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CHARACTERIZATIONS OF THE EXPONENTIAL FUNCTION BY THE VALUE DISTRIBUTION

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1. Introduction. Baker [1] has shown the following characterization of the exponential function.

Let $f(z)$ be a transcendental entire function. Assume that for every complex number w there is a straight line L_w of the complex plane on which all the solutions of $f(z)=w$ lie. Then $f(z)=a+b\cdot\exp(Az)$ with constants a, b, A.

Recently Kobayashi [6] has shown the following theorem.

THEOREM A. *Let $f(z)$ be a transcendental entire function. Assume that there are three distinct finite complex numbers a_j and three straight lines L_j of the complex plane on which all the solutions of $f(z)=a_j$ lie $(j=1, 2, 3)$. Assume further that these three values never lie on any straight line of the complex plane. Then $f(z)=P(\exp Az)$ with a quadratic polynomial $P(z)$ and a non-zero constant A.*

In this note we shall give a generalization of Baker's result. In what follows we shall mean a strip by the set $\{az+b;\ |\mathrm{Re}\, z|\leqq 1\}$, where $a\ (\neq 0)$ and b are constants.

THEOREM 1. *Let $f(z)$ be a transcendental entire function and k a positive number. Assume that for every complex number w there is a strip S_w of width k of the complex plane in which all the solutions of $f(z)=w$ lie. Then $f(z)=a+b\cdot\exp(Az)$ with constants $a, b, A, bA\neq 0$.*

THEOREM 2. *Let $f(z)$ be a transcendental real entire function of finite lower order and k a positive number. Assume that for every real number w there is a strip S_w of width k of the complex plane in which all the solutions of $f(z)=w$ lie. Then $f(z)=a+b\cdot\exp(Az)$ with real constants $a, b, A, bA\neq 0$.*

THEOREM 3. *Let $f(z)$ be a transcendental entire function and G an open subset of the complex plane. Assume that for every $w\in G$ there is a straight line L_w of the complex plane on which all but a finite number of the solutions of $f(z)=w$ lie. Then $f(z)=a+b\cdot\exp(Az)$ with constants $a, b, A, bA\neq 0$.*

THEOREM 4. *Let $f(z)$ be a transcendental entire function, G an open subset*

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of the complex plane and n a positive integer. Assume that for every $w \in G$ *there are n straight lines* $L_{w,1}, \cdots, L_{w,n}$ *of the complex plane, which are parallel with one another, on which all the solutions of* $f(z)=w$ *lie. Then* $f(z)=Q(\exp Az)$ *with a rational function Q of order at most n and a non-zero constant A.*

Remarks. The function e^z+e^{az} $(a>1)$ shows that the assumption "for every complex number" in Theorem 1 cannot be improved. If G is a straight line of the complex plane, then the conclusion of Theorem 3 cannot hold generally. This is shown by the function $f(z)=z\cos z$. It is easily seen that for every real number w all but a finite number of the solutions of $f(z)=w$ lie on the real axis.

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2. Statement of known results. We need the following theorems.

THEOREM A [5]. *Let* $f(z)$ *be an entire function of finite genus* q $(\geqq 1)$. *If its zeros* $\{a_n\}$ *satisfy*

$$\lim_{n\to\infty} \arg a_n=0 \qquad (|\arg a_n|\leqq \pi),$$

then $f(z)$ *has zero as a deficient value.*

THEOREM B [5]. *Let* $f(z)$ *be an entire function of finite genus* q $(\geqq 2)$. *If its zeros* $\{a_n\}$ *lie in a strip of the complex plane, then* $f(z)$ *has zero as a deficient value.*

THEOREM C [5]. *Let* $f(z)$ *be a non-constant entire function satisfying*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r}=0.$$

Then the smallest convex set which contains the zeros of $f(z)$ *also contains the zeros of* $f'(z)$.

LEMMA A [5]. *Let* $f(z)$ *be an entire function of genus at most one. Assume that 0 is not a lacunary value of* $f(z)$ *and all the zeros* $\{a_n\}$ *of* $f(z)$ *lie in the strip*

$$\{z\,;\, |\operatorname{Re} z|\leqq h\}.$$

Then

$$\operatorname{Re}\frac{f'(z)}{f(z)}=A+\sum_n \operatorname{Re}\frac{1}{z-a_n}$$

with a real constant A. Further if A is positive, zero, or negative, then for real number x

$$\lim_{x\to+\infty}|f(x)|=+\infty\,,$$

$$\lim_{x\to+\infty}|f(x)|=\lim_{x\to-\infty}|f(x)|=+\infty\,,\quad or\quad \lim_{x\to-\infty}|f(x)|=+\infty$$

respectively.

LEMMA B [5]. *If $f(z)$ is regular in the half plane $\{z\,;\,\mathrm{Re}\,z>h>0\}$ and fails to take there 0 and 1, then*

$$\log|f(z)|\leqq\frac{A(1+h+|z|)^2}{(x-h)}\qquad(\mathrm{Re}\,z=x>h)\,,$$

where A is a positive constant.

3. Lemmas. In this section we shall prove the following lemmas.

LEMMA 1. *Let $f(z)$ be an entire function. Assume that there exist four distinct finite complex numbers a_j and four strips S_j of the complex plane such that all the solutions of $f(z)=a_j$ lie in S_j $(j=1,\cdots,4)$. Assume further that the four strips S_j are parallel with one another and $S_i\cap S_j=\varnothing$ $(i\neq j)$. Then $f(z)$ has at most order one.*

The proof is essentially the same as that of Theorem 4 in [5], hence omitted.

LEMMA 2. *Let $f(z)$ be an entire function. Assume that there exist three distinct finite complex numbers a_j and three strips S_j of the complex plane such that all the solutions of $f(z)=a_j$ lie in S_j $(j=1,2,3)$. Assume further that no two of the three strips S_j run parallel with each other. Then the order ρ, of $f(z)$, is finite and*

$$\rho\leqq\max\left\{\frac{\pi}{\omega_1+\omega_2},\ \frac{\pi}{\omega_2+\omega_3},\ \frac{\pi}{\omega_3+\omega_1}\right\},$$

where ω_j $(j=1,2,3)$ are apertures of those three angular sectors which are components of $\boldsymbol{C}\backslash(S_1\cup S_2\cup S_3)$ and adjoin.

Proof. We assume, without loss of generality, that

$$S_1=\{z\,;\,|\mathrm{Im}\,z|\leqq k\},\qquad S_2=\{z\,;\,|\mathrm{Im}\,(ze^{-i\omega_2})|\leqq k\},$$

$$S_3=\{z\,;\,|\mathrm{Im}\,(ze^{i\omega_3})|\leqq k\},\qquad \left(0<\omega_2,\ \omega_3\leqq\frac{\pi}{2},\ k>0\right).$$

Let ε be an arbitrarily fixed positive number less than $\min(\omega_2,\omega_3)$. We choose real numbers α, β such that

$$\{z^\beta e^{i\alpha}\,;\,\mathrm{Re}\,z>0\}=\{z\,;\,\omega_2-\varepsilon>\arg z>-\omega_3+\varepsilon\}\,.$$

Let

$$F(z)=(f(z^{\beta}e^{i\alpha})-a_2)/(a_3-a_2)\,,$$

then for a suitable positive number h $F(z)$ fails to take 0 and 1 in $\operatorname{Re} z>h$. Thus by Lemma B

$$\log^+|F(z)|\leqq A\frac{(1+h+|z|)^2}{(x-h)}\qquad(\operatorname{Re} z=x>h)\,. \tag{3.1}$$

Let δ be an arbitrarily fixed number in $(0, \pi/2)$. We choose a positive number γ satisfying $\cos\delta-(h/r)>\gamma$ for every sufficiently large r. Then by (3.1)

$$\int_{-\delta}^{\delta}\log^+|F(re^{i\theta})|\,d\theta\leqq A(1+h+r)^2\frac{1}{r}\int_{-\delta}^{\delta}\frac{d\theta}{\cos\theta-h/r}$$

$$\leqq A\frac{2\delta}{\gamma}\cdot\frac{(1+h+r)^2}{r}=O(r)$$

for every sufficiently large r. Thus

$$\int_{-\beta\delta+\alpha}^{\beta\delta+\alpha}\log^+|f(re^{i\theta})|\,d\theta=O(r^{1/\beta})\,. \tag{3.2}$$

If ε is sufficiently small and δ is sufficiently close to $\pi/2$, then

$$\{e^{i\theta}\,;\ \beta\delta+\alpha>\theta>-\beta\delta+\alpha\}\supset\left\{e^{i\theta}\,;\ \frac{\omega_2}{2}>\theta>\frac{-\omega_3}{2}\right\}.$$

Further if ε tends to 0 from above, then $1/\beta$ tends to $\pi/(\omega_2+\omega_3)$ from above. Thus by (3.2)

$$\limsup_{r\to\infty}\,(\log r)^{-1}\log\left(\int_{-\omega_3/2}^{\omega_2/2}\log^+|f(re^{i\theta})|\,d\theta\right)\leqq\pi/(\omega_2+\omega_3)\,.$$

For other angular sectors we obtain similar results. Thus the order of $f(z)$ is at most

$$\max\left\{\frac{\pi}{\omega_1+\omega_2},\ \frac{\pi}{\omega_2+\omega_3},\ \frac{\pi}{\omega_3+\omega_1}\right\}.$$

LEMMA 3. *Let $f(z)$ be an entire function. Assume that there exist three distinct finite complex numbers a_j and three straight lines L_j of the complex plane on which all but a finite number of the solutions of $f(z)=a_j$ lie ($j=1, 2, 3$). Then the order of $f(z)$ is finite.*

Proof. It is sufficient to consider only the case that the three lines L_j are distinct and parallel with one another. Indeed in other cases the assertion of Lemma 3 follows at once from Theorem 1 in [2].

We can assume, without loss of generality, that

$$L_i=\{z\,;\ \operatorname{Re} z=h_i\}\qquad(h_1>0,\ h_2=0,\ h_3<0)\,,$$

and that $f(z)\neq a_i$ for every z in $\boldsymbol{C}\backslash(\{z\,;\ |z|<1\}\cup L_i)$ ($i=1, 2, 3$). Let $w=\phi(z)=$

$z-z^{-1}$, then the function $z=\phi^{-1}(w)$ maps the half plane $\operatorname{Re} w>0$ conformally onto the region $\{z ; \operatorname{Re} z>0, |z|>1\}$. Let $g(z)=(f(z)-a_2)/(a_3-a_2)$, then $g(\phi^{-1}(w))$ fails to take 0 and 1 in $\operatorname{Re} w>0$. Thus by Lemma B

$$\log^+|g(\phi^{-1}(w))| \leqq \frac{A(1+|w|)^2}{\operatorname{Re} w} \qquad (\operatorname{Re} w>0). \tag{3.3}$$

If z ($\operatorname{Re} z>0$) is sufficiently large, then $\operatorname{Re} w=\operatorname{Re}(z(1-|z|^{-2}))>\operatorname{Re}(z/2)$. From (3.3) we thus obtain

$$\log^+|g(z)| \leqq \frac{8A|z|^2}{\operatorname{Re} z} \qquad (\operatorname{Re} z>0)$$

for sufficiently large z. Therefore

$$\log^+|f(z)| \leqq \frac{9A|z|^2}{\operatorname{Re} z} \qquad (\operatorname{Re} z>0) \tag{3.4}$$

for sufficiently large z. Similarly, we obtain

$$\log^+|f(z)| \leqq \frac{9A|z|^2}{-\operatorname{Re} z} \qquad (\operatorname{Re} z<0) \tag{3.5}$$

for sufficiently large z. Applying the essentially same method as in the proof of Theorem 4 in [5] to (3.4) and (3.5), we conclude that $f(z)$ has at most order one. Lemma 3 is thus proved.

LEMMA 4. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-zero complex numbers such that $a_n \to \infty$ as $n \to \infty$, and $n(r, a_n)$ the counting function of the sequence $\{a_n\}$. Assume that*

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty .$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{|z-a_n|^2} \geqq \frac{1}{4|z|^2} n(|z|, a_n) \qquad (z \neq 0).$$

Proof. Let z be an arbitrarily fixed non-zero complex number. Let $|z|=r$, then

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{|z-a_n|^2} \geqq \sum_{n=1}^{\infty} \frac{1}{(r+|a_n|)^2} \\
&=\int_0^{\infty} \frac{1}{(t+r)^2} dn(t, a_n)=2\int_0^{\infty} \frac{1}{(t+r)^3} n(t, a_n)\, dt \\
&=\frac{2}{r^2}\int_0^{\infty} \frac{1}{(s+1)^3} n(rs, a_n) ds \geqq \frac{2}{r^2} n(r, a_n) \int_1^{\infty} \frac{1}{(s+1)^3} ds = \frac{1}{4r^2} n(r, a_n).
\end{aligned}$$

Lemma 4 is thus proved.

4. Proof of Theorem 1. Firstly by Lemma 1 and Lemma 2 the order of $f(z)$ must be finite. Let us denote

$$\boldsymbol{C}^* = \{a \in \boldsymbol{C}\,;\ \delta(a, f)=0\}.$$

Then by Theorem B the genus of $f(z)-a$ is at most one for every $a \in \boldsymbol{C}^*$. Further by the following lemma all the strips S_a $(a \in \boldsymbol{C}^*)$ are parallel with one another.

LEMMA 5. *Let $f(z)$ be a transcendental entire function of finite order, and A an infinite set of complex numbers containing no deficient value of $f(z)$. Assume that the genus of $f(z)-a$ is at most one for every $a \in A$, and that for every $a \in A$ there is a strip T_a of the complex plane in which all the solutions of $f(z)=a$ lie. Then all the strips T_a $(a \in A)$ are parallel with one another.*

Proof. We consider the following two cases.

$$(1)\quad \liminf_{r\to\infty} T(r, f)/r = 0\,. \qquad (2)\quad \liminf_{r\to\infty} T(r, f)/r \neq 0\,.$$

Firstly we consider the case (1). If there exist two values, a, b $(\in A)$ such that T_a and T_b are not parallel with each other, then by Theorem C $f'(z)$ has at most a finite number of zeros. Therefore by (1) $f(z)$ must be a polynomial. This is a contradiction. Thus all the strips T_a $(a \in A)$ are parallel with one another in this case.

Secondly we consider the case (2). Suppose that T_a and T_b are not parallel with each other for some a, b $(\in A)$. There exist infinitely many elements $\{a_n\}_{n=1}^{\infty}$ of A and a straight line

$$L = \{te^{i\omega}\,;\ t \in \boldsymbol{R}\} \qquad (0 \leqq \omega < \pi)$$

such that the direction of T_{a_n} approaches that of L as $n \to \infty$. Then using Lemma A and Lindelöf-Iversen-Gross' theorem [7] we deduce that

$$\lim_{r\to\infty} |f(re^{i\theta})| = +\infty$$

uniformly for $|\theta-\omega+\pi/2| \leqq \theta^* < \pi/2$ or for $|\theta-\omega-\pi/2| \leqq \theta^* < \pi/2$, where θ^* is an arbitrarily fixed number in $(0, \pi/2)$. Thus a-points or b-points of $f(z)$ must lie in a half strip. Therefore by Theorem A and (2) we conclude that a or b is a deficient value of $f(z)$. This is a contradiction. Thus all the strips T_a $(a \in A)$ are parallel with one another in this case. Lemma 5 is thus proved.

By Lemma 5 we can assume, without loss of generality, that all the strips S_a $(a \in \boldsymbol{C}^*)$ run parallel with the imaginary axis. We now consider a sequence $\{x_n\}_{n=-\infty}^{+\infty}$ of complex numbers such that $\mathrm{Re}(x_{n+1}) - \mathrm{Re}(x_n) \geqq 3k$, and that $f(x_n) = w_n \in \boldsymbol{C}^*$ $(n=0, \pm 1, \pm 2, \cdots)$. Then $S_{w_i} \cap S_{w_j} = \varnothing$ $(i \neq j)$. From Lemma A

$$\text{(4.1)} \qquad \operatorname{Re}\frac{f'(z)}{f(z)-w_n}=A_n+\sum_m \operatorname{Re}\frac{1}{z-a_m},$$

where A_n is a real constant and $\{a_m\}$ the w_n-points of $f(z)$ $(n=0, \pm 1, \pm 2, \cdots)$.

Next we prove

LEMMA 6. *The following two cases donot occur.*

$$(1) \quad A_n>0,\ A_m<0 \quad (n>m). \qquad (2) \quad A_n=A_m=0 \quad (n\neq m).$$

Proof. Firstly we assume that the case (1) occurs. Let us write $A_n=A$, $A_m=-B$, $w_n=a$, $w_m=b$, $\operatorname{Re}(x_n)+k=r$, $\operatorname{Re}(x_m)-k=s$. Let

$$L=\{a+t(a-b)\,;\ t\in(0, \infty)\}.$$

By (4.1) and (1) we obtain

$$\text{(4.2)} \qquad \begin{aligned}&\arg(f(x+iy)-a)-\arg(f(x)-a)\\ &=\int_0^y \operatorname{Re}\frac{f'(x+it)}{f(x+it)-a}\,dt\geqq Ay\end{aligned}$$

for every $(x, y)\in(r, \infty)\times\boldsymbol{R}$. Thus from (4.2)

$$\text{(4.3)} \qquad \{\arg(f(x+iy)-a)\,;\ y\in[0, 2\pi/A]\}\supset[0, 2\pi]$$

for every x $(>r)$. On the other hand, Lemma A and Lindelöf-Iversen-Gross' theorem [7] imply

$$\text{(4.4)} \qquad \lim_{x\to+\infty}|f(x+iy)|=+\infty$$

uniformly for y $(0\leqq y\leqq 2\pi/A)$. By (4.3) and (4.4) we see that for some point z_1 in $\operatorname{Re} z>r$, $f(z_1)$ lies on L. Hence using the same argument as in the proof of Lemma 9 in [5], we can deduce that every sufficiently large value on L can be taken by $f(z)$ in the half plane $\operatorname{Re} z>r$. For completeness we shall give a proof of this assertion.

Let $v_1=f(z_1)$, and $E(w, v_1)$ be the regular element of $f^{-1}(w)$ with center v_1 which satisfies $E(v_1, v_1)=z_1$. Put $v_1=a+e^{i\alpha}t_1$ with real constants α, t_1 $(t_1>0)$. We continue $E(w, v_1)$ analytically along the segment $\{a+te^{i\alpha}\,;\ t_1\leqq t<t_2<\infty\}$. Put

$$Z(t)=E(a+e^{i\alpha}(t_1+t), v_1), \qquad 0\leqq t<(t_2-t_1).$$

Then

$$\text{(4.5)} \qquad f(Z(t))=a+e^{i\alpha}(t_1+t).$$

If $Z(t_*)$ $(0\leqq t_*<t_2-t_1)$ is contained in the half plane $\operatorname{Re} z>r$, then $Z(t)$ is differentiable at t_* and from (4.5)

$$\text{(4.6)} \qquad f'(Z(t_*))Z'(t_*)=e^{i\alpha}.$$

From (4.1) and (1)

$$\operatorname{Re}\frac{f'(Z(t_*))}{f(Z(t_*))-a}\geqq A>0\,. \tag{4.7}$$

By (4.5), (4.6) and (4.7) we conclude

$$\operatorname{Re} Z'(t_*)>0\,.$$

Therefore $Z(t)$ must be contained in $\operatorname{Re} z>r$ for every t $(\in[0, t_2-t_1))$. If this analytic continuation defines a transcendental singularity at the point $a+t_2\exp(i\alpha)$, then the path $\Gamma=\{Z(t)\,;\, 0\leqq t<(t_2-t_1)\}$ must be an asymptotic path of $f(z)$ and as z tends to infinity along this path Γ, $f(z)$ approaches the value $a+t_2\exp(i\alpha)$. Thus, by Lindelöf-Iversen-Gross' theorem [7], we deduce that for real number x

$$\lim_{x\to+\infty} f(x)=a+t_2e^{i\alpha}\,.$$

This contradicts (4.4). Thus $E(w, v_1)$ can be continued analytically along the half line L up to infinity. Hence we conclude that every sufficiently large value on L can be taken by $f(z)$ in the half plane $\operatorname{Re} z>r$.

Similarly, every sufficiently large value on L can be taken by $f(z)$ in the half plane $\operatorname{Re} z<s$. Some sufficiently large value on L is in $\boldsymbol{C}^*$. Thus we have a contradiction, since every strip S_a $(a\in\boldsymbol{C}^*)$ is parallel with the imaginary axis. Thus (1) cannot occur.

We next show that the case (2) cannot occur. Indeed, if otherwise, then by (4.1) and (2) $f'(z)$ fails to take 0 in $\boldsymbol{C}$. Further by Lemma 1 the order of $f(z)$ is at most one. Thus $f(z)=a+b\cdot\exp(Az)$ with constants a, b, A. On the other hand by Lemma A and (2) we deduce that for real number x

$$\lim_{x\to+\infty}|f(x)|=\lim_{x\to-\infty}|f(x)|=+\infty\,.$$

However the function $f(z)=a+b\cdot\exp(Az)$ does not satisfy this asymptotic behavior. Thus (2) cannot occur. Lemma 6 is thus proved.

By Lemma 6 we have only the following two possibilities.

1) There exists an integer N such that $A_n>0$ for every $n\leqq N$.

2) There exists an integer N such that $A_n<0$ for every $n\geqq N$.

In each case by (4.1) $f'(z)$ fails to take 0 in $\boldsymbol{C}$. Further by Lemma 1 the order of $f(z)$ is at most one. Thus $f(z)=a+b\cdot\exp(Az)$ with constants a, b, A, $bA\neq 0$. The proof of Theorem 1 is now complete.

5. Proof of Theorem 2. Let L be the real axis. If a $(\in\boldsymbol{R})$ is not a Picard exceptional value of $f(z)$, then $S_a\supset L$ or S_a is at right angles to L. Thus by the theorem in [4] we see that S_a is at right angles to L for every sufficiently large a $(\in\boldsymbol{R})$. Further by Theorem 4 in [5] we see that the order of $f(z)$ is at most one.

If there exist two real numbers a, b satisfying $S_a\supset L$, $S_b\supset L$, then by

Lemma A and Lindelöf-Iversen-Gross' theorem [7] we deduce that

$$\lim_{r\to+\infty} |f(re^{i\theta})| = +\infty$$

uniformly for $|\theta-\pi/2|\leqq\theta^*<\pi/2$ and for $|\theta+\pi/2|\leqq\theta^*<\pi/2$, where θ^* is an arbitrarily fixed number in $(0, \pi/2)$. Hence every sufficiently large real number must be a Picard exceptional value of $f(z)$. This is a contradiction. Thus S_a is at right angles to L for every a ($\in \boldsymbol{R}$) with at most one exception.

We now choose a sequence $\{x_n\}_{n=-\infty}^{+\infty}$ of real numbers such that $\mathrm{Re}(x_{n+1}-x_n)\geqq 3k$ and that S_{w_n} ($w_n=f(x_n)$) is at right angles to L ($n=0, \pm1, \pm2, \cdots$). We define A_n by (4.1). Then as the proof of Theorem 1 we have only the following two possibilities.

1) There exists an integer N such that $A_n>0$ for every $n\leqq N$.

2) There exists an integer N such that $A_n<0$ for every $n\geqq N$.

In each case by (4.1) $f'(z)$ fails to take 0 in $\boldsymbol{C}$. Thus $f(z)=a+b\cdot\exp(Az)$ with real constants $a, b, A, bA\neq 0$. Theorem 2 is thus proved.

6. Proof of Theorem 4. Firstly we prove that the number of directions of the straight lines $L_{w,1}$ ($w\in G$) is finite. Indeed, if otherwise, by Lemma 2 the order of $f(z)$ must be finite. Therefore Theorem 1 in [3] implies that $f(z)$ has at most a finite number of deficient values. Thus without loss of generality we can assume that G contains no deficient value of $f(z)$. Hence by Theorem B, the genus of $f(z)-a$ is at most one for every a ($\in G$). Thus by Lemma 5 we conclude that all the straight lines $L_{w,1}$ ($w\in G$) are parallel with one another. This is a contradiction.

From the above result it is easily seen that there is an open subset G^* of G such that all the straight lines $L_{w,1}$ ($w\in G^*$) are parallel with one another. We can assume, without loss of generality, that they are also parallel with the imaginary axis. Further we assume that there exists a points α ($\in G^*$) such that α is not a Picard exceptional value of $f(z)$, and that $\{L_{\alpha,i}\}_{i=1}^{n}$ are n distinct straight lines each of which carries at least one α-point of $f(z)$.

We choose $n+1$ α-points $\{z_i\}_{i=1}^{n+1}$ such that z_1 and z_2 lie on one of $\{L_{\alpha,i}\}_{i=1}^{n}$, say $L_{\alpha,1}$, and that z_i lies on $L_{\alpha,i-1}$ ($i=3, \cdots, n+1$). By the assumption on α it is easily seen that $f'(z_i)\neq 0$ ($i=1, \cdots, n+1$). Thus there exist neighborhoods U_i of z_i ($i=1, \cdots, n+1$) and a neighborhood A of α satisfying the following conditions.

1) $f(z)$ is univalent in U_i $\quad (i=1, 2, \cdots, n+1)$.

2) $f(U_i)=A$ $\quad (i=1, 2, \cdots, n+1)$.

3) $A\subset G^*$.

4) $\{\mathrm{Re}\, z;\ z\in U_1\}\cap\{\mathrm{Re}\, z;\ z\in U_i\}=\phi \quad (i=3, \cdots, n+1)$,

$\{\mathrm{Re}\, z;\ z\in U_2\}\cap\{\mathrm{Re}\, z;\ z\in U_i\}=\phi \quad (i=3, \cdots, n+1)$,

$\{\mathrm{Re}\, z;\ z\in U_i\}\cap\{\mathrm{Re}\, z;\ z\in U_j\}=\phi \quad (i, j=3, \cdots, n+1)$.

If $w \in A$, then by 1) and 2) there exists one w-point of $f(z)$ in each $U_\imath$, which is denoted by $p_\imath$ $(\imath=1, 2, \cdots, n+1)$. By 3) and 4) we deduce that $\operatorname{Re} p_1 = \operatorname{Re} p_2$. Put $F=(f|_{U_2})^{-1}\circ(f|_{U_1})$, then F is holomorphic in U_1 and $F(z)-z$ is purely imaginary for every z in U_1. Thus $F(z)\equiv z+c$ in a neighborhood of z_1 with a constant c. Hence we have

$$f(z+c)\equiv f(z). \tag{6.1}$$

Therefore $f(z)=Q(\exp Bz)$, where Q is a regular function in $\boldsymbol{C}\setminus\{0\}$ and B is a non-zero constant. From the assumption of Theorem 4, it is easily seen that Q is a rational function.

We may assume that $Q(w)$ has no factorization of form $Q(w)=P(w^N)$, where P is a rational function and N is an integer $(\geqq 2)$. Let m be the order of $Q(w)$. Then by the assumption of α there are m distinct roots $\{a_i\}$ of $Q(w)=\alpha$, which lie on n distinct circles whose center is at the origin. If $m>n$, then there exist two of $\{a_i\}$, say $a_\imath$, $a_\jmath$, such that $|a_\imath|=|a_\jmath|$. Then using essentially the same method which is used in showing (6.1), we easily deduce that

$$Q(w)\equiv Q(we^{\imath\theta}) \qquad (\theta=\arg a_i-\arg a_\jmath).$$

Thus $Q(w)=P(w^N)$ with a rational function P and an integer N $(\neq 0, \pm 1)$. This is a contradiction. Thus $m=n$. Theorem 4 is thus proved.

7. Proof of Theorem 3. By Lemma 3 the order of $f(z)$ must be finite. Hence Theorem 1 in [3] and the Denjoy-Carleman-Ahlfors Theorem imply that $f(z)$ has at most a finite number of deficient values and asymptotic values. Thus we may assume that G contains neither deficient nor asymptotic values of $f(z)$. By Theorem B the genus of $f(z)-a$ is at most one for every a $(\in G)$. By Lemma 5 all the straight lines L_a $(a\in G)$ are parallel with the imaginary axis.

Let us write for every a $(\in G)$

$$h(a)=\operatorname{Re} x \qquad (x\in L_a), \tag{7.1}$$

$$\operatorname{Re}\frac{f'(z)}{f(z)-a}=A(a)+\sum_n \operatorname{Re}\frac{1}{z-a_n}, \tag{7.2}$$

where $A(a)$ is a real constant and $\{a_n\}$ the a-points of $f(z)$. (7.2) follows from Lemma A.

Next we show the following Lemma 7 and Lemma 8. Which are modifications of Lemma 5 and Lemma 7 in [5].

LEMMA 7. *If* $h(a)<h(b)$ *then* $A(a)<0$ *or* $A(b)>0$ $(a, b\in G)$.

Proof. Suppose that $A(a)\geqq 0$, $A(b)\leqq 0$. Let $\{a_n\}$ be the a-points of $f(z)$ which lie on $L_a\setminus\{0\}$. We choose a positive number R such that $f(z)\neq a$ for every z $(\operatorname{Re} z>R)$. Put $\varepsilon=(h(b)-h(a))/3$, $S=\{z;\ h(a)+\varepsilon\leqq \operatorname{Re} z\leqq R\}$,

$$\phi_a(z)=\operatorname{Re}\frac{f'(z)}{f(z)-a}.$$

Then by (7.2)

$$\phi_a(z)>0 \qquad (\operatorname{Re} z>R) \tag{7.3}$$

and

$$\phi_a(z)=A(a)+\sum_{n=1}^{\infty}\operatorname{Re}\frac{1}{z-a_n}+O\left(\frac{1}{z^2}\right) \qquad (z\in S). \tag{7.4}$$

By (7.4) and Lemma 4 it is easily seen that

$$\phi_a(z)>0 \tag{7.5}$$

for every sufficiently large z ($\in S$). From (7.3) and (7.5) $f'(z)$ has at most a finite number of zeros in $\operatorname{Re} z\geqq h(a)+\varepsilon$.

Similarly $f'(z)$ has at most a finite number of zeros in $\operatorname{Re} z\leqq h(b)-\varepsilon$. Thus $f'(z)$ has at most a finite number of zeros in $\boldsymbol{C}$. Therefore

$$f(z)=P(z)\cdot e^{\alpha z}+\beta \tag{7.6}$$

with constants α, β and a polynomial $P(z)$. The assumption "$A(a)\geqq 0$ and $A(b)\leqq 0$", Lemma A and Lindelöf-Iversen-Gross' theorem [7] imply that

$$\lim_{r\to+\infty}|f(re^{i\theta})|=+\infty$$

uniformly for $|\theta|\leqq\theta^*<\pi/2$ and for $|\theta-\pi|\leqq\theta^*<\pi/2$, where θ^* is an arbitrarily fixed number in $(0, \pi/2)$. This asymptotic behavior contradicts (7.6). Lemma 7 is thus proved.

LEMMA 8. *For every a ($\in G$) we have*

$$\overline{(f(\bar{z}+h(a))-a)}=(\cdot f(-z+h(a))-a)\exp(2A(a)z+iB(a))(\overline{P_a(\bar{z})}/P_a(-z))$$

with a polynomial $P_a(z)$ and a real constant $B(a)$.

Proof. Let $\{a_n\}$ be the a-points of $f(z)$ which lie on L_a and $\{a'_m\}$ the a-points of $f(z)$ other than $\{a_n\}$. Let us write

$$f(z+h(a))-a=z^p e^{Az+B}\prod_{a_n\neq h(a)}E\left(\frac{z}{a_n-h(a)}, 1\right)\prod_m E\left(\frac{z}{a'_m-h(a)}, 1\right), \tag{7.7}$$

$$P_a(z)=\begin{cases}\prod_m\left(1-\dfrac{z}{a'_m-h(a)}\right) & \text{if } a'_m \text{ exists},\\[2ex] 1 & \text{if } a'_m \text{ does not exist},\end{cases} \tag{7.8}$$

$$g_a(z)=(f(z+h(a))-a)/P_a(z), \tag{7.9}$$

where $E(z, 1)$ is the Weierstrass primary factor of genus 1. Since all the zeros of $g_a(z)$ are purely imaginary, by Lemma 5 in [5]

$$\overline{g_a(\bar{z})}=g_a(-z)\cdot\exp(\widetilde{A(a)}z+iB(a)) \tag{7.10}$$

with a real constant $B(a)$, where

$$\widetilde{A(a)}=2\,\mathrm{Re}\Big(A+\sum_m \frac{1}{a'_m-h(a)}\Big). \tag{7.11}$$

From (7.7)

$$\mathrm{Re}\,\frac{f'(z)}{f(z)-a}=\widetilde{A(a)}/2+\sum_n \mathrm{Re}\,\frac{1}{z-a_n}+\sum_m \mathrm{Re}\,\frac{1}{z-a'_m}. \tag{7.12}$$

By (7.2) and (7.12)

$$2A(a)=\widetilde{A(a)}. \tag{7.13}$$

From (7.9), (7.10) and (7.13) we have the desired result.

Next we prove

LEMMA 9. *Let $a, b\in G$. Assume that $h(a)>h(b)$, $A(a)A(b)>0$. If $A(a)>0$, then there exists a neighborhood E $(\subset G)$ of a, such that $h(p)$ is continuous in E and that $A(p)>0$ for every p in E. If $A(a)<0$, then there exists a neighborhood E' $(\subset G)$ of b, such that $h(p)$ is continuous in E' and that $A(p)<0$ for every p in E'.*

Proof. In what follows we assume that $A(a)>0$. When $A(a)<0$, we only have to consider the function $f(-z)$ instead of $f(z)$.

Let ε be an arbitrarily fixed positive number less than $(h(a)-h(b))/3$ such that the set

$$C=\{z\,;\ 0<|h(a)-(\mathrm{Re}\,z)|\leqq 2\varepsilon\}$$

contains no a-point of $f(z)$. Using the same argument as in the proof of Lemma 7, we deduce that

$$\mathrm{Re}\,\frac{f'(z)}{f(z)-a}\geqq A(a), \tag{7.14}$$

$$\mathrm{Re}\,\frac{f'(z)}{f(z)-b}\geqq A(b) \tag{7.15}$$

for every z satisfying $h(a)+\varepsilon\leqq\mathrm{Re}\,z\leqq h(a)+2\varepsilon$ and $|\mathrm{Im}\,z|\geqq R_0$, where R_0 is a suitable positive number.

Let $\gamma=|a-b|$ and δ a positive number satisfyihg

$$\log\frac{\gamma+\delta}{\gamma-\delta}\leqq\varepsilon A(b). \tag{7.16}$$

Put

$$L=\{z\,;\ \mathrm{Re}\,z=h(a)+\varepsilon\}.$$

(7.17) $$S=\{z\in L\,;\ |a-f(z)|<\delta\},$$

(7.18) $$T=\{z\in L\,;\ |a-f(z)|\geqq\delta\}.$$

Let $z\in S$ and $|\mathrm{Im}\,z|\geqq R_0$, then by (7.15)

(7.19) $$\log|f(z+\varepsilon)-b|-\log|f(z)-b|$$
$$=\int_0^\varepsilon \mathrm{Re}\,\frac{f'(x+t+\imath y)}{f(x+t+\imath y)-b}\,dt\geqq A(b)\varepsilon \qquad (z=x+\imath y).$$

By (7.16), (7.17) and (7.19)

$$\log|f(z+\varepsilon)-b|\geqq\log|f(z)-b|+A(b)\varepsilon$$
$$\geqq\log(\gamma-\delta)+A(b)\varepsilon\geqq\log(\gamma+\delta).$$

Thus $|f(z+\varepsilon)-b|\geqq\gamma+\delta$. Hence

(7.20) $$|f(z+\varepsilon)-a|\geqq\delta \qquad (z\in S,\ |\mathrm{Im}\,z|\geqq R_0),$$

Similarly, by (7.14) and (7.18)

(7.21) $$|f(z+\varepsilon)-a|\geqq\delta \qquad (z\in T,\ |\mathrm{Im}\,z|\geqq R_0).$$

Since $f(z)-a$ does not vanish on $\mathrm{Re}\,z=h(a)+2\varepsilon$, by (7.20) and (7.21) we see that for a suitable positive number η

(7.22) $$|f(z)-a|\geqq\eta \qquad (\mathrm{Re}\,z=h(a)+2\varepsilon).$$

By Lemma 8 and (7.22)

(7.23) $$|f(-z+h(a))-a|\geqq\eta e^{-4\varepsilon A(a)}(|P_a(-z)/P_a(\bar{z})|) \qquad (\mathrm{Re}\,z=2\varepsilon).$$

Since $f(z)-a$ does not vanish on $\mathrm{Re}\,z=h(a)-2\varepsilon$, by (7.22) and (7.23) we conclude that for a suitable positive number ζ

(7.24) $$|f(z)-a|\geqq\zeta \qquad (\mathrm{Re}\,z=h(a)\pm2\varepsilon).$$

Let $\{a_i\}$ be the distinct a-points of $f(z)$ outside $\overline{C}$ and d a positive number satisfying the following conditions.

1) $D_i\cap D_j=\phi$ $(\imath\neq j)$, where $D_\imath=\{z\,;\ |z-a_\imath|\leqq d\}$.

2) $D_i\cap\overline{C}=\phi$.

Put

(7.25) $$m=\min_{z\in\bigcup_i\partial D_i}|f(z)-a|.$$

Let r be an arbitrarily fixed positive number less than $\min(\zeta, m)$ and $E=\{w\,;\ |w-a|<r\}\subset G$. By (7.24) and (7.25)

(7.26) $$f^{-1}(E)\cap(\partial\bar{C}\cup(\bigcup_{\iota}\partial D_{\iota}))=\phi\,.$$

Since E contains no asymptotic value of $f(z)$, each component of $f^{-1}(E)$ contains at least one a-point of $f(z)$. Thus by (7.26)

(7.27) $$f^{-1}(E)\subset(\bar{C}\cup(\bigcup_{\iota}D_{\iota}))\,.$$

By (7.27) for every p in E

(7.28) $$|h(a)-h(p)|\leqq 2\varepsilon\,.$$

Since ε can be taken arbitrarily small, $h(p)$ is continuous at the point a. By (7.28)

(7.29) $$h(p)>h(b)\qquad(p\in E)\,.$$

By Lemma 7 and (7.29)

(7.30) $$A(p)>0\qquad(p\in E)\,.$$

Applying the same method to (7.29) and (7.30), we conclude that $h(p)$ is continuous in E. Lemma 9 is thus proved.

Let $H=\{p\in G\,;\,A(p)=0\}$. If $p,q\in H$, then by Lemma 7 $L_p=L_q$. Thus $H\subset f(L_p)$ $(p\in H)$, or $H=\phi$. Hence $G\backslash H$ has infinitely many elements. Thus there exist two elements a, b $(\in G)$ satisfying $A(a)A(b)>0$. There are the following two cases.

1) $h(a)\neq h(b)$. 2) $h(a)=h(b)$.

In the case 1), by Lemma 9, we easily see that for some two points α, β in E, or in E', $h(\alpha)=h(\beta)$ and $A(\alpha)A(\beta)>0$. Thus in both cases there exist two elements $\alpha,\beta\in G$ such that $h(\alpha)=h(\beta)$ and $A(\alpha)A(\beta)>0$.

In what follows we assume, without loss of generality, that $\alpha=0$, $\beta=1$, $h(\alpha)=h(\beta)=0$. From Lemma 8 we have

$$\overline{f(\bar{z})}=f(-z)e^{2A(0)z+iB(0)}\overline{(P_0(\bar{z})}/P_0(-z))\,,$$

$$\overline{f(\bar{z})-1}=(f(-z)-1)e^{2A(1)z+iB(1)}\overline{(P_1(\bar{z})}/P_1(-z))\,,$$

Put

$$X(-z)=e^{iB(0)}\overline{(P_0(\bar{z})}/P_0(-z))\,,$$

$$Y(-z)=e^{iB(1)}\overline{(P_1(\bar{z})}/P_1(-z))\,,$$

$$A=2A(0)\,,\qquad B=2A(1)\,.$$

Then

$$f(z)(X(z)e^{-Az}-Y(z)e^{-Bz})=1-Y(z)e^{-Bz}\,.$$

Since $B\neq 0$, we easily have

$$f(z)=(e^{Bz}-Y(z))/(X(z)e^{(B-A)z}-Y(z))\,. \tag{7.31}$$

We now consider the following two cases.

(1) $A=B$. (2) $A\neq B$.

Firstly we consider the case (1). There are the following two subcases.
(1.1) $X(z)$ and $Y(z)$ are both constants.
(1.2) $X(z)$ or $Y(z)$ is not a constant.

Case (1.1). In this case, the assertion of Theorem 3 follows at once from (7.31) and (1).

Case (1.2). For instance, we assume that $X(z)$ is not a constant. Another case, when $Y(z)$ is not a constant, can be treated by the same method.

Let $p\in G$, and $\{z_n\}$ be the p-points of $f(z)$. Put

$$F(z,\ p)=pX(z)+(1-p)Y(z)\,.$$

Then by (7.31) and (1)

$$\exp(Bz_n)=F(z_n,\ p)$$

for sufficiently large n. Thus

$$|F(z_n,\ p)|=\exp(B\ \mathrm{Re}\, z_n)=\exp(B\cdot h(p)) \tag{7.32}$$

for sufficiently large n. Since $F(z,\ p)$ is regular at $z=\infty$, by (7.32)

$$|F(z,\ p)|=|F(\infty,\ p)| \qquad (z\in L_p)\,, \tag{7.33}$$

$$L_p=\{z\ ;\ \mathrm{Re}\, z=(\log|F(\infty,\ p)|)/B\}\,. \tag{7.34}$$

Since $X(z)$ and $Y(z)$ have no common pole, any pole of $X(z)$ must be also a pole of $F(z,\ p)$ for every p in $G\setminus\{0\}$. Let t_0 be a fixed pole of $X(z)$. Then (7.33) and Schwarz' reflection principle imply that $F(z,\ p)$ vanishes at the point $t_0-2((\mathrm{Re}\, t_0)-h(p))$.

Put

$$c(p)=t_0-2((\mathrm{Re}\, t_0)-h(p))\,.$$

$F(\infty,\ p)$ is an analytic function of p. Hence there exists a point x in $G\setminus\{0,\ 1\}$ satisfying

$$|F(\infty,\ x)|=|F(\infty,\ 1)|\,. \tag{7.35}$$

By (7.34) and (7.35) we have $c(x)=c(1)$. Thus

$$xX(c(1))+(1-x)Y(c(1))=xX(c(x))+(1-x)Y(c(x))=0\,,$$

$$1\cdot X(c(1))+0\cdot Y(c(1))=X(c(1))=0\,.$$

Hence $(1-x)Y(c(1))=0$. Since $X(z)$ and $Y(z)$ have no common pole, $Y(c(1))\neq 0$. Thus $x=1$. This is a contradiction. Thus the case (1.2) cannot occur.

Secondly, we consider the case (2). In this case by (7.31) we easily conclude that $B/(B-A)$ must be an integer. Put $q=B/(B-A)$. Then from (7.31)

$$f(z)=(e^{q(B-A)z}-Y(z))/(X(z)e^{(B-A)z}-Y(z)). \tag{7.36}$$

Let $\{a_n\}$ be the zeros of $X(z)\cdot\exp((B-A)z)-Y(z)$. From (7.36)

$$e^{(B-A)a_n}=Y(a_n)/X(a_n), \qquad e^{q(B-A)a_n}=Y(a_n).$$

Thus

$$(Y(a_n)/X(a_n))^q=Y(a_n).$$

Therefore

$$Y(z)^{q-1}\equiv X(z)^q. \tag{7.37}$$

By (7.36) and (7.37), q cannot be 0 or 1. Therefore $X(z)$ and $Y(z)$ are both constants, since $X(z)$ and $Y(z)$ have no common pole. Let us write $X(z)\equiv x$, $Y(z)\equiv y$.

From (7.36) and (7.37)

$$f(z)=(e^{q(B-A)z}-y)/(xe^{(B-A)z}-y), \tag{7.38}$$

$$y^{q-1}=x^q. \tag{7.39}$$

If $q\neq 2, 1, 0, -1$, then the order of the rational function

$$Q(w)=(w^q-y)/(xw-y)$$

is at least two. In this case, by the same method in the proof of Theorem 4, the function $f(z)=Q(\exp(B-A)z)$ cannot fulfill the assumption of Theorem 3. Thus this case cannot occur. Hence $q=2$ or -1.

By (7.38) and (7.39) we have the following results.

(1) If $q=2$, then

$$f(z)=(e^{(B-A)z}+x)/x.$$

(2) If $q=-1$, then

$$f(z)=(e^{-(B-A)z})/(-y).$$

The proof of Theorem 3 is now complete.

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