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LOCAL MAXIMA OF THE SPHERICAL DERIVATIVE

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Abstract

Let a function f be nonconstant and meromorphic in a domain D in the plane, and let $M(f)$ be the set of points where the spherical derivative $|f'|/(1+|f|^2)$ has local maxima. The components of $M(f)$ are at most countable and each component is (i) an isolated point, (ii) a noncompact simple analytic arc terminating nowhere in D, or, (iii) an analytic Jordan curve. Tangents to a component of type (ii) or (iii) are expressed by the argument of the Schwarzian derivative of f. If Δ is the Jordan domain bounded by a component of type (iii) and if $\Delta \subset D$, then the spherical area of the Riemann surface $f(\Delta)$ can be expressed by the total number of the zeros and poles of f' in Δ. Solutions of a nonlinear partial differential equation will be considered in connection with the spherical derivative.

1. Introduction.

Let f be a nonconstant meromorphic function in a domain D in the complex plane $\boldsymbol{C}=\{|z|<+\infty\}$. The spherical derivative of f at $z\in D$ is defined by

$$f^{\#}(z)=\begin{cases} |f'(z)|/(1+|f(z)|^2) & \text{if } f(z)\neq\infty\,; \\ |(1/f)'(z)| & \text{if } f(z)=\infty\,. \end{cases}$$

We let $M(f)$ be the set of points $z\in D$ where $f^{\#}$ has local maxima, namely, $f^{\#}(z)\geqq f^{\#}(w)$ in $\{|w-z|<\delta\}\subset D$ for $\delta>0$ depending on f and z.

The purpose of the present paper is to investigate $M(f)$ in detail. We begin with a classification.

THEOREM 1. *Let f be nonconstant and meromorphic in a domain $D\subset \boldsymbol{C}$ with nonempty $M(f)$. Then, the connected components of $M(f)$ are at most countable and each component is one of the following*:

(I) *An isolated point.*

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(II) *A (noncompact) simple analytic are terminating nowhere in D.*
(III) *A simple closed analytic curve.*

All the cases of (I), (II), and (III) actually happen; see Section 2.

Let $C_1(f)$, $C_2(f)$, and $C_3(f)$ be the set of connected components of $M(f)$ of type (I), (II), and (III), respectively. Our next work is to observe them in detail.

The Schwarzian derivative of f is the meromorphic function

$$\sigma(f)=\lambda(f)'-2^{-1}\lambda(f)^2,$$

where $\lambda(f)=f''/f'$. Therefore, $f^{\#}(z)\neq 0$ if and only if $\sigma(f)(z)\neq\infty$ if and only if either z is a simple pole of f or $f(z)\neq\infty$ with $f'(z)\neq 0$. The derivative $\sigma(f)$ plays an important role in

THEOREM 2. *Let f be nonconstant and meromorphic in a domain $D\subset\boldsymbol{C}$ with nonempty $M(f)$. Then at each $z\in M(f)$,*

$$|\sigma(f)(z)|\leqq 2f^{\#}(z)^2. \tag{1.1}$$

Furthermore, we have the following.

(IV) *We can conclude $\{z\}\in C_1(f)$ if the inequality in* (1.1) *is strict.*

(V) *Suppose that $c\in C_2(f)\cup C_3(f)$ exists. Then, at each $z\in c$, the equality in* (1.1) *holds by* (IV). *Furthermore, the line $\{z+te^{-i\Theta(z)/2};\ -\infty<t<+\infty\}$ is tangent to c at z, where $\Theta(z)=\arg\sigma(f)(z)$. Moreover, there exists $\tau>0$ such that the function*

$$f^{\#}(z+ite^{-i\Theta(z)/2}),\qquad -\tau<t<\tau,$$

has the maximum at $t=0$ and this is convex from below, that is, the second derivative with respect to t is strictly negative.

The last statement in Theorem 2 is concerned with the behavior of $f^{\#}$ along the normal line of c at z.

THEOREM 3. *Let f be nonconstant and meromorphic in a domain $D\subset\boldsymbol{C}$. Suppose that $c\in C_3(f)$ exists and suppose further that the Jordan domain Δ bounded by c is contained in D. Then*

$$(2/\pi)\iint_{\Delta}f^{\#}(z)^2dxdy=Z_{\Delta}(f')+P_{\Delta}(f')-2n, \tag{1.2}$$

where $Z_{\Delta}(f')$ is the sum of all orders of all distinct zeros of f' in Δ and $P_{\Delta}(f')$ is that of all distinct n poles of f' in Δ.

The integral in the left-hand side of (1.2) is the spherical area of the Riemannian image of Δ by f, so that it is positive. In view of the right-hand side of (1.2) we can conclude that $f^{\#}$ must vanish at a finite number of points in Δ, or equivalently, $\sigma(f)$ must have a finite number of poles in Δ. If $f^{\#}$

never vanishes in D, then either $C_3(f)$ is empty or else each Jordan domain Δ bounded by $c\in C_3(f)$ is not contained in D.

2. Examples and Proofs of Theorems 1 and 2.

Before the proofs we observe that all the cases (I), (II), (III) can happen. If f is a Möbius transformation $(az+b)/(cz+d)$ $(ad-bc\neq 0)$ considered in $\boldsymbol{C}$, then $M(f)$ is a one-point set (see the remark in Section 4). If $f(z)=z^n$ $(n\geqq 2)$ is considered in $\boldsymbol{C}$, then $M(f)$ is the circle

$$c_n\equiv\left\{|z|=\left(\frac{n-1}{n+1}\right)^{1/(2n)}\right\}.$$

Consider the function $f(z)=z^n$, this time, in a domain D such that both D and $\boldsymbol{C}\setminus D$ have the nonempty intersection with the circle c_n. Then components of $M(f)$ are of type (II) in D.

For the proof of Theorems 1 and 2 we shall make use of the following lemmas.

LEMMA 1. *Let g be holomorphic and h be meromorphic in a domain $G\subset\boldsymbol{C}$. Suppose that*

$$L(g,\ h)=\{z\in G\,;\,\overline{g(z)}=h(z)\}$$

has an accumulation point $a\in G$ and $g'(a)\neq 0$. Then there exists an open disk $U(a)$ of center a such that $U(a)\cap L(g,\ h)$ is a simple analytic arc passing through a with both terminal points on the circle $\partial U(a)$.

Proof. The case $g(z)=z$. The proof is the same as in the proof of [RW, Lemma 1]. In the general case, let $V(a)$ be an open disk with center a where g is univalent. Regarding $g(V(a))$ as G, $g(z)$ as z, and h as $h\circ g^{-1}$ we can reduce this case into the case specified in the above.

LEMMA 2. *Let g be holomorphic in a domain $G\subset\boldsymbol{C}$. Suppose further that g' never vanishes in G. Then,*

$$M(g)\subset L(g,\ h), \tag{2.1}$$

where $h=\lambda(g)/\{2g'-g\lambda(g)\}$. Furthermore, on each component of $L(g,\ h)$ the function $g^{\#}$ is constant.

Proof. Suppose that $z\in M(g)$. Taking the logarithm of $g^{\#}$ and then partially differentiating it by w $(\partial/\partial w=2^{-1}(\partial/\partial u-i\partial/\partial v),\ w=u+iv)$, we have

$$\frac{(g^{\#})_w(w)}{g^{\#}(w)}=\frac{1}{2}\lambda(g)(w)-\frac{\overline{g(w)}g'(w)}{1+|g(w)|^2}. \tag{2.2}$$

The value of (2.2) at $w=z$ is zero. By a simple calculation we have h. It follows from Lemma 1 that each component of $L(g, h)$ is one of the three types described in Theorem 1. Suppose that Λ is a component of $L(g, h)$ which is not a point. Then Λ is a simple analytic curve $w=w(t)$ in the parametric form, $a<t<b$ or $a\leqq t\leqq b$. For $w(t)\in L(g, h)$,

$$\frac{d}{dt}g^{\#}(w(t))=2\,\mathrm{Re}[(g^{\#})_{w}(w(t))w'(t)]=0,$$

whence $g^{\#}$ is constant on Λ.

For the proofs of Theorems 1 and 2 we suppose that $z\in M(f)$. Then, there is θ such that $e^{2i\theta}\sigma(f)(z)=|\sigma(f)(z)|$; if $\sigma(f)(z)=0$, then we set $\theta=0$. There exists $\delta>0$ such that

$$g(w)=\frac{f(e^{i\theta}w+z)-f(z)}{1+\overline{f(z)}f(e^{i\theta}w+z)}$$

is holomorphic in $|w|<\delta$. If $f(z)=\infty$, then we set $g(w)=1/f(e^{i\theta}w+z)$. We now have

$$g^{\#}(w)=f^{\#}(e^{i\theta}w+z), \qquad \sigma(g)(w)=e^{2i\theta}\sigma(f)(e^{i\theta}w+z).$$

In particular,

$$g^{\#}(0)=f^{\#}(z)\equiv\alpha \quad\text{and}\quad \sigma(g)(0)=|\sigma(f)(z)|.$$

We may suppose that g' never vanishes in $|w|<\delta_1\leqq\delta$; actually, $|g'(0)|=\alpha>0$. We thus have (2.2) for the present g, which we call (2.2P). Further differentiation of (2.2P) by w yields

$$(2.3)\quad \frac{(g^{\#})_{ww}(w)}{g^{\#}(w)}-\Big(\frac{(g^{\#})_{w}(w)}{g^{\#}(w)}\Big)^{2}$$
$$=\frac{1}{2}\sigma(g)(w)+\frac{1}{4}\lambda(g)(w)^{2}-\frac{\overline{g(w)}g'(w)}{1+|g(w)|^{2}}\lambda(g)(w)+\Big(\frac{\overline{g(w)}g'(w)}{1+|g(w)|^{2}}\Big)^{2}.$$

Partial differentiation of (2.2P) by $\bar{w}$ ($\partial/\partial\bar{w}=2^{-1}(\partial/\partial u+i\partial/\partial v)$), on the other hand, yields

$$(2.4)\qquad \frac{(g^{\#})_{w\bar{w}}(w)}{g^{\#}(w)}-\Big|\frac{(g^{\#})_{w}(w)}{g^{\#}(w)}\Big|^{2}=-g^{\#}(w)^{2}.$$

Since $0\in M(g)$, it follows that $(g^{\#})_{w}(0)=0$, which, together with (2.2P) and $g(0)=0$, yields that $\lambda(g)(0)=0$. We therefore have $(g^{\#})_{ww}(0)=2^{-1}\alpha|\sigma(f)(z)|$ and $(g^{\#})_{w\bar{w}}(0)=-\alpha^{3}$, whence

$$A\equiv(g^{\#})_{uu}(0)=-2\alpha^{3}+\alpha|\sigma(f)(z)|,$$

$$(g^{\#})_{uv}(0)=0,$$

$$C\equiv(g^{\#})_{vv}(0)=-2\alpha^{3}-\alpha|\sigma(f)(z)|<0,$$

so that

$$AC=\alpha^{2}\{4\alpha^{4}-|\sigma(f)(z)|^{2}\}.$$

The Taylor expansion of $g^{\#}(w)-\alpha$ in u and v in $|w|<\delta_1$ now reads:

$$g^{\#}(w)-\alpha=\frac{1}{2}(Au^2+Cv^2)+\Gamma(u, v), \tag{2.5}$$

where the remaining term $\Gamma(u, v)$ is a power series of u, v of degree at least three. Since $g^{\#}$ has the local maximum 0 at $w=0$, and since $C<0$, it follows that $AC\geqq 0$. We therefore have (1.1).

If $AC>0$, then $g^{\#}(w)-\alpha<0$ for $0<|w|<\delta_2\leqq\delta_1$. If 0 is not an isolated point of $M(g)$, then 0 is an accumulation point of $M(g, h)$ in $G\equiv\{|w|<\delta_2\}$, where h is as in Lemma 2. Lemmas 1 and 2 show that there is a point $w_1\in G\setminus\{0\}$ such that $g^{\#}(w_1)=\alpha$. This is a contradiction. Therefore (IV) is proved.

Suppose that $AC=0$, or $A=0$, Suppose that 0 is an accumulation point of $M(g)$. Then 0 is an accumulation point of $L(g, h)$ considered in $\{|w|<\delta_1\}$. Lemmas 1 and 2 then show that there exists δ_3, $0<\delta_3\leqq\delta_1$, such that

$$\gamma\equiv M(g)\cap\{|w|<\delta_3\}=L(g, h)\cap\{|w|<\delta_3\}$$

is a simple analytic arc ending at points on $\{|w|=\delta_3\}$.

Returning to f we have observed that for each $z\in M(f)$, either (i) z is an isolated point of $M(f)$ or (ii) there exists an open disk $U(z)$ of center z such that $M(f)\cap U(z)$ is a simple analytic arc ending at points on $\partial U(z)$. This completes the proof of Theorem 1.

For the proof of Theorem 2 we further analyze the case where $AC=0$ and 0 is an accumulation point of $M(g)$. Set $\gamma=\{w(t);\, a<t<b\}$. Then, $\overline{g(w(t))}=h(w(t))$, so that a short calculation shows that

$$\overline{w'(t)}/w'(t)=h'(w(t))/\overline{g'(w(t))},$$

$$h'/\overline{g'}=2^{-1}\sigma(g)/g^{\#2}.$$

Therefore the slope of the tangent at $w\in\gamma$ is $\tan Q(w)$, where

$$e^{-2iQ(w)}=2^{-1}\sigma(g)(w)/g^{\#}(w)^2.$$

In particular, $Q(0)=0$. Thus, γ has the u-axis as the tangent at 0. We thus have the tangent to c described in Theorem 2. Furthermore,

$$g^{\#}(iv)-\alpha=(C/2)v^2+\cdots,\ C<0,$$

so that $(d^2/dv^2)F(iv)<0$ near $v=0$. This completes the proof of the theorem.

Remark. Let $z\in c\in C_2(f)\cup C_3(f)$. Rectilinear segments containing z with the exception of the normal and the tangent ones to c are expressed by

$$\Lambda(\beta)\equiv\{\varphi_\beta(t);\ -\tau(\beta)\leqq t\leqq\tau(\beta)\},$$

where $0<\beta<\pi/2$, and $\varphi_\beta(t)=z+(1+i\tan\beta)t\,e^{-i\Theta(z)/2}$. It is now easy to prove that $f^{\#}(\varphi_\beta(t))$ is convex, that is, $(d^2/dt^2)f^{\#}(\varphi_\beta(t))<0$ for $|t|\leqq\tau(\beta)$, for suitable $\tau(\beta)>0$. Actually, we have in (2.5) that

$$g^{\#}(t+it\tan\beta)-\alpha=\frac{1}{2}C\,t^2\tan^2\beta+\Gamma(t,\ t\tan\beta),$$

because $A=0$. This fact shows that even in case $z\in c$, the function $f^{\#}$ attains its maximum at z "in the strict sense" except along c.

As a further remark we let $M^*(f)$ be the set of points $z\in D$ where $f^{\#}$ attains the (global) maximum in $D: f^{\#}(z)\geqq f^{\#}(w)$ for all $w\in D$. Suppose that $a\in D$ is an accumulation point of $M^*(f)$. Then, $a\in M^*(f)$. Suppose that f is nonconstant. Then, there exists $c\in C_2(f)\cup C_3(f)$ such that $a\in c$ because $M^*(f)\subset M(f)$. Since $f^{\#}$ is constant on c it follows that $c\subset M^*(f)$. Therefore we have the analogous classification: $C_k^*(f)$, $k=1, 2, 3$ of components of $M^*(f)$.

Suppose that isolated points of $M(f)$ has an accumulation point $a\in D$. Then $f^{\#}(a)=0$ so that a is not a member of $M(f)$. For the proof we suppose that $f^{\#}(a)\neq 0$. If $f(a)\neq\infty$, then $f'(a)\neq 0$. It then follows from Lemmas 1 and 2 that there exists an open disk $U(a)$ of center a such that

$$L(f,\ h_f)=\{z\in U(a)\,;\ \overline{f(z)}=h_f(z)\}\,,$$

where

$$h_f=\lambda(f)/\{2f'-f\lambda(f)\}\,,$$

is a simple, analytic arc on which $f^{\#}$ is constant. A contradiction comes from $M(f)\cap U(a)\subset L(f,\ h_f)$. If $f(a)=\infty$, we apply the same argument to $1/f$ with $(1/f)'(a)\neq 0$ to arrive at a contradiction. Since $M^*(f)$ is a closed set in D, it is now easy to observe that the isolated points of $M^*(f)$ cluster nowhere in D for nonconstant f.

3. Proof of Theorem 3.

First of all, $f^{\#}$ never vanishes on $c=\partial\Delta$. Let α_k, $1\leqq k\leqq p$, be all the simple poles of f on c and let γ_k, $1\leqq k\leqq n$, be all the distinct poles of f of order ν_k in Δ. Thus, $P_\Delta(f')=n+\sum_{k=1}^{n}\nu_k$. Let $A=\{\alpha_1,\cdots,\alpha_p,\gamma_1,\cdots\gamma_n\}$. Let $\varepsilon>0$ be sufficiently small, and for each $\alpha\in A$, we set

$$\delta(\alpha)=\{|z-\alpha|\leqq\varepsilon\}\,,\qquad c(\alpha)=\{z\in\Delta\,;\ |z-\alpha|=\varepsilon\}\,,$$

and further

$$\Delta(\varepsilon)=\Delta\setminus\bigcup_{\alpha\in A}\delta(\alpha)\,;$$

this becomes a domain bounded by a finite number of Jordan curves for sufficiently small ε.

Set $\Phi=\bar{f}f'/(1+|f|^2)$, and $\Psi=i\Phi$. Then the Green formula

$$\iint_{\Delta(\varepsilon)}(\Psi_x-\Phi_y)dxdy=\int_{\partial\Delta(\varepsilon)}(\Phi dx+\Psi dy)$$

can be rewritten as:

(3.1) $$\frac{2}{\pi}\iint_{\Delta(\varepsilon)}f^{\#}(z)^2dxdy=\frac{1}{\pi i}\int_{\partial\Delta(\varepsilon)}\Phi(z)dz;$$

the line integral is in the positive sense with respect to $\Delta(\varepsilon)$.

The Laurent expansion of f about $\alpha\in A$ yields

$$f(z)=(z-\alpha)^{-N}g(z) \qquad \text{in} \quad \delta(\alpha)\setminus\{\alpha\},$$

where g is holomorphic and nonvanishing in $\delta(\alpha)$; $N=1$ if $\alpha=\alpha_k$, while $N=\nu_k$ if $\alpha=\gamma_k$. The differentiation yields that

(3.2) $$f'(z)=(z-\alpha)^{-N-1}h(z), \qquad h(z)=-Ng(z)+(z-\alpha)g'(z).$$

Since

$$\varepsilon e^{it}\Phi(\varepsilon e^{it}+\alpha)=\frac{\overline{g(\varepsilon e^{it}+\alpha)}h(\varepsilon e^{it}+\alpha)}{\varepsilon^{2N}+|g(\varepsilon e^{it}+\alpha)|^2}\longrightarrow -N \quad \text{as} \quad \varepsilon\to 0$$

uniformly for real t, it follows that

$$\int_{c(\alpha)}\Phi(z)dz\longrightarrow\begin{cases}\pi i & \text{if} \quad \alpha=\alpha_k;\\ 2\pi\nu_k i & \text{if} \quad \alpha=\gamma_k,\end{cases}$$

as $\varepsilon\to 0$, where the integral is in the clockwise sense. Letting $\varepsilon\to 0$ in (3.1), we now have

(3.3) $$\frac{2}{\pi}\iint_{\Delta}f^{\#}(z)^2dxdy=\frac{1}{\pi i}\int_{c}\Phi(z)dz+p+2\sum_{k=1}^{n}\nu_k$$
$$=\frac{1}{2\pi i}\int_{c}\lambda(f)(z)dz+p+2\sum_{k=1}^{n}\nu_k.$$

On the other hand, for ε small we have

(3.4) $$\frac{1}{2\pi i}\int_{\partial\Delta_0(\varepsilon)}\lambda(f)(z)dz=Z_\Delta(f')-P_\Delta(f'),$$

where

$$\Delta_0(\varepsilon)=\Delta\setminus\bigcup_{k=1}^{p}\delta(\alpha_k).$$

In view of (3.2) we have in $\delta(\alpha)\setminus\{\alpha\}$, $\alpha=\alpha_k$.

$$\lambda(f)(z)=-2/(z-\alpha)+h'(z)/h(z),$$

and further, for small ε, the holomorphic h has no zero in $\delta(\alpha)$. Therefore, letting $\varepsilon\to 0$ in the left-hand side of (3.4) we have the identity:

(3.5) $$\frac{1}{2\pi i}\int_{c}\lambda(f)(z)dz+p=Z_{\Delta}(f')-P_{\Delta}(f').$$

Combining (3.3) and (3.5) we finally have (1.2) in the theorem.

4. A lemma.

LEMMA 3. *Let f be nonconstant and meromorphic in D. Suppose that $z\in M(f)$ and suppose further that there exist real constants $\delta>0$ and θ such that*

(4.1) $$|\sigma(f)(z+te^{i\theta})|>2f^{\#}(z+te^{i\theta})^{2}$$

for $0<t<\delta$. Then, $f^{\#}(z+te^{i\theta})$ is strictly decreasing for $0<t<\delta$.

An immediate consequence is that if $z\in M(f)$ and if $|\sigma(f)(w)|<2f^{\#}(w)^{2}$ in $|w-z|<\delta$, then there is no point of $M(f)$ in $0<|w-z|<\delta$; the result already observed in Section 2.

Proof of Lemma 3. It suffices to consider the case $z=0$ and $\theta=0$. By a technical reason we consider $G=1/f^{\#}$. Calculations like (2.2)-(2.4) this time, yield

$$\frac{G_{ww}(w)}{G(w)}=-\frac{1}{2}\sigma(f)(w),\qquad \frac{G_{w\bar{w}}(w)}{G(w)}=\left|\frac{G_{w}(w)}{G(w)}\right|^{2}+f^{\#}(w)^{2},$$

whence

$$\frac{G_{uu}(w)}{G(w)}=2\left|\frac{G_{w}(w)}{G(w)}\right|^{2}+(2f^{\#}(w)^{2}-\mathrm{Re}\,\sigma(f)(w)).$$

Therefore,

$$\frac{d^{2}}{dt^{2}}G(z+t)=G_{uu}(z+t)>0,$$

together with $G_{u}(z+t)\equiv(d/dt)G(z+t)\to0$ as $t\to+0$, shows that $G_{u}(z+t)>0$ for $0<t<\delta$. Therefore G is strictly increasing on the line segment.

Remark. If T is a Möbius transformation, then a computation shows that $T^{\#}(w)\to0$ as $w\to\infty$. Furthermore, $\sigma(T)\equiv0$, so that $|\sigma(T)(w)|<2T^{\#}(w)^{2}$ at each point $w\in\boldsymbol{C}$. We can apply Lemma 2 to consider the maximum of $T^{\#}$ of a Möbius transformation without further direct calculation. Apparently, $M(T)$ is nonempty. Suppose that there are $z_{k}\in M(T)$, $k=1, 2$, and $z_{1}\neq z_{2}$. Then a contradiction follows from Lemma 3. Therefore there is only one point z such that $T^{\#}(w)<T^{\#}(z)$ for all $w\in\boldsymbol{C}\setminus\{z\}$.

5. A partial differential equation with an exponential nonlinearity.

Let ω be a real-valued solution of the differential equation

$$(\partial^2/\partial z\partial\bar{z})\omega+ae^{\omega}=0 \tag{5.1}$$

in a domain $D\subset \boldsymbol{C}$, where $a>0$ is a constant. If f is meromorphic with nonvanishing $f^{\#}$ in D, then

$$\omega=\log(2a^{-1}(f^{\#})^2) \tag{5.2}$$

is a solution. Conversely, the celebrated Liouville paper [L] shows that, if D is simply connected, then each solution ω can be expressed as (5.2) for a meromorphic function f with nonvanishing $f^{\#}$ in D. A concise proof of this is given in [W] and a detailed one is given in [B1, pp. 27-28] (see also [S]). Usually one supposes the boundary condition:

$$\lim_{z\to\zeta}\omega(z)=0, \qquad \zeta\in\partial D, \tag{5.3}$$

to (5.1), where ∂D is the boundary of D in $\boldsymbol{C}\cup\{\infty\}$. Since ω is superharmonic in D, the minimum principle shows that $\omega>0$ in D. For the existence of the solutions for (5.1) under (5.3), in case D is bounded and simply connected, see [B1, p. 197] for example.

Let $M^*(\omega)$ be the set of points $z\in D$ where ω has the (global) maximum:

$$\omega(z)\geqq\omega(w) \qquad \text{for all} \quad w\in D.$$

We show that $M^*(\omega)$ is a finite set if D is simply connected and (5.3) holds. First, no point-sequence extracted from $M^*(\omega)$ accumulates at any point of ∂D. Consider $M(f)$ for f in (5.2). Theorem 3 shows that $C_3(f)$ is empty. Suppose that $c\in C_2(f)$ exists. If $c\cap M^*(\omega)$ is nonempty, then $c\subset M^*(\omega)$ because $f^{\#}$ is constant on c. This is a contradiction. Since $M^*(\omega)\subset M(f)$, it follows that each point of $M^*(\omega)$ is an isolated point of $M^*(\omega)$. The isolated points of $M^*(\omega)=M^*(f)$ cannot accumulate at any point of D. Therefore $M^*(\omega)$ is a finite set.

We return to (5.1) for general D. Let $M(\omega)$ be the set of points $z\in D$ where ω attains local maxima: $\omega(z)\geqq\omega(w)$ for w in an open disk $U(z)\subset D$ with center z. Restricting (5.2) to $U(z)$ we have again a meromorphic function f in $U(z)$ where (5.2) is valid. By the local observation of $M(\omega)$ one can easily obtain the ω-counterpart of Theorem 1. Namely, the components of $M(\omega)$ are at most countable and are classified into the three types (I), (II), and (III), described in Theorem 1. Apparently $M(\omega)\supset M^*(\omega)$.

As C. Bandle [B2, p. 231] (see also [B1, p. 29]) pointed out, we have

$$\sigma(f)(z)=\frac{\partial^2\omega(z)}{\partial z^2}-\frac{1}{2}\left(\frac{\partial\omega(z)}{\partial z}\right)^2$$

for (5.2) in D, so that, the inequality

$$|\sigma(f)(z)|\leqq 2f^{\#}(z)^2 \qquad \text{at} \quad z\in D$$

reads

$$\left|\frac{\partial^2\omega(z)}{\partial z^2}-\frac{1}{2}\left(\frac{\partial\omega(z)}{\partial z}\right)^2\right|\leqq ae^{\omega(z)}.$$

6. The Gauss curvature.

Let u be a nonconstant, real-valued, and harmonic function in a domain $D\subset \boldsymbol{C}$. Then u defines the surface or the graph: $\{(x, y, u(x, y)); (x, y)\in D\}$ in the space. The Gauss curvature $K(z)$ at the point $(x, y, u(x, y))$, $z=x+iy\in D$, is then $-f^{\#}(z)^2$, where $f=u_x-iu_y$ is holomorphic in D. Our results are therefore applicable to the study of the set of points in D where K attains local minima. See [G, J, K, KP, T, Y] on the cited Gauss curvature.

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