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CERTAIN INTEGRAL EQUALITY AND INEQUALITY FOR HYPERSURFACES OF $S^n(R)$

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§ 0. Introduction.

We have the following well known isoperimetric inequality for any simply connected domain Ω in the sphere $S^2(R)$ of radius R with smooth boundary $\partial\Omega$:

Let $A=\text{area}(\Omega)$ and $L=\text{length}(\partial\Omega)$, then

$$L^2 \geqq A\left(4\pi - \frac{1}{R^2}A\right)$$

and the equality is true if and only if Ω is a geodesic circular disk.

We can prove this inequality by a method of the integral geometry in which for any integer k and positive real number r, the set of points y of $S^2(R)$ such that the spherical circle with center at y and of radius r intersects $\partial\Omega$ at k points are used effectively. In the present paper, the author will try to get analogous results to this fact in a higher dimensional sphere $S^n(R)$ by means of the same way.

In § 1, we state some preliminary facts. In § 2, we shall obtain an integral equality for oriented hypersurfaces (Theorem 1). Then, in § 3, we shall have an equality on the volumes of a convex domain and the r-neighborhood Ω_r of $\Omega((3\cdot5))$. Finally, in §§ 4 and 5, combining the results in §§ 2 and 3 and using the Fenchel-Borsuk's theorem:

For any closed curve C in a Euclidean space, $\int_C |k(s)|ds \geqq 2\pi$, where $k(s)$ is the curvature of C and s denotes the arclength of C, we shall obtain a kind of isoperimetric inequality for a convex domain in $S^3(R)$ (Theorem 3).

§ 1. Preliminaries.

Let $S^n(R)$ be the standard n-sphere in R^{n+1} of radius R and with its centor at the origin, and Ω a domain in $S^n(R)$ with smooth boundary $\partial\Omega = M^{n-1}$ $(=M)$. Let $\Phi = \{\xi \mid \xi \in T_x S^n(R),\ x \in M,\ |\xi|=1\}$ and $\pi:\ \Phi \rightarrow M$ be the projection of the sphere bundle (Φ, M, π). For a positive real number $r>0$, let ϕ_r be the mapping $\phi_r:\ \Phi \rightarrow S^n(R)$ with $\phi_r(\xi)=\exp_x r\xi$, where $\exp_x$ denotes the exponential mapping of $S^n(R)$ at $x=\pi(\xi)$.

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Let $(x, e_1, \cdots, e_{n-1}, e_n, e_{n+1})$ be a moving orthonormal frame of R^{n+1} such that

$$x\in M,\ e_1,\ e_2,\ \cdots,\ e_{n-1}\in T_x M,\ e_{n+1}=\frac{1}{R}x$$

and the orientation of $(e_1, e_2, \cdots, e_{n-1}, e_n, e_{n+1})$ coincides with the canonical one of R^{n+1}. Then, we have

$$(1.1)\qquad \begin{cases} dx=\sum_{\beta=1}^{n-1}\omega_\beta e_\beta, \\ de_\alpha=\sum_{\beta=1}^{n-1}\omega_{\alpha\beta}e_\beta+\omega_{\alpha n}e_n-\frac{1}{R}\omega_\alpha e_{n+1}, \\ \qquad \alpha=1,2,\cdots,n-1, \\ de_n=-\sum_{\beta=1}^{n-1}\omega_{\beta n}e_\beta,\ de_{n+1}=\frac{1}{R}\sum_{\beta=1}^{n-1}\omega_\beta e_\beta \end{cases}$$

and

$$(1.2)\qquad \begin{cases} \omega_{\alpha\beta}=-\omega_{\beta\alpha}, \\ \omega_{\alpha n}=-\omega_{n\alpha}=\sum_{\beta=1}^{n-1}A_{\alpha\beta}\omega_\beta, \end{cases}$$

where

$$(1.3)\qquad A_{\alpha\beta}=A_{\beta\alpha},\quad \alpha,\beta=1,2,\cdots,n-1,$$

are the components of the 2nd fundamental form of M in $S^n(R)$ for the unit normal vector e_n with respect to $(x, e_1, e_2, \cdots, e_{n-1})$.

Setting $y=\phi_r(\xi)$, $\xi\in\Phi$, $\pi(\xi)=x$ and $\xi=\sum_{i=1}^{n}\xi_i e_i$, we have easily

$$(1.4)\qquad y=R\left(e_{n+1}\cos\frac{r}{R}+\xi\sin\frac{r}{R}\right)$$

and

$$(1.5)\qquad dy=\cos\frac{r}{R}\sum_{\alpha=1}^{n-1}\omega_\alpha e_\alpha+R\sin\frac{r}{R}\sum_{i=1}^{n}D\xi_i e_i-\sin\frac{r}{R}\sum_{\alpha=1}^{n-1}\xi_\alpha\omega_\alpha e_{n+1}$$

$$=\sum_{\alpha=1}^{n-1}\omega_\alpha\left(\cos\frac{r}{R}e_\alpha-\xi_\alpha\sin\frac{r}{R}e_{n+1}\right)+R\sin\frac{r}{R}\sum_{i=1}^{n}D\xi_i e_i,$$

where D denotes the covariant differentiation $S^n(R)$ with respect to its Riemannian connection.

Now, when $\cos\frac{r}{R}\neq 0$ and $\sin\frac{r}{R}\neq 0$, noticing that $\cos\frac{r}{R}e_\alpha-\xi_\alpha\sin\frac{r}{R}e_{n+1}$, $\alpha=1,2,\cdots,n-1$, and $\sum_i D\xi_i e_i$ are all orthogonal to $e_{n+1}\cos\frac{r}{R}+\xi\sin\frac{r}{R}$, we obtain by a straightforward calculation

$$\left(\cos\frac{r}{R}e_1-\xi_1\sin\frac{r}{R}e_{n+1}\right)\wedge\left(\cos\frac{r}{R}e_2-\xi_2\sin\frac{r}{R}e_{n+1}\right)\wedge\cdots$$

$$\wedge\left(\cos\frac{r}{R}e_{n-1}-\xi_{n-1}\sin\frac{r}{R}e_{n+1}\right)\wedge\sum_i D\xi_i e_i\wedge\left(\cos\frac{r}{R}e_{n+1}+\sin\frac{r}{R}\xi\right)$$

$$=\left(\cos\frac{r}{R}\right)^{n-2}D\xi_n(e_1\wedge e_2\wedge\cdots\wedge e_{n+1}).$$

We denote the volume element of $S^n(R)$ by dV_S. Then from the above equalities we have

$$\psi_r^* dV_S=R\sin\frac{r}{R}\left(\cos\frac{r}{R}\right)^{n-2}\omega_1\wedge\omega_2\wedge\cdots\wedge\omega_{n-1}\wedge D\xi_n, \tag{1.6}$$

in which we may replace $D\xi_n$ by $d\xi_n$.

Then, when $\cos\frac{r}{R}=0$, (1.4) and (1.5) turn into

$$y=\varepsilon R\xi,\quad \varepsilon=\sin\frac{r}{R}=\pm 1 \tag{1.4$_0$}$$

and

$$dy=\varepsilon\left\{R\sum_{i=1}^{n}D\xi_i e_i-\sum_{\alpha=1}^{n-1}\xi_\alpha\omega_\alpha e_{n+1}\right\}. \tag{1.5$_0$}$$

For $\xi\in\Phi$ with $\xi_n\neq 0$, substituting $D\xi_n=-\frac{1}{\xi_n}\sum_\alpha\xi_\alpha D\xi_\alpha$ into the above equality, we get

$$dy=\varepsilon\left\{R\sum_\alpha D\xi_\alpha\left(e_\alpha-\frac{1}{\xi_n}\xi_\alpha e_n\right)-\sum_\alpha\xi_\alpha\omega_\alpha e_{n+1}\right\}. \tag{1.5$_0'$}$$

Noticing $e_\alpha-\frac{1}{\xi_n}\xi_\alpha e_n$, $\alpha=1, 2, \cdots, n-1$, and e_{n+1} are all orthogonal to ξ, we obtain

$$\left(e_1-\frac{1}{\xi_n}\xi_1 e_n\right)\wedge\cdots\wedge\left(e_{n-1}-\frac{1}{\xi_n}\xi_{n-1}e_n\right)\wedge e_{n+1}\wedge\xi$$

$$=-\frac{1}{\xi_n}e_1\wedge\cdots\wedge e_n\wedge e_{n+1}.$$

Hence we have in this case

$$\psi_r^* dV_S=\varepsilon^{n+1}\frac{1}{\xi_n}R^{n-1}D\xi_1\wedge\cdots\wedge D\xi_{n-1}\wedge\sum_\alpha\xi_\alpha\omega_\alpha,\quad \varepsilon=\sin\frac{r}{R}. \tag{1.6$_0$}$$

The induced Riemannian metric on M from R^{n+1} is written as

$$ds_M{}^2 = \sum_{\alpha=1}^{n-1} \omega_\alpha \omega_\alpha \tag{1.7}$$

and we define a natural Riemannian metric on Φ by

$$ds_\Phi{}^2 := \sum_{\alpha=1}^{n-1} \omega_\alpha \omega_\alpha + \sum_{i=1}^{n} D\xi_i D\xi_i. \tag{1.8}$$

Then, their volume elements dV_M and dV_Φ are clearly given by

$$dV_M = \omega_1 \wedge \cdots \wedge \omega_{n-1} \tag{1.9}$$

and

$$dV_\Phi = \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge \sum_{i=1}^{n} (-1)^{n-i} \xi_i D\xi_1 \wedge \cdots \wedge \widehat{D\xi_i} \wedge \cdots \wedge D\xi_n. \tag{1.10}$$

We can easily prove that the following is a differential $(n-1)$-form on Φ:

$$d\mu_{n-1} := \sum_{i=1}^{n} (-1)^{n-i} \xi_i D\xi_1 \wedge \cdots \wedge \widehat{D\xi_i} \wedge \cdots \wedge D\xi_n, \tag{1.11}$$

whose restriction on the unit $(n-1)$-sphere $\pi^{-1}(x)$ of $T_x S^n(R)$, $x \in M$, is its volume element. We have

$$dV_\Phi = dV_M \wedge d\mu_{n-1}.$$

We can also easily prove that

$$d\mu_{n-1} = (-1)^{n-j} \frac{1}{\xi_j} D\xi_1 \wedge \cdots \wedge D\xi_{j-1} \wedge D\xi_{j+1} \wedge \cdots \wedge D\xi_n \tag{1.12}$$

at $\xi \in \Phi$ with $\xi_j \neq 0$, by using $\sum_i \xi_i \xi_i = 1$ and $\sum_i \xi_i D\xi_i = 0$, and so especially

$$dV_\Phi = \frac{1}{\xi_n} \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge D\xi_1 \wedge \cdots \wedge D\xi_{n-1} \tag{1.13}$$

at ξ with $\xi_n \neq 0$.

§ 2. An integral equality for hypersurfaces $S^n(R)$.

In the following, we suppose that $0 < r < \pi R$. For any point $y \in S^n(R)$, we denote the $(n-1)$-sphere on $S^n(R)$ of geodesic radius $r (0 < r < \frac{\pi}{2} R)$ or $\pi R - r (\frac{\pi}{2} R \leqq r < \pi R)$ with its centor at y or $-y$, by $F_r^{n-1}(y)$. We can easily see that

$$F_r^{n-1}(y) = \{\exp_y v \mid v \in T_y S^n(R),\ |v| = r\}$$

and

$$\psi_r^{-1}(y)=\{\text{tangent unit vectors } \xi \text{ at } x\in F_r^{n-1}(y)\cap M$$
$$\text{such that } y=\exp_x r\xi\}.$$

Now, when $\cos\frac{r}{R}\neq 0$, from (1.5) we have

$$(2.1)\qquad \begin{cases} D\xi_\alpha=-\dfrac{1}{R}\cot\dfrac{r}{R}\omega_\alpha, \quad \alpha=1, 2, \cdots, n-1, \\ D\xi_n=0, \\ \sum\limits_{\alpha=1}^{n-1}\xi_\alpha\omega_\alpha=0 \qquad \text{along } \psi_r^{-1}(y). \end{cases}$$

Hence, the induced Riemannian metric on $\psi_r^{-1}(y)$ from (1.8) on Φ can be written as

$$ds^2=\left(1+\frac{1}{R^2}\cot^2\frac{r}{R}\right)\sum_{\alpha=1}^{n-1}\omega_\alpha\omega_\alpha,$$

which implies the following equality

$$(2.2)\qquad \operatorname{vol}(\psi_r^{-1}(y))=\left(1+\frac{1}{R^2}\cot^2\frac{r}{R}\right)^{n/2-1}\operatorname{vol}(F_r^{n-1}(y)\cap M).$$

On the other hand, we consider a differential $(n-2)$-form in Φ of the from as

$$\Theta_{n-2}=\sum_{\alpha=1}^{n-1}(-1)^{\alpha-1}\lambda_\alpha D\xi_1\wedge\cdots\wedge\widehat{D\xi_\alpha}\wedge\cdots\wedge D\xi_{n-1},$$

where λ_α will be determined so that

$$(2.3)\qquad dV_\Phi=\psi_r^*(dV_S)\wedge\Theta_{n-2}$$

By means of (1.6), where $\xi_n\neq 0$, the right-hand side of this equality becomes

$$-R\sin\frac{r}{R}\left(\cos\frac{r}{R}\right)^{n-2}\frac{1}{\xi_n}\omega_1\wedge\cdots\wedge\omega_{n-1}\wedge\sum_{\alpha=1}^{n-1}\xi_\alpha D\xi_\alpha$$

$$\wedge\sum_{\beta=1}^{n-1}(-1)^{\beta-1}\lambda_\beta D\xi_1\wedge\cdots\wedge\widehat{D\xi_\beta}\wedge\cdots\wedge D\xi_{n-1}$$

$$=-R\sin\frac{r}{R}\left(\cos\frac{r}{R}\right)^{n-2}\frac{1}{\xi_n}\sum_{\alpha=1}^{n-1}\xi_\alpha\lambda_\alpha\omega_1\wedge\cdots\wedge\omega_{n-1}.$$

Comparing this with (1.13), we see that it is sufficient to take λ_α as

$$\lambda_\alpha = \frac{-\xi_\alpha}{R\sin\frac{r}{R}\left(\cos\frac{r}{R}\right)^{n-2}(1-\xi_n\xi_n)},$$

where $\xi_n\xi_n \neq 1$.

Thus we define Θ_{n-2} by

$$\Theta_{n=2} := -\frac{1}{R\sin\frac{r}{R}\left(\cos\frac{r}{R}\right)^{n-2}}\cdot\frac{1}{1-\xi_n\xi_n}\sum_{\alpha=1}^{n-1}(-1)^{\alpha-1}\xi_\alpha D\xi_1\wedge\cdots\wedge\widehat{D\xi_\alpha}\wedge\cdots\wedge D\xi_{n-1}, \tag{2.4}$$

where $\xi_n\xi_n < 1$.

Let ι_y: $\psi_r^{-1}(y)\to\Phi$ be the inclusion map. Then, by (2.1) we have

$$\iota_y^*\Theta_{n-2} = -\frac{1}{\left(R\sin\frac{r}{R}\right)^{n-1}}\cdot\frac{1}{1-\xi_n\xi_n}\sum_{\alpha=1}^{n-1}(-1)^{n-1-\alpha}\xi_\alpha\omega_1\wedge\cdots\wedge\widehat{\omega_\alpha}\wedge\cdots\wedge\omega_{n-1} \tag{2.5}$$

and especially

$$\iota_y^*\Theta_{n-2} = -\frac{1}{\left(R\sin\frac{r}{R}\right)^{n-1}}\frac{1}{\xi_{n-1}}\omega_1\wedge\omega_2\cdots\wedge\omega_{n-2}, \tag{2.5$'$}$$

where $\xi_{n-1}\neq 0$.

Next, we observe the volume element of $F_r^{n-1}(y)\cap M$. On $F_r^{n-1}(y)\cap M$, we obtain from (2.1)

$$ds^2 = \sum_{\alpha=1}^{n-1}\omega_\alpha\omega_\alpha = \sum_{a=1}^{n-2}\omega_a\omega_a + \left(-\frac{1}{\xi_{n-1}}\sum_{a=1}^{n-1}\xi_a\omega_a\right)^2$$

$$= \sum_{a,b=1}^{n-2}\left(\delta_{ab} + \frac{1}{\xi_{n-1}\xi_{n-1}}\xi_a\xi_b\right)\omega_a\omega_b$$

and

$$\det\left(\delta_{ab} + \frac{1}{\xi_{n-1}\xi_{n-1}}\xi_a\xi_b\right) = \frac{1}{\xi_{n-1}\xi_{n-1}}\sum_{\alpha=1}^{n-1}\xi_\alpha\xi_\alpha = \frac{1-\xi_n\xi_n}{\xi_{n-1}\xi_{n-1}},$$

where $\xi_{n-1}\neq 0$. Hence, the volume element of $F_r^{n-1}(y)\cap M$ is given by

$$dV_{F_r^{n-1}(y)\cap M} = \frac{\sqrt{1-\xi_n\xi_n}}{\xi_{n-1}}\omega_1\wedge\cdots\wedge\omega_{n-2}, \tag{2.6}$$

where $\xi_{n-1}\neq 0$. In general, we have

$$(2.6') \qquad dV_{F_r^{n-1}(y)\cap M}=(-1)^{n-1-\beta}\frac{\sqrt{1-\xi_n\xi_n}}{\xi_\beta}\omega_1\wedge\cdots\wedge\omega_{\beta-1}\wedge\omega_{\beta+1}\wedge\cdots\wedge\omega_{n-1},$$

where $\xi_\beta\neq 0$, and

$$(2.6'') \qquad dV_{F_r^{n-1}(y)\cap M}=\frac{1}{\sqrt{1-\xi_n\xi_n}}\sum_{\alpha=1}^{n-1}(-1)^{n-1-\alpha}\xi_\alpha\omega_1\wedge\cdots\wedge\omega_{\alpha-1}\wedge\omega_{\alpha+1}\wedge\cdots\wedge\omega_{n-1}.$$

Since we have

$$(2.7) \qquad dV_{\phi_r^{-1}(y)}=\left(1+\frac{1}{R^2}\cot^2\frac{r}{R}\right)^{n/2-1}dV_{F_r^{n-1}(y)\cap M},$$

hence

$$(2.8) \qquad dV_{\phi_r^{-1}(y)}=\left(1+\frac{1}{R^2}\cot^2\frac{r}{R}\right)^{n/2-1}\frac{1}{\sqrt{1-\xi_n\xi_n}}\sum_{\alpha=1}^{n-1}(-1)^{n-1-\alpha}\xi_\alpha\omega_1\wedge\cdots\wedge\omega_{\alpha-1}\wedge\omega_{\alpha+1}\wedge\cdots\wedge\omega_{n-1}$$

and

$$(2.8') \qquad dV_{\phi_r^{-1}(y)}=\left(1+\frac{1}{R^2}\cot^2\frac{r}{R}\right)^{n/2-1}\frac{\sqrt{1-\xi_n\xi_n}}{\xi_{n-1}}\omega_1\wedge\cdots\wedge\omega_{n-2},$$

where $\xi_{n-1}\neq 0$. From (2.5) and (2.8), we obtain

$$(2.9) \qquad \iota_y^*\Theta_{n-2}=-\frac{1}{R\sin\frac{r}{R}\left(\cos\frac{r}{R}\right)^{n-2}\left(1+R^2\tan^2\frac{r}{R}\right)^{n/2-1}}\cdot\frac{1}{\sqrt{1-\xi_n\xi_n}}dV_{\phi_r^{-1}(y)}$$

Finally, we consider the case $\cos\frac{r}{R}=0$, i.e. $r=\frac{\pi R}{2}$. From (1.5_0) we have

$$(2.1_0) \qquad \begin{cases} D\xi_i=0, \quad i=1,2,\cdots,n, \\ \sum_{\alpha=1}^{n-1}\xi_\alpha\omega_\alpha=0 \qquad \text{along } \phi_r^{-1}(y). \end{cases}$$

We take a differential $(n-2)$-form in Φ of the form

$$\Psi_{n-2}=\sum_{\alpha=1}^{n-1}(-1)^{\alpha-1}\lambda_\alpha\omega_1\wedge\cdots\wedge\omega_{\alpha-1}\wedge\omega_{\alpha+1}\wedge\cdots\wedge\omega_{n-1},$$

where λ_α will be determined so that

$$(2.3_0) \qquad dV_\Phi=\phi_r^*(dV_S)\wedge\Psi_{n-2}.$$

By means of (1.6_0), at ξ with $\xi_n \neq 0$, the right-hand side of this equality becomes

$$\frac{1}{\xi_n} R^{n-1} D\xi_1 \wedge \cdots \wedge D\xi_{n-1} \wedge \sum_{\alpha=1}^{n-1} \xi_\alpha \omega_\alpha \wedge \sum_{\beta=1}^{n-1} (-1)^{\beta-1} \lambda_\beta \omega_1 \wedge \cdots$$

$$\wedge \omega_{\beta-1} \wedge \omega_{\beta+1} \wedge \cdots \wedge \omega_{n-1}$$

$$= (-1)^{n-1} R^{n-1} \frac{1}{\xi_n} \sum_{\alpha=1}^{n-1} \xi_\alpha \lambda_\alpha \omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge D\xi_1 \wedge \cdots \wedge D\xi_{n-1}$$

Comparing this equality with (1.13), we see that it is sufficient to take λ_α as

$$\lambda_\alpha = \frac{(-1)^{n-1} \xi_\alpha}{R^{n-1}(1-\xi_n \xi_n)},$$

when $\xi_n \xi_n < 1$. Thus, we define Ψ_{n-2} by

$$\Psi_{n-2} := -\frac{1}{R^{n-1}(1-\xi_n \xi_n)} \sum_{\alpha=1}^{n-1} (-1)^{n-1-\alpha} \xi_\alpha \omega_1 \wedge \cdots \tag{2.10}$$

$$\wedge \omega_{\alpha-1} \wedge \omega_{\alpha+1} \wedge \cdots \wedge \omega_{n-1},$$

where $\xi_n \xi_n < 1$. In this case, from (2.1_0) we see that

$$dV_{\phi_r^{-1}(y)} = dV_{F_r^{n-1}(y) \cap M} \tag{2.7$_0$}$$

through an isometry. We can also use the formula (2.6″) and obtain

$$\iota_y^* \Psi_{n-2} = -\frac{1}{R^{n-1}\sqrt{1-\xi_n \xi_n}} dV_{\phi_r^{-1}(y)}. \tag{2.9$_0$}$$

Making use of these formulas and noticing that these hold good for oriented hypersurfaces in $S^n(R)$ in general, we obtain the following

THEOREM 1. *Let $M^{n-1} = M \subset S^n(R)$ be a smooth oriented hypersurface and $0 < r < \pi R$. Then, we have the following integral equality:*

$$\int_{S^n(R)} \left(\int_{F_r^{n-1}(y) \cap M} \frac{1}{\sqrt{1-\xi_n \xi_n}} dV_{F_r^{n-1}(y) \cap M} \right) dV_S \tag{2.11}$$

$$= \left(R \sin \frac{r}{R} \right)^{n-1} \cdot c_{n-1} \operatorname{vol}(M),$$

where c_{n-1} is the volume of the unit $(n-1)$-sphere.

Proof. From (1.10), we see that

$$\int_\Phi dV_\Phi = \operatorname{vol}(\Phi) = c_{n-1} \cdot \operatorname{vol}(M).$$

We prove the case $r\neq\pi R/2$. By means of (2.3), (2.8) and (2.7), the left-hand side of the above equality can be also computed as follows:

$$\int_{\Phi} dV_{\Phi}=\int_{S^n(R)} \frac{dV_S}{R\sin\frac{r}{R}\left|\cos\frac{r}{R}\right|^{n-2}\left(1+R^2\tan^2\frac{r}{R}\right)^{n/2-1}} \int_{\varphi_r^{-1}(y)} \cdot$$

$$\frac{1}{\sqrt{1-\xi_n\xi_n}}\, dV_{\varphi_r^{-1}(y)}$$

$$=\frac{1}{\left(R\sin\frac{r}{R}\right)^{n-1}}\int_{S^n(R)}\left(\int_{F_r^{n-1}(y)\cap M}\frac{1}{\sqrt{1-\xi_n\xi_n}}\, dV_{F_r^{n-1}(y)\cap M}\right) dV_S,$$

from which we obtain immediately (2.11).

For the case $r=\pi R/2$, we can prove it analogously by (2.3_0) and (2.9_0).

q. e. d.

§ 3. An integral equality for a convex domain in $S^3(R)$.

We say a domain Ω in $S^n(R)$ *convex* if Ω contains no pair of points y and $-y$ of $S^n(R)$ and for any two points p and q of Ω it contains the minimum geodesic segment of $S^n(R)$ joining p and q.

If $\Omega\subset S^n(R)$ is convex, it must be contained in a half n-sphere of $S^n(R)$. We see this fact easily by considering a contacting great $(n-1)$-sphere of $S^n(R)$ to $\partial\Omega$. Hence we have

$$V=\mathrm{vol}\,(\Omega)\leqq\frac{c_n}{2}R^n. \tag{3.1}$$

In the following, we suppose that Ω is convex and has smooth boundary $M=\partial\Omega$. For $r>0$, we set

$$\Omega_r=\{x\,|\,x\in S^n(R),\ \mathrm{dis}_{S^n(R)}(x,\ \Omega)<r\},\quad V_r=\mathrm{vol}\,(\Omega_r). \tag{3.2}$$

In this case, M must be diffeomorphic to an $(n-1)$-sphere. Furthermore, we suppose $0<r\leqq\pi R/2$. Using the notation in §§ 1, 2 and taking notice of that for the orthonormal frame $(x, e_1, \cdots, e_n, e_{n+1})$, $x\in M$, e_n directs inward of Ω at x and $e_{n+1}=(1/R)x$, we see that $\Omega_r-\Omega$ is the set of points y written as

$$y=R\left(e_{n+1}\cos\frac{t}{R}-e_n\sin\frac{t}{R}\right),\quad 0\leqq t<r. \tag{3.3}$$

Hence, we have

$$dy=\sum_{\alpha=1}^{n-1}\left\{e_\alpha\cos\frac{t}{R}+R\left(\sum_{\beta=1}^{n-1}A_{\alpha\beta}\right)\sin\frac{t}{R}\right\}\omega_\alpha$$

$$-\left(e_{n+1}\sin\frac{t}{R}+e_n\cos\frac{t}{R}\right)dt.$$

If we choose especially $e_1, \cdots, e_{n-1}$ in the principal directions of M at x, then we can put $A_{\alpha\beta}=k_\alpha\delta_{\alpha\beta}$. Denoting the normal exponential map of M in $S^n(R)$ by $\exp^\perp$, we induce a volume element of the normal bundle NM from dV_S through $\exp^\perp$. From the above computation, we have

$$(\exp^\perp)^*_{(x,-t)}dV_s=-\prod_{\alpha=1}^{n-1}\left(\cos\frac{t}{R}+k_\alpha R\sin\frac{t}{R}\right)\omega_1\wedge\cdots\wedge\omega_{n-1}\wedge dt,$$

hence

$$V_r-V=\int_0^r\int_M\prod_{\alpha=1}^{n-1}\left(\cos\frac{t}{R}+k_\alpha R\sin\frac{t}{R}\right)dV_M\,dt$$

$$=\int_0^r\int_M\left\{\left(\cos\frac{t}{R}\right)^{n-1}+\sum_{m=1}^{n-1}R^m\sigma_m(k_1,\cdots,k_{n-1})\right.$$

$$\left.\cdot\left(\cos\frac{t}{R}\right)^{n-m-1}\left(\sin\frac{t}{R}\right)^m\right\}dV_M\,dt,$$

where $\sigma_m(u_1, \cdots, u_{n-1})$ denotes the fundamental symmetric polynomial of order m in $u_1, u_2\cdots, u_{n-1}$. Thus, we have

$$V_r=V+\int_0^r\left(\cos\frac{t}{R}\right)^{n-1}dt\cdot\int_M dV_M \tag{3.4}$$

$$+\sum_{m=1}^{n-1}R^m\int_0^r\left(\cos\frac{t}{R}\right)^{n-m-1}\left(\sin\frac{t}{R}\right)^m dt\cdot\int_M\sigma_m(k_1,\cdots,k_{n-1})\,dV_M.$$

For the 2nd fundamental form $II=\sum_{\alpha,\beta}A_{\alpha\beta}\omega_\alpha\omega_\beta$, we set

$$\det(I_{n-1}+uA)=1+\sum_{m=1}^{n-1}\binom{n-1}{m}u^m P_m(A),$$

where $A=(A_{\alpha\beta})$. Especially, we have

$$P_1(A)=\frac{1}{n-1}\sum_{\alpha=1}^{n-1}k_\alpha=H \quad \text{(mean curvature)}.$$

Using these $P_m(A)$, we can rewrite (3.4) as

$$V_r = V + \int_0^r \left(\cos\frac{t}{R}\right)^{n-1} dt \cdot \int_M dV_M + \sum_{m=1}^{n-1} \binom{n-1}{m} R^m \int_0^r \left(\cos\frac{t}{R}\right)^{n-m-1} \left(\sin\frac{t}{R}\right)^m dt \cdot \int_M P_m(A)\, dV_M. \tag{3.5}$$

Now, we compute the right-hand side of (3.4) in more exact form for the case $n=3$. Since we have

$$\int_0^r \cos^2\frac{t}{R}\, dt = \frac{1}{2}\left(R\cos\frac{r}{R}\sin\frac{r}{R} + r\right), \quad \int_0^r \cos\frac{t}{R}\sin\frac{t}{R}\, dt = \frac{R}{2}\sin^2\frac{r}{R},$$

$$\int_0^r \sin^2\frac{t}{R}\, dt = \frac{1}{2}\left(-R\cos\frac{r}{R}\sin\frac{r}{R} + r\right),$$

(3.4) becomes in this case

$$V_r = V + \frac{1}{2}\left(r + R\cos\frac{r}{R}\sin\frac{r}{R}\right)\int_M dV_M + R^2\sin^2\frac{r}{R}\int_M H\, dV_M + \frac{1}{2}R^2\left(r - R\cos\frac{r}{R}\sin\frac{r}{R}\right)\int_M k_1 k_2\, dV_M.$$

On the other hand, denoting the Gaussian curvature of M by K, we have easily

$$K = k_1 k_2 + \frac{1}{R^2}.$$

Hence, by means of the Gauss-Bonnet theorem we obtain

$$\int_M k_1 k_2\, dV_M = \int_M K\, dV_M - \frac{1}{R^2}\int_M dV_M = 2\pi\cdot\chi(M) - \frac{1}{R^2}\int_M dV_M = 4\pi - \frac{1}{R^2}\int_M dV_M,$$

since M is homeomorphic to S^2. Substituting this into the above equality, we obtain

$$V_r = V + R\cos\frac{r}{R}\sin\frac{r}{R}\int_M dV_M + 2\pi R^2\left(r - R\cos\frac{r}{R}\sin\frac{r}{R}\right) + R^2\sin^2\frac{r}{R}\int_M H\, dV_M. \tag{3.6}$$

§ 4. An isoperimetric inequality for a convex domain in $S^3(R)$.

First of all, we investigate the integral in (2.11):

$$\int_{F_r^{n-1}(y)\cap M} \frac{1}{\sqrt{1-\xi_n\xi_n}}\, dV_{F_r^{n-1}(y)\cap M}. \tag{4.1}$$

For any point $x\in F_r^{n-1}(y)\cap M$ and a frame $(x, e_1, e_2, \cdots, e_n, e_{n+1})$ as in § 1, we have

$$\xi=\frac{1}{R\sin\frac{r}{R}}\left(y-x\cos\frac{r}{R}\right), \tag{4.2}$$

$$\xi_i=\langle\xi, e_i\rangle=\frac{y_i}{R\sin\frac{r}{R}}, \quad y_i=\langle y, e_i\rangle, \quad i=1, 2, \cdots, n \tag{4.3}$$

and

$$y_{n+1}=\langle y, e_{n+1}\rangle=R\cos\frac{r}{R}. \tag{4.4}$$

Along $F_r^{n-1}(y)\cap M$, $\langle y, x\rangle=R^2\cos\frac{r}{R}$ implies

$$\langle y, \sum_\alpha \omega_\alpha e_\alpha\rangle=\sum_\alpha y_\alpha\omega_\alpha=0.$$

On the other hand, restricting the moving frame $(x, e_1, e_2, \cdots, e_{n+1})$, $x\in F_r^{n-1}(y)\cap M$, to the one such that $e_1, \cdots, e_{n-2}\in T_x(F_r^{n-1}(y)\cap M)$, we have

$$\omega_{n-1}=0, \tag{4.5}$$

and hence

$$y_1=y_2=\cdots=y_{n-2}=0 \tag{4.6}$$

and

$$y=y_{n-1}e_{n-1}+y_n e_n+R\cos\frac{r}{R}e_{n+1}.$$

Using these relations, $dy=0$ implies

$$dy_{n-1}=y_n\omega_{n-1,n}, \quad dy_n=-y_{n-1}\omega_{n-1,n}, \tag{4.7}$$

$$y_{n-1}\omega_{a,n-1}+y_n\omega_{an}=\cos\frac{r}{R}\cdot\omega_a, \quad a=1, \cdots, n-2. \tag{4.8}$$

From (4.5) and the structure equation we obtain

$$\sum_{a=1}^{n-2} \omega_{n-1,a} \wedge \omega_a = 0,$$

hence we can put

$$\omega_{a,n-1} = \sum_{b=1}^{n-2} B_{ab} \omega_b, \quad B_{ab} = B_{ba}. \tag{4.9}$$

B_{ab} are the components of the 2nd fundamental form of $F_r^{n-1}(y) \cap M$ with respect to the normal unit vector e_{n-1}. By (4.4) and (4.7), we can put

$$y_{n-1} = R \sin\frac{r}{R} \cos\theta, \qquad y_n = R \sin\frac{r}{R} \sin\theta. \tag{4.10}$$

Substituting these into (4.7) and (4.8), we get

$$\omega_{n-1,n} = -d\theta, \tag{4.11}$$

$$\cos\theta \cdot B_{ab} + \sin\theta \cdot A_{ab} = \frac{1}{R} \cot\frac{r}{R} \cdot \delta_{ab}, \tag{4.12}$$

$$a,\ b = 1,\ 2,\ \cdots,\ n-2.$$

From the equality:

$$\omega_{n-1,n} = \sum_{a=1}^{n-2} A_{n-1,a}\, \omega_a \qquad \text{along} \quad F_r^{n-1}(y) \cap M \tag{4.13}$$

and (4.11), we can put

$$A_{n-1,a} = -\nabla_{e_a} \theta, \tag{4.14}$$

where ∇ denotes the covariant derivation of $F_r^{n-1}(y) \cap M$.

Now, we suppose $n=3$ in the following. Then, $F_r^2(y) \cap M$ is composed of curves in general. Setting $\omega_1 = ds$, (4.11) and (4.13) imply

$$\theta = -\int A_{12}\, ds + \text{const.}. \tag{4.15}$$

We have also

$$\frac{dV_{F_r^2(y)\cap M}}{\sqrt{1-\xi_3\xi_3}} = \frac{ds}{\sqrt{1-\sin^2\theta}} = \frac{ds}{\cos\theta} = -\frac{d\theta}{A_{12}\cos\theta}. \tag{4.16}$$

In this case, (4.12) becomes

$$B_{11}\cos\theta + A_{11}\sin\theta = \frac{1}{R}\cot\frac{r}{R},$$

from which we get

$$\cos\theta = \frac{1}{A_{11}{}^2 + B_{11}{}^2} \left\{ \frac{B_{11}}{R} \cot\frac{r}{R} + A_{11}\sqrt{A_{11}{}^2 + B_{11}{}^2 - \frac{1}{R^2}\cot^2\frac{r}{R}} \right\}.$$

Along the curve $F_r^2(y)\cap M$, we have

$$\frac{de_1}{ds}=B_{11}e_2+A_{11}e_3-\frac{1}{R}e_4$$

and hence its curvature as a curve in R^4 is

$$k(s)=\left|\frac{de_1}{ds}\right|=\sqrt{B_{11}{}^2+A_{11}{}^2+\frac{1}{R^2}}\,. \tag{4.17}$$

Using $k(s)$, the right-hand side of the above expression of $\cos\theta$ can be written as

$$\cos\theta=\frac{1}{A_{11}{}^2+B_{11}{}^2}\cdot\frac{1}{R\sin\frac{r}{R}}\left\{B_{11}\cos\frac{r}{R}+A_{11}\sqrt{k^2R^2\sin^2\frac{r}{R}-1}\right\}. \tag{4.18}$$

Then, we have the following theorem which will be proved in § 5.

THEOREM 2. *Let $\Omega\subset S^3(R)$ be convex and for $\partial\Omega=M$ its normal curvature A with respect to the inner normal unit vector satisfy $A_0\leqq A\leqq A_1$. Then, supposing $0<r<\pi R/2$, for $\cos\theta$ given by (4.18) there exists a constant C_0 depending only on A_0, A_1 and r such that $1/\cos\theta\geqq C_0k(s)$.*

Now, for a domain Ω in $S^3(R)$, let $r_i(r_e)$ be the supremum (infinimum) of radius of 3-disk included in (containing) Ω. Then, we have

THEOREM 3. *Let Ω be a convex domain of $S^3(R)$ with a smooth boundary $\partial\Omega=M$. Let H be the mean curvature of M. Then, for a fixed number r ($r_i\leqq r\leqq r_e$) we have*

$$\begin{aligned} &R\cos\frac{r}{R}\sin\frac{r}{R}\left(\frac{2R}{C_0}\tan\frac{r}{R}-1\right)\operatorname{area}(M) \\ &\qquad\geqq \operatorname{vol}(\Omega)+2\pi R^2\left(r-R\cos\frac{r}{R}\sin\frac{r}{R}\right)+R^2\sin^2\frac{r}{R}\int_M H\,dV_M. \end{aligned} \tag{4.19}$$

Proof. Since Ω is convex, we have easily $r_e\leqq\pi R/2$. Therefore, we can utilize Theorem 2 for the domain Ω. For a general point $y\in S^3(R)$, let $n(y)$ be the number of the components of $F_r^2(y)\cap M$. Let C be one of them. For the curvature $k(s)$ of C as a curve in R^4, we have $\int_0^L k(s)ds\geqq 2\pi$ by the Fenchel-Borsuk theorem, where $L=\operatorname{length}(C)$. It is clear that the set of general points y is open and dense in $S^3(R)$, and the function $n(y)$ is lower semi-continuous. Therefore, the set of y with $n(y)=m$ is measurable with respect to the 3-dimensional measure of $S^3(R)$ for any integer m. Setting

$$F_m:=\operatorname{vol}(\{y\mid y\in S^3(R),\ n(y)=m\}),$$

we obtain from Theorem 1 with $n=3$

$$R^2\sin^2\frac{r}{R}\cdot 4\pi\cdot\operatorname{area}(M)=\int_{S^3(R)}\left(\int_{F_r^2(y)\cap M}\frac{dV_{F^2(y)\cap M}}{\sqrt{1-\xi_3\xi_3}}\right)dV_S(y)$$

$$\geqq C_0\int_{S^3(R)}\left(\int_{F_r^2(y)\cap M}k_{F_r^2(y)\cap M}(s)\,ds\right)dV_S(y)$$

$$\geqq 2\pi C_0\int_{S^3(R)}n(y)\,dV_S(y)=2\pi C_0(F_1+2F_2+3F_3+\cdots),$$

i. e.

$$\frac{2R^2}{C_0}\cdot\sin^2\frac{r}{R}\cdot\operatorname{area}(M)\geqq F_1+2F_2+3F_3+\cdots. \tag{4.20}$$

On the other hand, we see easily that

$$\Omega_r:=\{x\mid x\in S^3(R),\ \operatorname{dis}_{S^3(R)}(x,\ \Omega)<r\}$$

$$=\{y\mid n(y)>0,\ y\in S^3(R)\}\quad\text{(except a set of measure 0)}$$

and

$$V_r=\operatorname{vol}(\Omega_r)=F_1+F_2+F_3+\cdots. \tag{4.21}$$

From (4.20) and (4.21), we have

$$\frac{2R^2}{C_0}\cdot\sin^2\frac{r}{R}\cdot\operatorname{area}(M)-V_r=F_2+2F_3+3F_4+\cdots\geqq 0,$$

and furthermore using (3.6) we obtain

$$\frac{2R^2}{C_0}\cdot\sin^2\frac{r}{R}\cdot\operatorname{area}(M)\geqq V_r=\operatorname{vol}(\Omega)+R\cos\frac{r}{R}\sin\frac{r}{R}\cdot\operatorname{area}(M)$$

$$+2\pi R^2\left(r-R\cos\frac{r}{R}\sin\frac{r}{R}\right)+R^2\sin^2\frac{r}{R}\int_M H\,dV_M,$$

which is equivalent to (4.19). q. e. d.

§ 5. Proof of Theorem 2.

According to the formula (4.18), we have

$$k(s)\cos\theta=\sqrt{A_{11}{}^2+B_{11}{}^2+\frac{1}{R^2}}\cdot\frac{1}{A_{11}{}^2+B_{11}{}^2}$$

$$\times\left\{B_{11}\frac{1}{R}\cot\frac{r}{R}+A_{11}\sqrt{A_{11}{}^2+B_{11}{}^2-\frac{1}{R^2}\cot^2\frac{r}{R}}\right\}$$

and setting $A_{11}=A$, $B_{11}=B=u$ for simplicity we consider the following function of u

$$f(u):=\frac{\sqrt{A^2+\frac{1}{R^2}+u^2}}{A^2+u^2}\left\{\frac{u}{R}\cot\frac{r}{R}+A\sqrt{A^2-\frac{1}{R^2}\cot^2\frac{r}{R}+u^2}\right\}. \tag{5.1}$$

Since Ω is convex, $A_{11}\geqq 0$ everywhere on $M=\partial\Omega$. We shall try to find an upper bound of $f(u)$ for $u\geqq 0$.

First of all, we write the right-hand side of (5.1) as

$$f(u)=\frac{\sqrt{A^2+\frac{1}{R^2}+u^2}}{\sqrt{A^2+u^2}}\left\{\frac{1}{R}\cot\frac{r}{R}\frac{u}{\sqrt{A^2+u^2}}+\frac{A\sqrt{A^2-\frac{1}{R^2}\cot^2\frac{r}{R}+u^2}}{\sqrt{A^2+u^2}}\right\}.$$

We can easily see that the function $\dfrac{\sqrt{A^2+\frac{1}{R^2}+u^2}}{\sqrt{A^2+u^2}}$ is decreasing, $\dfrac{u}{\sqrt{A^2+u^2}}$ and $\dfrac{\sqrt{A^2-\frac{1}{R^2}\cot^2\frac{r}{R}+u^2}}{\sqrt{A^2+u^2}}$ are increasing for $u\geqq 0$. Hence we have

$$1<\frac{\sqrt{A^2+\frac{1}{R^2}+u^2}}{\sqrt{A^2+u^2}}\leqq\frac{\sqrt{A^2+\frac{1}{R^2}}}{A},\qquad 0\leqq\frac{u}{\sqrt{A^2+u^2}}<1$$

and

$$\frac{\sqrt{A^2-\frac{1}{R^2}\cot^2\frac{r}{R}}}{A}\leqq\frac{\sqrt{A^2-\frac{1}{R^2}\cot^2\frac{r}{R}+u^2}}{\sqrt{A^2+u^2}}<1\qquad\text{for}\quad u\geqq 0.$$

Thus, we obtain

$$f(u)<\frac{\sqrt{A^2+\frac{1}{R^2}}}{A}\left(\frac{1}{R}\cot\frac{r}{R}+A\right):=h(A). \tag{5.2}$$

The function $h(A)$ of A has the properties as follows:

$$\lim_{A\to+0}h(A)=\lim_{A\to+\infty}h(A)=+\infty$$

and

$$\frac{h'(A)}{h(A)}=\frac{1}{A\left(A+\frac{1}{R}\cot\frac{r}{R}\right)\left(A^2+\frac{1}{R^2}\right)}\left(A^3-\frac{1}{R^3}\cot\frac{r}{R}\right),$$

hence

$$h'(A)<0 \quad \text{for} \quad 0<A<\frac{1}{R}\left(\cot\frac{r}{R}\right)^{1/3},$$

$$h'(A)>0 \quad \text{for} \quad \frac{1}{R}\left(\cot\frac{r}{R}\right)^{1/3}<A.$$

Let us suppose from the convexity of Ω that

$$(0<)\,A_0 \leqq A \leqq A_1. \tag{5.3}$$

Setting
$$\max(h(A_0),\ h(A_1))=\frac{1}{C},$$

we have

$$f(u)<h(A)\leqq\frac{1}{C}, \tag{5.4}$$

that is

$$\frac{1}{\cos\theta}>Ck(s). \tag{5.5}$$

q. e. d.

In the following, we shall show that C in (5.4) can be replaced with a more sharper constant C_0. Setting

$$\begin{cases} f_1(u):=\dfrac{u}{A^2+u^2}\sqrt{A^2+\dfrac{1}{R^2}+u^2}, \\[2ex] f_2(u):=\dfrac{1}{A^2+u^2}\sqrt{A^2+\dfrac{1}{R^2}+u^2}\sqrt{A^2-\dfrac{1}{R^2}\cot^2\dfrac{r}{R}+u^2}, \end{cases} \tag{5.6}$$

we write $f(u)$ as

$$f(u)=\frac{1}{R}\cot\frac{r}{R}\cdot f_1(u)+Af_2(u). \tag{5.7}$$

First of all, since we have

$$f_1'(u)=\frac{1}{(A^2+u^2)^2\sqrt{A^2+\dfrac{1}{R^2}+u^2}}\left\{A^2\left(A^2+\frac{1}{R^2}\right)+\left(A^2-\frac{1}{R^2}\right)u^2\right\},$$

we obtain easily the following:

i) when $A<\frac{1}{R}$, $f_1(u)\leqq f_1\left(A\sqrt{\frac{1+R^2A^2}{1-R^2A^2}}\right)=\frac{1+R^2A^2}{2AR}$ for $u\geqq 0$;

ii) when $A\geqq\frac{1}{R}$, $f_1(u)$ is monotone increasing and so

$$f_1(u)<\lim_{u\to+\infty} f_1(u)=1 \qquad \text{for} \quad u\geqq 0.$$

Second, we have

$$f_2'(u)=\frac{u}{(A^2+u^2)^2\sqrt{A^2+\frac{1}{R^2}+u^2}\sqrt{A^2-\frac{1}{R^2}\cot^2\frac{r}{R}+u^2}}\times$$

$$\times\left\{-\frac{A^2}{R^2}\left(1-\cot^2\frac{r}{R}\right)+\frac{2}{R^4}\cot^2\frac{r}{R}-\frac{1}{R^2}\left(1-\cot^2\frac{r}{R}\right)u^2\right\}.$$

Hence we have the following:

a) Case $0<r\leqq\frac{\pi}{4}R$, $f_2(u)$ is monotone increasing and so

$$f_2(u)<\lim_{u\to+\infty} f_2(u)=1 \qquad \text{for} \quad u\geqq 0,$$

and

b) Case $\frac{\pi}{4}R<r\leqq\frac{\pi}{2}R$, from the equation

$$-\frac{A^2}{R^2}\left(1-\cot^2\frac{r}{R}\right)+\frac{2}{R^4}\cot^2\frac{r}{R}=0,$$

we obtain $A=\dfrac{\sqrt{2}}{R\sqrt{\tan^2\frac{r}{R}-1}}$, and so

i) when $0<A<\dfrac{\sqrt{2}}{R\sqrt{\tan^2\frac{r}{R}-1}}$, $f_2(u)\leqq f_2\left(\frac{1}{R}\sqrt{\dfrac{2-A^2R^2\left(\tan^2\frac{r}{R}-1\right)}{\tan^2\frac{r}{R}-1}}\right)$

$$=\frac{1}{\sin\frac{2r}{R}} \qquad \text{for} \quad u\geqq 0,$$

ii) when $A \geqq \dfrac{\sqrt{2}}{R\sqrt{\tan^2\frac{r}{R}-1}}$, $f_2(u)$ is monotone decreasing and so

$$f_2(u) \leqq f_2(0) = \frac{\sqrt{A^2R^2+1}\sqrt{A^2R^2-\cot^2\frac{r}{R}}}{A^2R^2} \qquad \text{for} \quad u \geqq 0.$$

On the other hand, we compare the separating values $\dfrac{\sqrt{2}}{R\sqrt{\tan^2\frac{r}{R}-1}}$ and $\dfrac{1}{R}$ for A with respect to $f_2(u)$ and $f_1(u)$ respectively. We see easily that

$$\frac{\sqrt{2}}{R\sqrt{\tan^2\frac{r}{R}-1}} \begin{cases} > \dfrac{1}{R} & \text{for} \quad \dfrac{\pi}{4}R < r < \dfrac{\pi}{3}R, \\ = \dfrac{1}{R} & \text{for} \quad r = \dfrac{\pi}{3}R, \\ < \dfrac{1}{R} & \text{for} \quad \dfrac{\pi}{3}R < r \leqq \dfrac{\pi}{2}R. \end{cases}$$

From the above arguments, we define the following functions $h_i(A)$, $i=1, 2, 3$, as follows:

1) Case $0 < r \leqq \dfrac{\pi}{4}R$,

$$h_1(A) := \begin{cases} \dfrac{1+A^2R^2}{2AR^2}\cot\dfrac{r}{R} + A & \text{for} \quad 0 < A < \dfrac{1}{R}, \\ \dfrac{1}{R}\cot\dfrac{r}{R} + A & \text{for} \quad A \geqq \dfrac{1}{R}\,; \end{cases}$$

2) Case $\dfrac{\pi}{4}R < r \leqq \dfrac{\pi}{3}R$,

$$h_2(A) := \begin{cases} \dfrac{1+A^2R^2}{2AR^2}\cot\dfrac{r}{R} + \dfrac{A}{\sin\frac{2r}{R}} & \text{for} \quad 0 < A < \dfrac{1}{R}, \\ \dfrac{1}{R}\cot\dfrac{r}{R} + \dfrac{A}{\sin\frac{2r}{R}} & \text{for} \quad \dfrac{1}{R} \leqq A < \dfrac{\sqrt{2}}{R\sqrt{\tan^2\frac{r}{R}-1}}, \end{cases}$$

$$
\left|\begin{array}{l}
\dfrac{r}{R}\cot\dfrac{1}{R}+\dfrac{\sqrt{A^2R^2+1}\sqrt{A^2R^2-\cot^2\dfrac{r}{R}}}{AR^2} \\
\qquad\qquad\qquad\text{for}\quad \dfrac{\sqrt{2}}{R\sqrt{\tan^2\dfrac{r}{R}-1}}\leqq A\,;
\end{array}\right.
$$

3) Case $\dfrac{\pi}{3}R<r\leqq\dfrac{\pi}{2}R$,

$$
h_3(A):=\begin{cases}
\dfrac{1+A^2R^2}{2AR^2}\cot\dfrac{r}{R}+\dfrac{A}{\sin\dfrac{2r}{R}} & \text{for}\quad 0<A<\dfrac{\sqrt{2}}{R\sqrt{\tan^2\dfrac{r}{R}-1}}, \\
\dfrac{1+A^2R^2}{2AR^2}\cot\dfrac{r}{R}+\dfrac{\sqrt{A^2R^2+1}\sqrt{A^2R^2-\cot^2\dfrac{r}{R}}}{AR^2} & \\
\qquad\qquad\text{for}\quad \dfrac{\sqrt{2}}{R\sqrt{\tan^2\dfrac{r}{R}-1}}\leqq A<\dfrac{1}{R}, & \\
\dfrac{1}{R}\cot\dfrac{r}{R}+\dfrac{\sqrt{A^2R^2+1}\sqrt{A^2R^2-\cot^2\dfrac{r}{R}}}{AR^2} & \text{for}\quad \dfrac{1}{R}\leqq A.
\end{cases}
$$

For each cases, we obtain from (5.7)

$$
f(u)\leqq h_i(A)\qquad\text{for}\quad u\geqq 0. \tag{5.8}
$$

Furthermore, we can prove easily that

1) Case $0<r\leqq\dfrac{\pi}{4}R$, $h_1(A)$ takes its minimum value at

$$
A_1^*=\frac{1}{R\sqrt{2\tan\dfrac{r}{R}+1}}<\frac{1}{R}
$$

and it is monotone decreasing in $[0, A_1^*]$ and increasing in $[A_1^*, \infty)$;

2) Case $\dfrac{\pi}{4}R<r\leqq\dfrac{\pi}{3}R$, $h_2(A)$ takes its minimum value at

$$
A_2^*=\frac{1}{R\sqrt{1+\sec^2\dfrac{r}{R}}}<\frac{1}{R}
$$

and it is monotone decreasing in $[0, A_2^*]$ and increasing in $[A_2^*, \infty)$;

3) Case $\frac{\pi}{3}R<r\leqq\frac{\pi}{4}R$, $h_3(A)$ has the same property as $h_2(A)$.

Thus, making use of these functions $h_\imath(A)$, $\imath=1, 2, 3$, for these three cases, we set

$$\max\{h_\imath(A_0),\ h_\imath(A_1)\}=\frac{1}{C_0}.$$

Then we have

$$f(u)<h_\imath(A)\leqq\frac{1}{C_0}\qquad\text{for}\quad A_0\leqq A\leqq A_1.$$

It is clear that this C_0 is more sharper than C for our purpose.

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