.

THEOREM A [10]. *Let M^n be a compact CR -minimal submanifold of CP^m . Then M^n is of 1-type via ϕ if and only if*

a) *n is even and M^n is congruent to the complex projective space $CP^{n/2}$ immersed as a totally geodesic complex submanifold of CP^m .*

b) *M^n is a totally real minimal submanifold of a complex totally geodesic CP^n in CP^m .*

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Ros' proof relies on the equivariancy of the embedding ϕ and uses both assumptions *CR* and minimal in an essential way. In this paper, using somewhat different and more refined approach, we give an extension of this result (Th. 1). It turns out that one can classify 1-type submanifolds of CP^m assuming them to be only *CR* (or with parallel mean curvature vector in CP^m) and in the case of submanifolds of low dimension or codimension, without any a priori assumptions at all. We also give some properties of 1-type submanifolds of CP^m in general, and produce an extrinsic bound on the first eigenvalue of the Laplacian.

§1. Preliminaries.

Let $S^{2m+1} = \{z \in \mathbf{C}^{m+1} | \bar{z}^t z = 1\}$ be the unit hypersphere centered at the origin. Then by identifying z with λz , where $\lambda \in \mathbf{C}$, $|\lambda| = 1$, we get the quotient space which defines CP^m . We actually obtain the fiber bundle $\pi: S^{2m+1} \rightarrow CP^m$ with structure group S^1 whose vertical space at $v \in S^{2m+1}$ is $\mathbf{R}\{iv\}$. Let \mathcal{H}_v be the orthogonal complement of the vertical space so that $T_v S^{2m+1} = \mathbf{R}\{iv\} \oplus \mathcal{H}_v$. One defines the metric on CP^m such that $d\pi_v: \mathcal{H}_v \rightarrow T_{\pi(v)} CP^m$ is an isometry for every v . The complex structure J of CP^m is defined by means of the complex structure of \mathbf{C}^{m+1} and the uniqueness of the horizontal lift. Then π is a Riemannian submersion with totally geodesic fibers and CP^m is Kähler manifold with constant holomorphic sectional curvature 4.

Let $H(m+1) = \{P \in GL(m+1; \mathbf{C}) | \bar{P} = P^t\}$ be the set of $(m+1) \times (m+1)$ Hermitian matrices. Equipped with the metric

$$\langle P, Q \rangle = (1/2) \operatorname{tr}(PQ), \quad P, Q \in H(m+1)$$

it becomes the usual Euclidean space E^N of dimension $N = (m+1)^2$. It is well known that the map $\check{\phi}: S^{2m+1} \rightarrow H(m+1)$ given by $\check{\phi}(z) = z\bar{z}^t$, where $z \in S^{2m+1}$ is regarded as a column matrix, induces an isometric immersion $\phi: CP^m \rightarrow H(m+1)$ such that

$$\phi(CP^m) = \{P \in H(m+1) | P^2 = P, \operatorname{tr} P = 1\}.$$

$\phi(CP^m)$ actually lies in the intersection of the hyperplane $H_1 = \{P \in H(m+1) | \operatorname{tr} P = 1\}$ and the hypersphere of E^N with radius $r = \sqrt{m/2(m+1)}$ centered at $I/(m+1)$, I being the identity matrix of $H(m+1)$. For more details on the embedding ϕ see [3], [10], [11], [13], [14].

By identifying CP^m with $\phi(CP^m)$, which we will do henceforth, the tangent and the normal space of CP^m at a point $P \in CP^m$ are given respectively by

$$T_{\phi} CP^m = \{X \in H(m+1) | XP + PX = X\}$$

$$T_{\phi}^{\perp} CP^m = \{Z \in H(m+1) | ZP = PZ\}.$$

The second fundamental form of the embedding ϕ at a point P is given by

$$\sigma(X, Y) = (XY + YX)(I - 2P)$$

and it is known that σ is parallel, $\bar{\nabla}\sigma = 0$. We have also

$$(1) \quad \langle \sigma(X, Y), I \rangle = 0, \quad \langle \sigma(X, Y), P \rangle = -\langle X, Y \rangle.$$

The complex structure J and the shape operator \bar{A} of ϕ at P are given respectively by

$$JX = \sqrt{-1}(I - 2P)X, \quad \bar{A}_Z X = (XZ - ZX)(I - 2P).$$

The following formula for the Weingarten map of ϕ in the direction of $\sigma(X, Y)$ is due to A. Ros [11]

$$(2) \quad \bar{A}_{\sigma(X, Y)}V = 2\langle X, Y \rangle V + \langle Y, V \rangle X + \langle X, V \rangle Y + \langle JY, V \rangle JX + \langle JX, V \rangle JY.$$

Let $f : M^n \rightarrow E^N$ be an isometric immersion of a compact Riemannian manifold into a Euclidean space, and let Δ be the Laplace operator acting on smooth functions on M . An immersion f (or a submanifold M) is said to be of 1-type if it can be decomposed as $f = f_0 + f_t$, where $f_0 = \text{const}$ and $\Delta f_t = \lambda_t f_t$. In particular, if $x : M^n \rightarrow CP^m$ is an isometric immersion of a compact manifold and $\tilde{x} = \phi \circ x : M^n \rightarrow H(m+1)$, we will study submanifolds of CP^m with 1-type \tilde{x} , i.e. those for which $\tilde{x} = \tilde{x}_0 + \tilde{x}_t$ where \tilde{x}_0 is a constant matrix in $H(m+1)$ and \tilde{x}_t a nonconstant eigenfunction of the Laplacian (more precisely, each entry of the matrix \tilde{x}_t is either zero or an eigenfunction from the same eigenspace).

For the immersions $M^n \xrightarrow{x} CP^m \xrightarrow{\phi} H(m+1)$ let $\tilde{\nabla}$ denote the Euclidean connection in $H(m+1)$; $\bar{\nabla}, \bar{A}, \bar{D}$ denote the (Levi-Civita) connection on CP^m , Weingarten map and the connection in the normal bundle of ϕ ; ∇, A, D denote the induced connection on M , Weingarten map and the connection in the normal bundle of the immersion x and $h, \tilde{h}, H, \tilde{H}$ denote the second fundamental forms and the mean curvature vectors of the immersions x and $\tilde{x} = \phi \circ x$ respectively. $\Gamma(TM)$ and $\Gamma(T^\perp M)$ will denote the set of all smooth sections (vector fields) of the tangent and normal bundle of M respectively. We consider a local frame of orthonormal vector fields $e_1, \dots, e_n, e_{n+1}, \dots, e_{2m}$ tangent to CP^m where the first n vectors are tangent to M and the remaining ones are normal to M . In general, index i will range from 1 to n and index r from $n+1$ to $2m$, so that e_i, e_r represent basis vectors tangent to M and normal to M respectively. All immersions are assumed smooth and all manifolds will be assumed compact, connected smooth Riemannian manifolds of dimension ≥ 2 .

A submanifold M of the complex projective space is called a *CR-submanifold* if the tangent bundle TM splits into an orthogonal direct sum of two (differentiable) distributions $TM = \mathcal{D} \oplus \mathcal{D}^\perp$ such that $J\mathcal{D} \subset \mathcal{D}$ and $J\mathcal{D}^\perp \subset T^\perp M$. If $\mathcal{D}^\perp = \{0\}$ we have a complex (invariant) submanifold and if $\mathcal{D} = \{0\}$ we have a totally real one.

§ 2. 1-Type submanifolds of CP^m

Given a submanifold $x : M^n \rightarrow CP^m$. Then $\tilde{x} = \phi \circ x$ is a 1-type immersion if $\tilde{x} = \tilde{x}_0 + \tilde{x}_t$ as above so that $\Delta \tilde{x} = \lambda_t(\tilde{x} - \tilde{x}_0)$. On the other hand,

$$\Delta \tilde{x} = -n\tilde{H} = -nH - \sum_i \sigma(e_i, e_i)$$

and therefore \tilde{x} is of 1-type if and only if

$$(3) \quad -nH - \sum_i \sigma(e_i, e_i) = \lambda_t(\tilde{x} - \tilde{x}_0).$$

Differentiating this formula along an arbitrary vector field $X \in \Gamma(TM)$, that is, taking $\tilde{\nabla}_X$ of both sides we obtain (we assume for simplicity $\nabla_X e_i = 0$ at a point where the calculation is done)

$$(4) \quad nA_H X - nD_X H - n\sigma(X, H) + \sum_i \bar{A}_{\sigma(e_i, e_i)} X - \sum_i \bar{D}_X \sigma(e_i, e_i) = \lambda_t X.$$

Because σ is parallel we have

$$(5) \quad \sum_i \bar{D}_X \sigma(e_i, e_i) = 2 \sum_i \sigma(\nabla_X e_i, e_i) = 2 \sum_i \sigma(h(X, e_i), e_i) = 2 \sum_r \sigma(A_r X, e_r),$$

$$(6) \quad n\sigma(X, H) = \sum_r (\text{tr } A_r) \sigma(X, e_r).$$

From (2) we also compute

$$(7) \quad \begin{aligned} \sum_i \bar{A}_{\sigma(e_i, e_i)} X &= \sum_i [2X + 2\langle e_i, X \rangle e_i - 2\langle e_i, JX \rangle J e_i] \\ &= 2(n+1)X - 2J(JX)_T \end{aligned}$$

where for a vector W tangent to CP^m , W_T and W_N denote its orthogonal projections to the tangent and normal spaces $TM, T^\perp M$ at a given point respectively. Therefore, by comparing parts tangent and normal to CP^m in (4) and using (5), (6) and (7) we obtain

LEMMA 1. *Let $x : M^n \rightarrow CP^m$ be an isometric immersion of a compact manifold. Then $\tilde{x} = \phi \circ x$ is of 1-type if and only if for every vector field $X \in \Gamma(TM)$*

$$(8) \quad nA_H X - nD_X H + 2(n+1)X - 2J(JX)_T = \lambda_t X \text{ for some constant } \lambda_t \text{ and}$$

$$(9) \quad \sum_r \sigma(B_r X, e_r) = 0, \text{ where } B_r = (\text{tr } A_r)I + 2A_r.$$

Because $\bar{A}_Z = 0 \Leftrightarrow Z = \rho I$ we obtain

LEMMA 2. *With the assumptions as above, \tilde{x} is of 1-type if and only if for any vector fields $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$ we have*

$$(8) \quad nA_H X - nD_X H + 2(n+1)X - 2J(JX)_T = \lambda_t X, \quad \lambda_t = \text{const}$$

$$(10) \quad \sum_r [\langle B_r X, Y \rangle e_r + \langle J e_r, Y \rangle J B_r X + \langle J B_r X, Y \rangle J e_r] = 0,$$

$$(11) \quad \sum_r [\langle e_r, \xi \rangle B_r X + \langle J e_r, \xi \rangle J B_r X + \langle J B_r X, \xi \rangle J e_r] = 0.$$

Proof. First we claim that $\bar{A}_Z = 0 \Leftrightarrow Z = \rho I$, where $\rho \in \mathbf{R}$ and I is the $(m+1) \times (m+1)$ identity matrix. If $Z = \rho I$ then obviously $\bar{A}_Z = 0$. Conversely, let us suppose that $\bar{A}_Z X = (XZ - ZX)(I - 2P) = 0$ for every $X \in T_P CP^m$. After multiplying this formula by $I - 2P$ on the right we see that $XZ = ZX$. Because the immersion is equivariant it is enough to prove the claim at the "origin"

$$P_0 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

Any vector X tangent to CP^m at P_0 can be represented as $X = \begin{pmatrix} 0 & \bar{b}^t \\ b & \bar{b} \end{pmatrix}$,

where $b \in C^m$, and $\mathbf{0}$ is the $m \times m$ zero matrix, and a vector Z normal to CP^m at P_0 has the form

$$Z = \begin{pmatrix} \rho & 0 \\ 0 & S \end{pmatrix}, \text{ where } \rho \in \mathbf{R} \text{ and } S \in H(m) \text{ (see [10]).}$$

Condition $XZ = ZX$ for every $X \in TCP^m$ implies then that $Sb = \rho b$ for every $b \in C^m$, hence $S = \rho I$ and in turn $Z = \rho I$. Now we show that (9) is equivalent to (10) and (11) together. First observe that by (2), formulas (10) and (11) are equivalent to $\sum_r \bar{A}_{\sigma(B_r X, e_r)} Y = 0$ and $\sum_r \bar{A}_{\sigma(B_r X, e_r)} \xi = 0$ respectively. However, if $\sum_r \bar{A}_{\sigma(B_r X, e_r)} = 0$ then from the above we have $\sum_r \sigma(B_r X, e_r) = \rho I$, but since $\langle \sigma(V, W), I \rangle = 0$ we obtain (9). If (9) is true then by tracing this argument backwards we obtain (10) and (11).

It is easy to see now that if \tilde{x} is of 1-type, the conditions that M has parallel mean curvature vector in CP^m and that it is a CR submanifold of CP^m are equivalent. Namely, if M is a CR submanifold then $J(JX)_T$ is always tangent to M and therefore from (8) $DH = 0$. Conversely, suppose $DH = 0$ (so that the mean curvature α is constant), and let X be a principal direction of A_H i.e. $A_H X = \mu X$. Then from (8) we get $2(JX)_T = [\lambda_t - 2(n+1) - \mu n]JX$, so that at any given point we have either

$$1^\circ. \quad \mu = \frac{1}{n}[\lambda_t - 2(n+1)], \text{ equivalently } (JX)_T = 0 \text{ or}$$

$$2^\circ. \quad \mu = \frac{1}{n}[\lambda_t - 2(n+2)], \text{ equivalently } (JX)_N = 0$$

Therefore, there are at most two distinct principal curvatures $\mu_1 = \frac{1}{n}[\lambda_t - 2(n+2)]$ with multiplicity p and $\mu_2 = \frac{1}{n}[\lambda_t - 2(n+1)]$ with multiplicity q . Moreover,

$$n\alpha^2 = \text{tr } A_H = \frac{p}{n}[\lambda_t - 2(n+2)] + \frac{q}{n}[\lambda_t - 2(n+1)] = \lambda_t - 2(n+2) + \frac{2q}{n},$$

and hence p and q are constant. Let

$$\mathcal{D} = \{X \in TM \mid A_H X = \mu_1 X\} = \{X \in TM \mid (JX)_N = 0\} \text{ and}$$

$$\mathcal{D}^\perp = \{X \in TM \mid A_H X = \mu_2 X\} = \{X \in TM \mid (JX)_T = 0\}.$$

Then $J\mathcal{D} = \mathcal{D}$ and $J\mathcal{D}^\perp \subset T^\perp M$ and consequently M is a CR submanifold. We are now ready to prove the main theorem

THEOREM 1. *Let $x : M^n \rightarrow \mathbb{C}P^m$ be an isometric immersion of a compact Riemannian manifold into $\mathbb{C}P^m$ with parallel mean curvature vector (or let M be a CR submanifold of $\mathbb{C}P^m$). Then $\tilde{x} = \phi \circ x$ is of 1-type if and only if one of the following cases occurs*

1) n is even, and M^n is congruent to the complex projective space $\mathbb{C}P^{n/2}$ immersed as a totally geodesic complex submanifold of $\mathbb{C}P^m$.

2) M^n is a totally real minimal submanifold of a complex totally geodesic $\mathbb{C}P^n$ in $\mathbb{C}P^m$.

3) n is odd, and M^n is congruent to the geodesic hypersphere

$$M_{0, (n-1)/2}^{\mathbb{C}^1} = \pi(S^1(\sqrt{1/(n+3)}) \times S^n(\sqrt{(n+2)/(n+3)}))$$

of the complex projective space $\mathbb{C}P^{(n+1)/2}$ immersed as a totally geodesic complex submanifold of $\mathbb{C}P^m$.

Proof. Any of the cases 1), 2) and 3) is indeed of 1-type as follows from Theorem A of Ros and the result of [8] (also from our Lemma 2). Now assume that the submanifold of $\mathbb{C}P^m$ with $DH=0$ is of 1-type. As shown above such submanifold is also CR, so that at each point of M we have the following orthogonal splitting of the tangent space of $\mathbb{C}P^m$.

$$(12) \quad TCP^m = \mathcal{D} \oplus \mathcal{D}^\perp \oplus J\mathcal{D}^\perp \oplus \nu$$

where \mathcal{D} and ν are holomorphic subbundles and $TM = \mathcal{D} \oplus \mathcal{D}^\perp$, $T^\perp M = J\mathcal{D}^\perp \oplus \nu$. Let $p = 2k = \dim \mathcal{D}$ and $q = \dim \mathcal{D}^\perp$ so that $p+q=n$ and let $e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q}, e_{n+1}, \dots, e_{n+q}, e_{n+2+1}, \dots, e_{2m}$ be a local orthonormal basis for TCP^m adapted to the decomposition (12) where $Je_{p+j} = e_{n+j}$ for $j=1, \dots, q$. Because ν is a holomorphic subbundle we can assume that $\{e_{n+q+1}, \dots, e_{2m}\}$ is a J -basis for ν i.e. it is equal to $\{e_s, e_{s*} = Je_s\}$, $s=n+q+1, \dots, m+k+q$. Now in formula (11) let $\xi = e_s$, where s is any index in the range $n+q+1, \dots, m+k+q$. Then (11) reduces to

$$0 = \sum_r [\langle e_r, e_s \rangle B_r X + \langle J e_r, e_s \rangle J B_r X] = B_s X - J B_{s*} X$$

Hence, $B_s X = J B_{s*} X$ and $B_{s*} X = -J B_s X$ for a J -basis vector e_s in ν . We now show that $A_s = A_{s*} = 0$. Using the above we compute the trace of B_s as

$$(n+2) \operatorname{tr} A_s = \operatorname{tr} B_s = \sum_i \langle B_s e_i, e_i \rangle = \sum_i \langle J B_{s*} e_i, e_i \rangle = 0$$

because B_{s*} is symmetric and J skew symmetric. So $\operatorname{tr} A_s = 0$ and similarly $\operatorname{tr} A_{s*} = 0$. Note that at this point we can conclude that $H \in J\mathcal{D}^\perp$ since $\operatorname{tr} A_r = 0$ for any $e_r \in \nu$. Moreover, from the above we get

$$(13) \quad A_{s*} X = -J A_s X.$$

Let $Y \in \mathcal{D}^\perp$ be arbitrary. The formula (10) becomes

$$(14) \quad \sum_r [\langle B_r X, Y \rangle e_r + \langle J e_r, Y \rangle J B_r X] = 0.$$

Fix $e_s \in \nu$ and take the inner product of the above formula with e_s to get $\langle B_s X, Y \rangle = 0$ for every $Y \in \mathcal{D}^\perp$. This implies $B_s X \in \mathcal{D}$ and hence $A_s X \in \mathcal{D}$ for any X tangent to M and any $e_s \in \nu$, so that $J A_s X \in \mathcal{D}$ as well.

On the other hand, because CP^m is Kähler we have for $e_s \in \nu$ (note also $e_{s*} \in \nu$) $\bar{\nabla}_X (J e_s) = J \bar{\nabla}_X e_s$, i. e.

$$(15) \quad -A_{s*} X + D_X e_{s*} = -J A_s X + J D_X e_s,$$

where both $A_{s*} X$ and $J A_s X$ are tangent to M and $D_X e_{s*}$ is normal to M . Again choosing $Y \in \mathcal{D}^\perp$ we obtain from (15)

$$\langle J D_X e_s, Y \rangle = \langle J D_X e_s - J A_s X, Y \rangle = \langle D_X e_{s*} - A_{s*} X, Y \rangle = 0.$$

Obviously $\langle J D_X e_s, Y \rangle = 0$ for every $Y \in \mathcal{D}$ so that $J D_X e_s$ is normal to M (actually it belongs to ν). Comparing parts tangent to M in (15) we conclude $A_{s*} X = J A_s X$. But together with (13) this implies $A_{s*} = A_s = 0$.

We can assume that the constant mean curvature α is not zero because if it is, we have a CR -minimal submanifold of 1-type and therefore by Theorem A cases 1) or 2) occur. Since $H \in J\mathcal{D}^\perp$ we also assume from now on that the unit normal vector e_{n+1} is in the direction of H so that in the basis of principal directions the Weingarten map A_{n+1} has the following block form

$$A_{n+1} = \begin{pmatrix} \beta I_p & 0 \\ 0 & \gamma I_q \end{pmatrix}, \text{ where } p = \dim \mathcal{D}, q = \dim \mathcal{D}^\perp$$

and $\beta = (1/n\alpha)[\lambda_t - 2(n+2)]$, $\gamma = (1/n\alpha)[\lambda_t - 2(n+1)]$. From $\operatorname{tr} A_{n+1} = n\alpha$ we get $\beta = \alpha - 2q/n^2\alpha$ and $\gamma = \alpha + 2p/n^2\alpha$. Next we show that $q = \dim \mathcal{D}^\perp = 1$. Let $Y \in \mathcal{D}$ in formula (10) so that

$$\sum_r [\langle B_r X, Y \rangle e_r + \langle J B_r X, Y \rangle J e_r] = 0.$$

Because $B_r=0$ for $r \geq n+q+1$ and $Je_r \in \mathcal{D}^\perp$ for $n+1 \leq r \leq n+q$ we obtain $\langle B_r X, Y \rangle = 0$ i.e. $B_r X \in \mathcal{D}^\perp$ for every $X \in \Gamma(TM)$ and every r . In particular, for $X \in \mathcal{D}$ we get (if $\mathcal{D} = \{0\}$ there is no β) $B_{n+1}X = n\alpha X + 2A_{n+1}X = (n\alpha + 2\beta)X \in \mathcal{D}^\perp$ so that $n\alpha + 2\beta = 0$ i.e.

$$(16) \quad \beta = \alpha - 2q/n^2\alpha = -n\alpha/2.$$

For $r = n+j$, $j = 2, \dots, q$ we obtain $B_{n+j}X = 2A_{n+j}X \in \mathcal{D}^\perp$ for every $X \in \Gamma(TM)$ and hence $0 = \langle A_{n+j}\mathcal{D}^\perp, \mathcal{D} \rangle = \langle \mathcal{D}^\perp, A_{n+j}\mathcal{D} \rangle$ so that $A_{n+j}\mathcal{D} = 0$, which implies that A_{n+j} has the block form

$$A_{n+j} = \begin{pmatrix} 0 & 0 \\ 0 & M_j \end{pmatrix}$$

where M_j is $q \times q$ symmetric matrix with $\text{tr } M_j = \text{tr } A_{n+j} = 0$. Furthermore, it is known from [2] that

$$(17) \quad A_{jX}Y = A_{jY}X \quad \text{for } X, Y \in \mathcal{D}^\perp.$$

Letting $X = e_{p+1}$, $Y = e_{p+j}$, $2 \leq j \leq q$ in this formula we get $A_{n+j}e_{p+1} = A_{n+1}e_{p+j} = \gamma e_{p+j}$ and therefore M_j has the form

$$(18) \quad M_j = \begin{pmatrix} 0 & \dots & \gamma & \dots & 0 \\ \vdots & & & & \\ \gamma & & * & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$$

where γ is in the j^{th} slot in the first row and the first column and all other entries there are 0. We now show that $q = \dim \mathcal{D}^\perp = 1$. Suppose that $q \geq 2$. Then for $j \geq 2$ we have $B_{n+j} = 2A_{n+j}$ and from the above $\langle B_{n+j}e_{p+j}, e_{p+1} \rangle = \langle e_{p+j}, B_{n+j}e_{p+1} \rangle = 2\gamma$. On the other hand, putting $X = e_{p+j}$, $Y = e_{p+1}$ in (10) we have

$$0 = \sum_r \langle B_r e_{p+j}, e_{p+1} \rangle e_r - J B_{n+1} e_{p+j} = \sum_r \langle B_r e_{p+j}, e_{p+1} \rangle e_r - (n\alpha + 2\gamma)e_{n+j}$$

and by comparing e_{n+j} -components we obtain $\langle B_{n+j}e_{p+j}, e_{p+1} \rangle = n\alpha + 2\gamma$ so that $\alpha = 0$. Since we assumed nonzero mean curvature this contradiction proves $q = 1$ i.e. purely real distribution is one-dimensional and $n = 2k + 1$ is odd. Since $\dim M \geq 2$ it follows that $\dim \mathcal{D} \neq 0$. Then from (16) we find $\alpha = 2/n\sqrt{n+2}$ and

$$A_{n+1} = \frac{1}{\sqrt{n+2}} \begin{pmatrix} -I_{n-1} & 0 \\ 0 & n+1 \end{pmatrix}, \quad A_r = 0 \quad \text{for } r \geq n+2.$$

Therefore, the first normal space equals $N_1 = \text{Im } h = J\mathcal{D}^\perp = \mathbf{R}\{H\}$ and since $DH = 0$, N_1 is parallel, i.e. $DN_1 \subset N_1$. Also, $J(TM \oplus N_1) = TM \oplus N_1$ and therefore by the reduction theorem of B. Opozda [12, Prop. 3.1] there exists a totally

geodesic complex submanifold \bar{M} of CP^m so that the image of M is contained in \bar{M} and $T_p\bar{M}=T_pM\oplus N_{1p}$ for every $p\in M$. Note that even though Opozda's theorem is formally proved for totally real submanifolds, it is clear that her proof holds for CR submanifolds verbatim. The idea is as follows. Choose $p\in M$ and $v\in\pi^{-1}(p)$ and define $K_v=(\pi_*|_{\mathcal{A}_v})^{-1}(T_pM\oplus N_{1p})\oplus\text{Span}_C\{v\}$. Then K_v is a complex subspace of C^{m+1} . Set $\bar{M}_v=\pi(K_v\cap S^{2m+1})$ so that \bar{M}_v is a totally geodesic complex submanifold of CP^m and $T_p\bar{M}=T_pM\oplus N_{1p}$. From the conditions above it follows then that \bar{M}_v does not depend on p and $v\in\pi^{-1}(p)$; see also [1]. This means that M is a (real) hypersurface of some complex totally geodesic $CP^{(n+1)/2}$ and by [8] M is the geodesic hypersphere of the case 3) which completes the proof.

At the present time we are not able to give the classification of 1-type submanifolds of CP^m without any a priori assumptions but some general properties of those are given in the next theorem.

THEOREM 2. *If M is a 1-type submanifold of CP^m with the mean curvature vector H , then the mean curvature α of the immersion is constant, JH is tangent to M and integral curves of JH are geodesics (if $H\neq 0$).*

Proof. As before, since B_r is symmetric and J skew-symmetric we have $\sum_i \langle JB_r e_i, e_i \rangle = 0$ and putting $X=Y=e_i$ in (10) and adding on $i=1, \dots, n$ we obtain

$$\begin{aligned} 0 &= \sum_r (\text{tr } B_r) e_r + \sum_{r,i} \langle J e_r, e_i \rangle J B_r e_i \\ &= n(n+2)H + \sum_r J B_r (J e_r)_T, \end{aligned}$$

and therefore

$$JH = \frac{1}{n(n+2)} \sum_r B_r (J e_r)_T \in TM.$$

Now taking the inner product of (8) with H we get

$$n \langle D_X H, H \rangle = -2 \langle J(JX)_T, H \rangle = 2 \langle (JX)_T, JH \rangle = 2 \langle JX, JH \rangle = 2 \langle X, H \rangle = 0,$$

hence $X \langle H, H \rangle = X \alpha^2 = 0$ for every tangent vector X , and therefore the mean curvature α is constant. Letting $X=JH$ in (8) we obtain

$$D_{JH} H = 0, \quad A_H JH = \frac{1}{n} [\lambda_t - 2(n+1)] JH.$$

Therefore,

$$\nabla_{JH}(JH) = J \bar{\nabla}_{JH} H = -J A_H JH + J D_{JH} H = \frac{1}{n} [\lambda_t - 2(n+1)] H.$$

On the other hand $\bar{\nabla}_{JH}(JH) = \nabla_{JH}(JH) + h(JH, JH)$, and we conclude $\nabla_{JH}(JH) = 0$ i.e. the integral curves of JH are geodesics.

By way of a remark we note that having proved that $\alpha = \text{const}$ for 1-type submanifolds of CP^m , in Th. 1 instead of $DH=0$ we could have made slightly weaker assumption (if $\alpha \neq 0$) that the normalized mean curvature vector is parallel i.e. $De_{n+1}=0$.

COROLLARY 1. *Let $x: M^2 \rightarrow CP^m$ be an isometric immersion of a closed surface in CP^m . Then $\tilde{x} = \phi \circ x$ is of 1-type if and only if one of the cases 1) or 2) of Theorem 1 occurs. The same conclusion holds if M is a 1-type submanifold of codimension two.*

Proof. We assume $H \neq 0$, otherwise M is minimal and we proceed as before to show that M is one of the cases 1) or 2). By Theorem 2, JH is tangent to M and is a principal direction of A_H . Let $U \perp JH$ be the other principal direction of A_H i.e. $A_H U = \mu U$. Then U and JH span TM , and $\langle JU, U \rangle = 0$, $\langle JU, JH \rangle = \langle U, H \rangle = 0$ and consequently $JU \in T^\perp M$. Since also $JJH \in T^\perp M$ we conclude that M is a totally real surface and hence of case 2) of Theorem 1.

If M is codimension two submanifold, let H, ξ span $T^\perp M (\xi \perp H)$. Then since JH is tangent to M we have $\langle J\xi, H \rangle = -\langle \xi, JH \rangle = 0$ and $\langle J\xi, \xi \rangle = 0$ so that $J\xi$ is also tangent to M . This implies that M is a CR-submanifold and therefore by Theorem 1 one of the cases 1) and 2).

We now want to carry out investigation of 1-type submanifolds of CP^m without any a priori assumptions. As before, we can show that there are at most two distinct principal curvatures of A_H and corresponding two distributions $\mathcal{D} = \text{Ker } B_{n+1}$ and $\mathcal{D}^\perp = \text{Im } B_{n+1}$ with $J\mathcal{D} \subset T^\perp M$, but we are not able to show $J\mathcal{D} \subset TM$.

Let $Y = Je_{n+1}$ in (10) which then becomes $\sum_r \langle B_r X, Je_{n+1} \rangle e_r + JB_{n+1} X = 0$ from where we obtain

$$(19) \quad JB_{n+1} X = -\sum_r \langle B_r X, Je_{n+1} \rangle e_r \in T^\perp M.$$

If we define $\mathcal{D}^\perp = \{X \in TM \mid JX \in T^\perp M\}$ then we see that $\text{Im } B_{n+1} \subset \mathcal{D}^\perp$. At each point of M we have the orthogonal decomposition

$$(20) \quad TM = \text{Ker } B_{n+1} \oplus \text{Im } B_{n+1}.$$

Let e_1, \dots, e_p be an orthonormal basis for $\text{Ker } B_{n+1}$; these are eigenvectors for eigenvalue 0. The remaining orthonormal eigenvectors e_{p+1}, \dots, e_{p+q} of B_{n+1} span $\text{Im } B_{n+1}$. Because $B_{n+1} = (n\alpha)I + 2A_{n+1}$ we see that e_1, \dots, e_p are eigenvectors of A_{n+1} for eigenvalue $\beta = -n\alpha/2$ and e_{p+1}, \dots, e_n are also eigenvectors of A_{n+1} corresponding to eigenvalue $\gamma = (1/n\alpha)[\lambda_i - 2(n+1)]$ (This follows from (8) and (19)). In other words

$$(21) \quad A_{n+1} = \begin{pmatrix} \beta I_p & 0 \\ 0 & \gamma I_q \end{pmatrix}, \text{ where } p = \dim \text{Ker } B_{n+1}, q = \dim \text{Im } B_{n+1}.$$

Moreover, p and q are constant because $p+q=n$ and $p\beta+q\gamma=n\alpha=\text{const}$ ($\beta\neq\gamma$, see below). If $\alpha=0$ then the submanifold is CR and therefore classified by Theorem A. From now on we assume $\alpha\neq 0$. Not all eigenvalues of A_{n+1} can be equal to $\beta=-n\alpha/2$ because if that is the case we would have $n\alpha=\text{tr} A_{n+1}=-n^2\alpha/2$ which implies $\alpha=0$. Also not all eigenvalues can be equal to $\gamma=(1/n\alpha)[\lambda_t-2(n+1)]$ because if it is so it would follow that $(JX)_T=0$ for every X tangent to M and hence M would be totally real submanifold and therefore minimal by Theorem 1. It is also clear now that $\text{Im} B_{n+1}=\mathcal{D}^\perp$ because if $\text{Im} B_{n+1}$ is a proper subspace of \mathcal{D}^\perp there would exist a vector $X\in\text{Ker} B_{n+1}$ satisfying $JX\in T^\perp M$. Then (8) implies $A_{n+1}X=(1/n\alpha)[\lambda_t-2(n+1)]X$. On the other hand for $X\in\text{Ker} B_{n+1}$ we have $A_{n+1}X=-(n\alpha/2)X$ which is a contradiction proving the claim. In particular we can set $e_{p+1}=-Je_{n+1}\in\text{Im} B_{n+1}$. Let us also denote $\mathcal{D}=\text{Ker} B_{n+1}$. With respect to the decomposition (20) B_{n+1} has the following block form

$$B_{n+1}=\begin{pmatrix} 0 & 0 \\ 0 & (n\alpha+2\gamma)I_q \end{pmatrix}.$$

We now prove the following lemma which enables us to classify 1-type submanifolds of CP^m for which $q=\dim\text{Im} B_{n+1}=1$.

LEMMA 3. *Let M^n be a 1-type submanifolds of CP^m for which $q=1$. Then M^n has parallel mean curvature vector.*

Proof. Let $X\in\mathcal{D}=\text{Ker} B_{n+1}$ and $\xi=Je_{p+1}=e_{n+1}$ in (11). We get

$$B_{n+1}X+\sum_r\langle B_rX, e_{p+1}\rangle Je_r=0$$

and consequently $B_rX\perp e_{p+1}$. It follows immediately that also

$$(22) \quad A_rX\perp e_{p+1}, \quad \text{for every } r=n+1, \dots, 2m \text{ and every } X\in\mathcal{D}.$$

Further, by putting $X=Y=e_{p+1}=-Je_{n+1}$ in (10) we get

$$\begin{aligned} 0 &= \sum_r \langle B_r e_{p+1}, e_{p+1} \rangle e_r - J B_{n+1} e_{p+1} \\ &= \sum_r \langle B_r e_{p+1}, e_{p+1} \rangle e_r - (n\alpha + 2\gamma) e_{n+1} \\ &= \sum_{r \geq n+2} \langle B_r e_{p+1}, e_{p+1} \rangle e_r \end{aligned}$$

and therefore $B_r e_{p+1} \perp e_{p+1}$ for $r \geq n+2$. Because $B_r = 2A_r$ we also have

$$(23) \quad A_r e_{p+1} \perp e_{p+1} \quad \text{for every } r = n+2, \dots, 2m$$

We recall the Codazzi equation (see e. g. [3])

$$(24) \quad (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) = (\bar{R}(X, Y)Z)_N, \quad X, Y, Z \in \Gamma(TM),$$

where the curvature tensor $\bar{R}(X, Y)Z$ of CP^m is given by.

$$(25) \quad \bar{R}(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle JY, Z \rangle JX - \langle JX, Z \rangle JY + 2\langle X, JY \rangle JZ.$$

Let $X=Z=e_{p+1}, Y=e_i, i=1, \dots, p$. Then from (25) we get

$$\bar{R}(e_{p+1}, e_i)e_{p+1} = -e_i + 3\langle Je_i, e_{p+1} \rangle e_{n+1} = -e_i$$

so that the Codazzi equation (24) yields

$$\begin{aligned} 0 &= (\nabla_{e_{p+1}}h)(e_i, e_{p+1}) - (\nabla_{e_i}h)(e_{p+1}, e_{p+1}) \\ &= De_{p+1}(h(e_i, e_{p+1})) - h(\nabla_{e_{p+1}}e_i, e_{p+1}) - h(e_i, \nabla_{e_{p+1}}e_{p+1}) \\ &\quad - De_i(h(e_{p+1}, e_{p+1})) + 2h(\nabla_{e_i}e_{p+1}, e_{p+1}). \end{aligned}$$

We have $\nabla_{e_{p+1}}e_{p+1}=0$ by Theorem 2, $h(e_i, e_{p+1}) = \sum_{\tau} \langle A_{\tau}e_i, e_{p+1} \rangle e_{\tau} = 0$ by (22), $h(e_{p+1}, e_{p+1}) = \gamma e_{n+1}$ by (21) and (23) and the equation above becomes

$$(26) \quad 0 = -\omega_i^k(e_{p+1})h(e_k, e_{p+1}) - De_i(\gamma e_{n+1}) + 2\omega_{p+1}^k(e_i)h(e_k, e_{p+1})$$

where ω_i^k denote the connection 1-forms and summation is on $k=1, \dots, n$. Because $q=1, p=n-1, \beta=-n\alpha/2$, we find $\gamma = \alpha n(n+1)/2 = \text{const} \neq 0$. Now using (21), (22), (23) and (26) we have

$$-\omega_i^{p+1}(e_{p+1})\gamma e_{n+1} = \gamma De_i e_{n+1},$$

and because $\gamma \neq 0$, and $e_{n+1} \perp De_i e_{n+1}$ we obtain $De_i e_{n+1} = 0, i=1, \dots, p$. In other words $D_X H = 0$ for $X \in \mathcal{D} = \text{Ker } B_{n+1}$ and from (8) we also have $D_X H = 0$ for $X \in \mathcal{D}^{\perp}$ and therefore $DH = 0$ and M is a CR-submanifold.

COROLLARY 2. *Let $x : M^3 \rightarrow CP^m$ be a compact 3-dimensional submanifold. Then $\tilde{x} = \phi \circ x$ is of 1-type if and only if M^3 is one of the cases 2) or 3) of Theorem 1 (with $n=3$).*

Proof. If M is minimal the classification follows as before. If $\alpha \neq 0$ suppose first that $p = \dim \mathcal{D} = 2, q = \dim \mathcal{D}^{\perp} = 1$. Then Lemma 3 and Theorem 1 give desired classification. If $p=1$, and $q=2$, let $\mathcal{D} = \text{Span}\{X\}, \mathcal{D}^{\perp} = \text{Span}\{Y, Z\}$. In that case $\langle JX, X \rangle = 0, \langle JX, Y \rangle = -\langle X, JY \rangle = 0$ and likewise $\langle JX, Z \rangle = 0$, which implies that M is a totally real submanifold and Theorem 1 proves the claim again.

The same idea can be used to classify codimension 3 submanifolds of CP^m which are of 1-type. Again, they are of three cases described in Theorem 1.

§ 3. Remarks and related results

1. The complete classification in Theorem 1 is not given because the classification of minimal totally real submanifolds M^n of CP^n is not known, and

seemingly there are quite a few of them. For example, in [7] it was shown that the torus $T^2=S^1 \times S^1$ can be minimally immersed in CP^2 as a totally real submanifold. Also, according to Naitoh [9, §5] there are totally real minimal submanifolds M^n of CP^n including $SU(k)/SO(k)$ ($n=(1/2)(k+2)(k-1)$), $SU(2k)/Sp(k)$ ($n=2k^2-k-1$), $SU(k) \times SU(k)/SU(k)$ ($n=k^2-1$), E_6/F_4 ($n=26$), and Ejiri [5] shows that certain circle bundles (tubes) over S^2 of sufficiently high even Chern number can be minimally immersed in CP^3 as a totally real submanifolds. Note however that there are some pinching theorems stating that under certain conditions on curvature (e.g. sectional curvature $K>0$) every totally real minimal submanifold M^n of CP^n is necessarily totally geodesic; see [16], [4], [6].

2. Study of finite type immersions often provides some information on the spectrum of the Laplacian on M . For example, if M^n is a compact manifold that can be immersed into CP^m with constant mean curvature α then the first nonzero eigenvalue of the Laplacian satisfies

$$\lambda_1 \leq n\alpha^2 + 2(n+2)$$

and the equality holds if and only if $M^n=CP^{n/2}$ and the immersion is totally geodesic in CP^m (in which case $\alpha=0$). Namely, if $\tilde{x} = \sum_{i=0}^{\infty} \tilde{x}_i$, $\Delta \tilde{x}_i = \lambda_i \tilde{x}_i$, $\lambda_0=0$ then

$$\int_M \langle \Delta \tilde{x}, \Delta \tilde{x} \rangle dV - \lambda_1 \int_M \langle \Delta \tilde{x}, \tilde{x} \rangle dV = \sum_{i=1}^{\infty} \lambda_i (\lambda_i - \lambda_1) \int_M \langle \tilde{x}_i, \tilde{x}_i \rangle dV \geq 0,$$

equality holding if and only if \tilde{x} is of 1-type of order $\{1\}$. On the other hand, because $\Delta \tilde{x} = -nH - \sum_i \sigma(e_i, e_i)$, by using the equality (2.11) of [8] and (1) we get $\langle \Delta \tilde{x}, \tilde{x} \rangle = n$ and

$$\langle \Delta \tilde{x}, \Delta \tilde{x} \rangle = n^2 \alpha^2 + 2n(n+1) + 2 \sum_i \langle (Je_i)_T, (Je_i)_T \rangle \leq n^2 \alpha^2 + 2n(n+2),$$

equality holding if and only if $(JX)_N=0$ for every $X \in \Gamma(TM)$, which implies that M is an invariant submanifold, thus CR , so that Theorem 1 proves the claim (cf. [3, p. 314], [10]). Actually, it is easy to see from the above that for any compact submanifold M^n of $CP^m(4)$ we have

$$\lambda_1 \leq 2(n+2) + \frac{n}{\text{vol}(M)} \int_M \alpha^2 dV.$$

3. Let $\phi: RP^m \rightarrow SM(m+1)$ be the first standard embedding of the real projective space into the set of symmetric matrices of degree $m+1$. There is a problem of determining all submanifolds $x: M^n \rightarrow RP^m$ which are of 1-type via ϕ , i.e. for which $\tilde{x} = \phi \circ x$ is a 1-type immersion. Chen [3] shows that the only complete 1-type submanifold of RP^m which is minimal in RP^m is the totally geodesic RP^n . Because of the corresponding result for CR submani-

folds of CP^m we can classify complete 1-type submanifolds of RP^m without assuming minimality, thus generalizing the above result of Chen. Namely, we can embed RP^m in CP^m as a totally real totally geodesic submanifold such that the Hopf fibrations

$$\begin{array}{ccc} S^m & \subset & S^{2m+1} \\ \downarrow \pi & & \downarrow \pi \\ RP^m & \subset & CP^m \end{array}$$

are compatible. Then a 1-type submanifold $M^n \subset RP^m$ becomes a CR (totally real) submanifold of CP^m and the 1st standard embedding of RP^m is just the restriction of the 1st standard embedding of CP^m . By Theorem 1, case (2), it follows that M^n is a totally real minimal submanifold of some totally geodesic complex CP^n in CP^m . Therefore, M^n is minimal in CP^m and since $RP^m \subset CP^m$ is totally geodesic it follows that M^n is also minimal in RP^m so that the above result of Chen classifies M^n as a totally geodesic RP^n .

As above, for a compact submanifold M^n of RP^m the first non-zero eigenvalue of the Laplacian satisfies

$$\lambda_1 \leq 2(n+1) + \frac{n}{\text{vol}(M)} \int_M \alpha^2 dV$$

with equality if and only if M^n is the totally geodesic RP^n .

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Added in Proof. It should be remarked that the classification results presented in this paper are essentially of local nature. Namely, Lemmas 2 and 3, Theorems 1 and 2 and Corollaries 1 and 2 are valid without assuming M compact but merely complete. That is because Lemma 1, Theorem A and the result of [8] used in the proof are valid for any submanifold M .

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