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ON THE INITIAL BOUNDARY-VALUE PROBLEM FOR VISCOUS HEAT CONDUCTING COMPRESSIBLE FLUIDS

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Abstract

The existence of a weak solution, local in time of an initial boundary-value problem for the basic system of partial differential equations of the theory of viscous, heat conducting compressible fluids is shown.

Introduction. The purpose of this paper is to show the existence of a weak solution, local in time, of an initial boundary-value problem for viscous, heat conducting compressible fluids.

Let $u=(u_1, u_2, u_3)$ be the velocity of the fluid, ρ and θ be the density and the absolute temperature of the fluid respectively. The motion of the fluid is described by the initial boundary-value problem:

$$(0.1)\quad \begin{cases} \rho\Big(\dfrac{\partial u}{\partial t}+u.\nabla u\Big)-\varepsilon\Delta u-\varepsilon\,\mathrm{grad}\,(\mathrm{div}\,u)+\mathrm{grad}\,(\rho+\theta)=\rho f \quad \text{on } (0, T)\times G\,, \\ u(x, t)=0 \text{ on } (0, T)\times\partial G,\ u(x, 0)=0 \quad \text{on } G\,. \end{cases}$$

G is a bounded open subset of R^3 with a smooth boundary ∂G.

The conservation of mass is expressed by the initial value problem:

$$(0.2)\quad \frac{\partial\rho}{\partial t}+\rho\,\mathrm{div}(u)+u.\,\mathrm{grad}\,\rho=0,\ \rho>0 \quad \text{on } (0, T)\times G,\ \rho(x, 0)=\rho^0(x) \quad \text{on } G\,.$$

The conservation of energy is described by the initial boundary-value problem:

$$(0.3)\quad \begin{cases} \rho\theta\Big(\dfrac{\partial\theta}{\partial t}+u.\,\mathrm{grad}\,\theta-\rho\,\mathrm{div}\,(u)\Big)-\chi\Delta\theta-\varepsilon Bu=0,\ \theta>0 \quad \text{on } (0, T)\times G\,, \\ \dfrac{\partial\theta}{\partial\nu}=0 \text{ on } (0, T)\times\partial G \text{ and } \theta(x, 0)=\theta^0(x) \quad \text{on } G\,. \end{cases}$$

ν is the unit exterior normal vector to ∂G and B is the nonlinear operator $Bu=\Big(\dfrac{\partial u_i}{\partial x_k}+\dfrac{\partial u_k}{\partial x_i}\Big)^2$ with the usual summation convention. For the derivations of

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(0.1)-(0.3), we shall refer to Tani [8].

In this paper we shall consider the case when the viscosity ε and the coefficient of heat conduction χ are both constants and for simplicity of notations we shall set $\varepsilon=\chi=1$.

There are few mathematical works on the theory of viscous *compressible* fluids. The pioneering work was done by Nash [7] in 1962 who proved the existence of a unique local solution of the Cauchy problem:

$$(0.4)\qquad \rho\Big(\frac{\partial u}{\partial t}+u.\nabla u\Big)-\varepsilon\Delta u-\varepsilon\operatorname{grad}(\operatorname{div} u)+\operatorname{grad}(\rho+\theta)=\rho f \qquad \text{on } (0,T)\times R^3,$$

$$u(x,0)=0 \qquad \text{on } R^3$$

and

$$(0.5)\qquad \frac{\partial\rho}{\partial t}+\rho\operatorname{div}(u)+u.\operatorname{grad}\rho=0,\ \rho>0 \qquad \text{on } (0,T)\times R^3,$$

$$\rho(x,0)=\rho^0(x) \qquad \text{on } R^3$$

with

$$(0.6)\qquad \rho\theta\Big(\frac{\partial\theta}{\partial t}+u.\operatorname{grad}\theta-\rho\operatorname{div}u\Big)-\chi\Delta\theta-\varepsilon Bu=0,\ \theta>0 \qquad \text{on } (0,T)\times R^3,$$

$$\theta(x,0)=\theta^0(x) \qquad \text{on } R^3.$$

He used a characteristic transformation, an iteration method and together with estimates for fundamental solutions of parabolic equations solved a parabolic system at each step. The validity of Nash's proof is, however, in doubt. Cf Tani [8].

Recently, mathematical works on compressible fluids have been done by Itaya, Matsumura and Nishida and by Tani. In [2], Itaya has shown, independently of Nash, the existence of a unique local solution of the Cauchy problem for (0.4)-(0.6). Tani has in [8] proved the existence of a unique local solution of the initial boundary value problem (0.1)-(0.3). In both works, as done earlier by Nash, a characteristic transformation and estimstes for fundamental solutions of a parabolic equation are used. The approaches taken by Itaya and by Tani involve very delicate computations.

Using energy estimates and an iteration method, Matsumura and Nishida [6] have proved the existence of a unique classical solution of the Cauchy problem for (0.4)-(0.6), the solution is global in time if the data are "small".

By a completely different approach, using equations of Sobolev-Galpern type as approximants, the writer has in [9] shown the existence of a unique local solution of the Cauchy problem for (0.4)-(0.6) and studied the convergence of the solution as the viscosity tends to zero. In [10], the writer has established the above results by a simpler argument using an iteration method and some simple properties of the quasi-norms of Leray and Ohya.

For the initial boundary-value problem (0.1)-(0.3), the only known result is due to Tani [8] who proved the existence of a unique local solution $\{u, \rho, \theta\}$ in the space

$$C^{2+\alpha, 1/2(2+\alpha)}(Q_{T^*}) \times C^{2+\alpha, 1/2(2+\alpha)}(Q_{T^*}) \times C^{2+\alpha, 1/2(2+\alpha)}(Q_{T^*})$$

for $0<\alpha<1$ and where $Q_{T^*}=(0, T^*)\times G$. In this paper we shall show the existence of a weak local solution of (0.1)-(0.3) with minimum regularity hypothesis on the data. The solution obtained is, in a sense reminiscent of the Hopf solution of the Navier-Stokes equations for incompressible fluids with constant density. The solution $\{u, \rho, \theta\}$ is such that u belongs to $L^\infty(0, T_*; L^2(G)\cap L^2(0, T_*; H_0^1)$, ρ is in $L^\infty(0, T_*; L^\infty(G))$ with θ in $L^\infty(0, T_*; L^2(G))\cap L^2(0, T_*; H^1)$. We shall use standard techniques, the proof is rather simple although long and is completely different from the usual approach to compressible fluids.

The notations, the main result of the paper as well as a detailed outline of the proof of the basic theorem are given in Section 1.

Section 1: Notations, Definitions and statement of the main result. Let G be a bounded open subset of R^3 with a smooth boundary ∂G. For each triple $\alpha=(\alpha_1, \alpha_2, \alpha_3)$ of non-negative integers we write:

$$D^\alpha=\prod_{j=1}^{3} D_j^{\alpha_j} \text{ with } |\alpha|=\sum_{j=1}^{3}\alpha_j \text{ and } D_j=\partial/\partial x_j.$$

The inner product and the norm in $H=L^2(G)$ are denoted by $(.,.)$ and by $\|\cdot\|$ respectively. The Sobolev space

$$W^{k,p}(G)=\{u: u \text{ in } L^p(G), D^\alpha u \text{ in } L^p(G) \text{ for } |\alpha|\leqq k\}$$

is a reflexive separable Banach space with the norm

$$\|u\|_{k,p}=\Big\{\sum_{|\alpha|\leqq k}\|D^\alpha u\|_{L^p(G)}^p\Big\}^{1/p}; \ 2\leqq p<\infty.$$

We shall write H^k for $W^{k,2}(G)$ and H_0^k is the closure in the H^k-norm of the set of all infinitely differentiable functions with compact support in G. We shall identify H with its dual by the inner product $(.,.)$.

H^{-k} is the dual of H_0^k and $W^{-k,q}(clG)$ is the dual of $W^{k,p}(G)$ with q as the conjugate exponent of p. By abuse of notations we shall also use $(.,.)$ for the various pairings.

The following results of the Sobolev imbedding theorem will be used throughout the paper.

$$H^1\subset L^4(G);\ H^2\subset L^\infty(G);\ W^{1,4}(G)\subset L^\infty(G).$$

The above natural injection mappings are all continuous and moreover H^2 is algebra with respect to pointwise multiplication.

$L^2(0, T; H^k)$ is the set of equivalence classes of functions $u(., t)$ from $(0, T)$ to H^k which are L^2-integrable over $(0, T)$. It is a Hilbert space with the norm

$$\|u\|_{L^2(0,T,H^k)}=\Big\{\int_0^T\|u(.,t)\|_{k,2}^2 dt\Big\}^{1/2}$$

and the usual inner product.

$L^\infty(0, T; H^k)$ is similarly defined with the obvious modification. The derivative of u with respect to t is written as $\frac{\partial u}{\partial t}$ or simply as u' when there is no confusion possible.

For the convenience of the reader we shall state the following two basic compactness theorems used in the paper.

THEOREM (Aubin [1]). *Let W, V and X be three real reflexive Banach spaces with $W \subset V \subset X$. The natural injection mapping of W into V is assumed to be compact and that of V into X is continuous. Suppose that:*

$$\|u_n\|_{L^p(0,T,W)} + \|u_n'\|_{L^p(0,T;X)} \leqq C, \quad 2 \leqq p < \infty .$$

C is independent of n. Then there exists a subsequence denoted again by $\{u_n\}$ such that: $u_n \to u$ in $L^p(0, T; V)$.

THEOREM (Murat's compensated compactness theorem. See [5] p. 72, relation (1.64)).

Suppose that: (i) $\|\rho_n\|_{L^2(0,T;H^{-1})} + \|\rho_n'\|_{L^2(0,T,H^{-2})} \leqq C$,

(ii) $\|u_n\|_{L^2(0,T,H^1)} \leqq C$.

C independent of n. Then there exists a subsequence $\{u_n, \rho_n\}$ such that $u_n \rho_n \to u\rho$ in the distribution sense on $(0, T) \times G$.

DEFINITION 1. Let v be a vector-valued function in $L^2(0, T; H_0^1)$ and ρ^0 be a scalar function with $0 < a \leqq \rho^0(x)$, $|\text{grad } \rho^0| \leqq b$ on G. Then a scalar function ρ in $L^\infty(0, T; L^\infty(G))$ with $0 < \rho(x, t)$ on $(0, T) \times G$ is said to be a weak solution of the initial-value problem:

(1.1) $\rho' + v.\,\text{grad } \rho + \rho \text{ div}(v) = 0, \ \rho > 0 \quad \text{on } (0, T) \times G, \ \rho(x, 0) = \rho^0(x) \quad \text{on } G,$

if

$$-\int_0^T (\rho, \phi')dt - \int_0^T (\rho v, \text{grad } \phi)dt = (\rho^0, \phi(.\,, 0))$$

for all scalar functions ϕ in $L^2(0, T, H^1)$ with ϕ' in $L^2(0, T; H)$ and $\phi(.\,, T) = 0$.

Let v be as above and consider the initial boundary value problem

$$(1.2) \quad \begin{cases} \rho\left(\dfrac{\partial u}{\partial t} + v.\nabla u\right) - \Delta u - \text{grad}(\text{div } u) + \text{grad}(\rho + \tilde{\theta}) = \rho f & \text{on } (0, T) \times G, \\ u(x, t) = 0 \text{ on } (0, T) \times \partial G, \ u(x, 0) = 0 & \text{on } G. \end{cases}$$

DEFINITION 2. Let v be as before, f be a vector-valued function in $L^\infty(0, T; H)$ and $\tilde{\theta}$ be a scalar function in $L^\infty(0, T; L^\infty(G))$. Let u be a vector-valued function in $L^\infty(0, T; H) \cap L^2(0, T; H_0^1)$. Then $\{\rho, u\}$ is said to be a weak solution of (1.1)-(1.2) if:

(i) ρ is a weak solution of (1.1) in the sense of Definition 1,
(ii)

$$-\int_0^T(\rho u, w')dt+\int_0^T(\nabla u, \nabla w)dt+\int_0^T(\operatorname{div} u, \operatorname{div} w)dt-\int_0^T(\rho+\tilde{\theta}, \operatorname{div}(w))dt$$

$$-\int_0^T(\rho v.\nabla w, u)dt=\int_0^T(\rho f, w)dt$$

for all vector-valued functions w in $L^2(0, T; H_0^1\cap H^2)$ with w' in $L^2(0, T, H)$ and $w(., T)=0$.

Let v, ρ and $\tilde{\theta}$ be as before and consider the initial boundary-value problem:

$$\text{(1.3)}\quad\begin{cases}\rho\theta(\theta'+v.\operatorname{grad}\theta)-\dfrac{1}{2}\nabla(\tilde{\theta}^{-1}\nabla\theta^2)-\rho^2\tilde{\theta}\operatorname{div}(v)-Bv=0,\ \theta>0 \quad \text{on } (0, T)\times G,\\ \dfrac{\partial\theta}{\partial\nu}=0 \text{ on } (0, T)\times\partial G,\ \theta(x, 0)=\theta^0(x) \quad \text{on } G.\end{cases}$$

DEFINITION 3. Let v, ρ and $\tilde{\theta}$ be as before with $\tilde{\theta}>0$ on $(0, T)\times G$. Then $\{\rho\ \theta\}$ is said to be a weak solution of (1.1)-(1.3) if:
(i) ρ is a weak solution of (1.1) in the sense of Definition 1,
(ii) θ is a scalar function in $L^\infty(0, T; L^\infty(G))\cap L^2(0, T; H^1)$ with

$$0<\theta \quad \text{on} \quad (0, T)\times G,$$

(iii)

$$-\int_0^T(\rho\theta^2, \phi')dt+\int_0^T(\tilde{\theta}^{-1}\nabla\theta^2, \nabla\phi)dt-2\int_0^T(Bv, \phi)dt$$

$$-2\int_0^T(\rho^2\tilde{\theta}\operatorname{div}(v), \phi)dt-\int_0^T(\rho\theta^2, v.\operatorname{grad}\phi)dt=(\rho^0\theta_0^2, \phi(., 0))$$

for all scalar functions ϕ in $C^1(0, T; H^2)$ with $\phi(., T)=0$.

DEFINITION 4. Let f, ρ^0, θ^0 be as before. Then $\{u, \rho, \theta\}$ in

$$\{L^\infty(0, T; H)\cap L^2(0, T; H_0^1)\}\times L^\infty(0, T; L^\infty(G))$$

$$\times\{L^\infty(0, T; L^\infty(G))\cap L^2(0, T; H^1)\}$$

is a weak solution of (0.1)-(0.3) if:
(i) ρ is a weak solution of (1.1) in the sense of Definition 1 with $v=u$ in (1.1)
(ii) u is a weak solution of (1.2) in the sense of Definition 2 with $v=u$, $\tilde{\theta}=\theta$ in (1.2)
(iii) θ is a weak solution of (1.3) in the sense of Definition 3 with

$$v=u,\ \tilde{\theta}=\theta \quad \text{in} \quad (1.1)\text{-}(1.3).$$

We shall now state the main result of the paper.

THEOREM 1.1 *Let f be a vector-valued function in $L^\infty(0, T; H)$, ρ^0 be a scalar function with $0<a\leqq\rho^0(x)$, $|\text{grad}\,\rho^0|\leqq b$ on G and θ_0 be a positive constant c. Then there exist:*

(1) *a non-empty interval* $(0, T^*)$,

(2) $\{u, \rho, \theta\}$ *in* $\{L^\infty(0, T^*; H)\cap L^2(0, T^*; H_0^1)\}\times L^\infty(0, T^*; L^\infty(G))$

$$\times\{L^\infty(0, T^*; L^\infty(G))\cap L^2(0, T^*; H^1)\},$$

weak solution of (0.1)-(0.3) *in the sense of Definition* 4.

We now give a detailed outline of the proof of the theorem.

Step 1. It will be carried out in Section 2. Let v and ρ^0 be as in Definition 1. By a standard method we show the existence of a weak local solution of

$$\rho'+v.\,\text{grad}\,\rho+\rho\,\text{div}\,(v)=0,\ \rho>0 \qquad \text{on } (0, T)\times G,\ \rho(v, 0)=\rho^0(x) \quad \text{on } G. \tag{1.4}$$

Moreover: $0<(1-\eta)a\leqq\rho(x, t)\leqq b+\eta a$ on $(0, T_*)\times G$ and

$$\|\rho'\|_{L^2(0,T_*;(H^1)^*)}+\|(\rho^2)'\|_{L^2(0,T_*;(H^1)^*)}\leqq C\{1+\|v\|_{L^2(0,T_*;(H_0^1)}\}.$$

C is a constant independent of v, ρ and depends only on a, b.

Step 2. It will be carried out in Section 3. Using the Galerkin approximation method and estimates for the kinetic energy, we prove the existence of a weak solution of (1.4)-(1.5) with

$$\begin{cases} \rho\Big(\dfrac{\partial u}{\partial t}+v.\,\nabla u\Big)-\Delta u-\text{grad}\,(\text{div}\,u)+\text{grad}\,(\rho+\tilde\theta)=\rho f \qquad \text{on } (0, T_*)\times G, \\ u(x, t)=0 \text{ on } (0, T_*)\times\partial G,\ u(x, 0)=a \qquad \text{on } G. \end{cases} \tag{1.5}$$

Moreover

$$\|u(\,.\,, t)\|^2+\int_0^t\|u(\,.\,, s)\|_{1,2}^2ds\leqq Ct\{1+\|\tilde\theta\|^2_{L^\infty(0,T_*:L^\infty(G))}\},\ \text{and}$$

$$\|(\rho u)'\|_{L^2(0,T^*;H^{-2})}\leqq C\{1+\|v\|_{L^2(0,T_*;H_0^1}+\|\tilde\theta\|_{L^\infty(0,T_*;L^\infty(G))}\},$$

C is aconstant independent of v, ρ, $\tilde\theta$, u and depends only on f and on the bounds of ρ^0.

Step 3. It will be carried out in Section 4. Using, first a discretisation of the time-variable, then a nonlinear elliptic perturbation of the discretised equation we show the existence of a weak local solution of (1.4)-(1.6) with:

$$\begin{cases} \rho\theta\Big(\dfrac{\partial\theta}{\partial t}+v.\,\text{grad}\,\theta\Big)-\dfrac{1}{2}\nabla(\tilde\theta^{-1}\nabla\theta^2)-\rho^2\tilde\theta\,\text{div}\,(v)-Bv=0,\ \theta>0 \\ \dfrac{\partial\theta}{\partial\nu}=0 \text{ on } (0, T_*)\times\partial G,\ \theta(x, 0)=\theta_0(x) \qquad \text{on } G. \end{cases} \quad \text{on } (0, T_*)\times G, \tag{1.6}$$

The existence of a non-empty subinterval $(0, T^*)$ where $\theta>0$ is shown. Moreover:

$$0<(1-\eta^2)^{1/2}c\leqq\theta\leqq(1+\eta^2)^{1/2}c \qquad \text{on } (0, T^*)\times G\,;\ 0<\eta<1 \text{ and } 0<c\leqq\theta_0\,.$$

Furthermore:

$$\|\theta\|_{L^2(0,T^*;H^1)}\leqq C\{1+\|v\|_{L^2(0,T^*;H_0^1)}+\|\tilde{\theta}\|_{L^\infty(0,T^*;L^\infty(G))}\}\,.$$

Let $\delta>0$, then an estimate of Kajikov's type holds:

$$\int_0^{T^*-\delta}\|\theta(\,.\,,\ t+\delta)-\theta(\,.\,,\ t)\|^2dt\leqq C\delta^{1/2}\{1+\|v\|^4_{L^2(0,T^*;H_0^1)}+\|\tilde{\theta}\|^4_{L^\infty(0,T^*;L^\infty(G))}\}$$

C is independent of δ, v, $\tilde{\theta}$, θ, ρ, u.

Step 4. It will be carried out in Section 5. We construct a sequence of successive approximations. Consider the initial-value problem:

$$\text{(1.7)}\qquad \begin{cases} \rho_n'+u_{n-1}\cdot\operatorname{grad}\rho_n+\rho_n\operatorname{div}(u_{n-1})=0 \qquad \text{on } (0, T)\times G\,, \\ \rho_n>0 \qquad \text{on } (0, T)\times G,\ \rho_n(x, 0)=\rho^0(x)\,;\ u_0=0,\ n=1, 2\cdots \end{cases}$$

and the initial boundary-value problem

$$\text{(1.8)}\qquad \begin{cases} \rho_n(u_n'+u_{n-1}.\,\nabla u_n)-\Delta u_n-\operatorname{grad}(\operatorname{div} u_n)+\operatorname{grad}(\rho_n+\theta_{n-1})=\rho_n f \quad \text{on } (0, T)\times G\,, \\ u_n(x, t)=0 \text{ on } (0, T)\times\partial G,\ u_n(x, 0)=0 \qquad \text{on } G\,. \end{cases}$$

together with the initial boundary-value problem

$$\text{(1.9)}\qquad \begin{cases} \rho_n\theta_n(\theta_n'+u_{n-1}\cdot\operatorname{grad}\theta_n)-\dfrac{1}{2}\nabla(\theta_{n-1}^{-2}\nabla\theta_n^2)-\rho_n^2\theta_{n-1}\operatorname{div}(u_{n-1})-Bu_{n-1}=0\,, \\ \dfrac{\partial\theta_n}{\partial\nu}=0 \text{ on } (0, T)\times\partial G,\ \theta_n(x, 0)=\theta_0(x) \qquad \text{on } G\,. \end{cases}$$

From steps 1—3, we show:

(i) there exists a non-empty interval $(0, T^*)$ independent of n,

(ii) $\{u_n,\ \rho_n,\ \theta\}$, solution of (1.7)-(1.9) in the sense of Definitions 1.3.

It is then not difficult to check that u_n, ρ_n and θ_n are all uniformly bounded in the appropriate norms. Using then Aubin's compactness theorem as well as the compensated compactness arguments of Murat as applied by Lions, we get the desired result.

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Section 2: Initial-value problem (1.4).

We shall now carry out Step 1 of the proof of Theorem 1.1. The main result of the section is the following theorem.

THEOREM 2.1. *Let v be a vector-valued function in $L^2(0, T; H_0^1)$ and let ρ^0 be a scalar function with $0<a\leqq\rho^0(x)$, $|\text{grad}\,\rho^0|\leqq b$ on G. Then there exist:*
(i) *a non-empty interval $(0, T_*)$,*
(ii) *a weak solution ρ in $L^\infty(0, T_*; L^\infty(G))$ of the initial-value problem (1.4).*
Moreover: $0<(1-\eta)a\leqq\rho(x, t)\leqq b+\eta a$ on $(0, T_*)\times G$ for $0<\eta<1$
Furthermore:

$$\|\rho'\|_{L^2(0,T_*;(H^1)^*)}+\|(\rho^2)'\|_{L^2(0,T_*;(H^1)^*)}\leqq C\{1+\|v\|_{L^2(0,T^*;H_0^1)}\}.$$

C is a constant independent of ρ, v and depends only on the bounds of ρ^0 on G.
If x^0 is the x-coordinate of the intersection of $y=x\exp(x)$ with $y=\eta a$, then T_* is determined by $CT_*^{1/2}\|v\|_{L^2(0,T;H_0^1)}\leqq x^0$. C is a constant independent of v, ρ.

First we have the lemma.

LEMMA 2.1. *Let v_n be in $C^1(0, T; C_0^\infty(G))$ with $v_n\to v$ in $L^2(0, T; H_0^1)$. Then there exist:*
1) *ρ_n in $C^1(0, T; C^1(G))$ such that*

$$\rho_n'+v_n\cdot\text{grad}\,\rho_n+\rho_n\,\text{div}\,(v_n)=0 \quad \text{on } (0, T)\times G,\ \rho_n(x, 0)=\rho^0(x) \quad \text{on } G. \tag{2.2}$$

2) *a non empty interval $(0, T_*)$ independent of n, v_n and as in Theorem 2.1, such that:*

$$0<(1-\eta)a\leqq\rho_n(x, t)\leqq b+\eta a \qquad on\ (0, T_*)\times G.$$

Furthermore:

$$\|\rho_n'\|_{L^2(0,T_*;(H^1)^*)}+\|(\rho_n^2)'\|_{L^2(0,T_*,(H^1)^*)}\leqq C\{1+\|v_n\|_{L^2(0,T_*;H_0^1)}\}.$$

C is independent of n and depends only on the bounds of ρ^0 on G.

Proof. The existence of ρ_n satisfying (2.2) is well-known. We shall now determine the time-interval where $\rho_n>0$ and establish some uniform estimates for ρ_n.

1) Set $h_n=\rho_n-\rho^0$. Then h_n is in $C^1(0, T; W^{1,\infty}(G))$ and

$$\begin{cases} h_n'+v_n\cdot\text{grad}\,h_n+h_n\,\text{div}\,(v_n)+\rho^0\,\text{div}\,(v_n)+v_n\cdot\text{grad}\,\rho^0=0 & \text{on } (0, T)\times G, \\ h_n(x, 0)=0 & \text{on } G. \end{cases} \tag{2.3}$$

Let s be a large positive integer, then h_n^{s-1} is also in $C^1(0, T; W^{1,\infty}(G))$. Multiplying (2.3) by h_n^{s-1} and integrating over G, we obtain:

$$s^{-1}\frac{d}{dt}\|h_n(.,t)\|_s^s+(v_n\cdot\text{grad}\,h_n+h_n\,\text{div}\,(v_n, h_n^{s-1}) \tag{2.4}$$
$$+(\rho^0\,\text{div}\,(v_n)+v_n\cdot\text{grad}\,\rho^0, h_n^{s-1})=0.$$

An integration by parts yields:

$$(v_n\cdot \mathrm{grad}\, h_n,\ h_n^{s-1})=-s^{-1}(\mathrm{div}\,(v_n),\ h_n^s)\,.$$

Hence:

$$\|h_n(.,\ t)\|_s^{s-1}\left|\frac{d}{dt}\|h_n(.,\ t)\|_s\right|\leqq(\mathrm{mes}\,(G))^{1/2}\|h_n\|_{L^\infty(G)}^{s-1}\{2\|h_n\|_{L^\infty(G)}\|v_n\|_{1,2}+2b\|v_n\|_{1,2}\}=C\|h_n\|_{L^\infty(G)}^{s-1}\{1+\|h_n\|_{L^\infty(G)}\}\,\|v_n\|_{1,2}\,. \tag{2.5}$$

C is independent of n, s, t and v_n. It depends on G and on the upper bound of ρ^0, grad ρ^0 on G.

Let

$$G_s=\{x:\ x \text{ in } G,\ 2^{-1/(s-1)}\|h_n\|_{L^\infty(G)}<|h_n(x)|\}\,.$$

Then:

$$2^{-1/(s-1)}\|h_n\|_{L^\infty(G)}\{\mathrm{mes}\, G_s\}^{1/s}\leqq\|h_n\|_s\,. \tag{2.6}$$

Using (2.6) in (2.5) we get:

$$\left|\frac{d}{dt}\|h_n(.,\ t)\|_s\right|\leqq C\,(\mathrm{mes}\, G_s)^{1-s/s}\|v_n\|_{1,2}\{1+\|h_n\|_{L^\infty(G)}\}\,.$$

Thus,

$$\|h_n(.,\ t)\|_s\leqq C\,(\mathrm{mes}\, G_s)^{1-s/s}\int_0^t\|v_n(.,\ \zeta)\|_{1,2}\{1+\|h_n(.,\ \zeta)\|_{L^\infty(G)}\}\,d\zeta\,.$$

Since h_n is in $L^\infty(0,\ T;\ L^\infty(G))$ we have by letting $s\to+\infty$:

$$\|h_n(.,\ t)\|_{L^\infty(G)}\leqq C\int_0^t\|v_n(.,\ \zeta)\|_{1,2}\{1+\|h_n(.,\ \zeta)\|_{L^\infty(G)}\}\,d\zeta\,.$$

It follows from the Gronwall lemma that:

$$\|h_n(.,\ t)\|_{L^\infty(G)}\leqq C\int_0^t\|v_n(.,\ \zeta)\|_{1,2}d\zeta\cdot\exp\,(C\int_0^t\|v_n(.,\ \zeta)\|_{1,2}d\zeta)\,. \tag{2.7}$$

2) Consider the curve $y=x\exp(x)$ and the line $y=\eta a$ with $0<\eta<1$. We know that they intersect at $x_0(\eta)>0$ and clearly $x\exp(x)\leqq\eta a$ for $0<x\leqq x_0(\eta)$. Thus, from (2.7) we obtain:

$$\|h_n(.,\ t)\|_{L^\infty(G)}\leqq\eta a\quad\text{if}\quad C\int_0^t\|v_n(.,\ \zeta)\|_{1,2}d\zeta\leqq x_0(\eta)\,. \tag{2.8}$$

The estimate (2.8) holds if in particular:

$$C\int_0^t\|v(.,\ \zeta)\|_{1,2}d\zeta\leqq x_0(\eta)\,.$$

Let T_* be such that

$$CT_*^{1/2}\|v\|_{L^2(0,T;H_0^1)}\leqq x_0(\eta).$$

Then (2.8) holds for all $0\leqq t\leqq T_*$. Therefore:

$$0<(1-\eta)a\leqq\rho_n=h_n+\rho^0\leqq b+\eta a \qquad \text{on } (0,\ T_*)\times G\,.$$

3) Let ϕ be a scalar function in H^1. We have:

$$(\rho_n',\ \phi)+(v_n\cdot\operatorname{grad}\rho_n+\rho_n\operatorname{div}(v_n),\ \phi)=0\,.$$

Since v_n is in $C^1(0,\ T\,;\ C_0^\infty(G))$, an integration by parts gives

$$(\rho_n',\ \phi)=(v_n,\ \rho_n\cdot\operatorname{grad}\phi)\,.$$

This,

$$|(\rho_n',\ \phi)|\leqq(b+\eta a)\|v_n\|\ \|\phi\|_{1,2}\,.$$

Hence:

$$\|\rho_n'\|_{L^2(0,T_*;(H^1)^*)}\leqq(b+\eta a)\|v_n\|_{L^2(0\ T_*,(H_0^1)}\leqq C\|v\|_{L^2(0,T_*,H_0^1)}\,.$$

C is independent of n, v and depends only the bounds of ρ^0 on G.
Finally with ϕ as above, we have:

$$((\rho_n^2)',\ \phi)+\frac{1}{2}(v_n\rho_n\cdot\operatorname{grad}\rho_n,\ \phi)+\frac{1}{2}(\rho_n^2\operatorname{div}(v_n),\ \phi)=0\,.$$

An elementary computation gives:

$$((\rho_n^2)\ ,\ \phi)=-\frac{1}{4}(\rho_n^2\operatorname{div}(v_n),\ \phi)+\frac{1}{4}(\rho_n^2,\ v_n\cdot\operatorname{grad}\phi)\,.$$

So:

$$|((\rho_n^2)',\ \phi)|\leqq\frac{1}{4}(\eta a+b)^2\|v_n\|_{1,2}\|\phi\|_{1,2}\,.$$

Therefore:

$$\|(\rho_n^2)'\|_{L^2(0,T_*;(H^1)^*)}\leqq C\|v\|_{L^2(0\ T_*;H_0^1)}\,.$$

C is as before independent of n, v, ρ_n and depends only on the bounds of ρ^0 on G.

LEMMA 2.2. *Let $\{\rho_n\}$ be a sequence of scalar functions with*

$$\|\rho_n\|_{L^\infty(0,T_*;L^\infty(G))}+\|\rho_n'\|_{L^2(0,T_*;(H^1)^*)}\leqq M\,.$$

M is a constant independent of n. Then there exists a subsequence denoted again by ρ_n such that:
(i) *$\rho_p\to\rho$ in $L^s(0,\ T_*\,;\ (H^1)^*)$; $2\leqq s<\infty$ and in the weak*-topology of $L^\infty(0,\ T_*\,;\ L^\infty(G))$,*
(ii) *$\rho_n^2\to\rho^2$ in the weak*-topology of $L^\infty(0,\ T_*\,;\ L^\infty(G))$.*

Proof. It is clear that there exists a subsequence such that $\rho_n \to \rho$ in the weak*-topology of $L^\infty(0, T_*; L^\infty(G))$ and $\rho'_n \to \rho'$ weakly in $L^2(0, T_*; (H^1)^*)$ with $\rho_n^2 \to \chi$ in the weak*-topology of $L^\infty(0, T_*; L^\infty(G))$.

Since the natural injection mapping of H^1 into H is compact, that of H into $(H^1)^*$ is also compact by Schauder's theorem. From the above estimates and from Aubin's theorem [1] we obtain:

$$\rho_n \to \rho \text{ in } L^s(0, T_*; (H^1)^*) \qquad \text{for } 2 \leqq s < \infty .$$

We now prove the key assertion of the lemma, namely that $\chi = \rho^2$.

1) From the theory of linear elliptic boundary-value problems we have a unique w_n in $H_0^1 \cap H^2$, solution of:

$$-\Delta w_n = \rho_n \text{ on } G,\ w_n = 0 \qquad \text{on } \partial G$$

Moreover:

$$\|w_n\|_{L^2(0, T_*; (H_0^1)} \leqq C \|\rho_n\|_{L^2(0, T_*; (H^1)^*)} ;\ \|w_n\|_{L^2(0, T_*, H^2)} \leqq C \|\rho_n\|_{L^2(0, T_*, H)} ,$$

C is independent of n. Hence: $w_n \to w$ in $L^2(0, T_*; H_0^1)$ and weakly in $L^2(0, T_*; H^2)$ as $n \to +\infty$. Moreover:

$$-\Delta w = \rho \text{ on } G ,\quad w = 0 \qquad \text{on } \partial G .$$

2) Let ϕ be a $C_0^\infty(G)$-function, then:

$$-\int_0^{T_*} (\Delta w_n,\ \rho_n \phi) dt = \int_0^{T_*} (\rho_n^2,\ \phi) dt \to \int_0^{T_*} (\chi,\ \phi) dt.$$

Since ρ_n is in $L^\infty(0, T_*; L^\infty(G))$, there exists $\{\psi_{nk}\}$ in $C_0^\infty(0, T_*; C_0^\infty(G))$ such that

$$\psi_{nk} \to \rho_n \text{ in } L^2(0, T_*; H),\ D_j \psi_{nk} \to D_j \rho_n \text{ in } L^2(0, T_*; H^{-1}) .$$

Hence:

$$\int_0^{T_*} (D_j^2 w_n,\ \psi_{n_k} \phi) dt = -\int_0^{T_*} (D_j w_n,\ D_j \psi_{n_k} \cdot \phi + \psi_{n_k} D_j \phi) dt .$$

As $k \to +\infty$, we obtain:

$$\int_0^{T_*} (D_j^2 w_n,\ \rho_n \phi) dt = -\int_0^{T_*} (D_j w_n,\ D_j \rho_n \cdot \phi + \rho_n D_j \phi) dt .$$

So:

$$-\sum_{j=1}^{3} \int_0^{T_*} (D_j w_n,\ \phi D_j \rho_n) dt - \sum_{j=1}^{3} \int_0^{T_*} (D_j w_n,\ \rho_n D_j \phi) dt \to \int_0^{T_*} (\chi,\ \phi) dt .$$

Since $D_j w_n \to D_j w$ weakly in $L^2(0, T_*; H^1)$, $\rho_n \to \rho$ in the weak*-topology of $L^\infty(0, T_*; L^\infty(G))$ with $\rho'_n \to \rho'$ weakly in $L^2(0, T_*; (H^1)^*)$, it follows from the compensated compactness argument of Murat as applied by Lions in [5], p. 72 relation 1.64 that:

$$\rho_n D_j w_n \longrightarrow \rho D_j w,\ D_j w_n \cdot D_j \rho_n \longrightarrow D_j w \cdot D_j \rho$$

both in the distribution sense on $(0, T_*)\times G$.

Hence:

$$-\sum_{j=1}^{3}\int_0^{T_*}(D_j w,\ \phi D_j \rho)dt - \sum_{j=1}^{3}\int_0^{T_*}(D_j w,\ \rho D_j \phi)dt = \int_0^{T_*}(\chi,\ \phi)dt\,.$$

But $D_j w$ is in $L^2(0, T_*; H^1)$ and ϕ is a testing function, thus $\phi D_j w$ is in $L^2(0, T_*; H_0^1)$ and hence:

$$-\int_0^{T_*}(D_j w,\ \phi D_j \rho)dt = \int_0^{T_*}(D_j^2 w \cdot \phi + D_j w \cdot D_j \phi,\ \rho)dt\,.$$

It follows that:

$$-\int_0^{T_*}(\Delta w,\ \rho\phi)dt = \int_0^{T_*}(\chi,\ \phi)dt$$

i. e.

$$-\rho\Delta w = \rho^2 = \chi\,.$$

Proof of Theorem 2.1. Let v be in $L^2(0, T; H_0^1)$. Then there exists $\{v_n\}$ in $C^1(0, T; C_0^\infty(G))$ such that $v_n \to v$ in $L^2(0, T; H_0^1)$.

Consider the initial-value problem:

$$\rho_n' + v_n \cdot \operatorname{grad} \rho_n + \rho_n \operatorname{div}(v_n) = 0,\ \rho_n > 0 \text{ on } (0, T)\times G,\ \rho_n(x, 0) = \rho^0(x) \quad \text{on } G\,.$$

From Lemma 2.1, we have a non-empty interval $(0, T_*)$ independent of n and ρ_n, solution of the above problem on $(0, T_*)\times G$. With the estimates of Lemma 2.1, we obtain by taking subsequences: $\rho_n \to \rho$ in the weak*-topology of $L^\infty(0, T_*; L^\infty(G))$, $\rho_n' \to \rho'$ weakly in $L^2(0, T_*; (H^1)^*)$ and in view of Lemma 2.2, $(\rho_n^2)' \to (\rho^2)'$ weakly in $L^2(0, T_*; (H^1)^*)$. Moreover: $0 < (1-\eta)a \leqq \rho(x, t) \leqq b + \eta a$ on $(0, T_*)\times G$.

Furthermore:

$$\|\rho'\|_{L^2(0, T_*; (H^1)^*)} + \|(\rho^2)'\|_{L^2(0, T_*; (H^1)^*)} \leqq C(1 + \|v\|_{L^2(0, T_*; H_0^1)})\,.$$

C is independent of ρ, v and depends only on the bounds of ρ^0 on Cs.

It is trivial to check that ρ is a solution of (1.4) in the sense of Definition 1.

Section 3. *The initial boundary-value problem* (1.5). We shall now carry out Step 2 of the proof of Theorem 1.1. We now state the main result of the section.

THEOREM 3.1. *Let v, ρ, T_* be as in Theorem* 2.1 *and let $\tilde{\theta}$ be a scalar function in $L^\infty(0, T_*; L^\infty(G))$. Let f be in $L^\infty(0, T_*; H)$. Then there exists u in $L^\infty(0, T_*; H) \cap L^2(0, T_*; H_0^1)$ such that $\{\rho, u\}$ is a weak solution of* (1.4)-(1.5) *in the sense of Definition* 2. *Moreover*:

$$\|u(\,.\,,\ t)\|^2 + \int_0^t \|u(\,.\,,\ s)\|_{1,2}^2 ds \leqq Ct\{1 + \|\tilde{\theta}\|^2_{L^\infty(0, T_*; L^\infty(G))}\}$$

and

$$\|(\rho u)'\|_{L^2(0,T_*;H^{-2})} \leqq C\{1+\|v\|_{L^2(0,T_*.H_0^1)}+\|\tilde{\theta}\|_{L^\infty(0,T_*.L^\infty(G))}\}.$$

C is a constant independent of t, ρ, v, $\tilde{\theta}$, u and depends only on f, the bounds of ρ^0 on G.

We shall use the standard Galerkin method and estimate the kinetic energy. Let $\{w_j\}$ be a vector basis of H_0^2 and set

$$u_k=\sum_{j=1}^{k} c_{jk}(t)w_j.$$

Consider the system of linear ordinary differential equations in $c_{jk}(t)$:

$$\text{(3.1)}\qquad \begin{cases} (\rho u_k', w_j)+(\nabla u_k, \nabla w_j)+(\operatorname{div} u_k, \operatorname{div} w_j)+(\rho v\cdot\nabla u_k, w_j) \\ \qquad\qquad -(\rho+\tilde{\theta}, \operatorname{div} w_j)=(\rho f, w_j), \\ c_{jk}(0)=0, \qquad 1\leqq j\leqq k. \end{cases}$$

LEMMA 3.1. *Suppose all the hypotheses of Theorem* 3.1 *are satisfied. Then there exists u_k in $C(0, T_*^k; H_0^2)$, solution of* (3.1).

Proof. We have:

$$(\rho u_k', w_j)=\sum_{s=1}^{k} c_{sk}'(t)(\rho w_s, w_j).$$

Since $\{w_j\}$ is linearly independent in H and $0<(1-\eta)a\leqq\rho\leqq b+\eta a$ on $(0, T_*)\times G$, it is clear the $\{\rho^{1/2}w_j\}$ is also linearly independent in H and thus $\det(\rho w_j, w_s)\neq 0$. The lemma is an immediate consequence of the Caratheodory theorem.

LEMMA 3.2. *Let u_k be as in Lemma* 3.1. *Then*:

$$\|u_k(\,.\,, t)\|^2+\int_0^t\|u_k(\,.\,, s)\|_{1,2}^2 ds\leqq Ct\{1+\|\tilde{\theta}\|_{L^\infty(0,T_*.L^\infty(G))}^2\}$$

for $0\leqq t\leqq T_*$.

C is a constant independent of t, k, ρ, v, $\tilde{\theta}$ and depends on the bounds of ρ^0 on G.

Proof. To show that the local solution u_k of Lemma 3.1 is in fact a global solution we shall estimate the kinetic energy.

1) Multiplying (3.1) by c_{jk} and taking the summation with respect to j from 1 to k we obtain:

$$\text{(3.2)}\qquad (\rho u_k', u_k)+\|\nabla u_k\|^2+\|\operatorname{div} u_k\|^2+(\rho v\cdot\nabla u_k, u_k)-(\rho+\tilde{\theta}, \operatorname{div} u_k)=(\rho f, u_k).$$

On the other hand since ρ is a weak solution of (1.4) we have:

$$\text{(3.3)}\qquad (\rho', \phi)-(v\cdot\operatorname{grad}\phi, \rho)=0$$

for all scalar functions ϕ in H^1.

Since u_k is in $C(0, T_*^k; H_0^2)$ and H_0^2 is an algebra, $|u_k|^2$ is also in $C(0, T_*^k; H_0^2)$ and thus by taking $\phi=\frac{1}{2}|u_k|^2$ in (3.3) we get:

$$\frac{1}{2}(\rho', |u_k|^2)-\frac{1}{2}(v\cdot\text{grad}|u_k|^2, \rho)=0. \tag{3.4}$$

Adding (3.4) to (3.2) we obtain:

$$\frac{d}{dt}(\rho u_k, u_k)+2\|\nabla u_k\|^2+2\|\text{div}\, u_k\|^2+2(\rho v\cdot\nabla u_k, u_k)$$
$$-(v\cdot\text{grad}|u_k|^2, \rho)-2(\rho+\tilde{\theta}, \text{div}\, u_k)=2(\rho f, u_k). \tag{3.5}$$

But:

$$2(\rho v\cdot\nabla u_k, u_k)=(v\cdot\text{grad}|u_k|^2, \rho).$$

So:

$$\frac{d}{dt}(\rho u_k, u_k)+2\|\nabla u_k\|^2+2\|\text{div}\, u_k\|^2\leqq 2\|u_k\|\ \|\rho f\|$$
$$+2\|\text{div}\, u_k\|\{\|\rho\|_{L^\infty(0,T_*;L^\infty(G))}+\|\tilde{\theta}\|_{L^\infty(0,T_*,L^\infty(G))}\}.$$

Hence:

$$\frac{d}{dt}(\rho u_k, u_k)+c_1\|u_k\|_{1,2}^2\leqq C\{1+\|\tilde{\theta}\|^2_{L^\infty(0,T_*;L^\infty(G))}+\|f\|^2_{L^\infty(0,T_*;H)}\}$$

C is a constant independent of k, t, θ, ρ, f, v but depends on the bounds of ρ^0 on G. Therefore:

$$(\rho(.,t)u_k(.,t), u_k(.,t))+c_1\int_0^t\|u_k(.,s)\|_{1,2}^2ds$$
$$\leqq Ct\{1+\|\tilde{\theta}\|^2_{L^\infty(0,T_*,L^\infty(G))}+\|f\|^2_{L^\infty(0,T_*;H)}\}$$

Since $0<(1-\eta)a\leqq\rho$ on $(0, T_*)\times G$, the lemma is proved.

Proof of Theorem 3.1. 1) Let u_k be as in Lemma 3.2. We have by taking subsequences if necessary: $u_k\to u$ weakly in $L^2(0, T_*; H_0^1)$ and in the weak*-topology of $L^\infty(0, T_*; H)$. Moreover:

$$\|(.,t)\|^2+c_1\int_0^t\|u(.,s)\|_{1,2}^2ds\leqq Ct\{1+\|\tilde{\theta}\|^2_{L^\infty(0,T_*;L^\infty(G))}+\|f\|^2_{L^\infty(0,T_*;H)}\}.$$

C is independent of t, u, v, ρ, $\tilde{\theta}$ and depends only on the bounds of ρ^0 on G.

2) We have:

$$(\rho u_k', w_j)+(\nabla u_k, \nabla w_j)+(\text{div}\, u_k, \text{div}\, w_j)+(\rho v\cdot\nabla u_k, w_j)$$
$$-(\rho+\tilde{\theta}, \text{div}\, w_j)=(\rho f, w_j). \tag{3.6}$$

With ρ as in Theorem 2.1, we get:

(3.7) $$(\rho', \phi)-(v\cdot\operatorname{grad}\phi, \rho)=0$$

for all ϕ in H^1.

Since u_k and w_j are in H_0^2, $u_k.w_j$ is also in H_0^1 and hence by taking $\phi=u_k.w_j$ in (3.7) we obtain:

(3.8) $$(\rho', u_k.w_j)-(v.\operatorname{grad}(u_k.w_j), \rho)=0.$$

It follows from (3.6) and (3.8) that

(3.9) $$((\rho u_k)', w_j)+(\nabla u_k, \nabla w_j)+(\operatorname{div} u_k, \operatorname{div} w_j)-(\rho v.\nabla w_j, u_k) -(\rho+\tilde{\theta}, \operatorname{div} w_j)=(\rho f, w_j).$$

Let ψ be a $C^1(0, T_*)$-function with $\psi(T_*)=0$. Then:

$$-\int_0^{T_*}(\rho u_k, \psi' w_j)dt+\int_0^{T_*}(\nabla u_k, \nabla(\psi w_j))dt+\int_0^{T_*}(\operatorname{div} u_k, \operatorname{div}(\psi w_j))dt -\int_0^{T_*}(\rho v.\nabla(\psi w_j), u_k)dt-\int_0^{T_*}(\rho+\tilde{\theta}, \operatorname{div}(\psi w_j))dt=\int_0^{T_*}(\rho f, \psi w_j)dt.$$

Keep j fixed and let $k\to+\infty$. We get

$$-\int_0^{T_*}(\rho u, \psi' w_j)dt+\int_0^{T_*}(\nabla u, \nabla(\psi w_j))dt+\int_0^{T_*}(\operatorname{div} u, \operatorname{div}(\psi w_j))dt -\int_0^{T_*}(\rho v.\nabla(\psi w_j), u)dt-\int_0^{T_*}(\rho+\tilde{\theta}, \operatorname{div}(\psi w_j))dt=\int_0^{T_*}(\rho f, \psi w_j)dt.$$

By a standard argument, we have:

(3.10) $$-\int_0^{T_*}(\rho u, w')dt+\int_0^{T_*}(\nabla u, \nabla w)dt+\int_0^{T_*}(\operatorname{div} u, \operatorname{div}(w))dt -\int_0^{T_*}(\rho v.\nabla w, u)dt-\int_0^{T_*}(\rho+\tilde{\theta}, \operatorname{div}(w))dt=\int_0^{T_*}(\rho f, w)dt$$

for all w in $L^2(0, T_*; H_0^2)$ with w' in $L^2(0, T_*; H)$ and $w(.,T_*)=0$. From (3.10) it is clear that:

$$\|(\rho u)'\|_{L^2(0,T_*;H^{-2})}\leqq C\{\|u\|_{L^2(0,T_*;H_0^1)}+\|\tilde{\theta}\|_{L^\infty(0,T_*;L^\infty(G))}+b+\eta a +(b+\eta a)\|f\|_{L^2(0,T_*;H)}+(b+\eta a)\|v\|_{L^\infty(0,T_*;H)}\|u\|_{L^2(0,T_*;H_0^1)}\}.$$

Applying (3.5) and Theorem 2.1 we get:

$$\|(\rho u)'\|_{L^2(0,T_*,H^{-2})}\leqq C\{1+\|\tilde{\theta}\|_{L^\infty(0,T_*,L^\infty(G))}+\|v\|_{L^2(0,T_*;H_0^1)}\}$$

C is a constant independent of ρ, u, v, $\tilde{\theta}$ but depends on f and on the bounds of ρ^0 on G.

Section 4. *The nonlinear initial boundary-value problem* (1.6). The proof of Step 3 of Theorem 1.1 will now be given. The main result of the section is the following theorem.

THEOREM 4.1. *Let* ρ, v, T_* *be as in Theorem* 2.1 *and let* $\tilde{\theta}$ *be a scalar function in* $L^\infty(0, T_*; L^\infty(G))$ *with* $0<(1-\eta^2)^{1/2}c\leqq\tilde{\theta}(x, t)\leqq(1+\eta^2)^{1/2}c$ *on* $(0, T_*)\times G$. *Then for any* $\theta^0(x)=c>0$ *on* G, *there exist*:
(i) *a non-empty interval* $(0, T^*)$ *with* $T^*\leqq T_*$,
(ii) *a scalar function* θ *in* $L^\infty(0, T^*; L^\infty(G))\cap L^2(0, T^*; H^1)$ *such that* $\{\rho, \theta\}$ *is a solution of* (1.4) *and* (1.6) *in the sense of Definition* 3.

Moreover: $0<(1-\eta^2)^{1/2}c\leqq\theta(x, t)\leqq(1+\eta^2)^{1/2}c$ *on* $(0, T^*)\times G$;

$$\|\theta\|_{L^2(0, T^*; H^1)}\leqq C\{1+\|\tilde{\theta}\|_{L^\infty(0, T^*; L^\infty(G))}+\|v\|_{L^2(0, T^*; H^1)}\}.$$

Furthermore for any $\delta>0$,

$$\int_0^{T^*-\delta}\|\theta(\,.\,, t+\delta)-\theta(\,.\,, t)\|^2 dt\leqq C\delta^{1/2}\{1+\|v\|^4_{L^2(0, T^*; H^1)}+\|\tilde{\theta}\|^4_{L^\infty(0, T^*; L^\infty(G))}\}.$$

C *is a constant independent of* δ, $\tilde{\theta}$, v, ρ. *It depends only on* f *the bounds of* ρ^0 *on* G *and on* c. *The interval* $(0, T^*)$ *is such that*

$$\int_0^{T^*}\{\|v\|^2_{1,2}+\|\tilde{\theta}\|^2_{1,2}\}\,dt\leqq\eta^2c^2/C.$$

The proof of the theorem is long and involved. We shall use a discretisation of the time-variable, then a nonlinear perturbation of the discretised equation to show the existence of θ and of a nonempty interval where $\theta>0$.

Set $w=\theta^2-c^2$ and consider the initial boundary-value problem:

$$\text{(4.1)}\quad \begin{cases} \rho w'-\nabla(\tilde{\theta}^{-1}\nabla w)-2\rho^2\tilde{\theta}\,\mathrm{div}(v)+\rho v.\,\mathrm{grad}\, w-2Bv=0 & \text{on } (0, T_*)\times G\,, \\ \partial w/\partial\nu=0 \text{ on } (0, T_*)\times\partial G\,, \quad \omega(x, 0)=0 & \text{on } G\,. \end{cases}$$

Let N be a large positive integer and let $h=T_*/N$ with T_* as in Theorem 2.1. Set·

$$\text{(4.2)}\qquad \rho_n(x)=h^{-1}\int_{nh}^{(n+1)h}\rho(x, t)dt\,,\quad 0\leqq n\leqq N-1\,.$$

Similarly for $\tilde{\theta}_n$ and for v_n. Let A be the nonlinear elliptic operator

$$A\phi=|\phi|^2\phi-\sum_{j=1}^3 D_j\{(D_j\phi)^2 D_j\phi\}$$

with

$$(A\phi, \psi)=a(\phi, \psi)=(|\phi|^2\phi, \psi)+\sum_{j=1}^3((D_j\phi)^2 D_j\phi, D_j\psi)$$

for all ϕ, ψ in $W^{1,4}(G)$. It is clear that A is a monotone, coercive operator mapping bounded sets of $W^{1,4}(G)$ into bounded sets of $W^{-1,4/3}(clG)=(W^{1,4}(G))^*$.

LEMMA 4.1. *For each $\varepsilon>0$ and each n, there exists ω_n^ε in $W^{1,4}(G)$ which we write as ω_n, solution of the nonlinear elliptic boundary value problem*:

$$(4.3)\quad \begin{cases} \rho_n\omega_n-\rho_n\omega_{n-1}+\varepsilon hA(\omega_n)-h\nabla(\tilde{\theta}_n^{-1}\nabla\omega_n)-2hBv_n-2h\rho_n^2\theta_n\operatorname{div}(v_n) \\ \qquad\qquad +h\rho_n v_n.\operatorname{grad}(\omega_n)=0 \qquad \text{on } G, \\ \dfrac{\partial\omega_n}{\partial\nu}=0 \qquad \text{on } \partial G,\ \omega_0=0,\ n\leqq N. \end{cases}$$

Proof. The lemma is an immediate consequence of the well-known theory of coercive pseudo-monotone operators mapping bounded sets of a reflexive Banach space into bounded sets of the dual space. e. g. Cf. [4].

LEMMA 4.2. *Let ω_n^ε be as in Lemma 4.1. Then there exists an integer N_*, independent of ε, ρ such that*

$$\|\omega_n^\varepsilon\|_{L^\infty(G)}\leqq\eta^2c^2;\ 0<\eta<1,\ n\leqq N_*.$$

Furthermore N_* is the largest integer with $N_*\leqq T^*/h$ and

$$\int_0^{T^*}\{\|v\|_{1,2}^2+\|\tilde{\theta}\|_{L^\infty(G)}^2\}\,dt\leqq\eta^2c^2/C$$

for some C independent of ε, v, ρ, $\tilde{\theta}$.

Proof. For simplicity of notations we shall write ω_n for ω_n^ε. Since G is a bounded open subset of R^3, $W^{1,4}(G)$ is contained in $L^\infty(G)$ and thus, $|\omega_n|^{s-2}\omega_n$ is in $W^{1,4}(G)$ for $s\geqq 2$. From (4.3) we have:

$$(4.4)\quad (\rho_n\omega_n-\rho_n\omega_{n-1},\ |\omega_n|^{s-2}\omega_n)+\varepsilon ha(\omega_n,\ |\omega_n|^{s-2}\omega_n)+h(\tilde{\theta}_n^{-1}\nabla\omega_n,\ \nabla(|\omega_n|^{s-2}\omega_n))$$
$$+h(\rho_n v_n.\operatorname{grad}\omega_n-2Bv_n-2\rho_n^2\tilde{\theta}_n\operatorname{div}(v_n),\ |\omega_n|^{s-2}\omega_n)=0.$$

It is easy to check that:

$$(4.5)\quad a(\omega_n,\ |\omega_n|^{s-2}\omega_n)\geqq 0,\ (\tilde{\theta}_n^{-1}\nabla\omega_n,\ \nabla(|\omega_n|^{s-2}\omega_n))\geqq(s-1)c_1\|\omega_n^{(s-2)/2}\nabla\omega_n\|^2.$$

Taking (4.5) into account in (4.4) we obtain by an elementary computation:

$$\|\rho_n^{1/s}\omega_n\|_s^s+h(s-1)c_1\|\omega_n^{(s-2)/2}\nabla\omega_n\|^2\leqq(\rho_n\omega_{n-1}|\omega_n|^{s-2}\omega_n)$$
$$+Ch\|\omega_n^{(s-2)/2}\nabla\omega\|\ \|\rho_n v_n\omega_n^{(s)/2}\|$$
$$+Ch\|\rho_n^{1/s}\omega_n\|_{L^\infty(G)}^{s-1}\{\|Bv_n\|_{L^1(G)}+\|v_n\|_{1,2}\|\tilde{\theta}_n\|_{L^\infty(G)}\}.$$

Thus,

$$\|\rho_n^{1/s}\omega_n\|_s^s+c_2(s-1)h\|\omega_n^{(s-2)2}\nabla\omega\|^2\leqq(\rho_n\omega_{n-1},\ |\omega_n|^{s-2}\omega_n)$$
$$(4.6)\quad +Ch\|\rho_n^{1/s}\omega_n\|_{L^\infty(G)}^{s-1}\{\|v_n\|_{1,2}^2+\|\tilde{\theta}_n\|_{L^\infty(G)}^2\}$$
$$+Ch\|v_n\|^2\|\rho_n^{1/s}\omega_n\|_{L^\infty(G)}^s(s-1)^{-1}$$

We note that:

$$(4.7)\qquad |(\rho_n\omega_{n-1},\ |\omega_n|^{s-2}\omega_n)|\leqq\|\rho_n^{1/s}\omega_n\|_s^{s-1}\|\rho_{n-1}^{1/s}\omega_{n-1}\|_s\|\rho_n\|_{L^\infty(G)}^{1/s}\|\rho_{n-1}\|_{L^\infty(G)}^{-1/s}.$$

Furthermore as done earlier in Section 2 (cf. relation 2.6)

$$(4.8)\qquad \|\rho_n^{1/s}\omega_n\|_{L^\infty(G)}\leqq(\text{mes } G)^{-1/s}\cdot 2^{1/s-1}\|\rho_n^{1/s}\omega_n\|_s.$$

So from (4.6)-(4.8), we have:

$$(4.9)\qquad \|\rho_n^{1/s}\omega_n\|_s\leqq\|\rho_{n-1}^{1/s}\omega_{n-1}\|_s\|\rho_n\|_{L^\infty(G)}^{1/s}\|\rho_{n-1}\|_{L^\infty(G)}^{-1/s}+Ch\{\|v_n\|_{1,2}^2+\|\tilde{\theta}_n\|_{L^\infty(G)}^2\}$$
$$+Ch\|v_n\|^2\|\rho_n^{1/s}\omega_n\|_{L^\infty(G)}/(s-1).$$

Since

$$\{(1-\eta)a\}^{1/s}\|\omega_n\|_s\leqq\|\rho_n^{1/s}\omega_n\|_s\leqq\{b+\eta a\}^{1/s}\|\omega_n\|_s$$

and ω_n is in $L^\infty(G)$ we obtain by letting $s\to+\infty$ in (4.9):

$$\|\omega_n\|_{L^\infty(G)}\leqq\|\omega_{n-1}\|_{L^\infty(G)}+Ch\{\|v_n\|_{1,2}^2+\|\tilde{\theta}_n\|_{L^\infty(G)}^2\}.$$

Therefore:

$$(4.10)\qquad \|\omega_n\|_{L^\infty(G)}\leqq Ch\sum_{j=1}^{n}\{\|v_j\|_{1,2}^2+\|\tilde{\theta}_j\|_{L^\infty(G)}^2\},\ n\leqq N.$$

The different constants C are all independent of ε, n, v, $\tilde{\theta}$, ρ.

Let T^* be such that:

$$(4.11)\qquad \int_0^{T^*}\{\|v\|_{1,2}^2+\|\tilde{\theta}\|_{L^\infty(G)}^2\}\,dt\leqq\eta^2c^2/C.$$

Since $\tilde{\theta}$ is in $L^\infty(0, T_*;\ L^\infty(G))$ and v is in $L^2(0, T_*;\ H^1)$, such T^* exists and is non-zero. It is clear that T^* is independent of ε, h.

With N_* as the largest integer such that $N_*\leqq T^*/h$, then it follows from (4.10)-(4.11) that:

$$\|\omega_n\|_{L^\infty(G)}\leqq\eta^2c^2\quad\text{for}\quad n\leqq N_*.$$

LEMMA 4.3. *Let N_* be as in Lemma 4.2. Then for each $n\leqq N_*$ there exists ω_n in $L^\infty(G)\cap H^1$, solution of the elliptic Neumann boundaryvalue problem*:

$$(4.12)\qquad \begin{cases}\rho_n w_n-\rho_n w_{n-1}-h\nabla(\tilde{\theta}_n^{-1}\nabla\omega_n)-2hBv_n-2h\rho_n^2\tilde{\theta}_n\,\text{div}(v_n)+h\rho_n v_n.\,\text{grad}\,\omega_n=0,\\ \partial\omega_n/\partial\nu=0\ \text{on}\ \partial G,\ \omega_0=0,\ n=1,\cdots,N_*.\end{cases}$$

Moreover: $\|\omega_n\|_{L^\infty(G)}\leqq\eta^2c^2$ *and*

$$\sum_{j=1}^{N_*}h\|\omega_n\|_{1,2}^2\leqq C\{1+\|v\|_{L^2(0,T^*;H^1)}^2+\|\tilde{\theta}\|_{L^\infty(0,T^*;L^\infty(G))}^2\}.$$

T^* is as in Lemma 4.2.

C is a constant independent of h, n, ρ, v, $\tilde{\theta}$ and ω_n.

Proof. Since $\|\omega_n^\varepsilon\|_{L^\infty(G)} \leqq \eta^2 c^2$, it is not difficult to get from (4.3)

$$\varepsilon h\|\omega_n^\varepsilon\|_{1,4}^4 + h\|\omega_n^\varepsilon\|_{1,2}^2 \leqq K(h, n)$$

$K(h, n)$ is independent of ε.

From the weak compactness of the unit ball in a reflexive Banach space we get by taking subsequences: $\omega_n^\varepsilon \to \omega_n$ weakly in H^1 and in the weak*-topology of $L^\infty(G)$, $\varepsilon^{1/4}\omega_n^\varepsilon \to 0$ weakly in $W^{1,4}(G)$ as $\varepsilon \to 0$. Moreover:

$$\|\omega_n\|_{L^\infty(G)} \leqq \eta^2 c^2 \qquad \text{for} \quad n \leqq N_*.$$

It is easy to check that:

$$\rho_n\omega_n - \rho_n\omega_{n-1} - h\nabla(\tilde{\theta}_n^{-1}\nabla\omega_n) - 2hBv_n - 2h\rho_n^2\tilde{\theta}_n \operatorname{div}(v_n) + h\rho_n v_n.\operatorname{grad}\omega_n = 0.$$

2) We now prove the crucial estimate of the lemma. From (4.12) we have:

$$(4.13) \qquad (\rho_n\omega_n, \omega_n) + c_1 h\|\nabla\omega_n\|^2 \leqq (\rho_n\omega_{n-1}, \omega_n) + 2h(Bv_n, \omega_n)$$
$$+ 2h(\rho_n^2\tilde{\theta}_n \operatorname{div}(v_n), \omega_n) - h(\rho_n v_n.\operatorname{grad}\omega_n, \omega_n).$$

On the other hand, it is easy to see that:

$$(4.14) \quad |(\rho_n\omega_{n-1}, \omega_n)| \leqq \frac{1}{2}(\rho_n\omega_n, \omega_n) + \frac{1}{2}(\rho_n\omega_{n-1}, \omega_{n-1}).$$
$$\leqq \frac{1}{2}(\rho_n\omega_n, \omega_n) + \frac{1}{2}(\rho_{n-1}\omega_{n-1}, \omega_{n-1}) + \frac{1}{2}((\rho_n - \rho_{n-1})\omega_{n-1}, \omega_{n-1}).$$

It follows from (4.13)-(4.14) that:

$$(\rho_n\omega_n, \omega_n) + hc_1\|\nabla\omega_n\|^2 \leqq (\rho_{n-1}\omega_{n-1}, \omega_{n-1}) + ((\rho_n - \rho_{n-1})\omega_{n-1}, \omega_{n-1})$$
$$+ Ch\|\omega_n\|_{L^\infty(G)}^2 \{\|v_n\|_{1,2}^2 + \|\tilde{\theta}_n\|_{L^\infty(G)}^2\}.$$

With our estimate for ω_n, we get:

$$(4.15) \qquad (\rho_n\omega_n, \omega_n) + hc_1\|\nabla\omega_n\|^2 \leqq (\rho_{n-1}\omega_{n-1}, \omega_{n-1}) + ((\rho_n - \rho_{n-1})\omega_{n-1}, \omega_{n-1})$$
$$+ Ch\{\|v_n\|_{1,2}^2 + \|\tilde{\theta}_n\|_{L^\infty(G)}^2\}.$$

C is a constant independent of n, h, ρ, v, $\tilde{\theta}$.

On the other hand

$$(4.16) \qquad |((\rho_n - \rho_{n-1})\omega_{n-1}, \omega_{n-1})| \leqq \|\omega_{n-1}^2\|_{1,2}\|(\rho_n - \rho_{n-1})\|_{(H^1)^*}.$$
$$\leqq C\|\nabla\omega_{n-1}\|\,\|(\rho_n - \rho_{n-1})\|_{(H^1)^*} \leqq Ch\|\nabla\omega_{n-1}\|\left\|\frac{\rho_n - \rho_{n-1}}{h}\right\|_{(H^1)^*}.$$
$$\leqq \frac{hc_1}{2}\|\nabla\omega_{n-1}\|^2 + Ch\left\|\left(\frac{\rho_n - \rho_{n-1}}{h}\right)\right\|_{(H^1)^*}^2.$$

In the above inequality we have used the estimate for ω_n in $L^\infty(G)$.

From (4.15)-(4.16) we obtain:

$$(\rho_n\omega_n,\ \omega_n)+c_1h\|\nabla\omega_n\|^2-c_1\frac{h}{2}\|\nabla\omega_{n-1}\|^2\leqq(\rho_{n-1}\omega_{n-1},\ \omega_{n-1})$$

$$+Ch\{\|(\rho_n-\rho_{n-1})h^{-1}\|^2_{(H^1)^*}+\|v_n\|^2_{1,2}+\|\tilde{\theta}_n\|^2_{L^\infty(G)}\}.$$

Taking the summation from 1 to n and noting that $\omega_0=0$, we have:

$$(4.17)\qquad (\rho_n\omega_n,\ \omega_n)+c_1\frac{h}{2}\sum_{j=1}^{n}\|\nabla\omega_j\|^2\leqq Ch\sum_{j=1}^{n}\{\|v_j\|^2_{1,2}+\|\tilde{\theta}_j\|^2_{L^\infty(G)}+\|(\rho_j-\rho_{j-1})h^{-1}\|^2_{(H^1)^*}\}.$$

Since v is in $L^2(0,\ T^*;\ H^1)$, $\tilde{\theta}$ is in $L^\infty(0,\ T^*;\ L^\infty(G))$ and ρ' is in $L^2(0,\ T^*;\ (H^1)^*)$, we obtain:

$$c_1\frac{h}{2}\sum_{j=1}^{n}\|\nabla\omega_j\|^2\leqq C\{\|v\|^2_{L^2(0,T^*;H^1)}+\|\tilde{\theta}\|^2_{L^\infty(0,T^*;L^\infty(G))}+\|\rho'\|^2_{L^2(0,T^*;(H^1)^*)}\}.$$

It then follows from Theorem 2.1 that:

$$h\sum_{j=1}^{n}\|\omega_j\|^2_{1,2}\leqq C\{\|v\|^2_{L^2(0,T^*;H^1)}+\|\tilde{\theta}\|^2_{L^\infty(0,T^*;L^\infty(G))}\}\qquad \text{for}\quad n\leqq N_*.$$

The different constants C are all independent of h, n, ρ, v, $\tilde{\theta}$.

LEMMA 4.4. *Suppose all the hypotheses of Theorem* 4.1 *are satisfied. Then there exists* θ *in* $L^\infty(0,\ T^*;\ L^\infty(G))\cap L^2(0,\ T^*;\ H^1)$, *solution of* (1.6). *Moreover*:

1) $0<(1-\eta^2)^{1/2}c\leqq\theta(x,\ t)\leqq(1+\eta^2)^{1/2}c$ *on* $(0,\ T^*)\times G$,

2) $\|\theta\|_{L^2(0,T^*;H^1)}\leqq C\{1+\|v\|_{L^2(0,T^*;H^1)}+\|\tilde{\theta}\|_{L^\infty(0,T^*;L(G))}\}$.

C is independent of ρ, v, $\tilde{\theta}$, θ *and depends on* ρ^0, c.

Proof. 1) Set $\omega_h(x,\ t)=\omega_h(x)$ for $nh\leqq t<(n+1)h$; $n=1,\cdots,N^*$. Similarly for $\tilde{\theta}_h$, v_h. Then from Lemma 4.3 we have:

$$\|\omega_h(\ .\ ,\ t)\|_{L^\infty(G)}\leqq\eta^2c^2;\ \|w_h\|_{L^2(0,T^*;H^1)}\leqq C\{1+\|v\|_{L^2(0,T^*;H^1)}+\|\tilde{\theta}\|_{L^\infty(0,T^*;L^\infty(G))}\}.$$

By taking subsequences if necessary we get: $w_h\to w$ weakly in $L^2(0,\ T^*;\ H^1)$ and in the weak*-topology of $L^\infty(0,\ T^*;\ L^\infty(G))$ as $h\to0$. Furthermore

$$\|w\|_{L^\infty(0,T^*;L^\infty(G))}\leqq\eta^2c^2\quad\text{and}$$

$$\|\omega\|_{L^2(0,T^*;H^1)}\leqq C\{1+\|v\|_{L^2(0,T^*;H^1)}+\|\tilde{\theta}\|_{L^\infty(0,T^*;L^\infty(G))}\}.$$

C is as in the lemma.

2) Let w be a vector-function in H^2_0 and let ϕ be a scalar function in $C^1(0,\ T^*)$ with $\phi(T^*)=0$. We have by a standard argument from (4.12) (e.g. cf [4] Chapter 4, 433-436):

$$(4.18)\qquad \sum_{n=1}^{N_*}\{-(\rho_h\omega_h,\ w)(\phi(nh)-\phi(nh-h))+h(\tilde{\theta}_h^{-1}\nabla\omega_h,\ \phi(nh)\nabla w)$$
$$-2h(Bv_h,\ \phi(nh)w)+h(\rho_h v_h.\operatorname{grad}\omega_h,\ \phi(nh)w)$$
$$-2h(\rho_h^2\tilde{\theta}_h\operatorname{div}(v_h),\ \phi(nh)w)+h(-(\rho_h-\rho_{h-1})h^{-1},\ \omega_{h-1}\phi(nh)w)\}=0\,.$$

Since $\rho_h\to\rho$, $\tilde{\theta}_h\to\tilde{\theta}$, $\tilde{\theta}_h^{-1}\to\tilde{\theta}^{-1}$ in $L^\infty(0,\ T^*;\ L^\infty(G))$,

$v_h\to v$ in $L^2(0,\ T^*;\ H_0^1)\cap L^\infty(0,\ T^*;\ H)$, $(\rho_h-\rho_{h-1})h^{-1}\to\rho'$ in $L^2(0,\ T^*;\ (H^1)^*)$,

$\omega_h\to\omega$ weakly in $L^2(0,\ T^*;\ H^1)$ and the weak*-topology of $L^\infty(0,\ T^*;\ L^\infty(G))$ we get:

$$(4.19)\qquad \sum_{n=1}^{N_*}-h(\rho_h\omega_h,\ w)h^{-1}\{\phi(nh)-\phi(nh-h)\}\to-\int_0^{T^*}(\rho\omega,\ \phi'w)dt\,,$$

and

$$(4.20)\qquad \sum_{n=1}^{N_*}-2(Bv_h,\ \phi(nh)w)h\to-\int_0^{T^*}2(Bv,\ \phi w)dt$$

with

$$(4.21)\qquad \sum_{n=1}^{N_*}h((\rho_h-\rho_{h-1})h^{-1},\ \omega_{h-1}\phi(nh)w)\to\int_0^{T^*}(\rho',\ \omega\phi w)dt\,.$$

On the other hand we have:

$$\|\rho_h v_h-\rho v\|_{L^2(0,T^*;H)}\leqq\|\rho_h(v_h-v)\|_{L^2(0,T^*;H)}+\|(\rho_h-\rho)v\|_{L^2(0,T^*;H)}\to0$$

and

$$\|\rho_h^2-\rho^2\|_{L^\infty(0,T^*;L^\infty(G))}\leqq\|\rho_h-\rho\|_{L^\infty(0,T^*;L^\infty(G))}2(b+\eta a)\to0$$
$$\|\rho_h^2\tilde{\theta}_h-\rho^2\tilde{\theta}\|_{L^\infty(0,T^*;H)}\leqq\|(\rho_h^2-\rho^2)\tilde{\theta}_h\|_{L^\infty(0,T^*;H)}+\|\rho^2(\tilde{\theta}_h-\tilde{\theta})\|_{L^\infty(0,T^*;H)}\to0\,.$$

Thus,

$$(4.22)\qquad \sum_{n=1}^{N_*}\{(\tilde{\theta}_h^{-1}\nabla\omega_h,\ \phi(nh)\nabla w)+(\rho_h v_h.\operatorname{grad}\omega_h-2\rho_h^2\tilde{\theta}_\eta\operatorname{div}(v_h),\ \phi(nh)w)\}\,h$$
$$\to\int_0^{T^*}\{(\tilde{\theta}^{-1}\nabla\omega,\ \phi\nabla\omega)+(\rho v.\operatorname{grad}\omega-2\rho^2\tilde{\theta}\operatorname{div}(v),\ \phi w)\}\,dt\,.$$

From (4.18)-(4.22), we get:

$$-\int_0^{T^*}(\rho w,\ \phi'w)dt+\int_0^{T^*}(\tilde{\theta}^{-1}\nabla\omega,\ \phi\nabla w)dt+\int_0^{T^*}(\rho v.\operatorname{grad}\omega-2\rho^2\tilde{\theta}\operatorname{div}(v),\ \phi w)dt$$
$$=\int_0^{T^*}(\rho',\ \omega\phi w)dt+2\int_0^{T^*}(Bv,\ \phi w)dt$$

Since ρ is a solution of (1.4), we have

$$\int_0^{T^*}(\rho'\phi\omega w)dt=-\int_0^{T^*}(\operatorname{div}(v\rho),\ \phi\omega w)dt\,.$$

Therefore by a standard argument we obtain:

$$(4.23)\quad -\int_0^{T^*}(\rho\omega,\ w')dt+\int_0^{T^*}(\tilde{\theta}^{-1}\nabla\omega,\ \nabla w)dt+\int_0^{T^*}(\rho v.\operatorname{grad}\omega-2\rho^2\tilde{\theta}\operatorname{div}(v),\ w)dt$$

$$-2\int_0^{T^*}(Bv,\ \phi w)dt+\int_0^{T^*}(\operatorname{div}(v\rho),\ \omega w)dt=0$$

for all w in $C^1(0,\ T^*;\ H^2)$ with $w(.,\ T^*)=0$.

Set $\theta^2=\omega+(\theta^0)^2$, then: $(1-\eta^2)c^2\leqq\theta^2\leqq(1+\eta^2)c^2$ on $(0,\ T^*)\times G$ and

$$\|\theta\|_{L^2(0,T^*;H^1)}\leqq C_1(1-\eta^2)^{-1/2}c^{-1}\|\omega\|_{L^2(0,T^*;H^1)}$$

$$\leqq C\{1+\|v\|_{L^2(0,T^*;H^1)}+\|\tilde{\theta}\|_{L^\infty(0,T^*,L^\infty(G))}\}\,.$$

Replacing ω by $\theta^2-(\theta^0)^2$ in (4.23) we get after an elementary computaiion the result stated it the lemma.

Proof of Theorem 4.1. In view of Lemma 4.4 it remains to show that:

$$\int_0^{T^*-\delta}\|\theta(.,\ t+\delta)-\theta(.,\ t)\|^2dt\leqq C\delta^{1/2}\{1+\|v\|^4_{L^2(0,T^*;H^1)}+\|\tilde{\theta}\|^4_{L^\infty(0,T^*;L^\infty(G))}\}$$

for any small $\delta>0$.

C is a constant independent of δ, v, $\tilde{\theta}$, ρ. It is an estimate of the type introduced by Kajikov [3] in the study of non-homogeneous incompressible fluids.

From Lemma 4.4 we get:

$$(\rho\theta^2)'-\nabla(\tilde{\theta}^{-1}\nabla\theta^2)-2Bv-\rho^2\tilde{\theta}\operatorname{div}(v)+\nabla.(\rho\theta^2v)=0\,.$$

Let w be in $L^\infty(G)\cap H^1$. Then:

$$((\rho\theta^2)',\ w)+(\tilde{\theta}^{-1}\nabla(\theta^2),\ \nabla w)-2(Bv,\ w)-(\rho^2\tilde{\theta}\operatorname{div}(v),\ w)-(\rho\theta^2v,\ \operatorname{grad}w)=0\,.$$

Let $F(.,\ t)$ be defined by:

$$(F(.,\ t),\ w)=2(Bv,\ w)+(\rho^2\tilde{\theta}\operatorname{div}(v),\ w)+(\rho\theta^2v,\ \operatorname{grad}w)-(\tilde{\theta}^{-1}\nabla(\theta^2),\ \nabla w)\,.$$

Then applying Theorem 2.1 and Lemma 4.4 we obtain:

$$|(F(.,\ t),\ w)|\leqq C\|w\|_{1,2}\{\|\rho\theta^2\|_{L^\infty(0,T^*;L^\infty(G))}\|v\|+\|\rho^2\tilde{\theta}\|_{L^\infty(0,T^*;L^\infty(G))}\|v\|_{1,2}$$

$$+\|\tilde{\theta}^{-1}\theta\|_{L^\infty(0,T^*;L^\infty(G))}\|\nabla\theta\|\}+C\|v\|^2_{1,2}\|w\|_{L^\infty(G)}\,.$$

$$\leqq C\{1+\|v\|^2_{1,2}+\|\tilde{\theta}\|_{L^\infty(0,T^*;L^\infty(G))}\}\|w\|_{1,2}+C\|v\|^2_{1,2}\|w\|_{L^\infty(G)}\,.$$

So:

$$((\rho\theta^2)',\ w)=(F(.,\ t),\ w)\,.$$

Integrating with respect to t from t to $t+\delta$ we get:

$$((\rho\theta^2)(.,t+\delta)-(\rho\theta^2)(.,t),w)=\left(\int_t^{t+\delta}F(.,s)ds,w\right).$$

With $w=\theta(.,t+\delta)-\theta(.,t)$, we have:

$$((\rho\theta^2)(.,t+\delta)-(\rho\theta^2)(.,t),\theta(.,t+\delta)-\theta(.,t))$$
$$=\left(\int_t^{t+\delta}F(.,s)ds,\theta(.,t+\delta)-\theta(.,t)\right). \tag{4.25}$$

Consider the left hand side of (4.25). We may write it as:

$$((\rho\theta^2)(.,t+\delta)-(\rho\theta^2)(.,t),\theta(.,t+\delta)-\theta(.,t))$$
$$=(\rho(.,t+\delta)\{\theta^2(.,t+\delta)-\theta^2(.,t)\},\theta(.,t+\delta)-\theta(.,t))+E(t)$$

(4.26) with $E(t)=((\rho(.,t+\delta)-\rho(.,t))\theta^2(.,t),\theta(.,t+\delta)-\theta(.,t))$.

Therefore from (4.25) we obtain:

$$(\rho(.,t+\delta)\{\theta^2(.,t+\delta)-\theta^2(.,t)\},\theta(.,t+\delta)-\theta(.,t))$$
$$=\left(\int_t^{t+\delta}F(.,s)ds,\theta(.,t+\delta)-\theta(.,t)\right)-E(t). \tag{4.27}$$

2) We consider the expression $E(t)$. Since ρ is a weak solution of (1.4) we have:

$$(\rho',\phi)=(v\rho,\operatorname{grad}\phi) \qquad \text{for } \phi \text{ in } H^1.$$

Thus,

$$(\rho(.,t+\delta)-\rho(.,t),\phi)=\left(\int_t^{t+\delta}v(.,s)\rho(.,s)ds,\operatorname{grad}\phi\right).$$

Since θ is in $L^\infty(0,T^*;L^\infty(G))\cap L^2(0,T^*;H^1)$, $\theta^2(.,t)\{\theta(.,t+\delta)-\theta(.,t)\}$ is in $L^2(0,T^*;H^1)$ and so with $\phi=\theta^2(.,t)\{\theta(.,t+\delta)-\theta(.,t)\}$ we get:

$$E(t)=\left(\int_t^{t+\delta}v(.,s)\rho(.,s)ds,\operatorname{grad}\{\theta^2(.,t)(\theta(.,t+\delta)-\theta(.,t)\}\right).$$

Applying the Sobolev imbedding theorem $H^1\subset L^4(G)$ and the Fubini theorem we obtain:

$$|E(t)|\leqq(b+\eta a)c\int_t^{t+\delta}\|v(.,s)\|_4ds\{\|\theta(.,t)\|_{1,2}\|\theta(.,t)\|_{L^\infty(G)}\|\theta(.,t+\delta)-\theta(.,t)\|_4$$
$$+\|\theta(.,t)\|^2_{L^\infty(G)}\|\theta(.,t+\delta)-\theta(.,t)\|_{1,2}\}.$$

Hence:

$$|E(t)|\leqq C\int_t^{t+\delta}\|v(.,s)\|_{1,2}ds.\ \{\|\theta(.,t)\|_{1,2}+\|\theta(.,t+\delta)\|_{1,2}\}(1+\eta^2)^{1/2}c.$$

C is a constant independent of δ, t, θ, v, ρ.

Applying the Holder inequality we have:

$$|E(t)|\leqq C\delta^{1/2}\|v\|_{L^2(0,T^*;H^1)}\{\|\theta(.,t)\|_{1,2}+\|\theta(.,t+\delta)\|_{1,2}\}.$$

Therefore by taking into account Lemma 4.4 we get:

$$(4.28)\quad \int_0^{T^*-\delta}|E(t)|\,dt\leqq C\delta^{1/2}\|v\|_{L^2(0,T^*;H^1)}\|\theta\|_{L^2(0,T^*;H^1)}$$
$$\leqq C\delta^{1/2}\{\|v\|_{L^2(0,T^*;H^1)}+\|\tilde{\theta}\|^2_{L^\infty(0,T^*;L^\infty(G))}+\|v\|^2_{L^2(0,T^*;H^1)}\}.$$

3) We now consider:

$$\int_0^{T^*-\delta}\left|\left(\int_t^{t+\delta}F(.,s)ds,\ \theta(.,t+\delta)-\theta(.,t)\right)\right|dt.$$

From (4.24) we have:

$$\int_0^{T^*-\delta}\left|\left(\int_t^{t+\delta}F(.,s)ds,\ \theta(.,t+\delta)-\theta(.,t)\right)\right|dt$$
$$\leqq C\int_0^{T^*-\delta}\int_t^{t+\delta}(1+\|v(.,s)\|^2_{1,2}+\|\tilde{\theta}\|^4_{L^\infty(0,T^*;L^\infty(G))})ds$$
$$\{\|\theta(.,t+\delta)\|_{1,2}+\|\theta(.,t)\|_{1,2}+\|\theta(.,t+\delta)_{L^\infty(G)}+\|\theta(.,t)\|_{L^\infty(G)}\}\,dt.$$

Now exactly as in [5] p. 67, we have by taking into account Lemma 4.4 and using a change of order of integration:

$$(4.29)\quad \int_0^{T^*-\delta}\left|\left(\int_t^{t+\delta}F(.,s)ds,\ \theta(.,t+\delta)-\theta(.,t)\right)\right|dt$$
$$\leqq C\delta^{1/2}\{1+\|v\|^4_{L^2(0,T^*;H^1)}+\|\tilde{\theta}\|^4_{L^\infty(0,T^*;L^\infty(G))}\}.$$

It follows from (4.25)-(4.29) that:

$$\int_0^{T^*-\delta}(\rho(.,t+\delta)(\theta^2(.,t+\delta)-\theta^2(.,t)),\ \theta(.,t+\delta)-\theta(.,t))dt$$
$$\leqq C\delta^{1/2}\{1+\|v\|^4_{L^2(0,T^*;H^1)}+\|\tilde{\theta}\|^4_{L^\infty(0,T^*;L^\infty(G))}\}.$$

The different constants C are all independent of δ, v, $\tilde{\theta}$, ρ. But; $\rho\geqq(1-\eta)a$ and $\theta\geqq(1-\eta^2)^{1/2}c$, thus:

$$2(1-\eta)a(1-\eta^2)^{1/2}c\int_0^{T^*-\delta}\|\theta(.,t+\delta)-\theta(.,t)\|^2dt$$
$$\leqq\int_0^{T^*-\delta}(\rho(.,t+\delta)\{\theta^2(.,t+\delta)-\theta^2(.,t)\},\ \theta(.,t+\delta)-\theta(.,t))dt.$$

The theorem is proved.

Section 5. *Successive approximations and proof of the main result.* We shall now carry out the last step of the proof of Theorem 1.1. First we shall construct a sequence of successive approximations to (0.1)-(0.3). Let u_0 be a fixed an arbitrary element of $L^2(0, T; H_0^1)$ and consider the initial-value problem.

$$\text{(5.1)}\quad \begin{cases} \rho_n' + u_{n-1}.\operatorname{grad}\rho_n + \rho_n \operatorname{div}(u_{n-1}) = 0,\ \rho_n > 0 & \text{on } (0, T)\times G, \\ \rho_n(x, 0) = \rho^0(x) \quad \text{on } G,\ n = 1, 2, \cdots \end{cases}$$

together with the initial boundary-value problem:

$$\text{(5.2)}\quad \begin{cases} (\rho_n u_n)' - \Delta u_n - \operatorname{grad}(\operatorname{div} u_n) + \operatorname{grad}(\rho_{n-1} + \theta_{n-1}) + \rho_n u_{n-1}\cdot\nabla u_n \\ \qquad\qquad + u_n(\operatorname{div}(\rho_n u_{n-1})) = \rho_n f \quad \text{on } (0, T)\times G, \\ u_n(x, t) = 0 \quad \text{on } (0, T)\times\partial G, \quad u_n(x, 0) = 0 \quad \text{on } G. \end{cases}$$

and the initial boundary-value problem:

$$\text{(5.3)}\quad \begin{cases} (\rho_n\theta_n^2)' - \nabla(\theta_{n-1}^{-1}\nabla\theta_n^2) - 2Bu_{n-1} - 2\rho_n^2\theta_{n-1}\operatorname{div}(u_{n-1}) + \nabla.(\rho_n\theta_n^2 u_{n-1}) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } (0, T)\times G, \\ \theta_n > 0 \text{ on } (0, T)\times G,\ \dfrac{\partial\theta_n}{\partial\nu} = 0 \text{ on } (0, T)\times\partial G, \quad \theta_n(x, 0) = \theta^0(x) = c \text{ on } G. \end{cases}$$

LEMMA 5.1. *Suppose all the hypotheses of Theorem* 1.1 *are satisfied. Then for each n, there exist*:
(i) *a non-empty interval* $(0, T^*)$ *independent of n*,

(ii) $\{\rho_n, u_n, \theta_n\}$ *in* $L^\infty(0, T^*; L^\infty(G))\times\{L^\infty(0, T^*; H)\cap L^2(0, T^*; H_0^1)\}\times \{L^\infty(0, T^*; L^\infty(G))\cap L^2(0, T^*; H^1)\}$,

solution of (5.1)-(5.3). *Moreover*:

1) $0 < (1-\eta)a \leqq \rho_n(x, t) \leqq b + \eta a \quad$ on $(0, T^*)\times G$,

2) $\|\rho_n'\|_{L^2(0,T^*;(H^1)^*)} + \|(\rho_n^2)'\|_{L^2(0,T^*;(H^1)^*)} \leqq C$,

3) $\|u_n\|_{L^\infty(0,T^*;H)} + \|u_n\|_{L^2(0,T^*;H_0^1)} + \|(\rho_n u_n)'\|_{L^2(0,T^*;H^{-1})} + \|\theta_n\|_{L^2(0,T^*;H^1)} \leqq C$,

4) $0 < (1-\eta^2)^{1/2}c \leqq \theta_n(x, t) \leqq (1+\eta^2)^{1/2}c \quad$ on $(0, T^*)\times G$,

5) $\displaystyle\int_0^{T^*-\delta} \|\theta_n(., t+\delta) - \theta_n(., t)\|^2 dt \leqq C\delta^{1/2} \quad$ for any $\delta > 0$.

C is a constant independent of n and of δ.

Proof. For $n=1$, the lemma follows from Theorems 2.1, 3.1 and 4.1. Suppose the lemma holds for $n-1$ and we shall show for n. With our inductive hypotheses, we have:

$$0<(1-\eta)a\leqq\rho_k\leqq b+\eta a\,;\ 0<(1-\eta^2)^{1/2}c\leqq\theta_k\leqq(1+\eta^2)^{1/2}c \qquad \text{on } (0,\ T^*)\times G\,,$$

(5.4) $$\|u_k(.,\ t)\|^2+\int_0^t\|u_k(.,\ s)\|_{1,2}^2 ds\leqq Ct\{1+\|\theta_{k-1}\|_{L^\infty(0,T_*,L^\infty(G))}^2\}$$
$$\leqq Ct\{1+(1+\eta^2)c^2\}, \quad \text{and}$$

(5.5) $$\|\theta_k(.,\ t)\|_{L^2(0,T^*;H^1)}^2\leqq C\{1+\|u_{k-1}\|_{L^2(0,T^*;H^1)}^2+\|\theta_{k-1}\|_{L^{2\infty}(0,T^*;L^\infty(G))}^2\}$$
$$\leqq C\{1+(1+\eta^2)c^2\}.$$

The interval T^* is such that:

(5.6) $$\int_0^{T^*}\{c^2(1+\eta^2)+\|u_k\|^2\}\,dt\leqq CT^*\{1+2(1+\eta^2)c^2\}\leqq\eta^2c^2\,;\ k\leqq n-1\,.$$

The constant C is independent of $k\leqq n-1$.

2) We now show for n and show that the same T^* as in (5.6) will hold. Applying Theorems 2.1, 3.1 and 4.1 with $v=u_{n-1}$, $\tilde{\theta}=\theta_{n-1}$ and we have from Theorem 2.1:

$$(1-\eta)a\leqq\rho_n(x,\ t)\leqq b+\eta a \qquad \text{on } (0,\ T^*)\times G\,.$$

With Theorem 3.1 we get:

(5.7) $$\|u_n(.,\ t)\|^2+\int_0^t\|u_n(.,\ s)\|_{1,2}^2 ds\leqq Ct\{1+\|\theta_{n-1}\|_{L^\infty(0,T^*,L^\infty(G))}^2\}$$
$$\leqq Ct\{1+(1+\eta^2)c^2\}$$

by the inductive hypothesis.

From Theorem 4.1 we obtain: $(1-\eta^2)^{1/2}c\leqq\theta_n\leqq(1+\eta^2)^{1/2}c$ on $(0,\ T_1)\times G$ with T_1 such that:

$$\int_0^{T_1}\{c^2(1+\eta^2)+\|u_n\|_{1,2}^2\}\,dt\leqq\eta^2c^2/C\,.$$

In view of (5.7), T_1 may be choosen such that

$$CT_1\{1+2(1+\eta^2)c^2\}\leqq\eta^2c^2\,;\ \text{i.e. } T_1=T^* \qquad \text{as in (5.6).}$$

Furthermore:

$$\|\theta_n\|_{L^2(0,T^*;H^1)}^2\leqq C\{1+\|u_{n-1}\|_{L(0,T^*;H^1)}^2+\|\theta_{n-1}\|_{L^\infty(0,T^*;L^\infty(G))}^2\}$$
$$\leqq C\{1+(\eta^2+1)c^2\}$$

by the inductive hypothesis again.

3) It remains to prove the estimates for ρ_n', $(\rho_n^2)'$, $(\rho_n u_n)'$ and for

$$\int_0^{T^*-\delta}\|\theta_n(.,\ t+\delta)-\theta_n(.,\ t)\|^2 dt\,.$$

A proof by induction as above gives the stated result without any difficulty. We shall not reproduce the proof here.

We need some technical convergence lemmas before going into the proof of Theorem 1.1.

LEMMA 5.2. *Let* $\{\rho_n, u_n\}$ *be as in Lemma* 5.1. *Then there exists a subsequence denoted again by* $\{\rho_n, u_n\}$ *such that*:

(i) $\rho_n \to \rho$ *in the weak*-topology of* $L^\infty(0, T^*; L^\infty(G))$ *and in* $L^2(0, T^*; (H^1)^*)$ *with* $0<(1-\eta)a \leqq \rho(x, t) \leqq b+\eta a$ *on* $(0, T^*)\times G$, $\rho_n' \to \rho'$ *and* $(\rho_n^2)' \to (\rho^2)'$ *both weakly in* $L^2(0, T^*; (H^1)^*)$.

(ii) $u_n \to u$ *weakly in* $L^2(0, T^*; H_0^1)$ *and in the weak*-topology of* $L^\infty(0, T^*; H)$,

(iii) $\rho_n u_n \to \rho u$ *weakly in* $L^2(0, T^*; H)$ *and in* $L^2(0, T^*; H^{-1})$,

(iv) $u_n \to u$ *in* $L^2(0, T^*; H)$.

Proof. 1) It is clear that $\rho_n \to \rho$ in the weak*-topology of $L^\infty(0, T^*; L^\infty(G))$ with $0<(1-\eta)a \leqq \rho(x, t) \leqq b+\eta a$ on $(0, T^*)\times G$ and that $\rho_n' \to \rho'$ weakly in $L^2(0, T^*; (H^1)^*)$. Since G is bounded, the natural injection mapping of H^1 into H is compact and hence by Schauder's theorem that of H into $(H^1)^*$ is also compact. It follows from Aubin's theorem [1] and from the estimates of Lemma 5.1 that $\rho_n \to \rho$ in $L^2(0, T^*; (H^1)^*)$. From Lemma 2.2 we know that $\rho_n^2 \to \rho^2$ in the weak*-topology of $L^\infty(0, T^*; L^\infty(G))$ and it is easy to see that $(\rho_n^2)' \to (\rho^2)'$ weakly in $L^2(0, T^*; (H^1)^*)$.

2) The second assertion of the lemma is trivial. It is not difficult to check that indeed $\rho_n u_n \to \rho u$ weakly in $L^2(0, T^*; H)$. On the other hand since $(\rho_n u_n)' \to (\rho u)'$ weakly in $L^2(0, T^*; H^{-2})$, the above argument of the first part yields: $\rho_n u_n \to \rho u$ in $L^2(0, T^*; H^{-1})$.

3) We now prove the key assertion of the lemma, namely that $u_n \to u$ in $L^2(0, T^*; H)$. Indeed we have:

$$(1-\eta)a\|u_n-u\|^2 \leqq (\rho_n(u_n-v),\ u_n-u) = (\rho_n u_n - \rho u,\ u_n-u)$$
$$+(\rho u,\ u_n-u)-(\rho_n u_n,\ u)+(\rho_n u,\ u).$$

Since $\rho_n u_n - \rho u \to 0$ in $L^2(0, T^*; H^{-1})$ and weakly in $L^2(0, T^*; H)$ with $u_n - u \to 0$ weakly in $L^2(0, T^*; H_0^1)$, it is clear that $u_n \to u$ in $L^2(0, T^*; H)$.

LEMMA 5.3. *Let* θ_n *be as in Lemma* 5.1. *Then there exists a subsequence denoted again by* θ_n *such that*:

1) $\theta_n \to \theta$ *in the weak*-topology of* $L^\infty(0, T^*; L^\infty(G))$ *and weakly in* $L^2(0, T^*; H^1)$ *with* $0<(1-\eta^2)^{1/2}c \leqq \theta \leqq (1+\eta^2)^{1/2}c$ *on* $(0, T^*)\times G$,

2) $\theta_n \to \theta$ *in* $L^2(0, T^*; H)$,

3) $\theta_n^2 \to \theta^2$ *in* $L^2(0, T^*; H)$.

Proof. The first assertion is trivial. Since:

$$\|\theta_n\|_{L^2(0,T^*;H^1)} \leqq C;\ \int_0^{T^*-\delta} \|\theta_n(.,\ t+\delta)-\theta_n(.,\ t)\|^2 dt \leqq C\delta^{1/2}$$

with C independent of n and of δ, it follows from Lions [5] p. 68 that $\theta_n \to \theta$

in $L^2(0, T^*; H)$.

We have:

$$\|\theta_n^2-\theta^2\|^2=((\theta_n-\theta)^2, (\theta_n+\theta)^2)\leqq\|\theta_n-\theta\|^2 4c^2(1+\eta^2).$$

Hence: $\theta_n^2\to\theta^2$ in $L^2(0, T^*; H)$.

LEMMA 5.4. *Let $\{u_n, \rho_n, \theta_n\}$ be as in Lemma 5.1. Then by taking subsequences we have:*

1) $\rho_n u_{n-1}\cdot u_n\to\rho u\cdot u$ *in the distribution sense on* $(0, T^*)\times G$,
2) $\rho_n\theta_n^2\to\rho\theta^2$ *in the weak*-topology of* $L^\infty(0, T^*; L^\infty(G))$
3) $\rho_n^2\theta_{n-1}\operatorname{div}(u_{n-1})\to\rho^2\theta\operatorname{div}(u)$ *weakly in* $L^2(0, T^*; H)$,
4) $\rho_n\theta_n^2 u_{n-1}\to\rho\theta^2 u$ *weakly in* $L^2(0, T^*; H)$.

Proof. 1) Let ϕ be a $C_0^\infty(0, T; C_0^\infty(G))$-function, then:

$$\int_0^{T^*}(\rho_n u_{n-1}\cdot u_n, \phi)dt+\int_0^{T^*}(\rho_n u_n, u_{n-1}\phi)dt.$$

It follows from Lemma 5.2 that:

$$\int_0^{T^*}(\rho_n u_n, u_{n-1}\phi)dt\longrightarrow\int_0^{T^*}(\rho u, u\phi)dt$$

i.e. $\rho_n u_{n-1}\cdot u_n\to\rho u\cdot u$ in the distribution sense on $(0, T^*)\times G$.

2) We have:

$$(\rho_n\theta_n^2-\rho\theta^2, \phi)=(\theta_n^2-\theta^2, \rho_n\phi)+(\rho_n-\rho, \theta^2\phi).$$

It is clear that $\theta^2\phi$ is in $L^2(0, T^*; H_0^1)$. It now follows from Lemmas 5.2-5.3 that $\rho_n\theta_n^2\to\rho\theta^2$ in the distribution sense on $(0, T^*)\times G$. On the other hand since $\rho_n\theta_n^2\to g$ in the weak*-topology of $L^\infty(0, T^*; L^\infty(G))$, $g=\rho\theta^2$.

3) We now show that $\rho_n\theta_n^2 u_{n-1}\to\rho\theta^2 u$ weakly in $L^2(0, T^*; H)$. First we note that $\rho_n\theta_n^2 u_{n-1}\to h$ weakly in $L^2(0, T^*; H)$. Let ϕ be as before. Then:

$$(\rho_n\theta_n^2 u_{n-1}, \phi)=(\rho_n u_{n-1}, \theta_n^2\phi).$$

From Lemma 5.2, $\rho_n u_{n-1}\to\rho u$ weakly in $L^2(0, T^*; H)$ and from Lemma 5.3 we have $\theta_n^2\to\theta^2$ in $L^2(0, T^*; H)$. Therefore: $h=\rho\theta^2 u$.

4) It remains to show that $\rho_n^2\theta_{n-1}\operatorname{div}(u_{n-1})\to\rho^2\theta\operatorname{div}(u)$ weakly in $L^2(0, T^*; H)$. Let ϕ be as above and consider

$$(\rho_n^2\operatorname{div}(u_{n-1}), \phi)=-\sum_{j=1}^{3}\{u_{n-1,j}\cdot D_j\rho_n^2, \phi)+(\rho_n^2 u_{n-1,j}, D_j\phi)\}.$$

A rigorous justification of the computations may be done in exactly the same way as in Lemma 2.2 (part 2).

Since $u_{n-1}\to u$ weakly in $L^2(0, T^*; H^1)$ and

$$\|D_j(\rho_n^2)\|_{L^2(0,T^*;H^{-1})}+\|D_j(\rho_n^2)'\|_{L^2(0,T^*;H^{-2})}\leqq C$$

with C independent of n, it follows from the compensated compactness agrument of Murat as applied by Lions in [5] p. 72 relation 1.64 that $u_{n-1,j}\cdot D_j(\rho_n^2)\to u_j\cdot D_j(\rho^2)$ in the distribution sense on $(0, T^*)\times G$. So:

$$(\rho_n^2 \operatorname{div}(u_{n-1}), \phi)\to -\sum_{j=1}^{3}\{(u_j D_j(\rho^2), \phi)+(\rho^2 u_j, D_j\phi)\}=(\rho^2 \operatorname{div} u, \phi)$$

by applying Lemmas 5.2-5.3. On the other hand $\rho_n^2 \operatorname{div}(u_{n-1})\to g$ weakly in $L^2(0, T^*; H)$ and hence $g=\rho^2 \operatorname{div} u$.

Applying now Lemma 5.3 we get:

$$\rho_n^2\theta_{n-1} \operatorname{div}(u_{n-1}) \longrightarrow \rho^2 \operatorname{div}(u)\ \theta \text{ weakly in } L^2(0, T^*; H).$$

LEMMA 5.5. *Let θ_n be as in Lemma 5.1. Then there exists a subsequence such that*

$$\theta_{n-1}^{-1}\nabla(\theta_n^2) \longrightarrow 2\nabla\theta \textit{ weakly in } L^2(0, T^*; H).$$

Proof. We have:

$$\|\theta_{n-1}^{-1}-\theta^{-1}\|\leqq\|(\theta-\theta_{n-1})/\theta\theta_{n-1}\|\leqq(1-\eta^2)^{-1}c^2\|\theta_{n-1}-\theta\|.$$

It follows from Lemma 5.3 that $\theta_{n-1}^{-1}\to\theta^{-1}$ in $L^2(0, T^*; H)$. On the other hand: $\|\theta_n^2\|_{L^2(0,T^*;H^1)}\leqq C$. Since Lemma 5.3 gives: $\theta_n^2\to\theta^2$ in $L^2(0, T^*; H)$, we get: $\nabla(\theta_n^2)\to\nabla(\theta^2)$ weakly in $L^2(0, T^*; H)$. It is now easy to check that $\theta_{n-1}^{-1}\nabla(\theta_n^2)\to\theta^{-1}\nabla(\theta^2)=2\nabla\theta$ weakly in $L^2(0, T^*; H)$.

LEMMA 5.6. *Let u_n be as in Lemma 5.1. Then:*

$$\int_0^{T^*}(Bu_n, \phi)dt \longrightarrow \int_0^{T^*}(Bu, \phi)dt$$

for all ϕ in $C^1(0, T^*; C^1(G))$.

Proof. Let ϕ be a testing function on $(0, T^*)\times G$. We have to show that:

$$\int_0^{T^*}(D_j u_{nk}+D_k u_{nj}, \phi(D_j u_{nk}+D_k u_n))dt \longrightarrow \int_0^{T^*}(D_j u_k+D_k u_j, \phi(D_j u_k+D_k u_j))dt.$$

Thus, it suffices to show that:

$$\int_0^{T^*}(D_j u_n D_k u_n, \phi)dt \longrightarrow \int_0^{T^*}(D_j u D_k u, \phi)dt$$

for all ϕ in $C^1(0, T^*; C^1(G))$.

We have:

$$(D_j u_n D_k u_n, \phi)=-(u_n\phi, D_j D_k u_n)-(u_n, D_k u_n D_j\phi).$$

In view of Lemma 5.2, it is clear that:

$$-\int_0^{T^*}(u_n, D_k u_n D_j\phi)dt \longrightarrow -\int_0^{T^*}(u, D_k u D_j\phi)dt.$$

It remains to show that: $\int_0^{T^*}(u_n D_j D_k u_n, \phi) \longrightarrow \int_0^{T^*}(u D_j D_k u, \phi)dt$.

1) From Lemma 5.2 we know that $u_n - u \to 0$ in $L^2(0, T^*; H)$. By taking subsequences, we have:

$$\|u_n(.,t)-u(.,t)\| \to 0 \text{ a.e. on } (0, T^*) \text{ as } n \to +\infty .$$

Let S be the set of all t on $(0, T^*)$ such that

$$\|u_n(.,t)-u(.,t)\| \to 0 \qquad \text{for all} \quad t \text{ on } S .$$

Then mes$(S)=T^*$. For $\varepsilon>0$ consider the sets:

$$E_n(\varepsilon)=\{t: t \text{ in } S,\ \varepsilon \leqq |(u_n(.,t)-u(.,t), \phi D_j D_k\{u_n(.,t)-u(.,t)\})|\}.$$

Set $S_n(\varepsilon)=\bigcup_{p=n}^{\infty} E_p(\varepsilon)$. Then clearly: $\cdots \subset S_{n+1}(\varepsilon) \subset S_n(\varepsilon) \subset \cdots$ and $\lim_{n\to\infty} \mathrm{mes}(S_n(\varepsilon)) = \lim_{n\to\infty} \mathrm{mes}(\bigcup_{p=n}^{\infty} E_p(\varepsilon)) = \mathrm{mes}(S_*)$ with

$$S_* = \bigcap_{n=1}^{\infty} S_n(\varepsilon) .$$

Suppose that $(u_n(.,t)-u(.,t), \phi D_j D_k\{u_n(.,t)-u(.,t)\})$ does not converge to 0 almost everywhere on S, then for any $\eta>0$ there exists $N(\eta)$ such that mes(S_n) $\geqq \eta$ for all $n \geqq N_\eta$. Hence:

$$0<\eta \leqq \mathrm{mes}(S_*),\ 0<\varepsilon \leqq |(u_n(.,t)-u(.,t), \phi D_j D_k\{u_n(.,t)-u(.,t)\})|$$

on S_* for all $n \geqq N_\eta$.

Since $\|u_n-u\|_{L^2(0,T^*;H^1)} \leqq C$, for almost all t there exists a subsequence (depending on t) such that:

$$\|u_n(.,t)-u(.,t)\|_{1,2} \leqq C(t) .$$

It follows that there exists t_0 in S_* with:

(i) $$\|u_n(.,t_0)-u(.,t_0)\|_{1,2} \leqq C(t_0)$$

(ii) $$\varepsilon \leqq |(u_n(.,t_0)-u(.,t_0), \phi D_j D_k\{u_n(.,t_0)-u(.,t_0)\})| .$$

From the Sobolev imbedding theorem, we have:

$$u_n(.,t_0)-u(.,t_0) \longrightarrow g \text{ in } H \text{ and weakly in } H_0^1 .$$

Since t_0 is in S_*; $g=0$. Thus, $D_j D_k\{u_n(.,t_0)-u(.,t_0)\} \to 0$ weakly in H^{-1}.

From the compensated compactness argument of Murat as applied by Lions in [5] p. 72 relation 1.56 (but now with time independent functions) we obtain

$$(u_n(.,t_0)-u(.,t_0), \phi D_j D_k\{u_n(.,t_0)-u(.,t_0)\}) \longrightarrow 0$$

which is a contradiction.

Hence $(u_n(.,t)-u(.,t),\ \phi D_j D_k\{u_n(.,t)-u(.,t)\}) \longrightarrow 0$

a.e. on $(0, T^*)$ and thus,

$$\int_0^{T^*}(u_n(.,t)-u(.,t),\ \phi D_j D_k\, u_n(.,t)-u(.,t))dt \longrightarrow 0.$$

The lemma is proved.

Proof of Theorem 1.1. Let u_n, ρ_n, θ_n be as in Lemma 5.1. Then from (5.1) and from Lemma 5.2 it is clear that

$$-\int_0^{T^*}(\rho, \phi')dt-\int_0^{T^*}(u \operatorname{grad}\phi, \rho)dt=(\rho^0, \phi(.,0))$$

for all ϕ in $L^2(0, T^*; H^1)$ with ϕ' in $L^2(0, T^*; H)$ and $\phi(., T^*)=0$.

Again from (5.2) and Lemmas 5.2-5.3, we get:

$$-\int_0^{T^*}(\rho u, w')dt+\int_0^{T^*}(\nabla u, \nabla w)dt+\int_0^{T^*}(\operatorname{div} u, \operatorname{div} w)dt$$

$$-\int_0^{T^*}(\rho+\theta, \operatorname{div}(w))-\int_0^{T^*}(\rho u.\nabla w, u)dt=\int_0^{T^*}(\rho f, w)dt$$

for all w in $L^2(0, T^*; H_0^2)$ with w' in $L^2(0, T^*; H)$ and $w(., T^*)=0$.

Finally from (5.3) and Lemmas 5.4-5.6 we obtain:

$$-\int_0^{T^*}(\rho\theta^2, \phi')dt+2\int_0^{T^*}(\nabla\theta, \nabla\phi)dt-2\int_0^{T^*}(Bu, \phi)dt$$

$$-2\int_0^{T^*}(\rho^2 \operatorname{div} u, \phi)dt-\int_0^{T^*}(\rho\theta^2 u, \operatorname{grad}\phi)dt=(\rho^0\theta_0^2, \phi(.,0))$$

for all ϕ in $L^2(0, T^*; H^2)$ with ϕ' in $L^2(0, T^*; H)$ and $\phi(., T^*)=0$.

Hence $\{u, \rho, \theta\}$ is a solution of (0.1)-(0.3) in the sense of Definition 4. The theorem is proved.

References

[1] J. P. Aubin, Un théorème de compacité. CR Acad. Sc. Paris **256** (1963), 5042-5044.

[2] N. Itaya, On the Cauchy problem for the system of fundamental equations describing the movement of compressible fluids. Kōdai Math. Sem. Report **23** (1971), 60-120.

[3] A. V. Kajikov, Resolution of boundary-value problems for non-homogeneous viscous fluids. Dokl. Akad. Nauk **216** (1974), 1008-1010.

[4] J. L. Lions, Quelques methodes de resolution des problemes aux limites non lineaires. Dunod. Paris (1969).

[5] J. L. Lions, On some problems connected with Navier-Stokes equations. Nonlinear evolution equations. M. Grandall editor. Academic Press (1978), 59-84.

[6] A. Matsumura and T. Nishida, The initial-value problem for equations of motion of viscous and heat conductive gases. J. Math. Kyoto University (to

appear).

[7] J. Nash, Le probleme de Cauchy pour les equations differentielles d'un fluide general. Bull. Soc. Math. France **90** (1962), 487-497.

[8] A. Tani, On the first initial boundary value problem of compressible viscous fluid motion: Publ. RIMS. Kyoto Univ. **13** (1977), 193-253.

[9] B. A. Ton, On the initial-value problem for compressible fluid flows with vanishing viscosity. Kodai Math. Journal **3** (1980), 96-113.

[10] B. A. Ton, On the initial-value problem for compressible fluids. Submitted to Publ. RIMS. Kyoto.

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