Q. HE AND Y.-B. SHEN
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DARBOUX TRANSFORMATIONS AND ISOMETRIC IMMERSIONS OF RIEMANNIAN PRODUCTS OF SPACE FORMS

Qun He and Yi-Bing Shen

Abstract

By using the Darboux transformation in Soliton theory, we give the explicit construction for local isometric immersions of the Riemannian product $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ into space forms $M^m(c)$ with flat normal bundle via purely algebraic algorithm.

§0. Introduction

The problem on isometric immersions of Riemannian manifolds into space forms is an interesting classical problem. There are a lot of nonexistence results in this area ([CK], [Hi], [Pe], [Xa], etc.). Recently, it has been found that the integrability condition for isometric immersions of space forms, i.e., Gauss-Codazzi-Ricci equations, is equivalent to the condition of a family of connections to be flat ([FP], [Ter]). This enable us to apply the soliton theory to the study of some problems on isometric immersions of space forms. For instance, some Bäcklund transformations for such isometric immersions were considered in [FP] and [TU]. The Darboux transformation method for the explicit expressions of such isometric immersions via purely algebraic algorithm has been given in [Zh, HS], respectively. It is natural to consider the problem on isometric immersions of Riemannian product $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ into space forms $M^m(c)$.

The purpose of this paper is to apply the Darboux transformation method to the study of local isometric immersions from the Riemannian product of space forms into space forms with flat normal bundle. Some fundamental theory on local isometric immersions of $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ into $M^m(c)$ with $c = c_1c_2/(c_1 + c_2)$ is developed in §1. In §2, a zero curvature condition for such local isometric immersions is given, i.e., a family of connection 1-forms including one parameter are flat. It is different from [FP]. In §3, the Darboux transformations for the explicit expressions of such isometric immersions are shown. This is

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a purely algebraic algorithm. Finally, in §4, we give an explicit construction of such isometric immersions from a trivial (degenerated) isometric immersion via the Darboux transformation for the twisted $so(p,q,r)$-hierarchy. It is possible that the method of this paper may be used to study local isometric immersions of the Riemannian product of several space forms into space forms.

§1. Isometric immersions of Riemannian products

Let $M^n(c)$ denote an n-dimensional space form of constant curvature c. Let $\mathbf{r}_c : M^m(c) \to \boldsymbol{R}_c^{m+1}$ be the following standard isometric embedding:

$$M^m(c) = \left\{ (x_0, x_1, \dots, x_m) \in \boldsymbol{R}^{m+1} \mid \sum_{A=1}^{m} x_A^2 + x_0^2 = \frac{1}{c^2} \right\} \quad \text{for } c > 0,$$

$$M^m(c) = \left\{ (x_0, x_1, \dots, x_m) \in \boldsymbol{R}^{m,1} \mid \sum_{A=1}^{m} x_A^2 - x_0^2 = -\frac{1}{c^2} \right\} \quad \text{for } c < 0,$$

$$M^m(0) = \{ (x_0, x_1, \dots, x_m) \in \boldsymbol{R}^{m+1} \mid x_0 = 0 \}.$$

Consider a locally isometric immersion $\varphi : M_1^{n_1}(c_1) \times M_2^{n_2}(c_2) \supset U \to M^m(c)$ with $c_2 \neq 0$ and $c = c_1c_2/(c_1+c_2)$, where $m > n_1 + n_2 = n$ and, without loss of generality, $c = \pm 1$ or 0. We shall make use of the following convention on the ranges of indices unless otherwise stated:

$$i, j, k, \ldots = 1, \ldots, n_1; \quad q, s, t, \ldots = n_1 + 1, \ldots, n;$$
$$I, J, K, \ldots = 1, \ldots, n; \quad r = 1, 2; \quad \alpha, \beta, \ldots = 1, \ldots, m - n.$$

Then the composition map $\mathbf{r} = \mathbf{r}_c \circ \varphi : U \to \boldsymbol{R}_c^{m+1}$ is a local isometric immersion into $\boldsymbol{R}_c^{m+1}$. Set

$$J_c = \begin{pmatrix} c & 0 \\ 0 & I_m \end{pmatrix}, \quad so_c(m+1) = \{ X \in sl(m+1, \boldsymbol{R}) \mid XJ_c + J_cX^T = 0 \}.$$

Denote by $SO_c(m+1)$ the Lie group of which the Lie algebra is $so_c(m+1)$. Consider a framing field $\Psi = (e_0, e_1, \ldots, e_m) : U \to SO_c(m+1)$ in $\boldsymbol{R}_c^{m+1}$ so that $\mathbf{r} = J_c^2 e_0$, $\{e_i\}$ and $\{e_s\}$ are tangent to $M_1^{n_1}(c_1)$ and $M_2^{n_2}(c_2)$ respectively, and $\{e_{n+\alpha}\}$ are normal to $M_1 \times M_2$ in $M^m(c)$. Clearly, e_0 is normal to $M^m(c)$ for $c \neq 0$. Let $\Xi = \Psi^{-1}\, d\Psi$ be the pull back of the Maurer-Cartan form of $SO_c(m+1)$ by Ψ, which is an $so_c(m+1)$-valued 1-form. We then have

$$\text{(1.1)} \qquad \begin{cases} d\Psi = \Psi\Xi, \\ \Psi(0) = I_{m+1}, \end{cases} \quad \text{where } \Xi = \begin{pmatrix} 0 & -c\theta_1^T & -c\theta_2^T & 0 \\ \theta_1 & \omega_1 & 0 & \beta_1 \\ \theta_2 & 0 & \omega_2 & \beta_2 \\ 0 & -\beta_1^T & -\beta_2^T & \eta \end{pmatrix},$$

where $\theta_1 = (\theta^1, \ldots, \theta^{n_1})^T$ and $\theta_2 = (\theta^{n_1+1}, \ldots, \theta^n)^T$ are dual fields of $\{e_i\}$ and $\{e_s\}$ respectively, $\omega_1 = (\omega_{ij})$ and $\omega_2 = (\omega_{st})$ are the Levi-Civita connection 1-forms of

$M_1^{n_1}(c_1)$ and $M_2^{n_2}(c_2)$ respectively, $\beta_1 = (\omega_{i,n+\alpha})$ and $\beta_2 = (\omega_{s,n+\alpha})$ are the second fundamental form of the isometric immersion φ, $\eta = (\omega_{n+\alpha,n+\beta})$ is the normal connection of φ.

The integrability condition for the existence of such a framing field Ψ is that Ξ satisfies the Maurer-Cartan equation

$$d\Xi + \Xi \wedge \Xi = 0,$$

i.e.,

$$\begin{aligned}
&d\theta_r + \omega_r \wedge \theta_r = 0,\\
&d\omega_r + \omega_r \wedge \omega_r - c\theta_r \wedge \theta_r^T - \beta_r \wedge \beta_r^T = 0,\\
&d\beta_r + \beta_r \wedge \eta + \omega_r \wedge \beta_r = 0,\\
&d\eta + \eta \wedge \eta - \beta_1^T \wedge \beta_1 - \beta_2^T \wedge \beta_2 = 0,\\
&\beta_1 \wedge \beta_2^T + c\theta_1 \wedge \theta_2^T = 0,\\
&\theta_1^T \wedge \beta_1 + \theta_2^T \wedge \beta_2 = 0.
\end{aligned} \tag{1.2}$$

Since $M_1^{n_1}(c_1)$ and $M_2^{n_2}(c_2)$ have constant curvatures c_1 and c_2 respectively, then

$$d\omega_r + \omega_r \wedge \omega_r = c_r\theta_r \wedge \theta_r^T. \tag{1.3}$$

It follows from $(1.2)_2$ and (1.3) that

$$\beta_r \wedge \beta_r^T + (c - c_r)\theta_r \wedge \theta_r^T = 0. \tag{1.4}$$

If the normal bundle of φ is flat, then

$$d\eta + \eta \wedge \eta = \beta_1^T \wedge \beta_1 + \beta_2^T \wedge \beta_2 = 0. \tag{1.5}$$

Set

$$\begin{aligned}
&\varepsilon_r = \operatorname{sgn}(c_r), \quad \kappa_r = \begin{cases} 1 & \text{for } c_r = 0,\\ \sqrt{|c_r|} & \text{for } c_r \neq 0,\end{cases}\\
&\varepsilon = \operatorname{sgn}(c - c_2), \quad v_1 = \varepsilon c\sqrt{|c - c_1|}, \quad v_2 = \sqrt{|c - c_2|}.
\end{aligned} \tag{1.6}$$

Clearly, we see that $\varepsilon_2, v_2, \varepsilon, \kappa_r \neq 0$, and $\varepsilon = \operatorname{sgn}(c - c_1) = \operatorname{sgn}(c - c_2)$ when $c \neq 0$. Noting that $c^2 = 1$ or 0, thus, $(1.2)_5$ and (1.4) can be rewritten as

$$\begin{aligned}
&\beta_1 \wedge \beta_2^T + \varepsilon v_1 v_2 \theta_1 \wedge \theta_2^T = 0,\\
&\beta_r \wedge \beta_r^T + \varepsilon v_r^2 \theta_r \wedge \theta_r^T = 0.
\end{aligned} \tag{1.7}$$

Definition 1.1. Let $\varphi : M_1^{n_1} \times M_2^{n_2} \to M^m$ be an isometric immersion. If Weingarten endomorphisms for φ preserve TM_1 and TM_2 invariant, respectively, i.e., β_1 and β_2 can be expressed linearly by θ_1 and θ_2 respectively, then the second fundamental form of φ is called to be *separable*.

For $c \neq 0$ and $\varepsilon > 0$, (1.7) implies that the second fundamental form $\psi_\alpha = \sum_I \omega_{I,n+\alpha} \otimes \theta^I$ and the symmetric bilinear form $\psi = v_1\theta_1^T \otimes \theta_1 + v_2\theta_2^T \otimes \theta_2$ are exteriorly orthogonal, and they can be simultaneously diagonalized for $m = 2n-1$ by virtue of Cartan's theorem [Mo]. Thus, the immersion φ has flat normal bundle. Hence, in the way similar to the case of isometric immersions of space forms [Mo], we have the following

PROPOSITION 1.2. *There is no isometric immersion* $\varphi : M_1^{n_1}(c_1) \times M_2^{n_2}(c_2) \to M^{2n-2}(c)$ *with* $c = c_1c_2/(c_1+c_2) > c_1$. *Moreover, if* $\varphi : M_1^{n_1}(c_1) \times M_2^{n_2}(c_2) \to M^{2n-1}(c)$ *is a local isometric immersion with* $c = c_1c_2/(c_1+c_2) > c_1$, *then the normal connection of* φ *is flat.*

In general, if the isometric immersion φ has flat normal bundle, then the second fundamental of φ can be simultaneously diagonalized. In addition, if the second fundamental form of φ is separable, then we can choose the tangent frame fields $\{e_i\}$ to $M_1^{n_1}(c_1)$ and $\{e_s\}$ to $M_2^{n_2}(c_2)$ such that $\omega_{i,n+\alpha} = b_{i\alpha}\theta^i$ and $\omega_{s,n+\alpha} = b_{s\alpha}\theta^s$. Moreover, we can choose a parallel normal frame fields $\{e_\alpha\}$ so that $\eta = 0$.

On putting

$$\omega_{ij} = \sum_k \Gamma_{ij}^k \theta^k, \quad \omega_{st} = \sum_q \Gamma_{st}^q \theta^q,$$

we have from (1.2), (1.4) and (1.7)

$$\begin{aligned}
&\sum_\alpha b_{i\alpha}b_{j\alpha} + \varepsilon v_1^2 = 0, \quad (i \neq j)\\
&\sum_\alpha b_{s\alpha}b_{t\alpha} + \varepsilon v_2^2 = 0, \quad (s \neq t)\\
&\sum_\alpha b_{i\alpha}b_{s\alpha} + \varepsilon v_1 v_2 = 0,\\
&(b_{i\alpha} - b_{j\alpha})\Gamma_{ij}^k = (b_{i\alpha} - b_{k\alpha})\Gamma_{ik}^j, \quad (i,j,k \neq)\\
&(b_{s\alpha} - b_{t\alpha})\Gamma_{st}^q = (b_{s\alpha} - b_{q\alpha})\Gamma_{sq}^t, \quad (q,s,t \neq)\\
&e_j(b_{i\alpha}) = (b_{j\alpha} - b_{i\alpha})\Gamma_{ij}^i, \quad (i \neq j)\\
&e_t(b_{s\alpha}) = (b_{t\alpha} - b_{s\alpha})\Gamma_{st}^s, \quad (s \neq t)\\
&e_s(b_{i\alpha}) = e_i(b_{s\alpha}) = 0.
\end{aligned} \tag{1.8}$$

Set

$$B_1 = (b_{i\alpha}), \quad B_2 = (b_{s\alpha}), \quad V_1 = (\underbrace{v_1, \dots, v_1}_{n_1})^T, \quad V_2 = (\underbrace{v_2, \dots, v_2}_{n_2})^T,$$

$$B = \begin{pmatrix} B_1 & V_1 \\ B_2 & V_2 \end{pmatrix}, \quad \tilde{J}_\varepsilon = \begin{pmatrix} I_{m-n} & 0 \\ 0 & \varepsilon \end{pmatrix}. \tag{1.9}$$

Then we see from $(1.8)_{1-3}$ that $B\tilde{J}B^T = \mathrm{diag}(\rho_1, \dots, \rho_n)$, where

$$\rho_i = \sum_\alpha b_{i\alpha}^2 + \varepsilon v_1^2, \quad \rho_s = \sum_\alpha b_{s\alpha}^2 + \varepsilon v_2^2. \tag{1.10}$$

Lemma 1.3. *Let $\varphi : M_1^{n_1}(c_1) \times M_2^{n_2}(c_2) \supset U \to M^m(c)$ be a locally isometric product immersion with flat normal bundle and the separable second fundamental form. Assume that $\rho_I \neq 0$ for all I where ρ_I are smooth functions defined by* (1.10). *Then there exist a line of curvature coordinates (x_i, x_s) on U such that the first and second fundamental forms of φ can be given by*

$$\begin{aligned} I &= \sum_i a_i^2\, dx_i^2 + \sum_s a_s^2\, dx_s^2, \\ II &= \sum_\alpha \left(\sum_i a_i^2 b_{i\alpha}\, dx_i^2 + \sum a_s^2 b_{s\alpha}\, dx_s^2 \right) e_{n+\alpha}. \end{aligned} \tag{1.11}$$

Proof. Since $\rho_I \neq 0$ then we can write $\rho_I = \pm(a_I)^{-2}$ with $a_I > 0$. It follows from (1.8) and (1.10)

$$e_j(a_i) = a_i \Gamma_{ij}^i, \quad e_t(a_s) = a_s \Gamma_{st}^s. \tag{1.12}$$

For any point $x \in U$, if we choose e_{n+1} at x so that $b_{11}(x) \neq 0$, $b_{12}(x) = \cdots = b_{1,m-n}(x) = 0$, then it follows from $(1.8)_1$ and $(1.10)_1$ that

$$b_{i1}(x) = -\frac{\varepsilon v_1^2}{b_{11}(x)} \quad (i \neq 1), \quad b_{11}^2(x) + \varepsilon v_1^2 = \rho_1 \neq 0. \tag{1.13}$$

By taking $\alpha = j = 1$ in $(1.8)_4$, we see from (1.13) that $\Gamma_{i1}^k(x) = 0$. Since x is arbitrary, then it follows that $\Gamma_{i1}^k = 0$ for $i, k, 1$ distinct. We know that the components Γ_{ij}^k of ω_{ij} are independent of the choice of the fields of normal frames. Thus, we have $\Gamma_{ij}^k = 0$ for i, j, k distinct. By the same reason, we have also $\Gamma_{st}^q = 0$ for q, s, t distinct. Hence, by using (1.12) and the skew-symmetry, we conclude that

$$\omega_{ij} = \frac{e_j(a_i)}{a_i}\theta^i - \frac{e_i(a_j)}{a_j}\theta^j, \quad \omega_{st} = \frac{e_t(a_s)}{a_s}\theta^s - \frac{e_s(a_t)}{a_t}\theta^t. \tag{1.14}$$

As the same as in [Mo], it is easy from (1.14) to see that there exist a line of curvature coordinates (x_i, x_s) on U so that $\partial/\partial x_i = a_i e_i$, $\partial/\partial x_s = a_s e_s$, $\theta^i = a_i\, dx_i$, $\theta^s = a_s\, dx_s$. Thus, the first and second fundamental forms of φ are given by (1.11). □

Definition 1.4. Let $\varphi : M^n \to \tilde{M}^m$ be an isometric immersion. Denote by A_ξ the Weingarten endomorphism with respect to $\xi \in T^\perp M$. If for any $\xi \in T^\perp M$, the rank of A_ξ is equal to $\min\{n, m-n\}$, then the normal bundle of φ is said to be *nondegenerate*.

Definition 1.5. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, and $\varphi : M_1 \times M_2 \to \tilde{M}$ an isometric immersion of Riemannian product. If for a $\xi \in T^{\perp}(M_1 \times M_2)$, there exist functions λ_1 and λ_2 such that $h_\xi = \lambda_1 g_1 + \lambda_2 g_2$ where h_ξ is the second fundamental tensor of φ with respect to ξ, then φ is called to be *special quasiumbilical* with respect to ξ. In particular, a special quasiumbilical immersion φ with respect to $\xi \in T^{\perp}(M_1 \times M_2)$ is umbilical when $\lambda_1 = \lambda_2$.

In the case that $\varepsilon > 0$, if $c \neq 0$, then we see from $(1.8)_{1-3}$ that rank $B = n$, which implies that $m - n \geq n - 1$ and $\rho_I > 0$ for all I. If $c = c_1 = 0$, then we see that $m - n \geq n_2 - 1$. In particular, when either $m - n = n - 1$ for $c \neq 0$ or $m - n = n_2 - 1$ for $c = c_1 = 0$, then the normal bundle of φ is nondegenerate and φ satisfies one of the following conditions:

(1.15)
(i) φ is not special quasiumbilical;
(ii) φ is special quasiumbilical with respect to some $\xi \in T^{\perp}(M_1 \times M_2)$, i.e., $h_\xi = \lambda_1 g_1 + \lambda_2 g_2$, and $\lambda_1 v_1 \neq \lambda_2 v_2$.

In the case that $\varepsilon < 0$, if either $m - n < n - 1$ for $c \neq 0$ or $m - n < n_2 - 1$ for $c = c_1 = 0$, then φ is special quasiumbilical with respect to some $\xi \in T^{\perp}(M_1 \times M_2)$, satisfying $\lambda_1 v_1 = \lambda_2 v_2$. Hence, if the immersion φ satisfies one of (1.15), then we have either $m - n \geq n - 1$ for $c \neq 0$ or $m - n \geq n_2 - 1$ for $c = c_1 = 0$.

For simplicity, in the following, we assume always that $m = 2n - 1 = 2(n_1 + n_2) - 1$. In such a case, B of (1.9) is a non-degenerate $n \times n$ matrix, and $\rho_I \neq 0$ where only one of $\{\rho_I\}$ has the same sign as ε, and the remains are positive. Without loss of generality, suppose that $\rho_\alpha = (a_\alpha)^{-2}$, $\rho_n = \varepsilon(a_n)^{-2}$. On putting

$$a_{I\alpha} = a_I b_{I\alpha}, \quad a_{in} = v_1 a_i, \quad a_{sn} = v_2 a_s, \quad A = (a_{IJ}),$$

we see that A satisfies $A\tilde{J}_\varepsilon A^T = \tilde{J}_\varepsilon$. So, by Lemma 1.3, we have immediately

Proposition 1.6. *Let $\varphi : M_1^{n_1}(c_1) \times M_2^{n_2}(c_2) \supset U \to M^{2n-1}(c)$ be a locally isometric immersion with the flat normal bundle and the separable second fundamental form. If φ satisfies one of* (1.15), *then there exist a line of curvature coordinates (x_i, x_s) such that the first and second fundamental forms of φ can be written as*

$$\begin{aligned} I &= \sum_i a_i^2\, dx_i^2 + \sum_s a_s^2\, dx_s^2, \\ II &= \sum_\alpha \left(\sum_i a_i a_{i\alpha}\, dx_i^2 + \sum_s a_s a_{s\alpha}\, dx_s^2 \right) e_{n+\alpha}, \end{aligned} \tag{1.16}$$

and

$$A = (a_{IJ}) : \boldsymbol{R}^n \to SO_\varepsilon(n), \tag{1.17}$$

where $a_{in} = v_1 a_i$, $a_{sn} = v_2 a_s$.

Corollary 1.7. *Under the same hypothesis as in Proposition* 1.5, *if* $c \neq 0$, *then there exist a line of curvature spherical* (*hyperbolic*) *coordinates* $\{x_i, x_s\}$ *such that the first and second fundamental forms of* φ *can be written as*

$$\begin{aligned} I &= \sum_i a_i^2\, dx_i^2 + \sum_s a_s^2\, dx_s^2, \\ II &= \sum_\alpha \left(\sum_i v_1 a_i a_{i\alpha}\, dx_i^2 + \sum_s v_2 a_s a_{s\alpha}\, dx_s^2 \right) e_{n+\alpha}, \end{aligned} \tag{1.18}$$

where $A = (a_{IJ}) : \boldsymbol{R}^n \to SO_\varepsilon(n)$, $a_{in} = a_i$ (*resp.* $a_{sn} = a_s$) *are dependent only on* x_i (*resp.* x_s).

§2. The zero-curvature condition

Consider the Lie algebra

$$so_{ex}(m+3) = \{X \in sl(m+3) \mid XJ + JX^T = 0\}, \quad J = \begin{pmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & I_m & \\ & & & \varepsilon \end{pmatrix}. \tag{2.1}$$

The Lie group $SO_{ex}(m+3)$ corresponding to $so_{ex}(m+3)$ is

$$SO_{ex}(m+3) = \{A \in SL(m+3) \mid AJA^T = J\}.$$

We now define a family of $so_{ex}(m+3, \boldsymbol{C})$-valued 1-forms parameterized by $\lambda \in \boldsymbol{C}^* = \boldsymbol{C} \backslash \{0\}$ as follows

$$\tilde{\Theta}_\lambda = \begin{pmatrix} 0 & 0 & -\varepsilon_1 \kappa_1 \theta_1^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_2 \kappa_2 \theta_2^T & 0 & 0 \\ \kappa_1 \theta_1 & 0 & \omega_1 & 0 & \lambda \beta_1 & \lambda v_1 \theta_1 \\ 0 & \kappa_2 \theta_2 & 0 & \omega_2 & \lambda \beta_2 & \lambda v_2 \theta_2 \\ 0 & 0 & -\lambda \beta_1^T & -\lambda \beta_2^T & \eta & 0 \\ 0 & 0 & -\lambda \varepsilon v_1 \theta_1^T & -\lambda \varepsilon v_2 \theta_2^T & 0 & 0 \end{pmatrix}. \tag{2.2}$$

Lemma 2.1. *There exists a locally isometric immersion* $\varphi : M_1^{n_1}(c_1) \times M_2^{n_2}(c_2) \supset U \to M^m(c)$ *with flat normal bundle and the separable second fundamental form if and only if*

$$d\tilde{\Theta}_\lambda + \tilde{\Theta}_\lambda \wedge \tilde{\Theta}_\lambda = 0 \tag{2.3}$$

for $\lambda \in \boldsymbol{C}^*$.

Proof. By (2.2), it is easy to see that (2.3) is equivalent to

$$(2.4)\qquad \begin{aligned} &d\theta_r + \omega_r \wedge \theta_r = 0,\\ &d\omega_r + \omega_r \wedge \omega_r - c_r\theta_r \wedge \theta_r^T = 0,\\ &\beta_r \wedge \beta_r^T + \varepsilon v_r^2 \theta_r \wedge \theta_r^T = 0,\\ &\beta_1 \wedge \beta_2^T + \varepsilon v_1 v_2 \theta_1 \wedge \theta_2^T = 0,\\ &d\eta + \eta \wedge \eta = 0, \quad \theta_r^T \wedge \beta_r = 0,\\ &\beta_1^T \wedge \beta_1 + \beta_2^T \wedge \beta_2 = 0,\\ &d\beta_r + \beta_r \wedge \eta + \omega_r \wedge \beta_r = 0. \end{aligned}$$

On the other hand, since β_1 and β_2 may be expressed by θ_1 and θ_2, respectively, then $(1.2)_6$ is equivalent to $(2.4)_6$. Hence, (2.4) is equivalent to (1.2)~(1.4). □

In the following, we assume $m = 2n - 1$. Under the same hypothesis as in Proposition 1.6, we set

$$f_{ij} = \frac{\partial_j a_i}{a_j} \quad (i \neq j), \quad f_{ii} = 0, \quad f_{st} = \frac{\partial_t a_s}{a_t} \quad (s \neq t), \quad f_{ss} = 0,$$

$$(2.5)\qquad b_1 = \kappa_1(a_1, \dots, a_{n_1})^T, \quad b_2 = \kappa_2(a_{n_1+1}, \dots, a_n)^T, \quad b = \begin{pmatrix} \kappa_1^{-1} v_1 b_1 \\ \kappa_2^{-1} v_2 b_2 \end{pmatrix},$$

$$\delta_1 = \operatorname{diag}(dx_1, \dots, dx_{n_1}), \quad \delta_2 = \operatorname{diag}(dx_{n_1+1}, \dots, dx_n),$$

$$A_1 = (a_{i\alpha}), \quad A_2 = (a_{s\alpha}), \quad F_1 = (f_{ij}), \quad F_2 = (f_{st}).$$

We then see that

$$b = AE_n \in S^{n-1}(\varepsilon), \quad F_r \in gl(n_r)_* = \{Y = (y_{ij}) \in gl(n_r) \mid y_{ii} = 0\},$$

where $E_n = \operatorname{diag}(\underbrace{0, \dots, 0}_{n-1}, 1)$. Choose parallel frame fields in the normal bundle so that $\eta = 0$. Thus, Ξ of (1.1) and $\tilde{\Theta}_\lambda$ of (2.2) are reduced as

$$(2.6)\qquad \Xi = \begin{pmatrix} 0 & -c\kappa_1^{-1} b_1^T \delta_1 & -c\kappa_2^{-1} b_2^T \delta_2 & 0 \\ \kappa_1^{-1}\delta_1 b_1 & \omega_1 & 0 & \delta_1 A_1 \\ \kappa_2^{-1}\delta_2 b_2 & 0 & \omega_2 & \delta_2 A_2 \\ 0 & -A_1^T\delta_1 & -A_2^T\delta_2 & 0 \end{pmatrix},$$

$$(2.7)\qquad \tilde{\Theta}_\lambda = \begin{pmatrix} 0 & 0 & -\varepsilon_1 b_1^T\delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_2 b_2^T \delta_2 & 0 & 0 \\ \delta_1 b_1 & 0 & \omega_1 & 0 & \lambda\delta_1 A_1 & \lambda\kappa_1^{-1} v_1 \delta_1 b_1 \\ 0 & \delta_2 b_2 & 0 & \omega_2 & \lambda\delta_2 A_2 & \lambda\kappa_2^{-1} v_2 \delta_2 b_2 \\ 0 & 0 & -\lambda A_1^T \delta_1 & -\lambda A_2^T \delta_2 & 0 & 0 \\ 0 & 0 & -\lambda\varepsilon\kappa_1^{-1} v_1 b_1^T \delta_1 & -\lambda\varepsilon\kappa_2^{-1} v_2 b_2^T \delta_2 & 0 & 0 \end{pmatrix}.$$

LEMMA 2.2. *Let* $h : \boldsymbol{R}^n \to \boldsymbol{R}^{2n+2}$ *satisfy the equation* $dh = h\tilde{\Theta}_1$. *Write* h *as a row vector*

$$\begin{matrix} 1 & 1 & n_1 & n_2 & n-1 & 1 \\ (\xi_1, & \xi_2, & \eta_1, & \eta_2, & \zeta, & \tilde{\zeta}), \end{matrix}$$

where $\tilde{\zeta}$ *satisfies that* $\tilde{\zeta}(0) = \kappa_1^{-1}v_1\xi_1(0) + \kappa_2^{-1}v_2\xi_2(0)$. *Then* $\tilde{h} = (\xi, \eta_1, \eta_2, \zeta) : \boldsymbol{R}^n \to \boldsymbol{R}^{2n}$ *satisfies the equation* $d\tilde{h} = \tilde{h}\Xi$, *where* $\xi = \kappa_1^{-1}\xi_1 + \kappa_2^{-1}\xi_2$.

Proof. Since h satisfies $dh = h\tilde{\Theta}_1$, then we have from (2.7)

$$\begin{aligned} d\xi_r &= \eta_r\delta_r b_r, \\ d\eta_r &= \eta_r\omega_r - \zeta A_r^T\delta_r - (\varepsilon_r\xi_r + \varepsilon\kappa_r^{-1}v_r\tilde{\zeta})b_r^T\delta_r, \\ d\zeta &= \eta_1\delta_1 A_1 + \eta_2\delta_2 A_2, \\ d\tilde{\zeta} &= \kappa_1^{-1}v_1\eta_1\delta_1 b_1 + \kappa_2^{-1}v_2\eta_2\delta_2 b_2 = \kappa_1^{-1}v_1\, d\xi_1 + \kappa_2^{-1}v_2\, d\xi_2. \end{aligned}$$

By the last equation and the initial condition, we see that

$$\tilde{\zeta} = \kappa_1^{-1}v_1\xi_1 + \kappa_2^{-1}v_2\xi_2.$$

It follows that

$$\begin{aligned} \varepsilon_1\xi_1 + \varepsilon\kappa_1^{-1}v_1\tilde{\zeta} &= \kappa_1^{-1}\{(\kappa_1\varepsilon_1 + \varepsilon v_1^2\kappa_1^{-1})\xi_1 + \varepsilon v_1 v_2\kappa_2^{-1}\xi_2\} \\ &= c\kappa_1^{-1}(\kappa_1^{-1}\xi_1 + \kappa_2^{-1}\xi_2) = c\kappa_1^{-1}\xi. \end{aligned}$$

In the similar way, we can obtain $\varepsilon_2\xi_2 + \varepsilon\kappa_2^{-1}v_2\tilde{\zeta} = c\kappa_2^{-1}\xi$. Hence, we have

$$\begin{aligned} d\xi &= \kappa_1^{-1}\eta_1\delta_1 b_1 + \kappa_2^{-1}\eta_2\delta_2 b_2, \\ d\eta_r &= \eta_r\omega_r - \zeta A_r^T\delta_r - c\kappa_r^{-1}\xi b_r^T\delta_r, \\ d\zeta &= \eta_1\delta_1 A_1 + \eta_2\delta_2 A_2, \end{aligned}$$

i.e., $\tilde{h} = (\xi, \eta_1, \eta_2, \zeta)$ satisfies $d\tilde{h} = \tilde{h}\Xi$. □

For simplicity, we write a $p \times (2n+2)$ matrix $\mathscr{M}$ as a row matrix

$$\begin{matrix} 1 & 1 & n_1 & n_2 & n_1 & n_2 \\ (\mathscr{M}^{(1)} & \mathscr{M}^{(2)} & \mathscr{M}^{(3)} & \mathscr{M}^{(4)} & \mathscr{M}^{(5)} & \mathscr{M}^{(6)}). \end{matrix}$$

Particularly, we write a $(2n+2) \times (2n+2)$ matrix $\mathscr{M}$ as a block matrix

$$\begin{matrix} 1 & 1 & n_1 & n_2 & n_1 & n_2 & \\ \end{matrix}$$
$$\begin{pmatrix} \mathscr{M}^{(11)} & \mathscr{M}^{(12)} & \mathscr{M}^{(13)} & \mathscr{M}^{(14)} & \mathscr{M}^{(15)} & \mathscr{M}^{(16)} \\ \mathscr{M}^{(21)} & \mathscr{M}^{(22)} & \mathscr{M}^{(23)} & \mathscr{M}^{(24)} & \mathscr{M}^{(25)} & \mathscr{M}^{(26)} \\ \mathscr{M}^{(31)} & \mathscr{M}^{(32)} & \mathscr{M}^{(33)} & \mathscr{M}^{(34)} & \mathscr{M}^{(35)} & \mathscr{M}^{(36)} \\ \mathscr{M}^{(41)} & \mathscr{M}^{(42)} & \mathscr{M}^{(43)} & \mathscr{M}^{(44)} & \mathscr{M}^{(45)} & \mathscr{M}^{(46)} \\ \mathscr{M}^{(51)} & \mathscr{M}^{(52)} & \mathscr{M}^{(53)} & \mathscr{M}^{(54)} & \mathscr{M}^{(55)} & \mathscr{M}^{(56)} \\ \mathscr{M}^{(61)} & \mathscr{M}^{(62)} & \mathscr{M}^{(63)} & \mathscr{M}^{(64)} & \mathscr{M}^{(65)} & \mathscr{M}^{(66)} \end{pmatrix} \begin{matrix} 1 \\ 1 \\ n_1 \\ n_2 \\ n_1 \\ n_2 \end{matrix}.$$

By using the guage transformation

$$\Theta_\lambda = H\tilde{\Theta}_\lambda H^{-1} - dHH^{-1}, \quad \text{where } H = \begin{pmatrix} I_{n+2} & 0 \\ 0 & A \end{pmatrix} \in SO_{ex}(2n+2), \tag{2.8}$$

we obtain

$$\Theta_\lambda = \begin{pmatrix} 0 & 0 & -\varepsilon_1 b_1^T \delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_2 b_2^T \delta_2 & 0 & 0 \\ \delta_1 b_1 & 0 & \omega_1 & 0 & \lambda\delta_1 & 0 \\ 0 & \delta_2 b_2 & 0 & \omega_2 & 0 & \lambda\delta_2 \\ 0 & 0 & -\lambda\delta_1 & 0 & \vartheta_1 & 0 \\ 0 & 0 & 0 & -\lambda J_\varepsilon \delta_2 & 0 & \vartheta_2 \end{pmatrix}, \tag{2.9}$$

where

$$\begin{gathered} \omega_r = \delta_r F_r - F_r^T \delta_r, \quad \vartheta_1 = \delta_1 F_1^T - F_1 \delta_1, \quad \vartheta_2 = J_\varepsilon \delta_2 F_2^T J_\varepsilon - F_2 \delta_2, \\ J_\varepsilon = \begin{pmatrix} I_{n_2-1} & 0 \\ 0 & \varepsilon \end{pmatrix}. \end{gathered} \tag{2.10}$$

It is easy to see that $d\Theta_\lambda + \Theta_\lambda \wedge \Theta_\lambda = 0$ if and only if $d\tilde{\Theta}_\lambda + \tilde{\Theta}_\lambda \wedge \tilde{\Theta}_\lambda = 0$, which is equivalent to that (F_1, F_2, A) satisfies the following system of PDE:

$$\begin{cases} dA_r = -\vartheta_r A_r, \\ db_r = -\vartheta_r b_r, \\ d\omega_r + \omega_r \wedge \omega_r - \varepsilon_r \delta_r b_r \wedge b_r^T \delta_r = 0, \end{cases} \tag{2.11}$$

i.e., the Gauss-Codazzi-Ricci equations for the isometric immersion φ.

On putting

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_2 \\ 0 & 0 & -\delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_2 & 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 0 & 0 & 0 & -\varepsilon_1 b_1^T & 0 \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon_2 b_2^T J_\varepsilon \\ 0 & 0 & 0 & 0 & -F_1^T & 0 \\ 0 & 0 & 0 & 0 & 0 & -F_2^T J_\varepsilon \\ b_1 & 0 & F_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & F_2 & 0 & 0 \end{pmatrix},$$

$$\sigma = \begin{pmatrix} I_{n+2} & 0 \\ 0 & -I_n \end{pmatrix}, \quad \sigma' = \begin{pmatrix} 1 & & & & & \\ & -1 & & & 0 & \\ & & I_{n_1} & & & \\ & & & -I_{n_2} & & \\ & 0 & & & I_{n_1} & \\ & & & & & -I_{n_2} \end{pmatrix}, \tag{2.12}$$

we have

$$\begin{gathered}\Theta_\lambda = a\lambda + [a, v],\\ b_1 = v^{(51)}, \quad b_2 = v^{(62)}, \quad F_1 = v^{(53)}, \quad F_2 = v^{(64)}.\end{gathered} \tag{2.13}$$

With respect to σ and σ', the Lie algebra $\mathscr{G} = so_{ex}(2n+2)$ has the Cartan decompositions $\mathscr{G} = \mathscr{P} \oplus \mathscr{K} = \mathscr{P}' \oplus \mathscr{K}'$, respectively. Let

$$\mathscr{G}_a = \{y \in \mathscr{G} \mid [a, y] = 0\}, \quad \mathscr{G}_a^\perp = \{z \in \mathscr{G} \mid \operatorname{tr}(zy) = 0 \text{ for } y \in \mathscr{G}_a\},$$
$$\wedge_{\sigma,\sigma'}\mathscr{G} = \{X(\lambda) \in \wedge\mathscr{G} \mid \sigma X(\lambda)\sigma = X(-\lambda), \sigma' X(\lambda)\sigma' = X(\lambda)\}.$$

Clearly, a is $(\mathscr{P} \cap \mathscr{K}')$-valued 1-form, $v : U \to g_a^\perp \cap \mathscr{P} \cap \mathscr{K}'$ is a smooth map. Thus, Θ_λ is a $\wedge_{\sigma,\sigma'}\mathscr{G}$-valued 1-form.

Consider the system

$$\begin{cases} d\Phi_\lambda = \Phi_\lambda(a\lambda + [a, v]) = \Phi_\lambda\Theta_\lambda,\\ \Phi_\lambda(0) = I_{2n+2},\end{cases} \tag{2.14}$$

of which the integrability condition is (2.11). For the solution Φ_λ to (2.14), we have

$$A = \begin{pmatrix} \Phi_0^{(55)} & \Phi_0^{(56)} J_\varepsilon \\ J_\varepsilon \Phi_0^{(65)} & J_\varepsilon \Phi_0^{(66)} J_\varepsilon \end{pmatrix}^T A(0). \tag{2.15}$$

Set $\tilde{\Phi}_\lambda = Q_1 H^{-1}(0)\Phi_\lambda H$, where

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \kappa_2^{-1} v_2 \\ 0 & 1 & 0 & 0 & \kappa_1^{-1} v_1 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & A_2(0) & 0 \\ 0 & 0 & 0 & A_1(0) & 0 \end{pmatrix}.$$

Then $\tilde{\Phi}_\lambda$ satisfies

$$d\tilde{\Phi}_\lambda = \tilde{\Phi}_\lambda \tilde{\Theta}_\lambda.$$

Write

$$\tilde{\Phi}_1 = (\mathbf{r}_1, \mathbf{r}_2, \tilde{e}_1, \ldots, \tilde{e}_{2n}), \quad \tilde{\mathbf{r}} = \frac{1}{\kappa_1}\mathbf{r}_1 + \frac{1}{\kappa_2}\mathbf{r}_2, \quad \tilde{\Psi} = (\tilde{\mathbf{r}}, \tilde{e}_1, \ldots, \tilde{e}_{2n-1}).$$

By a straightforward calculation, one can see that $\tilde{e}_{2n}(0) = v_1\kappa_1^{-1}\mathbf{r}_1(0) + v_2\kappa_2^{-1}\mathbf{r}_2(0)$. So, it follows from Lemma 2.2 that $\tilde{\Psi}$ satisfies the following system

$$d\tilde{\Psi} = \tilde{\Psi}\Xi, \quad \tilde{\Psi}(0) = \begin{pmatrix} \kappa_1^{-1} & \kappa_2^{-1} & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & A_1(0)^T & A_2(0)^T \end{pmatrix}^T,$$

where Ξ is defined in (2.6). Set

$$\Psi = Q_2\tilde{\Psi} \quad \text{with } Q_2 = \begin{pmatrix} \frac{1}{2}\kappa_1 & \frac{1}{2}\kappa_2 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & A_1(0)^T & A_2(0)^T J_\varepsilon \end{pmatrix}.$$

Then $\Psi : U \to SO_c(2n)$ satisfies the system (1.1). Hence, we obtain

$$\begin{aligned} \mathbf{r} &= J^2 Q_2 \tilde{\mathbf{r}} = J^2 Q_2 Q_1 H^{-1}(0)\Phi_1(\kappa_1^{-1}H^{(1)} + \kappa_2^{-1}H^{(2)}) \\ &= Q(\kappa_1^{-1}\Phi_1^{(1)} + \kappa_2^{-1}\Phi_1^{(2)}), \end{aligned} \tag{2.16}$$

where

$$Q = \begin{pmatrix} \frac{c^2}{2}\kappa_1 & \frac{c^2}{2}\kappa_2 & 0 & \frac{c(2-cc_1)}{2\kappa_1} b_1^T(0) & \frac{c(2-cc_2)}{2\kappa_2} b_2^T(0)J_\varepsilon \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & A_1(0)^T & A_2(0)^T J_\varepsilon \end{pmatrix} \tag{2.17}$$

is a constant $2n \times (2n+2)$ matrix. Summing up and combining Proposition 1.6, we have proved the following

THEOREM 2.3. *Let $U \subset M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ be a simply-connected domain around the origin $x = 0$, and $\varphi : U \to M^{2n-1}(c)$, $n = n_1 + n_2$, a locally isometric immersion, of which the first and second fundamental forms can be written as* (1.16). *Then there exists a smooth map $(F_1, F_2, b) : U \to gl(n_1)_* \times gl(n_2)_* \times S^{n-1}(\varepsilon)$ such that Θ_λ defined by* (2.9) *is a flat connection and the system* (2.14) *has a unique solution Φ_λ satisfying*

$$\mathbf{r} = \mathbf{r}_c \circ \varphi = Q(\kappa_1^{-1}\Phi_1^{(1)} + \kappa_2^{-1}\Phi_1^{(2)}). \tag{2.18}$$

Conversely, for a map $(F_1, F_2, b) : \mathbf{R}^n \to gl(n_1)_* \times gl(n_2)_* \times S^{n-1}(\varepsilon)$, if (2.14) *has a unique solution Φ_λ, then there exists a smooth map $A = (a_{ij}) : U \to O_\varepsilon(n)$ such that $b = AE_n$. Moreover, if $U = \{x \in \mathbf{R}^n \mid a_i(x) \neq 0, a_s(x) \neq 0 \text{ for all } i, s\}$ is not empty, then there exists an isometric immersion $\varphi : U \to M^{2n-1}(c)$ with flat normal bundle such that the first and second fundamental forms of φ are given by* (1.16), *and* (2.18) *holds.*

§3. Darboux transformation

We now consider the Darboux transformation for solutions of the system (2.14). Since $\Theta_\lambda = a\lambda + [a, u]$ is a $\wedge_{\sigma,\sigma'}\mathscr{G}$-valued 1-form, then Φ_λ satisfies the following $K'/(K \cap K')$-reality condition (cf. [TU]):

$$f(\lambda)Jf(\bar{\lambda})^* = J, \quad \overline{f(\bar{\lambda})} = f(\lambda), \quad \sigma f(\lambda)\sigma = f(-\lambda), \quad \sigma' f(\lambda)\sigma' = f(\lambda). \tag{3.1}$$

Let O_∞ be an open neighborhood around ∞ in $\mathbf{C} \cup \{\infty\} = S^2$, and let

$$G_-^m = \{f : O_\infty \to GL(N, \mathbf{C}) \mid f \text{ is a holomorphic rational fraction satisfying } (3.1)_1 \text{ and } f(\infty) = I_N\},$$

$$(G_-^m)_{\sigma,\sigma'} = \{f(\lambda) \in G_-^m \mid f(\lambda) \text{ satisfies } (3.1)\}.$$

A map $\pi : V \to \boldsymbol{C}^N$ is called a *J-Hermitian projection* if

$$\pi^2 = \pi, \quad J\pi^* = \pi J.$$

Clearly, $\pi' = I - \pi$ is also a J-Hermitian projection if π is one. Thus, a simple element of G_-^m is of the form [TU]:

$$h_{\alpha,\pi}(\lambda) = \pi' + \frac{\lambda - \alpha}{\lambda - \bar{\alpha}}\pi = I - \frac{\alpha - \bar{\alpha}}{\lambda - \bar{\alpha}}\pi \tag{3.2}$$

for $\alpha \in \boldsymbol{C}^* = \boldsymbol{C}\backslash\{0\}$. Let τ be a diagonal complex matrix satisfying $\tau^2 = \sigma$. A direct computation yields the following

LEMMA 3.1. *Let π_0 be a J-Hermitian projection in $\boldsymbol{C}^N$ satisfying*

$$\bar{\pi}_0 = \pi_0, \quad \sigma\pi_0\sigma\pi_0 = \pi_0\sigma\pi_0\sigma, \quad \sigma'\pi_0\sigma' = \pi_0.$$

If $\pi = \tau^{-1}\pi_0\tau$, then $f(\lambda) = h_{\alpha,\pi}h_{-\alpha,\sigma\pi\sigma} \in (G_-^m)_{\sigma,\sigma'}$ for $\alpha \in \sqrt{-1}\boldsymbol{R}$.

Let Φ_λ be a solution to (2.14), L a constant complex $s \times N$ matrix satisfying

$$LJ\sigma L^T = 0, \quad \mathrm{span}\{L\sigma'\} = \mathrm{span}\{L\}, \quad \det(LJL^T) \neq 0. \tag{3.3}$$

Set

$$\pi_0 = JL^T(LJL^T)^{-1}L, \quad \pi = \tau^{-1}\pi_0\tau,$$

$$h = L\tau\Phi_\alpha, \quad \tilde{\pi}_1 = Jh^*(hJh^*)^{-1}h,$$

$$\Psi_\lambda = h_{\bar{\alpha},\pi}\Phi_\lambda h_{\alpha,\tilde{\pi}_1}, \quad \tilde{h} = L\tau\sigma\Psi_{-\alpha}, \quad \tilde{\pi}_2 = J\tilde{h}^*(\tilde{h}J\tilde{h}^*)^{-1}\tilde{h}.$$

Let

$$\alpha = \sqrt{-1}\mu \quad \text{for } \mu \in \boldsymbol{R}\backslash\{0\},$$

$$\Delta = \frac{1}{2}hJh^*, \quad \tilde{\Delta} = \frac{1}{2}\tilde{h}J\tilde{h}^*, \quad h' = \frac{dh}{d\mu}, \quad \rho = -h'\sigma\tau h^*.$$

Then it is easy to see that π_0 satisfies conditions in Lemma 3.1, ρ is a real skew-symmetric $s \times s$ matrix, Δ and $\tilde{\Delta}$ all are real symmetric invertible $s \times s$ matrices. Hence, we may write the following Darboux matrix [HS]:

$$\begin{aligned} D_\lambda = h_{\sqrt{-1}\mu,\tilde{\pi}_1}h_{-\sqrt{-1}\mu,\tilde{\pi}_2} &= \left(I - \frac{2\sqrt{-1}\mu}{\lambda + \sqrt{-1}\mu}\tilde{\pi}_1\right)\left(I + \frac{2\sqrt{-1}\mu}{\lambda - \sqrt{-1}\mu}\tilde{\pi}_2\right) \\ &= I - \frac{\sqrt{-1}\mu}{\lambda + \sqrt{-1}\mu}Jh^*\Delta^{-1}h \\ &\quad + \frac{\sqrt{-1}\mu}{\lambda - \sqrt{-1}\mu}J(\sigma h^* + \mu h^*\Delta^{-1}\rho)\tilde{\Delta}^{-1}(h\sigma - \mu\rho\Delta^{-1}h) \\ &\quad + \frac{2\mu^3}{\lambda^2 + \mu^2}Jh^*\Delta^{-1}\rho\tilde{\Delta}^{-1}(h\sigma - \mu\rho\Delta^{-1}h), \end{aligned} \tag{3.4}$$

$$d_1 = \left(\frac{dD_{\lambda^{-1}}}{d\lambda}\right)\Big|_{\lambda=0} = 2\sqrt{-1}\mu(\tilde{\pi}_2^{\perp} - \tilde{\pi}_1^{\perp}) \tag{3.5}$$
$$= \sqrt{-1}\mu J\{(\sigma h^* + \mu h^*\Delta^{-1}\rho)\tilde{\Delta}^{-1}(h\sigma - \mu\rho\Delta^{-1}h) - h^*\Delta^{-1}h\}.$$

Thus, in the similar way as in [HS], we have following

THEOREM 3.2. *Let Φ_λ be a solution of the system* (2.14), *and L a real constant $s \times N$ matrix. Set $h = L\tau\Phi_{\sqrt{-1}\mu}$ for $\mu \in \boldsymbol{R}\backslash\{0\}$. Then there is an open neighbourhood U around the origin* 0 *such that on U, $\tilde{\Phi}_\lambda = D_\lambda(0)^{-1}\Phi_\lambda D_\lambda$ satisfies the system* (2.14) *with $\tilde{v} = v + (d_1)_{\mathscr{G}_a^{\perp}} : U \to \mathscr{G}_a^{\perp} \cap \mathscr{P} \cap \mathscr{K}'$, namely, $a\lambda + [a, \tilde{v}]$ is a $\wedge_{\sigma,\sigma'}\mathscr{G}$-valued* 1*-form, where D_λ and d_1 are defined by* (3.4) *and* (3.5), *respectively.*

In the following, we take $N = 2(n+1)$. By Theorem 2.3 and Theorem 3.2, it is sufficient to find the Darboux matrix (3.4) preserving $b(x) \in S^{n-1}(\varepsilon)$.

LEMMA 3.3. *Let Φ_λ be a solution of the system* (2.14), *L a complex constant $s \times 2(n+1)$ matrix. Assume that $\lambda_0 \in \boldsymbol{C}$ and $h = L\Phi_{\lambda_0} = (\xi_1, \xi_2, \eta_1, \eta_2, \zeta_1, \zeta_2)$. Then $d(\zeta_r b_r - \lambda_0\xi_r) = 0$.*

Proof. Since h satisfies the equation $dh = h\Theta_{\lambda_0}$, i.e.,

$$\begin{aligned} d\xi_r &= \eta_r\delta_r b_r, \\ d\eta_1 &= -\varepsilon_1\xi_1 b_1^T\delta_1 + \eta_1\omega_1 - \lambda_0\zeta_1\delta_1, \\ d\eta_2 &= -\varepsilon_2\xi_2 b_2^T\delta_2 + \eta_2\omega_2 - \lambda_0 J_\varepsilon\zeta_2\delta_2, \\ d\zeta_r &= \lambda_0\eta_r\delta_r + \zeta_r\vartheta_r, \end{aligned}$$

then, by (2.11), we get

$$\begin{aligned} d(\zeta_r b_r - \lambda_0\xi_r) &= (d\zeta_r)b_r + \zeta_r\, db_r - \lambda_0\, d\xi_r \\ &= (\lambda_0\eta_r\delta_r + \zeta_r\vartheta_r)b_r - \zeta_r\vartheta_r b_r - \lambda_0\eta_r\delta_r b_r = 0. \end{aligned}$$ □

Let $\mu \in \boldsymbol{R}\backslash\{0\}$, $h = L\tau\Phi_{\sqrt{-1}\mu} = (\xi_1, \xi_2, \eta_1, \eta_2, \sqrt{-1}\zeta_1, \sqrt{-1}\zeta_2)$ satisfying

$$\begin{aligned} d\xi_r &= \eta_r\delta_r b_r, \\ d\eta_1 &= -\varepsilon_1\xi_1 b_1^T\delta_1 + \eta_1\omega_1 + \mu\zeta_1\delta_1, \\ d\eta_2 &= -\varepsilon_2\xi_2 b_2^T\delta_2 + \eta_2\omega_2 + \mu\zeta_2 J_\varepsilon\delta_2, \\ d\zeta_r &= \mu\eta_r\delta_r + \zeta_r\vartheta_r. \end{aligned} \tag{3.6}$$

By Theorem 3.2 and Lemma 3.3, if we choose L such that L satisfies (3.3) and

$$L^{(5)}b_1(0)^T - \mu L^{(1)} = 0, \quad L^{(6)}b_2(0)^T - \mu L^{(2)} = 0, \tag{3.7}$$

then there exists an open neighbourhood U around the origin 0 such that $h = (\xi_1, \xi_2, \eta_1, \eta_2, \sqrt{-1}\zeta_1, \sqrt{-1}\zeta_2)$ satisfies

$$\begin{gathered} hJ\sigma h^* = \varepsilon_1\xi_1\xi_1^T + \varepsilon_2\xi_2\xi_2^T + \eta_1\eta_1^T + \eta_2\eta_2^T - \zeta_1\zeta_1^T - \zeta_2 J_\varepsilon \zeta_2^T = 0, \\ \det(hJh^*) \neq 0, \quad \zeta_r b_r^T - \mu\xi_r = 0, \\ \operatorname{span}\{h\sigma'\} = \operatorname{span}\{h\}, \end{gathered} \tag{3.8}$$

in U. Thus, (3.4) can be written as

(3.9)

$$D_\lambda = I - \frac{2\mu}{\lambda^2+\mu^2} \cdot \begin{pmatrix} \mu\varepsilon_1\xi_1^T\Delta_1\xi_1 & 0 & \mu\varepsilon_1\xi_1^T\Delta_1\eta_1 & 0 & -\lambda\varepsilon_1\xi_1^T\Delta_2\zeta_1 & 0 \\ 0 & \mu\varepsilon_2\xi_2^T\Delta_1\xi_2 & 0 & \mu\varepsilon_2\xi_2^T\Delta_1\eta_2 & 0 & -\lambda\varepsilon_2\xi_2^T\Delta_2\zeta_2 \\ \mu\eta_1^T\Delta_1\xi_1 & 0 & \mu\eta_1^T\Delta_1\eta_1 & 0 & -\lambda\eta_1^T\Delta_2\zeta_1 & 0 \\ 0 & \mu\eta_2^T\Delta_1\xi_2 & 0 & \mu\eta_2^T\Delta_1\eta_2 & 0 & -\lambda\eta_2^T\Delta_2\zeta_2 \\ \lambda\zeta_1^T\Delta_1\xi_1 & 0 & \lambda\zeta_1^T\Delta_1\eta_1 & 0 & \mu\zeta_1^T\Delta_2\zeta_1 & 0 \\ 0 & \lambda J_\varepsilon\zeta_2^T\Delta_1\xi_2 & 0 & \lambda J_\varepsilon\zeta_2^T\Delta_1\eta_2 & 0 & \mu J_\varepsilon\zeta_2^T\Delta_2\zeta_2 \end{pmatrix},$$

where

$$\Delta_1 = (\Delta + \mu\rho)^{-1}, \quad \Delta_2 = (\Delta - \mu\rho)^{-1},$$

$$\begin{gathered} \Delta = \frac{1}{2}hJh^* = \zeta_1\zeta_1^T + \zeta_2 J_\varepsilon\zeta_2^T = \varepsilon_1\xi_1\xi_1^T + \varepsilon_2\xi_2\xi_2^T + \eta_1\eta_1^T + \eta_2\eta_2^T, \\ \rho = -h'\sigma Jh^* = \zeta_1'\zeta_1^T + \zeta_2' J_\varepsilon\zeta_2^T - \varepsilon_1\xi_1'\xi_1^T - \varepsilon_2\xi_2'\xi_2^T - \eta_1'\eta_1^T - \eta_2'\eta_2^T. \end{gathered} \tag{3.10}$$

Here $h' = dh/d\mu = (\xi_1', \xi_2', \eta_1', \eta_2', \sqrt{-1}\zeta_1', \sqrt{-1}\zeta_2')$.

Set $\tilde{\Phi}_\lambda = \Phi_\lambda D_\lambda$. It is easy from Theorem 3.2 to see that $\tilde{\Phi}_\lambda$ satisfies the system (2.14) with $\tilde{v} = v + (d_1)_{\mathscr{G}_a^\perp}$. Thus, by (2.13) and (2.15), we have

$$\begin{gathered} \tilde{F}_1 = (\tilde{v}^{(53)})_{\text{off}} = F_1 - 2\mu(\zeta_1^T\Delta_1\eta_1)_{\text{off}}, \\ \tilde{F}_2 = (\tilde{v}^{(64)})_{\text{off}} = F_2 - 2\mu(J_\varepsilon\zeta_2^T\Delta_1\eta_2)_{\text{off}}, \\ \tilde{b}_1 = \tilde{v}^{(51)} = b_1 - 2\mu\zeta_1^T\Delta_1\xi_1 = \kappa_1^{-1}(\tilde{a}_1, \ldots, \tilde{a}_{n_1})^T, \\ \tilde{b}_2 = \tilde{v}^{(64)} = b_2 - 2\mu J_\varepsilon\zeta_2^T\Delta_1\xi_2 = \kappa_2^{-1}(\tilde{a}_{n_1+1}, \ldots, \tilde{a}_n)^T, \\ \tilde{A} = \begin{pmatrix} \tilde{\Phi}_0^{(55)} & \tilde{\Phi}_0^{(56)}J_\varepsilon \\ J_\varepsilon\tilde{\Phi}_0^{(65)} & J_\varepsilon\tilde{\Phi}_0^{(66)}J_\varepsilon \end{pmatrix}^T A(0) = A - 2\begin{pmatrix} \zeta_1^T\Delta_1\zeta_1 & 0 \\ 0 & J_\varepsilon\zeta_2^T\Delta_1\zeta_2 \end{pmatrix} A. \end{gathered} \tag{3.11}$$

Since $\zeta_r b_r = \mu\xi_r$, then $\tilde{b}^T = (\kappa_1^{-1}v_1\tilde{b}_1^T, \kappa_2^{-1}v_2\tilde{b}_2^T) = \tilde{A}E_n$. Noting that $\tilde{A} \in O_\varepsilon(n)$, we see that $\tilde{b} \in S^{n-1}(\varepsilon)$. Hence, we have the following

THEOREM 3.4. *Let* $\varphi : M_1^{n_1}(c_1) \times M_2^{n_2}(c_2) \to M^{2n-1}(c)$, $n = n_1 + n_2$, *be a local isometric immersion whose first and second fundamental forms can be written as* (1.16). *Suppose that* Φ_λ *is a solution of the system* (2.14), $\mu \in \boldsymbol{R}\backslash\{0\}$, *and* L *is a real constant* $s \times 2(n+1)$ *matrix satisfying* (3.7). *Let* $h = L\tau\Phi_{\sqrt{-1}\mu}$ *and* $\tilde{\Phi}_\lambda = \Phi_\lambda D_\lambda$ *where* D_λ *is the Darboux matrix given by* (3.4). *If* $\tilde{a}_j(0) \neq 0$,

$\tilde{a}_s(0) \neq 0$ *for all* j, s, *then there exist an open neighbourhood* U *around the origin* $0 \in M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ *and a local isometric immersion* $\tilde{\varphi} : U \to M^{2n-1}(c)$ *such that* $\tilde{\mathbf{r}} = \mathbf{r}_c \circ \tilde{\varphi}$ *can be expressed explicitly by*

$$\tilde{\mathbf{r}} = \tilde{Q} D_1^{-1}(0)(\kappa_1^{-1}\tilde{\Phi}_1^{(1)} + \kappa_2^{-1}\tilde{\Phi}_1^{(2)}) = \tilde{Q} D_1^{-1}(0)\Phi_1(\kappa_1^{-1}D_1^{(1)} + \kappa_2^{-1}D_1^{(2)}), \tag{3.12}$$

where $\tilde{Q}$ *is a constant matrix defined by* (2.17) *with* $\tilde{A}(0)$, $\tilde{b}_1(0)$ *and* $\tilde{b}_2(0)$.

Remark. The above process of the Darboux transformation is purely algebraic. Hence, starting from a special solution Φ_λ to (2.14) for which the corresponding $\mathbf{r} = Q(\kappa_1^{-1}\Phi_1^{(1)} + \kappa_2^{-1}\Phi_1^{(2)})$ may be degenerated, we can repeat the processes via the purely algebraic algorithm and obtain a sequence of solutions to (2.14): $\Phi_\lambda \to \tilde{\Phi}_\lambda \to \tilde{\tilde{\Phi}}_\lambda \to \cdots$, from which we obtain a sequence of local isometric immersions from $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ to $M^{2n-1}(c)$.

§4. The construction of local isometric immersions from a trivial solution

For $c \neq 0$ we may take the following trivial solution of (2.11):

$$F_r = 0, \quad A = \begin{pmatrix} I_{n_1-1} & 0 & 0 & 0 \\ 0 & \hat{\varepsilon} & 0 & \dfrac{1}{\sqrt{2}} \\ 0 & 0 & I_{n_2-1} & 0 \\ 0 & \dfrac{-1}{\sqrt{2}} & 0 & \varepsilon\hat{\varepsilon} \end{pmatrix}, \quad b = AE_n, \quad \hat{\varepsilon} = \sqrt{1 - \frac{\varepsilon}{2}}. \tag{4.1}$$

Set

$$k_1 = \frac{\kappa_1}{\nu_1\sqrt{2}}, \quad k_2 = \frac{\kappa_2 \varepsilon\hat{\varepsilon}}{\nu_2}. \tag{4.2}$$

For $c = c_1 = 0$, we may take the following trivial solution of (2.11):

$$F_r = 0, \quad A = I_n, \quad b = AE_n,$$

and set

$$k_1 = 1, \quad k_2 = \nu_2^{-1}\kappa_2.$$

Thus, by writting $b_r^T = (0, \dots, k_r)$, the solution to (2.14) can be expressed as a block matrix

$$\Phi_\lambda = (\Phi_\lambda^{(ij)}), \quad \text{with } \Phi_\lambda^{(ij)} = 0 \quad \text{for } i + j = \text{odd}, \quad i, j = 1, \dots, 6, \tag{4.3}$$

where

$$\Phi_\lambda^{(11)} = \frac{1}{\chi_1^2}(\varepsilon_1 k_1^2 X_{n_1} + \lambda^2), \quad \Phi_\lambda^{(22)} = \frac{1}{\chi_2^2}(\varepsilon_2 k_2^2 X_n + \varepsilon\lambda^2),$$

$$\Phi_\lambda^{(13)} = -\varepsilon_1(\Phi_\lambda^{(31)})^T = -\varepsilon_1\left(0, \dots, 0, \frac{k_1}{\chi_1} Y_{n_1}\right),$$

$$\Phi_\lambda^{(24)} = -\varepsilon_2(\Phi_\lambda^{(42)})^T = -\varepsilon_2\left(0,\dots,0,\frac{k_2}{\chi_2}\,Y_n\right),$$

$$\Phi_\lambda^{(15)} = \varepsilon_1(\Phi_\lambda^{(15)})^T = \varepsilon_1\left(0,\dots,0,\frac{\lambda k_1}{\chi_1^2}(X_{n_1}-1)\right),$$

$$\Phi_\lambda^{(26)} = \frac{\varepsilon_2}{\varepsilon}(\Phi_\lambda^{(62)})^T = \varepsilon_2\left(0,\dots,0,\frac{\lambda k_2}{\chi_2^2}(X_n-1)\right),$$

$$\Phi_\lambda^{(33)} = \operatorname{diag}(X_1,\dots,X_{n_1}),\quad \Phi_\lambda^{(44)} = \operatorname{diag}(X_{n_1+1},\dots,X_n),$$

$$\Phi_\lambda^{(35)} = -\Phi_\lambda^{(53)} = \operatorname{diag}\left(Y_1,\dots,Y_{n_1-1},\frac{\lambda}{\chi_1}\,Y_{n_1}\right),$$

$$\Phi_\lambda^{(64)} = -J_\varepsilon\Phi_\lambda^{(46)} = -\operatorname{diag}\left(Y_{n_1+1},\dots,Y_{n-1},\frac{\varepsilon\lambda}{\chi_2}\,Y_n\right),$$

$$\Phi_\lambda^{(55)} = \operatorname{diag}\left(X_1,\dots,X_{n_1-1},\frac{1}{\chi_1^2}(\lambda^2 X_{n_1}+\varepsilon_1 k_1^2)\right),$$

$$\Phi_\lambda^{(66)} = \operatorname{diag}\left(X_{n_1+1},\dots,X_{n-1},\frac{1}{\chi_2^2}(\varepsilon\lambda^2 X_n+\varepsilon_2 k_2^2)\right),$$

$$\chi_1=\chi_1(\lambda)=\sqrt{\lambda^2+\varepsilon_1 k_1^2},\quad \chi_2=\chi_2(\lambda)=\sqrt{\varepsilon\lambda^2+\varepsilon_2 k_2^2},$$

$$X_I=\cos(\lambda x_I)\ \ (I\neq n_1,n),\quad X_{n_1}=\cos(\chi_1 x_{n_1}),\quad X_n=\cos(\chi_2 x_n),$$

$$Y_I=\sin(\lambda x_I)\ \ (I\neq n_1,n),\quad Y_{n_1}=\sin(\chi_1 x_{n_1}),\quad Y_n=\sin(\chi_2 x_n).$$

For $\mu\in\boldsymbol{R}$, we choose the following constant $2\times 2(n+1)$ matrix L:

$$L=\begin{pmatrix} L_1 & 0 & L_3 & 0 & L_5 & 0\\ 0 & L_2 & 0 & L_4 & 0 & L_6\end{pmatrix}, \tag{4.4}$$

$$=\begin{pmatrix} l_0 & 0 & l_1\cdots l_{n_1} & 0 & l_{n+1}\cdots l_{n+n_1} & 0\\ 0 & l_0' & 0 & l_{n_1+1}\cdots l_n & 0 & l_{n+n_1+1}\cdots l_{2n}\end{pmatrix},$$

such that

$$\begin{gathered}\sum_j l_{n+j}^2=\sum_j l_j^2+\varepsilon_1 l_0^2\neq 0,\quad \sum_{s\neq n} l_{n+s}^2+\varepsilon l_{2n}^2=\sum_s l_s^2+\varepsilon_2 l_0'^2\neq 0,\\ k_1 l_{n+n_1}-\mu l_0=0,\quad k_2 l_{2n}-\mu l_0'=0.\end{gathered} \tag{4.5}$$

It is easy to see that L defined by (4.4) satisfies (3.3) and (3.7). Thus, we have

$$h=L\tau\Phi_{\sqrt{-1}\mu}=\begin{pmatrix} h_1 & 0 & h_3 & 0 & \sqrt{-1}h_5 & 0\\ 0 & h_2 & 0 & h_4 & 0 & \sqrt{-1}h_6\end{pmatrix}$$

$$=\begin{pmatrix} \xi_1 & 0 & \eta_1\cdots\eta_{n_1} & 0 & \sqrt{-1}\zeta_1\cdots\sqrt{-1}\zeta_{n_1} & 0\\ 0 & \xi_2 & 0 & \eta_{n_1+1}\cdots\eta_n & 0 & \sqrt{-1}\zeta_{n_1+1}\cdots\sqrt{-1}\zeta_n\end{pmatrix},$$

where

$$
\begin{aligned}
&\xi_1 = l_0 \cos(\gamma_1\sqrt{-1}x_{n_1}) - l_{n_1}\sqrt{-1}\gamma_1^{-1}k_1 \sin(\gamma_1\sqrt{-1}x_{n_1}) = \mu^{-1}k_1\zeta_{n_1},\\
&\xi_2 = l_0' \cos(\gamma_2\sqrt{-1}x_n) - l_n\sqrt{-1}\gamma_2^{-1}k_2 \sin(\gamma_2\sqrt{-1}x_n) = \mu^{-1}k_2\zeta_n,\\
&\eta_I = l_I \operatorname{ch}(\mu x_I) + l_{n+I} \operatorname{sh}(\mu x_I), \quad (I \neq n_1, n)\\
&\eta_{n_1} = l_{n_1} \cos(\gamma_1\sqrt{-1}x_{n_1}) - l_0\sqrt{-1}k_1^{-1}\gamma_1 \sin(\gamma_1\sqrt{-1}x_{n_1}),\\
&\eta_n = l_n \cos(\gamma_2\sqrt{-1}x_n) - l_0'\sqrt{-1}k_2^{-1}\gamma_2 \sin(\gamma_2\sqrt{-1}x_n),\\
&\zeta_I = l_I \operatorname{sh}(\mu x_I) + l_{n+I} \operatorname{ch}(\mu x_I), \quad (I \neq n_1, n),\\
&\gamma_r = -\sqrt{-1}\chi_r(\sqrt{-1}\mu).
\end{aligned}
\tag{4.6}
$$

It is clear that

$$h_i(0) = L_i \quad \text{for } i = 1, \ldots, 6.$$

When $\gamma_r = 0$, we have

$$\xi_1 = l_0, \quad \xi_2 = l_0', \quad \zeta_n = l_{2n}, \quad \zeta_{n_1} = l_{n+n_1}.$$

It is easy to see that $\rho = -h'\sigma Jh^*$ is a 2×2 diagonal matrix. Since ρ is skew-symmetric, then we get $\rho = 0$. Thus, by (3.9)~(3.11), we have

$$
\begin{aligned}
&\Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, \quad \Delta_1 = h_5 h_5^T = \sum_i \zeta_i^2, \quad \Delta_2 = h_6 h_6^T \sum_s \zeta_s^2,\\
&D_1^{(1)} = \frac{1}{\Delta_1(1+\mu^2)}((1+\mu^2)\Delta_1 - 2\varepsilon_1 k_1^2 \zeta_{n_1}^2, 0, -2\mu k_1 \zeta_{n_1} h_3, 0, -2k_1\zeta_{n_1} h_5, 0)^T,\\
&D_1^{(2)} = \frac{1}{\Delta_2(1+\mu^2)}(0, (1+\mu^2)\Delta_2 - 2\varepsilon_2 k_2^2 \zeta_n^2, 0, -2\mu k_2 \zeta_n h_4, 0, -2k_2\zeta_n h_6)^T,\\
&\tilde{b}_1 = \frac{k_1}{\Delta_1}(-2\zeta_{n_1}\hat{h}_5, \hat{h}_5\hat{h}_5^T - \zeta_{n_1}^2)^T, \quad \tilde{b}_2 = \frac{k_2}{\Delta_2}(-2\zeta_n\hat{h}_6, \hat{h}_6\hat{h}_6^T - \zeta_n^2)^T,
\end{aligned}
\tag{4.7}
$$

and

$$
\tilde{A} = \begin{pmatrix}
I_{n_1-1} - \frac{2}{\Delta_1}\hat{h}_5^T\hat{h}_5 & -\frac{2\hat{\varepsilon}}{\Delta_1}\zeta_{n_1}\hat{h}_5^T & 0 & -\frac{\sqrt{2}}{\Delta_1}\zeta_{n_1}\hat{h}_5^T \\
-\frac{2}{\Delta_1}\zeta_{n_1}\hat{h}_5 & \hat{\varepsilon}\left(1 - \frac{2}{\Delta_1}\zeta_{n_1}^2\right) & 0 & \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{\Delta_1}\zeta_{n_1}^2 \\
0 & \frac{\sqrt{2}}{\Delta_2}\zeta_n\hat{h}_6^T & I_{n_2-1} - \frac{2}{\Delta_2}\hat{h}_6^T\hat{h}_6 & -\frac{2\varepsilon\hat{\varepsilon}}{\Delta_2}\zeta_n\hat{h}_6^T \\
0 & \frac{\sqrt{2}}{\Delta_2}\zeta_n^2 - \frac{\sqrt{2}}{2} & -\frac{2}{\Delta_2}\zeta_n\hat{h}_6 & \varepsilon\hat{\varepsilon}\left(1 - \frac{2}{\Delta_2}\zeta_n^2\right)
\end{pmatrix} \quad (c \neq 0),
\tag{4.8}
$$

$$\tilde{A} = \begin{pmatrix} I_{n_1} - \dfrac{2}{\Delta_1} h_5^T h_5 & 0 \\ 0 & I_{n_2} - \dfrac{2}{\Delta_2} h_6^T h_6 \end{pmatrix} \quad (c = 0),$$

where

$$\hat{h}_1 = (\zeta_1, \ldots, \zeta_{n_1 - 1}), \quad \hat{h}_2 = (\zeta_{n_1+1}, \ldots, \zeta_{n-1}).$$

By Theorem 3.4, if $\tilde{a}_j(0) \neq 0$, $\tilde{a}_s(0) \neq 0$ for all j, s, then there exist an open neighbourhood U around the point $x = 0$ in $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ and a non-degenerated locally isometric immersion $\tilde{\varphi} : U \to M^{2n-1}(c)$, such that its position vector in $\boldsymbol{R}^{2n}$ is

$$\tilde{\mathbf{r}} = \mathbf{r}_c \circ \tilde{\varphi} = \tilde{Q} D_1^{-1}(0)(\kappa_1^{-1} \tilde{\Phi}_1^{(1)} + \kappa_2^{-1} \tilde{\Phi}_1^{(2)}) = \tilde{Q} D_1^{-1}(0) \Phi_1 (\kappa_1^{-1} D_1^{(1)} + \kappa_2^{-1} D_1^{(2)}),$$

where $\tilde{Q}$ is a constant matrix defined by (2.17) with $\tilde{A}(0)$, $\tilde{b}_1(0)$ and $\tilde{b}_2(0)$. Hence, it is sufficient to choose suitably L such that

$$l_{n+I} \neq 0, \quad \sum_{i \neq n_1} l_{n+i}^2 - l_{n+n_1}^2 \neq 0, \quad \sum_{s \neq n} l_{n+s}^2 - l_{2n}^2 \neq 0. \tag{4.9}$$

Thus, starting from a trivial solution Φ_λ and using the Darboux transformation $\tilde{\Phi}_\lambda = \Phi_\lambda D_\lambda$ and (3.12), we can construct a series of locally isometric immersions from $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ to $M^{2n-1}(c)$.

Remark. It should be remarked that we may choose constant matrices L satisfying (4.5) and (4.9) in a quite arbitrary way. For example, according to the signs of $\gamma_1^2 = \mu^2 - \varepsilon_1 k_1^2$ and $\gamma_2^2 = \varepsilon \mu^2 - \varepsilon_2 k_2^2$, we can choose L as follows. If $\gamma_1^2 \geq 0$ and $\gamma_2^2 < 0$, then L may be taken as

$$L = \begin{pmatrix} \mu^{-1} k_1 l_{n_1} & 0 & L_3 & 0 & L_5 & 0 \\ 0 & \mu^{-1} k_2 l_{n-1} & 0 & L_4 & 0 & L_6 \end{pmatrix},$$

where

$$L_3 = (l_1, \ldots, l_{n_1 - 1}, \mu^{-1} \gamma_1 l_{n_1}), \quad L_5 = (l_1, \ldots, l_{n_1}),$$
$$L_4 = (l_{n_1+1}, \ldots, l_{n-2}, 0, 0), \quad L_6 = (l_{n_1+1}, \ldots, l_{n-2}, \sqrt{-1} \mu^{-1} \gamma_2 l_{n-1}, l_{n-1}),$$

such that

$$l_I \neq 0, \quad \sum_{i \neq n_1} l_i^2 - l_{n_1}^2 \neq 0, \quad \sum_{s \neq n-1, n} l_s^2 + (\varepsilon_2 \mu^{-2} k_2^2 - \varepsilon - 1) l_{n-1}^2 \neq 0.$$

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Department of Applied Mathematics
Tongji University
Shanghai 200092, China
e-mail: Qun_he@163.com

Department of Mathematics
West Brook Campus
Zhejiang University
Hangzhou 310028, China
e-mail: ybsh@mail.hz.zj.cn