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HYPERSURFACE SECTIONS OF TORIC SINGULARITIES

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Introduction

As is well-known, we can obtain much information about hypersurface singularities $\{f=0\}$ in $\boldsymbol{C}^{n+1}$ by the Newton polyhedra $\Gamma_+(f)\subset\boldsymbol{R}^{n+1}$ of the defining equations f. (For instance, see [5] and [11].) In this paper, we define the Newton polyhedra also for hypersurface sections (X, x) of any toric singularity (Y, y) and show that a part of the results in [11] are valid. On the other hand, as we see in the last of §2 and in §3, we obtain as (X, x) many singularities, a part of which are not complete intersections. For instance, 2-dimensional cusp singularities with multiplicities greater than 4 and a 3-dimensional singularity with a resolution whose exceptional set is an Enriques surface. Moreover, in the case that the ambient space Y has only an isolated singularity, these singularities (X, x) are obviously smoothable. Hence we can obtain examples of smoothable cusp singularities (see §3). In this paper, we are mainly concerned about singularities (X, x) with the plurigenera $\delta_m(X, x)$ which are not greater than 1 and at least one of which is equal to 1. (For the definition of plurigenera, see [11].) We call such singularities, *periodically elliptic singularities*, following Ishii [2].

In Section 1, we recall some facts about toric singularities, necessary in this paper.

In Section 2, we show a sufficient condition on the Newton polyhedra of defining equations f of X, under which (X, x) are periodically elliptic singularities and give some examples.

In Section 3, we show a sufficient condition on a 3-dimensional non-terminal Gorenstein toric singularity (Y, y), under which hyperplane sections (X, x) of (Y, y) are simple elliptic singularities or cusp singularities. We can determine the multiplicities of these singularities.

In Section 4, we show that if $H^1(X\setminus\{x\}, i^*\Theta_Y)=0$ and $\dim X\geqq 3$, then we can concretely construct a locally semiuniversal family of deformations of (X, x) and that any small deformation of (X, x) is also a hypersurface section of Y, where $i: X\hookrightarrow Y$ is the inclusion map and Θ_Y is the tangent sheaf of Y. The above condition is satisfied, if Y is a quotient of $\boldsymbol{C}^{n+1}$, by torus actions.

We use the notation and the terminology in [4] freely.

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that hypersurface sections (X, x) of toric singularities (Y, y) are Cohen-Macaulay and that (X, x) are smoothable, if (Y, y) is an isolated singularity.

§1. Toric singularities

Let N be a free $\boldsymbol{Z}$-module of rank $n+1$ and let $N_{\boldsymbol{R}}=N\otimes_{\boldsymbol{Z}}\boldsymbol{R}$. Let $M=\mathrm{Hom}(N, \boldsymbol{Z})$ be the $\boldsymbol{Z}$-module dual to N with the canonical pairing $\langle\ ,\ \rangle: M\times N\to\boldsymbol{Z}$. Let $\sigma=\boldsymbol{R}_{\geq 0}u_1+\boldsymbol{R}_{\geq 0}u_2+\cdots+\boldsymbol{R}_{\geq 0}u_s$ be an $(n+1)$-dimensional strongly convex rational polyhedral cone in $N_{\boldsymbol{R}}$. Here we may assume that $\boldsymbol{R}_{\geq 0}u_i$ are 1-dimensional faces of σ, for $i=1$ through s. Let Y be the complex space associated to $\mathrm{Spec}(\boldsymbol{C}[M\cap\sigma^*])$ and let $\boldsymbol{e}(v): Y\to\boldsymbol{C}$ be the natural extension to Y of the character $v\otimes 1_{\boldsymbol{C}^\times}: T_N\to\boldsymbol{C}^\times$ for each v in $M\cap\sigma^*$, where $\sigma^*:=\{v\in M_{\boldsymbol{R}}\,|\,\langle v, u\rangle\geqq 0$ for all $u\in\sigma\setminus\{0\}\}$ is the dual cone of σ and $T_N=\mathrm{Spec}(\boldsymbol{C}[M])$ $(\cong(\boldsymbol{C}^\times)^{n+1})$. Then any holomorphic function f on a neighborhood U of $y=\mathrm{orb}(\sigma)$ is expressed as the series:

$$f=\textstyle\sum_{v\in\sigma^*\cap M} c_v\boldsymbol{e}(v).$$

Hence we can define the Newton polyhedron $\Gamma_+(f)$ and the Newton boundary $\Gamma(f)$ of f in the same way as in the case of $Y=\boldsymbol{C}^{n+1}$. More precisely, $\Gamma_+(f)$ is the convex hull of $\cup_{c_v\neq 0}v+\sigma^*$ and $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$. Let $D=D_1+D_2+\cdots+D_s$, where D_i is the closure of $\mathrm{orb}(\boldsymbol{R}_{\geq 0}u_i)$. Here we note that $Y\setminus D=T_N$ and that Y is a Cohen-Macaulay space by [4, Corollary 3.9]. Let $\{v_1, v_2, \cdots, v_{n+1}\}$ be a basis of M and let $w_i=\boldsymbol{e}(v_i)$ for $i=1$ through $n+1$. Then $(w_1, w_2, \cdots, w_{n+1})$ is a global coordinate of T_N. Let $\nu=(dw_1/w_1)\wedge(dw_2/w_2)\wedge\cdots\wedge(dw_{n+1}/w_{n+1})$. Then ν is a nowhere vanishing holomorphic $(n+1)$-form on T_N whose natural extension to Y has poles of order 1 along D.

DEFINITION 1.1. (Y, y) is said to be r-Gorenstein, if there exists a nowhere vanishing holomorphic r-ple $(n+1)$-form on $U\setminus\mathrm{Sing}(U)$ for an open neighborhood U of y, where $\mathrm{Sing}(U)$ is the singular locus of U.

Since (Y, y) is a Cohen-Macaulay singularity, (Y, y) is Gorenstein, if it is 1-Gorenstein.

PROPOSITION 1.2. ([6, the footnote of p294]) *(Y, y) is r-Gorenstein, if and only if there exists an element v_0 in $M_{\boldsymbol{Q}}$ such that $rv_0\in M$ and that $\langle v_0, u_i\rangle=1$ for $i=1$ through s, where we assume that $u_1, u_2, \cdots$ and u_s are primitive elements in N. (Here we note that the above v_0 is uniquely determined by σ, if it exists.)*

Proof. Let v_0 be an element in $M_{\boldsymbol{Q}}$ satisfying the above condition. Then $\theta:=\boldsymbol{e}(rv_0)\nu^r$ is a nowhere vanishing holomorphic r-ple $(n+1)$-form on $Y\setminus\mathrm{Sing}(Y)$, because $\boldsymbol{e}(rv_0)$ has zeros of order $\langle rv_0, u_i\rangle=r$ only along D. Conversely, assume that (Y, y) is r-Gorenstein, i.e., there exists a nowhere vanishing holomorphic r-ple $(n+1)$-form θ on $U\setminus\mathrm{Sing}(U)$ for an open neighborhood U of y. Then $f:=\theta/\nu^r$ is a holomorphic function on $U\setminus\mathrm{Sing}(U)$ which does not vanisn on

$T_N \cap U$ and whose vanishing order at D_i is equal to r. Since the codimension of $\mathrm{Sing}(Y)$ is greater than 1, f is extended to U, by [1, Chapter II, Corollary 3.12]. Hence f ie expressed as the series $\sum_{v \in (\sigma^* \setminus \{0\}) \cap M} c_v e(v)$. Suppose that $\Gamma_+(f)$ has a compact face Δ with $\dim \Delta \geqq 1$. Then there exist a primitive element u_0 in $\mathrm{Int}(\sigma) \cap N$ and a positive integer t such that $\langle v, u_0 \rangle = t$ (resp. $> t$) for any element v in Δ (resp. $\Gamma_+(f) \setminus \Delta$). Let Y_0 be the complex space associated to $\mathrm{Spec}(\boldsymbol{C}[(\boldsymbol{R}_{\geq 0} u_0)^* \cap M]) \ (\cong \boldsymbol{C} \times (\boldsymbol{C}^\times)^n)$ and let $D_0 = \mathrm{orb}(\boldsymbol{R}_{\geq 0} u_0)$. Then we have a holomorphic map $\pi : Y_0 \to Y$ such that $\pi_{|T_N} = id$ and that $\pi^{-1}(y) = D_0$, because $\boldsymbol{R}_{>0} u_0 \subset \mathrm{Int}(\sigma)$. Take a basis $\{v_1', v_2', \cdots, v_{n+1}'\}$ of M so that $\langle v_1', u_0 \rangle = 1$ and that $\langle v_i', u_0 \rangle = 0$ for $i = 2$ through $n+1$. Let $z_i = e(v_i')$ for $i = 1$ through $n+1$. Then $D_0 = \{z_1 = 0\}$ and $f = z_1^t g_0 + z_1^{t+1} g_1 + \cdots + z_1^{t+i} g_i + \cdots$ on $U \cap \boldsymbol{T}_N$, where $g_i = \sum_{v \in L_i} c_v e(v - (t+i) v_1')$ and $L_i = \{v \in \Gamma_+(f) \cap M \mid \langle v, u_0 \rangle = t+i\}$. Here we note that g_i are polynomials with variables $z_2, \cdots, z_{n+1}$ and that $g_0 = \sum_{v \in \Delta \cap M} c_v e(v - t v_1')$ is not a monomial, because the cardinal number of $\{v \in \Delta \cap M \mid c_v \neq 0\}$ is greater than 1. Hence $\{y' \in U \cap T_N \mid (g_0 + z_1 g_1 + \cdots)(y') = 0\} \neq \emptyset$, because $Y \setminus D_0 = T_N$. Then f must vanish at a point of $U \cap T_N$, a contradiction. Therefore, any compact face of $\Gamma_+(f)$ is a point. This implies that $\Gamma(f)$ consists of only one point v_0'. Hence $\Gamma_+(f) = v_0' + \sigma^*$. Therefore, $\langle v_0', u_i \rangle \leqq \langle v, u_i \rangle$ for any element v in $\Gamma_+(f) \cap M$ and for $i = 1$ through $n+1$. Since the vanishing order of f at D_i is r, we have $\langle v_0', u_i \rangle = r$. Hence the point $v_0 = (1/r) v_0'$ satisfies the condition of the proposition.

q. e. d.

Remark. If $N = \boldsymbol{Z}^{n+1}$ and $\sigma = (\boldsymbol{R}_{\geq 0})^{n+1}$, then Y is isomorphic to $\boldsymbol{C}^{n+1}$ and the point y corresponds to the origin. Clearly $v_0 = (1, 1, \cdots, 1)$ satisfies the condition of the above proposition, if we identify M with N, by the canonical inner product.

§ 2. Hypersurface sections

Let f be an element of the maximal ideal $\mathfrak{m}_{Y,y}$ of Y at y, let $X = \{f = 0\}$ and let $x = y$. Throughout the rest of this paper, we assume that $n = \dim X \geqq 2$, that X is irreducible reduced, that (X, x) is an isolated singularity and that $X \cap \mathrm{Sing}(Y) = \{x\}$. By [1, Chapter I, Proposition 1.6 (ii) and Corollary 4.4], we have:

PROPOSITION 2.1. *(X, x) is a Cohen-Macaulay and normal singularity.*

Assume that $f = \sum_{v \in (\sigma^* \setminus \{0\}) \cap M} c_v e(v)$ is non-degenerate, i.e.,

$$\partial f_\Delta / \partial w_1 = \partial f_\Delta / \partial w_2 = \cdots = \partial f_\Delta / \partial w_{n+1} = 0$$

has no solutions in $T_N = Y \setminus D (\cong (\boldsymbol{C}^\times)^{n+1})$, for each face Δ of $\Gamma(f)$, where $f_\Delta = \sum_{v \in \Delta \cap M} c_v e(v)$ and $(w_1, w_2, \cdots, w_{n+1})$ is a global coodinate of T_N.

THEOREM 2.2. *Assume that (Y, y) is r-Gorenstein, (that (Y, y) is not r'-Gorenstein for $1 \leqq r' < r$) and let v_0 be the element satisfying the condition of*

Proposition 1.2. *Then* (X, x) *is r-Gorenstein. Moreover, if* v_0 *is on* $\Gamma(f)$, *then*

$$\delta_m(X, x)=\begin{cases} 1 & \text{for } m\equiv 0 \bmod r \\ 0 & \text{for } m\not\equiv 0 \bmod r . \end{cases}$$

Conversely, if $\max\{\delta_m(X, x)\,|\,m\in \boldsymbol{Z}, m>0\}=1$, *then* v_0 *is on* $\Gamma(f)$. (*See* [11], *for the definition of* $\delta_m(X, x)$.)

For the proof, we need some preparations. For $u\in\sigma$, let $d(u)=\min\{\langle v, u\rangle\,|\, v\in\Gamma_+(f)\}$ and let $\Delta(u)=\{v\in\Gamma_+(f)\,|\,\langle v, u\rangle=d(u)\}$. For a face Δ of $\Gamma_+(f)$, let $\Delta^*=\{u\in\sigma\,|\,\Delta(u)\supset\Delta\}$. Then $\Gamma^*(f):=\{\Delta^*\,|\,\Delta$ is a face of $\Gamma_+(f)\}\cup\{0\}$ is an r. p. p. decomposition of $N_{\boldsymbol{R}}$ with $|\Gamma^*(f)|$ $(:=U_{\Delta^*\in\Gamma^*(f)}\Delta^*)=\sigma$. Let Σ^* be a subdivision of $\Gamma^*(f)$ consisting of non-singular cones and let $\tilde{Y}=T_N\operatorname{emb}(\Sigma^*)$. Then we have a resolution $\Pi:\tilde{Y}\to Y$ of Y. Let $\tilde{X}$ be the proper transformation of X under Π and let $E=\tilde{X}\cap\Pi^{-1}(x)$. Then $\pi(:=\Pi_{|\tilde{X}}):\tilde{X}\to X$ is a resolution of X whose exceptional set is E. Assume that u is a primitive element in N and that $\boldsymbol{R}_{\geqq 0}u$ is a 1-dimensional cone in Σ^* with $\dim\Delta(u)\geqq 1$. Then we denote by $E(u)$ the closure of $\operatorname{orb}(\boldsymbol{R}_{\geqq 0}u)\cap E(\neq\phi)$. Recall that $\theta:=\boldsymbol{e}(rv_0)\nu^r$ is a nowhere vanishing r-ple $(n+1)$-form on $Y\setminus\operatorname{Sing}(Y)$. Let $\omega=\operatorname{Res}(\theta/f^r)$, i.e., $\omega=g_{|X\cap U}(dw_1\wedge\cdots\wedge dw_n)^r$ on $X\cap U$, if θ is expressed as $g(df\wedge dw_1\wedge\cdots\wedge dw_n)^r$ on an open set U of Y.

LEMMA 2.3. $\pi^*\omega^l$ *has zeros of order* $lr(\langle v_0, u\rangle-1-d(u))$ *along* $E(u)$.

Proof. The lemma follows from the fact that $\boldsymbol{e}(rv_0)$, ν^r and $(\pi^*f)^r$ have zeros of order $r\langle v_0, u\rangle$, $-r$ and $rd(u)$, respectively, along $\operatorname{orb}(\boldsymbol{R}_{\geqq 0}u)$. q. e. d.

Proof of Theorem 2.2. Since ω is a nowhere vanishing holomorphic r-ple n-form on $X\setminus\{x\}$, we see that (X, x) is r-Gorenstein. Assume that v_0 is on $\Gamma(f)$. Then $\langle v_0, u\rangle\geqq d(u)$ for any u in $\operatorname{Int}(\sigma)\cap N$. Hence the nowhere vanishing holomorphic lr-ple n-form $\pi^*\omega^l$ has poles of order at most lr along each irreducible component of the exceptional set E, by Lemma 2.3. On the other hand, $\Gamma_+(f)$ has a compact face Δ_0 containing v_0 with $\dim\Delta_0\geqq 1$. Otherwise, $\Gamma_+(f)=v_0+\sigma^*$ and hence $f=\boldsymbol{e}(v_0)g$ for a holomorphic function g on Y. Then since $[\boldsymbol{e}(v_0)]=rD$, we get a contradiction to the assumption that X is irreducible. Hence we can take a subdivision Σ^* of $\Gamma^*(f)$ so that $\Delta(u_0)=\Delta_0$ for a 1-dimensional cone $\Delta^*=\boldsymbol{R}_{\geqq 0}u_0$ in Σ^*. Then $u_0\in\operatorname{Int}(\sigma)$, $\operatorname{orb}(\Delta^*)\cap\tilde{X}\neq\emptyset$ and $\langle v_0, u_0\rangle=d(u_0)$. Hence $\pi^*\omega^l$ has poles of order lr along the irreducible component $E(u_0)$ of E. Therefore, $\delta_{lr}(X, x)=1$. Next, assume that $m\not\equiv 0 \bmod r$ and let η be an element in $H^0(X\setminus\{x\}, \mathcal{O}_X(mK_X))$. In the following, we show that η is in $L^{2/m}(X\setminus\{x\})$. We note that rv_0 is a primitive element in M. Otherwise, (Y, y) is r'-Gorenstein for a positive integer $r'<r$. Hence we can take n elements $v_1, v_2, \cdots$ and v_n in M so that $\{rv_0, v_1, \cdots, v_n\}$ is a basis of M. Let $w_0=\boldsymbol{e}(rv_0)$ and let $w_i=\boldsymbol{e}(v_i)$ for $i=1$ through n. Then $(w_0, w_1, \cdots, w_n)$ is a global coordinate of T_N. Let $M'=M+\boldsymbol{Z}v_0$ and let $N'=\{u\in N\,|\,\langle v', u\rangle\in\boldsymbol{Z}$ for any $v'\in M'\}$ $(=\{u$

$\in N|\langle v_0, u\rangle \in \boldsymbol{Z}\}$). Then the inclusion $N'\to N$ induces a holomorphic map $\varphi: Y'\to Y$, where Y' is the complex space associated to $\mathrm{Spec}(\boldsymbol{C}[M'\cap\sigma^*])$. Since $\{v_0, v_1, \cdots, v_n\}$ is a basis of M', $(z_0, z_1, \cdots, z_n)$ is a global coordinate of $T_{N'}=\mathrm{Spec}(\boldsymbol{C}[M'])$, where $z_i=\boldsymbol{e}(v_i)$ for $i=0$ through n. Clearly, $\varphi^*w_0=(z_0)^r$ and $\varphi^*w_i=z_i$ for $i=1$ through n. Hence φ is the quotient map under the group $\langle t\rangle$ generated by the element $t=(\xi, 1, \cdots, 1)$ in $T_{N'}$, where ξ is a primitive r-th root of 1. Moreover, φ is unramified over $Y\setminus\mathrm{Sing}(Y)$, because $\theta:=w_0((dw_0/w_0)\wedge(dw_1/w_1)\wedge\cdots\wedge(dw_n/w_n))^r$ (resp. $\theta':=z_0(dz_0/z_0)\wedge(dz_1/z_1)\wedge\cdots\wedge(dz_n/z_n)$) is a nowhere vanishing holomorphic r-ple $(n+1)$-form on $Y\setminus\mathrm{Sing}(Y)$ (resp. $(n+1)$-form on $Y'\setminus\mathrm{Sing}(Y')$) and $\varphi^*\theta=(r\theta')^r$. Hence $\mathrm{Sing}(X')=\{x'\}$, where $X':=\varphi^{-1}(X)$ and $x':=\varphi^{-1}(x)$. Let $\omega'=\mathrm{Res}(\theta'/\varphi^*f)$. Then ω' is a nowhere vanishing holomorphic n-form on $X'\setminus\{x'\}$ with $t^*\omega'=\xi\omega'$, because $t^*z_0=\xi z_0$ and $t^*(dz_i/z_i)=dz_i/z_i$ for $i=0$ through n. Hence $\varphi^*\eta=g(\omega')^m$ for a holomorphic function g on X'. Since $t^*(\varphi^*\eta)=\varphi^*\eta$ and $t^*(g(\omega')^m)=t^*g\xi^m(\omega')^m$, we have $t^*g=\xi^{-m}g$. Since $\xi^{-m}\neq 1$, we have $g(x')=0$. Hence $\varphi^*\eta=g(\omega')^m\in\mathcal{L}^{2/m}(X'\setminus\{x'\})$, because $(\pi')^*\omega'\in H^0(\tilde{X}', \mathcal{O}(K_{\tilde{X}'}+E'))$, for any resolution $\pi':(\tilde{X}', E')\to(X', x')$ of (X', x'). Therefore, $\eta\in\mathcal{L}^{2/m}(X\setminus\{x\})$. Thus we conclude that $\delta_m(X, x)=0$. Finally, note that if $v_0\notin\Gamma_+(f)$ (resp. $\in\mathrm{Int}(\Gamma_+(f))$), then $\delta_m(X, x)\geqq 2$ for certain positive integers m (resp. $=0$ for all positive integers m), by Lemma 2.3 and that if $v_0\in\partial\Gamma_+(f)\setminus\Gamma(f)$, then (X, x) is not isolated. Thus we obtain the last assertion of the theorem. q.e.d.

We can obtain a system of defining equations of X from those of Y and f.

PROPOSITION 2.4. *If* $f\notin\mathfrak{m}_{Y,y}^2$ *(resp.* $f\in\mathfrak{m}_{Y,y}^2$*), then* $\dim\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2=\dim\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2-1$ *(resp.* $\dim\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2$*).*

Proof. We have the following exact sequence.

$$0\longrightarrow f\cdot\mathcal{O}_{Y,y}/(f\cdot\mathcal{O}_{Y,y}\cap\mathfrak{m}_{Y,y}^2)\longrightarrow\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2\longrightarrow\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2\longrightarrow 0.$$

We easily see that $\dim f\cdot\mathcal{O}_{Y,y}/(f\cdot\mathcal{O}_{Y,y}\cap\mathfrak{m}_{Y,y}^2)=1$ or 0, according as $f\notin\mathfrak{m}_{Y,y}^2$ or $f\in\mathfrak{m}_{Y,y}^2$. q.e.d.

Assume that $\sigma^*\cap M$ is generated by m elements $v_1, v_2, \cdots, v_m$ and let $z_i=\boldsymbol{e}(v_i)$, for $i=1$ through m. Then we have the embedding $i: Y\ni p\mapsto(z_1(p), z_2(p), \cdots, z_m(p))\in\boldsymbol{C}^m$. Assume that $i(Y)$ is defined by $g_1(z)=g_2(z)=\cdots=g_l(z)=0$, where $z=(z_1, z_2, \cdots, z_m)$. If $f\in\mathfrak{m}_{Y,y}^2$, then $X=\{f=0\}$ is isomorphic to the subvariety in $\boldsymbol{C}^m$ defined by $\tilde{f}(z)=g_1(z)=\cdots=g_l(z)=0$, where $\tilde{f}(z)$ is a holomorphic function on $\boldsymbol{C}^m$ with $i^*\tilde{f}=f$. Next, assume that we can express $\tilde{f}(z)=z_1-h(z_2, \cdots, z_m)$. Hence $f\notin\mathfrak{m}_{Y,y}^2$. Then X is isomorphic to the subvariety in $\boldsymbol{C}^{m-1}$ defined by $g_1'(w)=g_2'(w)=\cdots=g_l'(w)=0$, where $w=(z_2, \cdots, z_m)$ and $g_i'(w)=g_i(h(w), z_2, \cdots, z_m)$.

Example 1. Let $n=2$, let $\{u_1, u_2, u_3\}$ be a basis of N and let $\{v_1, v_2, v_3\}$ be the basis of M dual to $\{u_1, u_2, u_3\}$. Let $\sigma=\boldsymbol{R}_{\geqq 0}(u_1+u_3)+\boldsymbol{R}_{\geqq 0}(u_1+u_2+u_3)+$

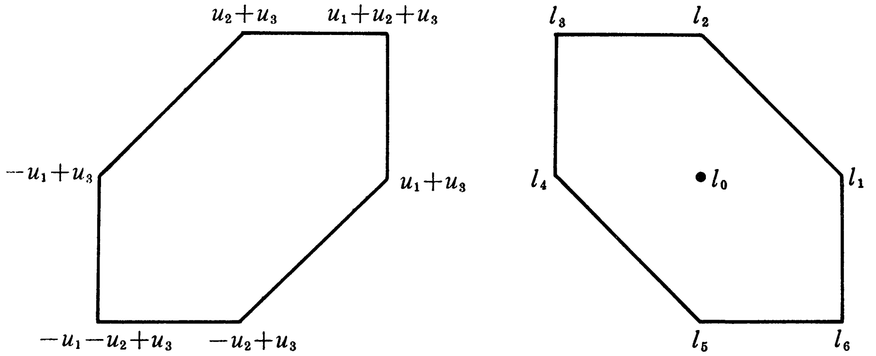

Figure 1.

$\boldsymbol{R}_{\geqq 0}(u_2+u_3)+\boldsymbol{R}_{\geqq 0}(-u_1+u_3)+\boldsymbol{R}_{\geqq 0}(-u_1-u_2+u_3)+\boldsymbol{R}_{\geqq 0}(-u_2+u_3)$. Then (Y, y) is Gorenstein and $v_0=v_3$ satisfies the condition of Proposition 1.2. We see that $\sigma^*\cap M$ is generated by $l_0=v_3$, $l_1=v_1+v_3$, $l_2=v_2+v_3$, $l_3=-v_1+v_2+v_3$, $l_4=-v_1+v_3$, $l_5=-v_2+v_3$ and $l_6=v_1-v_2+v_3$. (See Figure 1.) Hence Y is isomorphic to the subvariety in $\boldsymbol{C}^7$ defined by the equations (1) $z_0z_1-z_6z_2=z_0z_2-z_1z_3=z_0z_3-z_2z_4=z_0z_4-z_3z_5=z_0z_5-z_4z_6=z_0z_6-z_5z_1=z_0^2-z_1z_4=z_0^2-z_2z_5=z_0^2-z_3z_6=0$, where $z_i=\boldsymbol{e}(l_i)$, for $i=0$ throuh 6. Let $f=z_0-z_1^2-z_2^2-\cdots-z_6^2$. Then (X, x) is a cusp singularity with a resolution $\pi:(\tilde{X}, E)\rightarrow(X, x)$ such that the exceptional set E is a cycle of six rational curves whose self-intersection numbers are all -3. Since $f\notin \mathfrak{m}_{Y,y}^2$, we see that X is isomorphic to the subvariety in $\boldsymbol{C}^6$ defined by the equations obtained from the above equations (1), replacing z_0 by $z_1^2+z_2^2+\cdots+z_6^2$.

Example 2. Let n, $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ be the same as in Example 1. Let $\sigma=\boldsymbol{R}_{\geqq 0}(u_1+2u_3)+\boldsymbol{R}_{\geqq 0}(u_2+2u_3)+\boldsymbol{R}_{\geqq 0}(u_1+2u_2+2u_3)+\boldsymbol{R}_{\geqq 0}(2u_1+u_2+2u_3)$. Then (Y, y) is 2-Gorenstein and $v_0=(1/2)v_3$ satisfies the condition of Proposition 1.2. We see that $\sigma^*\cap M$ is generated by $l_1=-2v_1-2v_2+3v_3$, $l_2=-v_1+v_3$, $l_3=-2v_1+2v_2+v_3$, $l_4=v_2$, $l_5=2v_1+2v_2-v_3$, $l_6=v_1$, $l_7=2v_1-2v_2+v_3$ and $l_8=-v_2+v_3$. (See Figure 2.) Let $z_i=\boldsymbol{e}(l_i)$ for $i=1$ through 8 and let $f=z_2-z_4+z_6+z_8$. Then f is non-degenerate, (X, x) is an isolated singularity and $X\cap\mathrm{Sing}(Y)=\{x\}$. Moreover, (X, x) is a quotient of a simple elliptic singularity.

Example 3. Let $n=3$, let $\{u_1, u_2, u_3, u_4\}$ be a basis of N and let $\{v_1, v_2, v_3, v_4\}$ be the basis of M dual to $\{u_1, u_2, u_3, u_4\}$. Let $\sigma=\boldsymbol{R}_{\geqq 0}(u_1+u_2+2u_4)+\boldsymbol{R}_{\geqq 0}(u_1+u_3+2u_4)+\boldsymbol{R}_{\geqq 0}(u_2+u_3+2u_4)+\boldsymbol{R}_{\geqq 0}(u_1+u_2+2u_3+2u_4)+\boldsymbol{R}_{\geqq 0}(u_1+2u_2+u_3+2u_4)+\boldsymbol{R}_{\geqq 0}(2u_1+u_2+u_3+2u_4)$. Then (Y, y) is 2-Gorenstein and $v_0=(1/2)v_4$ satisfies the condition of Proposition 1.2. We see that $\sigma^*\cap M$ is generated by $l_1=v_1-v_2-v_3+v_4$, $l_2=v_1-v_2+v_3$, $l_3=v_1+v_2+v_3-v_4$, $l_4=v_1+v_2-v_3$, $l_5=-v_1-v_2-v_3+2v_4$, $l_6=-v_1-v_2+v_3+v_4$, $l_7=-v_1+v_2+v_3$, $l_8=-v_1+v_2-v_3+v_4$, $l_9=v_1$, $l_{10}=-v_2+v_4$, $l_{11}=v_3$, $l_{12}=v_2$, $l_{13}=-v_3+v_4$ and $l_{14}=-v_1+v_4$. (See Figure 3.) Let $z_i=\boldsymbol{e}(l_i)$, for $i=1$ through 14 and let $f=\sum_{1\leqq i\leqq 14}z_i$. Then f is non-degenerate, (X, x) is an isolated singularity

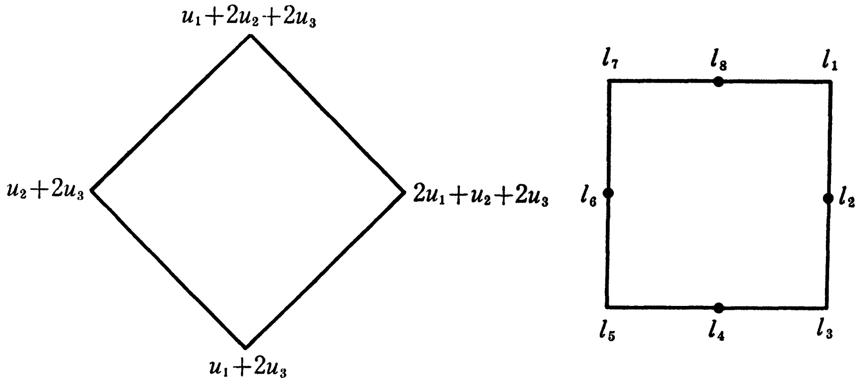

Figure 2.

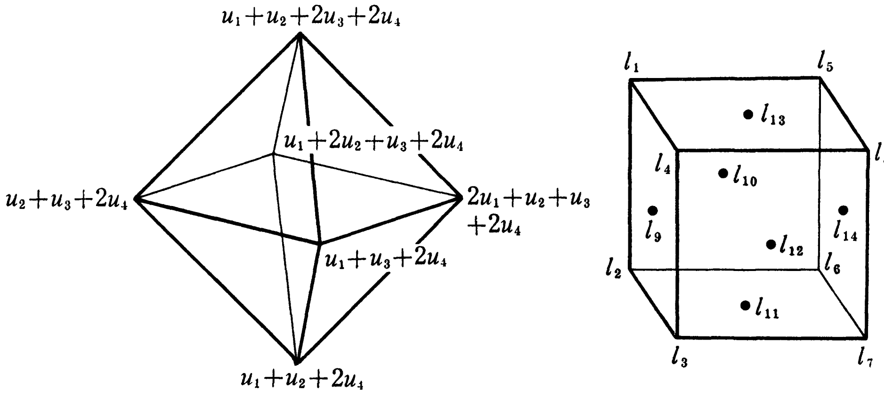

Figure 3.

and $X\cap\mathrm{Sing}(Y)=\{x\}$. Let $\Sigma=\{$faces of $\boldsymbol{R}_{\geq 0}(u_1+u_2+u_3+2u_4)+\tau\,|\,\tau$ are 3-dimensional faces of $\sigma\}$ and let $\tilde{Y}=T_N\mathrm{emb}(\Sigma)$. Then $\Sigma=\Gamma^*(f)$ and $\tilde{Y}$ is the blowing up of Y along $y=\mathrm{orb}(\sigma)$. Although $\tilde{Y}$ has singularities, $\tilde{X}\cap\mathrm{Sing}(\tilde{Y})=\phi$, where $\tilde{X}$ is the proper transformation of X under the blowing up $\Pi:\tilde{Y}\to Y$. Moreover, $\Pi^{-1}(y)\cap\tilde{X}=E(u_1+u_2+u_3+2u_4)$ is an Enriques surface. Each of small deformations $X_\varepsilon=\{f=\varepsilon\}$ of X has eight isolated quotient singularities.

§ 3. Hyperplane sections of Gorenstein toric singularities

We keep the notations of the previous section and throughout this section, we assume that (Y, y) is an isolated (, i.e., each n-dimensional face of σ is non-singular), non-terminal and Gorenstein singularity. Hence (Y, y) is a canonical singularity of index 1 and the set $\mathcal{L}:=\{u\in\mathrm{Int}(\sigma)\cap N\,|\,\langle v_0, u\rangle=1\}$ is nonempty. Moreover, we assume that $X=\{f=0\}$ is a generic hyperplane section, i.e., $f=\Sigma c_v \boldsymbol{e}(v)$ with $c_v \neq 0$, for the generators v of $\sigma^* \cap M$.

PROPOSITION 3.1. *Under the above assumptions, (X, x) is a purely elliptic singularity, i.e., $\delta_m(X, x)=1$ for each positive integer m.*

Proof. Let u_0 be an element in $\mathcal{L}$. Then $\langle v_0, u_0\rangle=1$ and $\{v\in\sigma^* \mid \langle v, u_0\rangle \geqq 1\} \supset$the convex hull of $(\sigma^*\setminus\{0\})\cap M=\Gamma_+(f)\ni v_0$. Hence the set $\{v\in\sigma^* \mid \langle v, u_0\rangle =1\}\cap\Gamma_+(f)$ is a compact face of $\Gamma_+(f)$ and contains v_0. Therefore, $\delta_m=1$ for each positive integer m, by Theorem 2.2. q.e.d.

Remark. (1) If (Y, y) is non-terminal and canonical of index $r>1$, then (Y, y) is r-Gorenstein and $v_0\in\sigma^*\setminus\mathrm{Int}(\Gamma_+(f))$. There are examples with $v_0\notin \Gamma_+(f)$, as well as examples with $v_0\in\Gamma(f)$. Hence (X, x) may not be a periodically elliptic singularity in contrast with the above proposition.

(2) In the case that (Y, y) is not isolated, if $\mathcal{L}=\phi$, then (X, x) may be an isolated canonical singularity, even though (Y, y) is a non-terminal Gorenstein singularity. For instance, let σ be the cone generated by $(\pm 1, 0, 0, 1)$, $(0, \pm 1, 0, 1)$ and $(1, 1, 2, 1)$ in $\boldsymbol{Z}^4$.

Ishii [3] and Koyama independently showed that a 2-dimensional purely elliptic singularity is a simple elliptic singularity or a cusp singularity.

PROPOSITION 3.2. *When $n=2$, (X, x) is a simple elliptic singularity (resp. a cusp singularity), if the cardinal number of $\mathcal{L}$ is equal to (resp. greater than) 1.*

Proof. First, we consider the case that $\mathcal{L}$ consists of one element u_0. For each 2-dimensional face $\tau=\boldsymbol{R}_{\geqq 0}u'+\boldsymbol{R}_{\geqq 0}u''$ ($\{u', u''\}\subset\{u_1, u_2, \cdots, u_s\}$) of σ, $\{u_0, u', u''\}$ is a basis of N, because $\langle v_0, u_0\rangle=\langle v_0, u'\rangle=\langle v_0, u''\rangle=1$ and the triangle spanned by u_0, u' and u'' contains no elements in N except u_0, u' and u''. Let $\{v_\tau, v', v''\}$ be the basis of M dual to $\{u_0, u', u''\}$. Then $\langle v_\tau, u_0\rangle=1$ and $\langle v_\tau, u'\rangle=\langle v_\tau, u''\rangle=0$. Hence σ^* is generated by v_τ and $\Gamma(f)$ consists of one face which is the polygon spanned by v_τ, for all 2-dimensional faces τ of σ. Therefore, $\Gamma^*(f)=\{$faces of $\boldsymbol{R}_{\geqq 0}u_0+\tau \mid \tau$ are 2-dimensional faces of $\sigma\}$ and the exceptional set $E=E(u_0)$ of the resolution of (X, x) obtained from $\Gamma^*(f)$ is a non-singular curve. It should be elliptic, because $\delta_m(X, x)=1$.

When the cardinal number of $\mathcal{L}$ is greater than 1, we easily see that there exist at least two 2-dimensional compact faces of $\Gamma_+(f)$ containing v_0, which we denote by Δ_1 and Δ_2. Then $\Delta_1^*=\boldsymbol{R}_{\geqq 0}u_1'$ and $\Delta_2^*=\boldsymbol{R}_{\geqq 0}u_2'$ for primitive elements u_1' and u_2' in $\mathrm{Int}(\sigma)\cap N$ such that $\langle v_0, u_1'\rangle=d(u_1')$ and that $\langle v_0, u_2'\rangle=d(u_2')$. Hence the exceptional set E of the resolution $\pi:(\tilde{X}, E)\to(X, x)$ of (X, x) obtained from any subdivision of $\Gamma^*(f)$ contains two irreducible components $E(u_1')$ and $E(u_2')$ along which $\pi^*\omega$ has poles of order 1, by Lemma 2.3, where $\omega=\mathrm{Res}(\boldsymbol{e}(v_0)((dw_1/w_1)\wedge\cdots\wedge(dw_{n+1}/w_{n+1}))/f)$. Therefore, (X, x) is not a simple elliptic singularity. q.e.d.

Since (Y, y) is an isolated singularity, (X, x) is smoothable. On the other hand, Wahl [9, 10] showed that if a simple elliptic singularity (resp. a cusp singularity) (X, x) is smoothable, then $m(X) \leqq 9$, (resp. $m(X)-l(X) \leqq 9$), where $l(X)$ is the number of the irreducible components of the exceptional set E of the minimal resolution of (X, x) and $m(X)$ is the multiplicity of (X, x), which is equal to $-E^2$, if $-E^2 \geqq 3$.

PROPOSITION 3.3. *Assume that the cardinal number of $\mathcal{L}$ is equal to* 1. *If σ is an s-gonal cone, $-E^2=12-s$. (Therefore, $-E^2 \leqq 9$.)*

Proof. Let $\mathcal{L}=\{u_0\}$. Then $\Gamma^*(f)=\{$faces of $\boldsymbol{R}_{\geq 0}u_0+\tau \mid \tau$ are 2-dimensional faces of $\sigma\}$ consists of non-singular cones, by the proof of Proposition 3.2. Hence we obtain resolutions $\Pi:(\tilde{Y}, F) \to (Y, y)$ and $\pi=\Pi_{|\tilde{X}}:(\tilde{X}, E) \to (X, x)$, where $\tilde{Y}=T_N\,\mathrm{emb}(\Gamma^*(f))$, F is the closure of $\mathrm{orb}(\boldsymbol{R}_{\geq 0}u_0)$, $\tilde{X}$ is the proper transformation of X under Π and $E=\tilde{X}\cdot F$. Let $\tilde{D}_i$ be the proper transformation of D_i under Π and let $E_i=F\cdot\tilde{D}_i$. Since $F+\tilde{X}=[\Pi^*f]$ and $F+\tilde{D}_1+\tilde{D}_2+\cdots+\tilde{D}_s=[\Pi^*\boldsymbol{e}(v_0)]$ are principal divisors, we have $-E^2_{|\tilde{X}}=-F^2\cdot\tilde{X}=F\cdot\tilde{X}^2=\sum_{1\leq i\leq s}F\cdot\tilde{D}_i^2+2\sum_{0\leq i<j\leq s}F\cdot\tilde{D}_i\cdot\tilde{D}_j=(\sum_{1\leq i\leq s}E^2_{i|F})+2s=3(4-s)+2s=12-s$, because F is a non-singular toric variety whose 1-dimensional orbits are $E_1, E_2, \cdots$ and E_s.

q.e.d.

PROPOSITION 3.4. *Assume that the convex hull of $\mathcal{L}$ is a polygon. If σ is an s-gonal cone, then, $-E^2-l(X)=12-s$. (Therefore, $-E^2-l(X) \leqq 9$.)*

Proof. Let P (resp. Q) be the convex hull of $\mathcal{L}$ (resp. $\{u\in\sigma\cap N \mid \langle v_0, u\rangle=1\}$). Then $Q=\{u\in\sigma \mid \langle v_0, u\rangle=1\}$ and $\mathrm{Int}(Q)\supset P$. Take a triangulation Δ (resp. Δ') of P (resp. $Q\setminus\mathrm{Int}(P)$) so that the set of the vertices of Δ (resp. Δ') agrees with $P\cap N=\mathcal{L}$ (resp. $(Q\setminus\mathrm{Int}(P))\cap N$). Let e_0, e_1 and e_2 (resp. e_0', e_1' and e_2') be the numbers of the vertices, edges and faces, respectively, of Δ (resp. Δ'). Then $e_0-e_1+e_2=1$ and $e_0'-e_1'+e_2'=0$, because P and Q are polygons. Let l be the number of the vertices on the boundary ∂P of P. Then $e_0'=l+s$ and $3e_2'=2e_1'-(l+s)$, because the number of the vertices (resp. edges) on the boundary of $Q\setminus\mathrm{Int}(P)$ is equal to $l+s$. Hence by an easy calculation, we have $e_1'=2(l+s)$. Since $\Box:=\Delta\cup\Delta'$ is a triangulation of Q, we see that $\Sigma^*:=\{\boldsymbol{R}_{\geq 0}\tau \mid \tau$ are simplexes of $\Box\}\cup\{0\}$ is a subdivision of $\Gamma^*(f)$ and consists of non-singular cones. Hence we have a resolution $\Pi:(\tilde{Y}, F)\to(Y, y)$, where $\tilde{Y}=T_N\,\mathrm{emb}(\Sigma^*)$. Let $\tilde{D}_i$ be the proper transformation of D_i under Π and let $\tilde{D}=\tilde{D}_1+\tilde{D}_2+\cdots+\tilde{D}_s$. Then Δ and $\Box$ are the dual graphs of $F=F_1+F_2+\cdots+F_{e_0}$ and $F+\tilde{D}$, respectively. Since $\tilde{X}+F=[\Pi^*f]$ and $\tilde{D}+F=[\Pi^*\boldsymbol{e}(v_0)]$ are principal divisors, we have $0=F_i\cdot F_j\cdot(\tilde{D}+F)=F_i^2\cdot F_j+F_i\cdot F_j^2+2$, if $F_i\cap F_j\neq\phi$ and $-E^2_{|\tilde{X}}=-F^2\cdot\tilde{X}=F\cdot\tilde{D}^2=\sum_{1\leq i\leq e_0}(\sum_{1\leq j\leq s}F_i\cdot\tilde{D}_j^2+2\sum_{1\leq j<k\leq s}F_i\cdot\tilde{D}_j\cdot\tilde{D}_k)=\sum_{1\leq i\leq e_0, 1\leq j\leq s}F_i\cdot\tilde{D}_j^2+2s$, where $\tilde{X}$ is the proper transformation of X under Π and $E=\tilde{X}\cdot F$. On the other hand, since each irreducible component F_i of F is a non-singular toric variety with $F_i\cdot(F+\tilde{D}-F_i)$ as the union of 1-dimensional orbits, we have $\sum_{i\neq j}F_i\cdot F_j^2+\sum_{1\leq k\leq s}F_i\cdot\tilde{D}_k^2=3(4-d_i)$, where d_i is the number of the double curves on F_i. Hence by taking the sum

of the self-intersection numbers of the double curves $F_i \cdot F_j$ and $F_i \cdot \tilde{D}_k$ on all the irreducible components F_i of F, we have $-2e_1+\sum_{1\leq i\leq e_0, 1\leq j\leq s} F_i \cdot \tilde{D}_j^2=\sum_{1\leq i\leq e_0} 3(4-d_i)=12e_0-3(2e_1+l+s)=12e_0-6e_1-3l-3s$. Therefore, $-E^2_{|\tilde{X}}=12e_0-6e_1-3l-3s+2e_1+2s=12e_0-4e_1-3l-s=12e_0-12e_1+12e_2+l-s=12+l-s$, because $3e_2=2e_1-l$. Thus we obtain $-E^2-l=12-s$. Here we note that l is equal to the number of the irreducible components of E, because $\tilde{X}\cap F_i \neq \phi$, if and only if $\tilde{D}\cap F_i \neq \phi$ (, i.e., the vertex of $\square$ corresponding to F_i is on ∂P) and then $\tilde{X}\cap F_i$ is irreducible. Moreover, $E_i \cdot E_j=\tilde{X}\cdot F_i \cdot F_j=\tilde{D}\cdot F_i \cdot F_j \leqq 1$ and the equality holds, if and only if the vertices of $\square$ corresponding to F_i and F_j are joined by an edge on ∂P. Hence E forms a cycle. Therefore, although $(\tilde{X}, E)$ is not a minimal resolution, the contraction of a rational curve E_i with $E_i^2=-1$ does not change the number $-E^2-l$. Thus we complete the proof.

Examples. In the following table, $E=E_1+E_2+\cdots+E_l$ is the exceptional set of the minimal resolution of (X, x) such that $E_i \cdot E_{i+1}=1$ for each $i\in \boldsymbol{Z}/l\boldsymbol{Z}$.

generators of σ	l	$-E_1^2, -E_2^2, \cdots, -E_l^2$
(0, 0, 1), (5, 2, 1), (3, 5, 1)	6	5, 4, 5, 4, 5, 4
(0, 0, 1), (4, 1, 1), (3, 4, 1)	6	7, 2, 7, 2, 7, 2
(0, 0, 1), (8, 3, 1), (5, 8, 1)	9	5, 4, 3, 5, 4, 3, 5, 4, 3
(0, 0, 1), (7, 2, 1), (5, 7, 1)	9	5, 5, 2, 5, 5, 2, 5, 5, 2
(0, 0, 1), (7, 3, 1), (4, 7, 1)	9	6, 4, 2, 6, 4, 2, 6, 4, 2

§ 4. Deformations

We assume that $n=\dim X\geqq 3$, throughout this section. Let $U=X\setminus\{x\}$ and let $W=Y\setminus\{y\}$. Then we have the isomorphism $T_X^1\cong H^1(U, \Theta_U)$, by Proposition 2.1 and [7, Theorem 2], where $T_X^1=H^0(X, \Theta_X^1)$ is the tangent space to the formal moduli space of X and Θ_U is the tangent sheaf of U. Consider the long exact sequence arising from the short exact sequence of sheaves:

$$0 \longrightarrow \Theta_U \longrightarrow i^*\Theta_W \longrightarrow \mathcal{N} \longrightarrow 0,$$

where $i: U\hookrightarrow W$ is the inclusion map. Here we note that the normal sheaf $\mathcal{N}\cong \mathcal{O}_U(U)$ is isomorphic to the structure sheaf $\mathcal{O}_U$, because X is a principal divisor on Y. Let $\{\theta_1, \theta_2, \cdots, \theta_l\}$ be a basis of the image of the map $\delta: H^0(U, \mathcal{N})\to H^1(U, \Theta_U)$ and let g_i be an element of $H^0(Y, \mathcal{O}_Y)$ whose image is θ_i under the composite of the surjective maps $H^0(Y, \mathcal{O}_Y)=H^0(W, \mathcal{O}_W)\to H^0(U, \mathcal{O}_U)\cong H^0(U, \mathcal{N})$ sending h to $h_{|U}\cdot\partial/\partial f$ and $H^0(U, \mathcal{N})\to \mathrm{Im}(\delta)$. Let $\mathcal{X}=\{(z, t)\in Y\times\Delta \mid f(z)+t_1g_1(z)+t_2g_2(z)+\cdots+t_lg_l(z)=0\}$ and let π be the restriction to $\mathcal{X}$ of the projection

$Y\times\Delta\to\Delta$, where $\Delta=\{(t_1, t_2, \cdots, t_l)\in \boldsymbol{C}^l \mid |t_j|<\varepsilon\}$. Then π is flat, by [1, Chapter V, Corollary 1.5]. Let $\mathcal{U}$ be the open set of $\mathcal{X}$ on which π is smooth. Then we obtain a family $\pi_{|\mathcal{U}}: \mathcal{U}\to\Delta$ of deformations of the complex manifold U. Moreover, by an easy calculation, we have $\rho(\partial/\partial t_j)=\theta_j$ for $j=1$ through l, where $\rho: T_0(\Delta)\to H^1(U, \Theta_U)$ is the infinitesimal deformation map. Hence ρ is injective and if $H^1(U, i^*\Theta_W)=0$, then ρ is surjective.

THEOREM 4.1. *If $H^1(U, i^*\Theta_W)=0$, then $\pi: \mathcal{X}\to\Delta$ is a locally semiuniversal family of X.*

Proof. Recall that T_X^1 is defined by the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\Omega_X^1, \mathcal{O}_X) \longrightarrow \mathrm{Hom}(j^*\Omega^1_{\boldsymbol{C}^N}, \mathcal{O}_X) \longrightarrow \mathrm{Hom}(I/I^2, \mathcal{O}_X) \longrightarrow T_X^1 \longrightarrow 0$$

obtained by the exact sequence of sheaves: $I/I^2 \xrightarrow{d} j^*\Omega^1_{\boldsymbol{C}^N}\to\Omega_X^1\to 0$, for an inclusion $j: X\hookrightarrow \boldsymbol{C}^N$ with the ideal sheaf I. On the other hand, we have the exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}(\Omega_X^1, \mathcal{O}_X) \longrightarrow \mathrm{Hom}(j^*\Omega^1_{\boldsymbol{C}^N}, \mathcal{O}_X) \longrightarrow \mathrm{Hom}(\mathrm{Im}(d), \mathcal{O}_X) \\ &\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) \longrightarrow 0, \end{aligned}$$

by the short exact sequence of sheaves: $0\to\mathrm{Im}(d)\to j^*\Omega^1_{\boldsymbol{C}^N}\to\Omega_X^1\to 0$. Since the support of $\ker(d)$ is $\{x\}$, we have $\mathrm{Hom}(\mathrm{Im}(d), \mathcal{O}_X)=\mathrm{Hom}(I/I^2, \mathcal{O}_X)$. Thus we have the canonical isomorphism $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)\cong T_X^1$. Hence the infinitesimal deformation map $T_0(\Delta)\to\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ for the family $\pi: \mathcal{X}\to\Delta$ is bijective. Then by [8, Theorem 6.1], $\pi: \mathcal{X}\to\Delta$ is locally semiuniversal. q.e.d.

COROLLARY 4.2. *If $H^1(U, i^*\Theta_W)=0$, then any small deformation of X is also a hypersurface section of Y.*

PROPOSITION 4.3. *If σ is a simplicial cone (hence Y is a quotient space of $\boldsymbol{C}^{n+1}$), then $H^1(U, i^*\Theta_W)=0$.*

Proof. Let $l_1, l_2, \cdots$ and l_{n+1} be the generators of σ and let $N'=\boldsymbol{Z}l_1+\boldsymbol{Z}l_2+\cdots+\boldsymbol{Z}l_{n+1}$. Here we may assume that $l_1, l_2, \cdots$ and l_{n+1} are primitive elements in N. Then the inclusion $N'\hookrightarrow N$ induces a holomorphic map $\varphi: Y'\to Y$, where $Y'=T_{N'}\,\mathrm{emb}(\{\text{faces of } \sigma\})\cong\boldsymbol{C}^{n+1}$. Let $U'=\varphi^{-1}(U)$. Then $\varphi_{|U'}: U'\to U$ is unramified, by the assumption $X\cap\mathrm{Sing}(Y)=\{x\}$. Hence $H^1(U, i^*\Theta_W)=H^1(U', h^*\Theta_{Y'})^G=0$, where $h: U'\hookrightarrow Y'$ is the inclusion map and G is the covering transformation group of φ. q.e.d.

Example. Let X' be the hypersurface of $\boldsymbol{C}^4$ defined by $z_1^2+z_2^6+z_3^6+z_4^6=0$ and let $X=X'/G$ be the quotient space of X' under the group G generated by $(1, \xi, \xi, \xi)$, where ξ is a primitive cube root of 1. Then X is a hypersurface section of $Y=\boldsymbol{C}^4/G$, which is a toric singularity, and whose singular locus

Sing(Y) is 1-dimensional. We easily see that X has an isolated singularity obtained by contracting a K3 surface. By Corollary 4.2 and Proposition 4.3, any small deformation of X is also a hypersurface section $X_t=\pi^{-1}(t)$ of Y. Since X_t intersect Sing(Y) at finitely many points, X_t has singularities, i.e., X is not smoothable.

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