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CONVERGENCE AND EXTENSION THEOREMS IN GEOMETRIC FUNCTION THEORY

Dedicated to Professor Shoshichi Kobayashi on the occasion of his 70th birthday

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Abstract

In this article we show several convergence and extension theorems for analytic hypersurfaces (not necessarily with normal crossings) and for closed pluripolar sets of complex manifolds. Moreover, a generalization of theorem of Alexander to complex spaces is given.

1. Introduction

Extending holomorphic mappings is one of the most important problems of Analysis in several complex variables. Much attention has been given to this problem from the viewpoint of Hyperbolic complex analysis since S. Kobayashi introduced the notion of the Kobayashi pseudodistance and used it to study Geometric function theory. In this direction, we recall here the remarkable theorem of Noguchi (see [14] or [17]).

Let X be relatively compact hyperbolically imbedded into Y. Let M be a complex manifold and A a complex hypersurface of M with only normal crossings.

If $\{f_j : M \backslash A \to X\}_{j=1}^{\infty}$ is a sequence of holomorphic mappings which converges uniformly on compact subsets of $M \backslash A$ to a holomorphic mapping $f : M \backslash A \to X$, then $\{\bar{f}_j\}_{j=1}^{\infty}$ converges uniformly on compact subsets of M to $\bar{f}$, where $\bar{f}_j : M \to Y$ and $\bar{f} : M \to Y$ are the unique holomorphic extensions of f_j and f over M.

The above theorem of Noguchi opened the new view in studying problems of extending holomorphic mappings. That is to study the Noguchi-type convergence-extension theorems. More precisely, a "Noguchi-type convergence-extension theorem" means a theorem on mappings analogous to the theorem of Noguchi of extending holomorphic mappings, which would keep the local uniform convergence. Recently, several Noguchi-type convergence-extension theo-

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rems for analytic hypersurfaces of complex manifolds have been obtained by various authors (see [11], [12], [14]).

The first aim of this article is to give several convergence and extension theorems for analytic hypersurfaces (not necessarily with normal crossings) of complex manifolds.

It is much to be regretted, therefore, that while a substantial amount of information has been amassed concerning convergence and extension theorems for analytic hypersurfaces through the years, the present knowledge of these theorems for pluripolar subsets, has remained extremely meagre.

For the convenient presentation, we give the following

Definition. Let X be a complex space. We say that X has the convergence-extension property through closed pluripolar sets (shortly X has (PEP)) if the following holds.

Let M be any complex manifold and A be any closed pluripolar subset of M. Let $f_j : M\backslash A \to X$, $j = 1, 2, \dots$ be holomorphic mappings which converge uniformly on compact subsets of $M\backslash A$ to a holomorphic mapping $f : M\backslash A \to X$. Then there are the unique holomorphic extensions $\bar{f}_j : M \to X$ of f_j and $\bar{f} : M \to X$ of f over M, and $\{\bar{f}_j\}_{j=1}^{\infty}$ converges uniformly on compact subsets of M to $\bar{f}$.

Up to now we only know the following three classes of complex manifolds having (PEP)

a) Every Siegel domain of the second kind in $\boldsymbol{C}^n$ has (PEP) [22]. This result of Sibony was generalized to the Siegel domain of the second kind in a Banach space by Thai D. D. [25].

b) Every hyperbolic compact Riemann surface has (PEP) (see [10] and [25]).

c) Every compact manifold whose universal covering is a polynomially convex bounded domain of $\boldsymbol{C}^n$, also has (PEP) (see [23] and [25]).

The second aim of this paper is to show the new class of complex spaces which also have (PEP). These are the class of pseudoconvex complex spaces having the weak 1-EP.

The last aim of this paper is to generalize the theorem of Alexander [1, Thm 6.2.] to complex spaces.

Here then is a brief outline of the content of this paper.

In §2 we review some basic properties of Geometric Function Theory needed for our purpose.

In §3 we are going to prove the following

Theorem 1. *Let X be a pseudoconvex complex space having Δ^*-EP. Let A be any analytic hypersurface of a complex manifold M.*

Let $\{f_j : M\backslash A \to X\}_{j=1}^{\infty}$ be a sequence of holomorphic mappings which converges uniformly on compact subsets of $M\backslash A$ to a holomorphic mapping $f : M\backslash A \to X$. Then there are unique holomorphic extensions $\bar{f}_j : M \to X$ and $\bar{f} : M \to X$ of f_j and f over M, and $\{\bar{f}_j\}_{j=1}^{\infty}$ converges uniformly on compact subsets of M to $\bar{f}$.

THEOREM 2. *Let X be a complex subspace of a hyperbolic complex space Y such that X has Δ^*-EP for Y. Let A be any analytic hypersurface of a complex manifold M.*

Let $\{f_j : M \backslash A \to X\}_{j=1}^{\infty}$ be a sequence of holomorphic mappings which converges uniformly on compact subsets of $M \backslash A$ to a holomorphic mapping $f : M \backslash A \to X$. Then there are unique holomorphic extensions $\bar{f}_j : M \to Y$ and $\bar{f} : M \to Y$ of f_j and f over M, and $\{\bar{f}_j\}_{j=1}^{\infty}$ converges uniformly on compact subsets of M to $\bar{f}$.

THEOREM 3. *Let X be a weakly disc-convex complex space. Let M be a complex manifold of dimension m, and let A be a subset which is nowhere dense in a complex subspace $B \subset M$ of dimension $\leq m-1$. Then every holomorphic mapping $f : M \backslash A \to X$ extends to a holomorphic mapping $F : M \to X$.*

In §4 we are going to prove the following

THEOREM 4. *Let X be a pseudoconvex complex space having the weak 1-EP. Then X has (PEP).*

In §5 we are going to prove the following

THEOREM 5. *Let M be a complex space of Stein type and $\mathscr{F} \subset \mathrm{Hol}(B^n, M)$. If the restriction of $\mathscr{F}$ to each complex line through the origin is a normal family, then $\mathscr{F}$ is a normal family.*

2. Basic notions and auxiliary results

2.1. In this article, we shall make use of properties of complex spaces as in Gunning-Rossi [8].

2.2. We denote the Kobayashi pseudodistance on a complex space X by d_X. X is called to be hyperbolic if d_X is a distance. For details concerning Hyperbolic complex analysis we refer the readers to the books of S. Kobayashi [14] and S. Lang [16].

2.3. A complex space is called to be taut if whenever Y is a complex space and $f_j : Y \to X$ is a sequence of holomorphic mappings, then either there exists a subsequence which is compactly divergent or a subsequence which converges uniformly on compact subsets to a holomorphic mapping $f : Y \to X$. It suffices that this condition should hold when $Y = \Delta$ [3], where Δ is the unit disc in $\boldsymbol{C}$.

2.4. A meromorphic map f from complex space X into a complex space Y is defined by its graph Γ_f, which is an analytic subset of the product $X \times Y$, satisfying the following conditions

(i) The graph Γ_f is a locally irreducible analytic subset of $X \times Y$

(ii) The restriction $\pi|_{\Gamma_f} : \Gamma_f \to X$ of the natural projection $\pi : X \times Y \to X$ to Γ_f is proper, surjective and generically one to one.

Kodama A. [15] showed that every meromorphic mapping from a nonsingular complex manifold X into a hyperbolic complex space Y is holomorphic (also see [14, Thm 6.3.19, p. 288]).

2.5. Let X be a complex space. A plurisubharmonic function on X is a function $\varphi : X \to [-\infty, \infty)$ having the following property. For every $x \in X$ there exist an open neighbourhood U with a biholomorphic map $h : U \to V$ onto a closed complex subspace V of some domain $G \subset \boldsymbol{C}^m$ and a plurisubharmonic function $\tilde{\varphi} : G \to [-\infty, \infty)$ such that $\varphi|_U = \tilde{\varphi} \circ h$ (see Peternell [18, p. 225]).

Some remarks should be made at this point. First, the definition of plurisubharmonicity does not depend on the choice of local charts. Second, Fornaess and Narasimhan proved [6] that the upper semi-continuous function $\varphi : X \to [-\infty, \infty)$ on a complex space X is plurisubharmonic iff $\varphi \circ f$ is either subharmonic or $-\infty$ for all holomorphic maps $f : \Delta \to X$.

Denote $\mathrm{PSH}(X)$ the set of all plurisubharmonic functions on X.

2.6. Let X be a complex space and K be a compact subset of X. The plurisubharmonically convex hull of K (in X) is the set $K^{\wedge}_{PSH(X)} = \{x \in X : u(x) \leq \sup u(K) \text{ for all } u \in PSH(X)\}$.

The complex space X is said to be pseudoconvex if for each compact subset K of X, $K^{\wedge}_{PSH(X)}$ so is compact in X.

2.7. Let M be a complex manifold and $S \subset M$ a subset. We say that S is pluripolar if for any $x_0 \in S$ there are an open neighbourhood U of x_0 in M and a plurisubharmonic function $\varphi : U \to [-\infty, \infty)$ such that $S \cap U \subset \{\varphi = -\infty\}$. If $\dim M = 1$, then S is polar.

2.8. Let X be a complex space. We say that X has the Hartogs extension property (shortly X has (HEP)) if every holomorphic mapping, from a Riemann domain D over a Stein manifold into X, can be extended holomorphically to $\hat{D}$, the envelope of holomorphy of D.

Let $H_2(r) = \{(z_1, z_2) \in \Delta^2 \mid |z_1| < r \text{ or } |z_2| > 1 - r\}$ $(0 < r < 1)$ denote the 2-dimensional Hartogs domain.

It is well-known ([20] or [9]) that X has (HEP) iff every holomorphic mapping $f : H_2(r) \to X$ extends holomorphically over Δ^2.

The class of complex spaces having (HEP) is large. This contains taut complex spaces [7], complex Lie group [2], complete hermitial complex manifolds with non-positive holomorphic sectional curvature [20]. In particular, Ivashkovich [9] showed that a holomorphically convex Kahler manifold has (HEP) iff it contains no rational curves. This was generalized to holomorphically convex Kahler spaces by Thai D. D. [24].

2.9. Let X be a complex space. We say that X has the n-extension property through closed pluripolar sets (shortly X has n-EP) if the restriction $R : H(Z, X) \to H(Z \backslash S, X)$ is a homeomorphism for every closed pluripolar set S in any complex manifold Z of dimension n.

It is easy to see that X has (PEP) if and only if X has n-EP for each $n \geq 1$.

2.10. For every positive number r, put $\Delta_r = \{z \in \boldsymbol{C} : |z| < r\}$, $\Delta_1 = \Delta$.

3. Convergence-extension theorems for analytic subsets

First we give the following

3.1. Definition. Modifying the definition of the disc-convexity (see [20] or [14]), a complex space X is called to be weakly disc-convex if every sequence $\{f_n\} \subset H(\Delta, X)$ converges in $H(\Delta, X)$ whenever the sequence $\{f_n|_{\Delta^*}\} \subset H(\Delta^*, X)$ converges in $H(\Delta^*, X)$. Here, denote $H(X, Y)$ the space of holomorphic mappings from a complex space X into a complex space Y equipped with the compact-open topology and $\Delta^* = \Delta \backslash \{0\}$.

The theorems of Montel and Kiernan (see [19]) follow the following implications

$$\text{complete hyperbolic} \Rightarrow \text{taut} \Rightarrow \text{weakly disc-convex.}$$

The converse assertions are not true in general (see [19]).

For details concerning the (weak) disc-convexity we refer the readers to [26].

3.2. Definition. Let X be a complex subspace of a complex space Y. We say that X has the Δ^*-EP for Y if every holomorphic mapping f from Δ^* into X extends to a holomorphic mapping F from Δ into Y. If X has the Δ^*-EP for itself then X is said to have the Δ^*-EP (shortly X has Δ^*-EP).

Example. i) By a theorem of Kobayashi [14, Thm 6.3.7, p. 284], if X is relatively compact and hyperbolically imbedded into Y then X has the Δ^*-EP for Y.

ii) It is easy to see from the Riemann extension theorem that if D is a bounded domain in $\boldsymbol{C}^n$ and Ω is an open neighbourhood of $\bar{D}$ in $\boldsymbol{C}^n$ then D has the Δ^*-EP for Ω.

For details concerning the Δ^*-EP we refer the readers to [24], [27] and [28].

3.3. Proof of Theorem 1.

(i) First we prove that X is weakly disc-convex. Indeed, let $\{f_k\} \subset H(\Delta, X)$ be such that the sequence $\{f_k|_{\Delta^*}\}$ converges, uniformly on compact subsets, to a mapping $f \in H(\Delta^*, X)$. Let $\{f_{k_j}\}$ be any subsequence of the sequence $\{f_k\}$.

Put $K = \bigcup_{j \geq 1} f_{k_j}(\partial \Delta_s)$, where $0 < s < 1$.

By the hypothesis and by the maximum principle, it follows that $(\bar{K})^{\wedge}_{PSH(X)}$ is compact and $\bigcup_{j \geq 1} f_{k_j}(\Delta_s) \subset (\bar{K})^{\wedge}_{PSH(X)}$. Since X has Δ^*-EP, X contains no complex lines (see [24]). Therefore, by the theorem of Brody [4] and Urata [29] and Zaidenberg [30], there exists a hyperbolic neighbourhood W of $(\bar{K})^{\wedge}_{PSH(X)}$ in X. This implies that the family $\{f_{k_j}|_{\Delta_s}\}$ is equicontinuous.

On the other hand, since $\{f_{k_j}(\lambda)\}$ is relatively compact for each $\lambda \in \Delta_s$, by the Ascoli theorem the family $\{f_{k_j} : j \geq 1\}$ is relatively compact in $H(\Delta_s, X)$.

Thus there exists a subsequence $\{f_{k_{j_l}}\}$ of $\{f_{k_j}\}$ which converges, uniformly on compact subsets, to the mapping F in $H(\Delta, X)$. The equality $F|_{\Delta^*} = f$ determines F uniquely, hence independently of the choices of subsequences $\{f_{k_j}\}$ of the sequence $\{f_k\}$. It follows that the sequence $\{f_k\}$ converges, uniformly on compact sets, to the mapping F in $H(\Delta, X)$.

(ii) We now prove that every holomorphic mapping $f : M \backslash A \to X$ extends holomorphically over M. First we note that we may assume that A is nonsingular; i.e., we extend f to $M \backslash S(A)$ then to $M \backslash S(S(A))$ and so on. Here $S(Z)$ denote the singular locus of Z.

By localizing the mapping f, we may assume that $M = \Delta^m = \Delta^{m-1} \times \Delta$ and $A = \Delta^{m-1} \times \{0\}$.

For each $z' \in \Delta^{m-1}$, consider the holomorphic mapping $f_{z'} : \Delta^* \to X$ given by $f_{z'}(z) = f(z', z)$ for each $z \in \Delta^*$. By the hypothesis, there exists a holomorphic extension $\bar{f}_{z'} : \Delta \to X$ of $f_{z'}$ for each $z' \in \Delta^{m-1}$. Define the mapping $\bar{f} : \Delta^{m-1} \times \Delta \to X$ by $\bar{f}(z', z) = \bar{f}_{z'}(z)$ for all $(z', z) \in \Delta^{m-1} \times \Delta$. It suffices to prove that $\bar{f}$ is continuous at $(z'_0, 0) \in \Delta^{m-1} \times \Delta$.

Indeed, assume that $\{(z'_k, z_k)\} \in \Delta^{m-1} \times \Delta$ such that $\{(z'_k, z_k)\} \to (z'_0, 0)$.

Put $\sigma_k = \bar{f}_{z'_k}$ for each $k \geq 1$ and $\sigma_0 = \bar{f}_{z'_0}$. Then the sequence $\{\sigma_k|_{\Delta^*}\}$ converges uniformly to the mapping $\{\sigma_0|_{\Delta^*}\}$ in $H(\Delta^*, X)$. Since X is weakly disc-convex, the sequence $\{\sigma_k\}$ converges uniformly to the mapping σ_0 in $H(\Delta, X)$. Therefore, $\{\sigma_k(z_k) = \bar{f}(z'_k, z_k)\} \to \sigma_0(0) = \bar{f}(z'_0, 0)$ and hence, $\bar{f}$ is continuous at $(z'_0, 0)$.

(iii) Let $\{f_k\} \subset H(M \backslash A, X)$ be such that $\{f_k\} \to f_0$ in $H(M \backslash A, X)$.

We will show that $\{\bar{f}_k\} \to \bar{f}_0$ in $H(M, X)$.

As above, we may assume that A is nonsingular and by localizing the mappings, we may assume that $M = \Delta^m = \Delta^{m-1} \times \Delta$ and $A = \Delta^{m-1} \times \{0\}$. Let $\{(z'_k, z_k)\} \subset \Delta^{m-1} \times \Delta$ be any sequence converging to $(z'_0, z_0) \in \Delta^{m-1} \times \Delta$. We now prove the sequence $\{\bar{f}_k(z'_k, z_k)\}$ converges to $\bar{f}_0(z'_0, z_0)$.

Indeed, for each $k \geq 0$ consider the holomorphic mapping $\varphi_k : \Delta \to X$ given by $\varphi_k(z) = \bar{f}_k(z'_k, z)$ for all $z \in \Delta$. Then $\{\varphi_k|_{\Delta^*}\} \to \varphi_0|_{\Delta^*}$ in $H(\Delta^*, X)$. Since X is weakly disc-convex, we have $\{\varphi_k\} \to \varphi_0$ in $H(\Delta, X)$. Hence $\{\varphi_k(z_k) = \bar{f}_k(z'_k, z_k)\} \to \varphi_0(z_0) = \bar{f}_0(z'_0, z_0)$. Q.E.D.

3.4. Remark. In [27] we conjectured that any complex manifold, or more generally any complex space, having Δ^*-EP must be pseudoconvex; but as far as we know, this is an interesting open problem.

3.5. Proof of Theorem 2.

a) First we show that every holomorphic mapping $f : M \backslash A \to X$ extends to a holomorphic mapping $F : M \to Y$.

We consider two cases.

Case 1. The singularities of A are normal crossings.

By assumption, we may assume that $M = \Delta^n \times \Delta^l$ and $M \backslash A = (\Delta^*)^n \times \Delta^l$. The proof is by induction on n. We do it in three steps.

(i) If $M\backslash A = \Delta^*$ then the assertion is deduced immediately from Definition 2.12.

(ii) Assume that we can extend f when $M\backslash A = (\Delta^*)^n$ for some n. We show that this implies we can extend f if $M\backslash A = (\Delta^*)^n \times \Delta^l$ for any l.

Let $f : \Delta^{*n} \times \Delta^l \to X$ be holomorphic. For each u let $f_u(t) = f(t,u)$. Then by assumption we can extend f_u to a holomorphic mapping $\tilde{f}_u : \Delta^n \to Y$ for each u. Define the mapping $F : \Delta^n \times \Delta^l \to Y$ by $F(t,u) = \tilde{f}_u(t)$. By the Riemann extension theorem, it sufficies to prove that the mapping F is continuous.

Indeed, assume that a sequence $\{(t^k, u^k)\} \subset \Delta^n \times \Delta^l$ converges to a point $(0, u^0)$. Take some sequence $\{\tilde{t}^k\} \in \Delta^{*n}$ such that $\lim_{n\to+\infty} d_{\Delta^n}(t^k, \tilde{t}^k) = 0$.

We have

$$\begin{aligned}
&d_Y(F(t^k,u^k), F(0,u^0)) \\
&\quad \le d_Y(F(t^k,u^k), F(\tilde{t}^k,u^k)) + d_Y(F(\tilde{t}^k,u^k), F(\tilde{t}^k,u^0)) + d_Y(F(\tilde{t}^k,u^0), F(0,u^0)) \\
&\quad = d_Y(\tilde{f}_{u^k}(t^k), \tilde{f}_{u^k}(\tilde{t}^k)) + d_Y(f(\tilde{t}^k,u^k), f(\tilde{t}^k,u^0)) + d_Y(\tilde{f}_{u^0}(\tilde{t}^k), \tilde{f}_{u^0}(0)) \\
&\quad \le d_{\Delta^n}(t^k,\tilde{t}^k) + d_{\Delta^l}(u^k,u^0) + d_{\Delta^n}(\tilde{t}^k,0) \quad \text{for all } k \ge 1.
\end{aligned}$$

Thus $\lim_{k\to+\infty} d_Y(F(t^k,u^k), F(0,u^0)) = 0$, i.e., $\{F(t^k,u^k)\} \to F(0,u^0)$ as $k \to +\infty$. This concludes the second step of the proof.

(iii) Assume that f can be extended if $M\backslash A = \Delta^{*n} \times \Delta^l$ for any l. We then show that f can be extended if $M\backslash A = \Delta^{*n+1}$.

Indeed, by induction, f extends to a holomorphic mapping $f_1 : \Delta^{n+1}\backslash\{(0, 0,\dots,0)\} \to Y$. The holomorphic mapping $g : \Delta^* \to X$, given by $g(z) = f(z,\dots,z)$ for each $z \in \Delta^*$, extends to a holomorphic mapping $\tilde{g} : \Delta \to Y$. Define the mapping $F : \Delta^{n+1} \to Y$ by $F(0,0,\dots,0) = \tilde{g}(0)$ and $F|_{\Delta^{n+1}\backslash\{(0,0,\dots,0)\}} = f_1$. As above, it suffices to show that F is continuous.

Assume that a sequence $\{(t_1^k, t_2^k, \dots, t_{n+1}^k)\} \subset \Delta^{n+1}$ converges to $(0,0,\dots,0)$. Without loss of generality we may assume that $t_1^k \ne 0$ for all $k \ge 1$.

Then

$$\begin{aligned}
&d_Y(F(t_1^k,t_2^k,\dots,t_{n+1}^k), F(0,0,\dots,0)) \\
&\quad \le d_Y(F(t_1^k,t_2^k,\dots,t_{n+1}^k), F(t_1^k,t_1^k,\dots,t_1^k)) + d_Y(F(t_1^k,t_1^k,\dots,t_1^k), F(0,0,\dots,0)) \\
&\quad = d_Y(f_1(t_1^k,t_2^k,\dots,t_{n+1}^k), f_1(t_1^k,t_1^k,\dots,t_1^k)) + d_Y(\tilde{g}(t_1^k), \tilde{g}(0)) \\
&\quad \le d_{\Delta^*\times\Delta^n}((t_1^k,t_2^k,\dots,t_{n+1}^k), (t_1^k,t_1^k,\dots,t_1^k)) + d_Y(\tilde{g}(t_1^k), \tilde{g}(0)) \\
&\quad \le \max_{j=\overline{2,n+1}} d_\Delta(t_j^k, t_1^k) + d_\Delta(t_1^k, 0) \\
&\quad \le \max_{j=\overline{2,n+1}} (d_\Delta(t_j^k,0) + d_\Delta(t_1^k,0)) + d_\Delta(t_1^k,0) \\
&\quad \le \left(\max_{j=\overline{2,n+1}} d_\Delta(t_j^k,0)\right) + 2d_\Delta(t_1^k,0) \quad \text{for all } k \ge 1.
\end{aligned}$$

Thus every subsequence of the sequence contains some subsequence converging to $F(0,0,\dots,0)$. Then the sequence $\{F(t_1^k, t_2^k, \dots, t_{n+1}^k)\}$ converges to $F(0,0,\dots,0)$.

Case 2. A is any closed analytic set of M.

By the Hironaka theorem of singularities, there is (at least locally) a triple (Z, B, θ) such that B is an analytic set with normal crossings of a complex manifold Z and $\theta : Z \to M$ is a proper holomorphic mapping onto M with $B = \theta^{-1}(A)$.

Define $g : Z \backslash B \to X$ by $g = f \circ \theta$. By Case 1, g extends to a holomorphic mapping $G : Z \to Y$. Then f extends meromorphically to M by defining $F = G \circ \theta^{-1}$. By the theorem of Kodama [15] (see [14, Thm 6.3.19, p. 288]), F is holomorphic.

b) Assume that $\{f_j\} \subset H(M\backslash A, X)$ such that $\{f_j\} \to f \in H(M\backslash A, X)$ in $H(M\backslash A, X)$. We will show that $\{\bar{f}_j\} \to \bar{f}$ in $H(M, Y)$.

First we note that we may assume that A is nonsingular; i.e., our assertion holds up to $M\backslash S(A)$ then to $M\backslash S(S(A))$ and so on. Here $S(Z)$ denote the singular locus of Z.

Let z_0 be an arbitrary point of A. By localizing our assertion, we may assume that $M = \Delta^m$ and $A = \Delta^{m-1} \times \{0\}$ and $z_0 = (0,0)$. Put $a_0 = \bar{f}(z_0)$. For a point $y \in Y$ and a positive real number r, we set $B_Y(y, r) = \{y' \in Y : d_Y(y, y') < r\}$. Similarly, for a point $z \in M$ and a positive real number r, we set $B_M(z, r) = \{z' \in M : d_M(z, z') < r\}$.

We first show that for an arbitrary number $\varepsilon > 0$ there exists a neighbourhood V_0 of z_0 in M such that $\bar{f}(V_0) \subset B_Y(a_0, \varepsilon)$ and $\bar{f}_j(V_0) \subset B_Y(a_0, \varepsilon)$ for all $j \geq j_0$.

Indeed, take a point $z_1 \in B_M(z_0, \varepsilon/3)\backslash A$. Then $f(z_1) \in B_Y(a_0, \varepsilon/3)$. There is an integer j_0 such that $f_j(z_1) \in B_Y(a_0, 2\varepsilon/3)$ for all $j \geq j_0$. Then we have $\bar{f}_j(B_M(z_1, \varepsilon/3)) \subset B_Y(a_0, \varepsilon)$. Put $V_0 = B_M(z_0, \varepsilon/3) \cap B_M(z_1, \varepsilon/3)$. Then $z_0 \in V_0$ and $\bar{f}(V_0) \subset B_Y(a_0, \varepsilon)$ and $\bar{f}_j(V_0) \subset B_Y(a_0, \varepsilon)$ for all $j \geq j_0$.

Take $\varepsilon > 0$ so small that $B_Y(a_0, \varepsilon)$ is contained in a holomorphic local coordinate neighbourhood of a_0 in Y. Choose $\delta > 0$ small enough such that $\bar{\Delta}_\delta^m \subset V_0$. Since $\{\bar{f}_j|_{(\partial\Delta_\delta)^m}\}_{j=1}^\infty$ converges uniformly to $\bar{f}|_{(\partial\Delta_\delta)^m}$, the maximum principle implies the uniform convergence of $\{\bar{f}_j|_{\Delta_\delta^m}\}_{j=1}^\infty$ with limit $\bar{f}|_{\Delta_\delta^m}$. Q.E.D.

3.6. Proof of Theorem 3.

First we also note that we may assume B is non-singular.

Take an arbitrary point $a \in A$. By localizing the mapping f, we may assume that $M = \Delta^m = \Delta^{m-1} \times \Delta$, $A = A' \times \{0\}$, where A' is a nowhere dense subset of Δ^{m-1}, and $a = (t_0, 0) \in A' \times \{0\}$. For every point $z \in \Delta^m$ denote $z = (t, u)$ with $t \in \Delta^{m-1}$ and $u \in \Delta$.

Assume that a sequence $\{a_j = (t_j, u_j)\} \subset (\Delta^{m-1}\backslash A') \times \Delta$ converges to a. Consider the holomorphic mappings $f_j : \Delta \to X$, $u \mapsto f_j(u) = f(t_j, u)$ for each $j \geq 1$, and $f_{t_0} : \Delta^* \to X$, $u \mapsto f_{t_0}(u) = f(t_0, u)$. It is easy to see that $\{f_j|_{\Delta^*}\} \to f_{t_0}$ in $H(\Delta^*, X)$. Since X is weakly disc-convex, the sequence $\{f_j\}$ converges uniformly to the holomorphic mapping $g \in H(\Delta, X)$, where $g|_{\Delta^*} = f_{t_0}$. Put $g(0) = p$. Then $\{f_j(u_j)\} \to g(0)$, i.e., $\{f(a_j)\} \to p$. Thus, the sequence $\{f(a_j)\}$ converges to p for any sequence $\{a_j\} \subset (\Delta^{m-1} \backslash A') \times \Delta$ converging to a (*). Choose a relatively compact neighbourhood V_p of p in X such that $\overline{V}_p$ is contained in a holomorphic local coordinate neighbourhood of p in Y. By (*) there exists an open neighbourhood $T_0 \times U_0$ of $a = (t_0, 0)$ in $\Delta^{m-1} \times \Delta$ such that $f((T_0 \backslash A') \times U_0) \subset V_p$.

For every point $u \in U_0 \backslash \{0\}$, consider the holomorphic mapping $f_u : \Delta^{m-1} \to X$, $t \mapsto f_u(t) = f(t, u)$.

Since $f_u(T_0 \backslash A') \subset V_p$, it follows that $f_u(\overline{T_0 \backslash A'}) = f_u(T_0) \subset \overline{V}_p$. Thus $f(T_0 \times (U_0 \backslash \{0\})) \subset \overline{V}_p$. By the Riemann extension theorem, the mapping f extends holomorphically to $T_0 \times U_0$. Q.E.D.

Theorem 3 contains the following results of Kobayashi [14] and Noguchi-Ochiai [17], which were proved by different methods.

3.7. Corollary ([14, Thm 6.2.3, p. 281]).
Let X be a complete hyperbolic space. Let M be a complex manifold of dimension m, and let A be a subset which is nowhere dense in a complex subspace $B \subset M$ of dimension $\leq m-1$. Then every holomorphic mapping $f : M \backslash A \to X$ extends to a holomorphic mapping $F : M \to X$.

3.8. Corollary ([17, Thm 1.6.28, p. 35]).
Let X be a complete hyperbolic space, M a complex manifold and $A \subset M$ an analytic subset of codimension ≥ 2. Then all holomorphic mappings from $M \backslash A$ into X extend holomorphically over the whole M. Furthermore, if $\{f_j\}_{j=1}^{\infty} \to f$ in $H(M \backslash A, X)$, then $\{\bar{f}_j\}_{j=1}^{\infty} \to \bar{f}$ in $H(M, X)$, where $\bar{f}_j$ and $\bar{f}$ stand for the extended holomorphic mappings from M into X.

3.9. Remark. In [26] we showed a bounded pseudoconvex domain D in $\boldsymbol{C}^2$ such that D is not taut. Thus Theorem 3 is strictly stronger than the above-mentioned results of Kobayashi and Noguchi-Ochiai.

4. Convergence-extension theorems for closed pluripolar sets

First we give the following

4.1. Definition. Let X be a complex space. We say that X has the weak 1-EP if every holomorphic mapping $f : \Delta \backslash S \to X$ extends holomorphically over Δ, where S is any closed polar subset of Δ.

Example. Every bounded hyperconvex domain in $\boldsymbol{C}^n$ has the weak 1-EP.
Indeed, let Ω be a bounded hyperconvex domain in $\boldsymbol{C}^n$ and $f : \Delta \backslash S \to \Omega$ any holomorphic mapping, where S is any closed polar subset of Δ. Put $f = (f_1, \ldots, f_n)$ and $f_j = u_j + iv_j$, where u_j, and v_j are bounded harmonic on $\Delta \backslash S$.

By the theorem on removable singularities of harmonic functions, u_j and v_j extend to the harmonic functions $\tilde{u}_j$ and $\tilde{v}_j$ on Δ. Since they are of class $\mathscr{C}^2$ and satisfy the Cauchy-Riemann equations on a dense subset of Δ, they satisfy these equations at each point of Δ. Then $\tilde{f}_j = \tilde{u}_j + i\tilde{v}_j \in H(\Delta)$. Thus the mapping $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n) \in H(\Delta, \bar{\Omega})$. Let ρ be a plurisubharmonic exhaustion function of Ω. Put $h = \rho \circ \tilde{f}$ on $\Delta \backslash S$. Then h is subharmonic and negative on $\Delta \backslash S$. Therefore h extends to a subharmonic function $\tilde{h}$ over Δ and $\tilde{h} \leq 0$ over Δ. Suppose that there exists $z_0 \in S$ such that $\tilde{f}(z_0) \in \partial\Omega$. Then $\tilde{h}(z_0) = 0$. The maximum principle implies $\tilde{h} = 0$ on Δ. This is a contradiction.

In order to prove Theorem 4 we need the following

4.2. Lemma. *Let A be a subset of Δ^n_{2n} such that $H^{2n-2+2/n}(A) = 0$. Then there exists a new orthogonal coordinate system of $\boldsymbol{C}^n$ with the same origin such that in this new orthogonal coordinate system the set $S'' := \{w \in \Delta_2 : \Delta_2^{n-1} \times \{w\} \subset A \cap \Delta_2^n\}$ is empty.*

Proof. Denote by $\{e_1, \dots, e_n\}$ the canonical base of $\boldsymbol{C}^n$.
For every $a = (a_1, a_2, \dots, a_{n-1}, b) \in \boldsymbol{C}^n$ we put

$$H_{a,b} = \{(z_1, z_2, \dots, z_n) : z_n = a_1 z_1 + \cdots + a_{n-1} z_{n-1} + b\}$$

$$\tilde{H}_{a,b} = \{z : z \in H_{a,b} \text{ and } |z_i| < 1 \text{ for each } i = 1, \dots, n-1\}$$

Put $B = \{(a,b) : \tilde{H}_{a,b} \subset \Delta^n_{2n} \cap A\}$.
Consider two cases.

Case 1. $H^2(B) > 0$

Suppose that there exist $\boldsymbol{C}$-linear maps $\sigma_i : \boldsymbol{C}^n \to \boldsymbol{C}$, $i = 1, \dots, n$ such that $\{\sigma_i\}$ is linearly independent and $H^{2/n}(\sigma_i(B)) = 0$, $i = 1, \dots, n$ (1). Choose a new base $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ of $\boldsymbol{C}^n$ such that σ_i is a canonical projection onto i^{th}-axis. Consider the $\boldsymbol{C}$-isomorphism $\phi : \boldsymbol{C}^n \to \boldsymbol{C}^n$ given by $\phi(e_i) = \tilde{e}_i$, $i = 1, \dots, n$. Put $\tilde{B} = \phi(B)$. By (1), we have $H^2(\prod_{i=1}^n \sigma_i(\tilde{B})) = 0$. Since $\tilde{B} \subset \prod_{i=1}^n \sigma_i(\tilde{B})$, this follows that $H^2(\tilde{B}) = 0$. Hence $H^2(B) = 0$. This is a contradiction. Thus $H^{2/n}(\sigma(B)) > 0$ for each generic $\boldsymbol{C}$-linear mapping $\sigma : \boldsymbol{C}^n \to \boldsymbol{C}$ (2). Remark that if σ satisfies (2) then $c\sigma$ also satisfies (2) for all $c \in \boldsymbol{C}^*$.

This implies that $H^{2/n}(\sigma_\alpha(B)) > 0$ for almost $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in \boldsymbol{C}^{n-1}$, where $\sigma_\alpha(a,b) = \alpha_1 a_1 + \cdots + \alpha_{n-1} a_{n-1} + b = a.\alpha + b$.

For every $\alpha \in \boldsymbol{C}^{n-1}$, consider the line $d_\alpha = \{(\alpha, z_n) : z_n \in \boldsymbol{C}\}$. It is easy to see that $d_\alpha \cap A \supset \{(\alpha, z_n) : z_n \in \sigma_\alpha(B)\}$. This implies that $H^{2/n}(d_\alpha \cap A) \geq H^{2/n}(\{(\alpha, z_n) : z_n \in \sigma_\alpha(B)\} = H^{2/n}(\sigma_\alpha(B)) > 0$ for almost $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in \boldsymbol{C}^{n-1}$, and hence, $H^{2n-2+2/n}(A) > 0$ ([5, Lemma A.6.3, p. 350]). This is a contradiction.

Case 2. $H^2(B) = 0$

By [5, Lemma A.6.3, p. 350], for every $k \geq 1$ there exists $\tilde{a}_k \in \Delta^{n-1}_{1/k}$ such that $(\{\tilde{a}_k\} \times \boldsymbol{C}) \cap B = \emptyset$ (3).

For every $k \geq 1$ consider a hyperplane $H_k = \{z_n = \tilde{a}_k.z'\} \subset (\boldsymbol{C}^n, \{e_j\})$. Choose a new orthogonal coordinate system $\{e_1^{(k)}, \dots, e_n^{(k)}\}$ of $\boldsymbol{C}^n$ such that

• $\{e_1^{(k)}, \dots, e_{n-1}^{(k)}\} \subset H_k$

• $\lim_{k\to\infty} e_j^{(k)} = e_j$ for each $1 \leq j \leq n$, where limitations are considered in $(\boldsymbol{C}^n, \{e_j\})$.

In $(\boldsymbol{C}^n, \{e_j^{(k)}\}_{j=1}^n)$, we consider the polydisc $\Delta_2^n(k)$. It is easy to see that in the set-theoretic side $\Delta_2^n(k) \subset \Delta_{2n}^n$ for k large enough.

Consider the isomorphism $\phi_k : (\boldsymbol{C}^n, \{e_j\}) \to (\boldsymbol{C}^n, \{e_j^{(k)}\})$ given by $z = (z_1, z_2, \dots, z_n) \mapsto (z_1^{(k)}, \dots, z_n^{(k)})$, where $z = (z_1^{(k)}, \dots, z_n^{(k)})$ in $(\boldsymbol{C}^n, \{e_j^{(k)}\})$. Then

• ϕ_k is isometric

• $\phi_k(H_k) = \{z_n^{(k)} = 0\}$

• $\phi_k = \mathrm{Id}_{\boldsymbol{C}^n}$ if omiting algebraic structures in $\boldsymbol{C}^n$

Put $S_k'' = \{b_k \in \Delta_2(k) : \Delta_2(k)^{n-1} \times \{b_k\} \subset A\}$.

Suppose that $S_k'' \neq \emptyset$ for each $k \geq 1$. Then there is $b_k \in \Delta_2(k)$ such that $T_{b_k} := \Delta_2^{n-1}(k) \times \{b_k\} \subset A$. Without loss of generality we may assume that $\{b_k\} \to b_0 \in \boldsymbol{C}$.

Consider the canonical projection $p : (\boldsymbol{C}^n, \{e_j\}_{j=1}^n) \to (\boldsymbol{C}^{n-1}, \{e_j\}_{j=1}^{n-1})$.

Then $\{p(T_{b_k})\} \to \Delta_2^{n-1} \supset \Delta_1^{n-1}$ in $(\boldsymbol{C}^n, \{e_j\})$. Without loss of generality we may assume that $p(T_{b_k}) \supset \Delta_1^{n-1}$ for all $k \geq 1$ (4).

In $(\boldsymbol{C}^n, \{e_j^{(k)}\}_{j=1}^n)$, we consider the hyperplane $K_{b_k} = \{z_n^{(k)} - b_k = 0\} \supset T_{b_k}$. Then in $(\boldsymbol{C}^n, \{e_j\}_{j=1}^n)$ the set K_{b_k} is given by $\{z_n = \tilde{a}_k.z' + \tilde{b}_k\}$

Indeed, since H_k consided in $(\boldsymbol{C}^n, \{e_j^{(k)}\}_{j=1}^n)$ is given by the equation $\{z_n^{(k)} = 0\}$, it implies that $\phi_k^{-1}(\{z_n^{(k)} = 0\}) = H_k$. Thus we have

$$\begin{aligned}
\phi_k^{-1}(z_1^{(k)}, \dots, z_{n-1}^{(k)}, b_k) &= \phi_k^{-1}(z_1^{(k)}, \dots, z_{n-1}^{(k)}, 0) + \phi_k^{-1}(0, \dots, 0, b_k)\\
&= (z_1, \dots, z_{n-1}, \tilde{a}_k.z') + (u(k)', u(k))\\
&= (z', \tilde{a}_k.z') + (u(k)', u(k)) = (z' + u(k)', \tilde{a}_k.z' + u(k))\\
&= (z' + u(k)', \tilde{a}_k.(z' + u(k)') + u(k) - \tilde{a}_k.u(k)')\\
&= (z' + u(k)', \tilde{a}_k.(z' + u(k)') + \tilde{b}_k),
\end{aligned}$$

where $\tilde{b}_k = u(k) - \tilde{a}_k.u(k)'$.

Hence in $(\boldsymbol{C}^n, \{e_j\}_{j=1}^n)$ $\phi_k^{-1}(K_{b_k})$ is defined by $\{z_n = \tilde{a}_k.z' + \tilde{b}_k\}$. This follows that $T_{b_k} \subset H_{\tilde{a}_k, \tilde{b}_k}$ in $(\boldsymbol{C}^n, \{e_j\}_{j=1}^n)$. From (4) we get $T_{b_k} \supset \{z \in H_{\tilde{a}_k, \tilde{b}_k} : p(z) \in \Delta_1^{n-1}\}$.

In $(\boldsymbol{C}^n, \{e_j\}_{j=1}^n)$ consider the set

$$\begin{aligned}
\tilde{H}_{\tilde{a}_k, \tilde{b}_k} &= \{z : z \in H_{\tilde{a}_k, \tilde{b}_k} \text{ and } |z_j| < 1 \text{ for each } 1 \leq j \leq n-1\}\\
&\quad \{z : z \in H_{\tilde{a}_k, \tilde{b}_k} \text{ and } p(z) \in \Delta_1^{n-1}\} \subset T_{b_k} \subset A
\end{aligned}$$

Hence $\tilde{H}_{\tilde{a}_k, \tilde{b}_k} \subset A$.

On the other hand, we have $T_{b_k} \subset \Delta_2^n(k) \subset \Delta_{2n}^n$, i.e., $\tilde{H}_{\tilde{a}_k, \tilde{b}_k} \subset \Delta_{2n}^n$. Thus $\tilde{H}_{\tilde{a}_k, \tilde{b}_k} \subset \Delta_{2n}^n \cap A$, and hence, $(\tilde{a}_k, \tilde{b}_k) \in B$. This implies that $(\{\tilde{a}_k\} \times \boldsymbol{C}) \cap B \neq \emptyset$. This is a contradiction. Q.E.D.

4.3. Lemma. *Let X be a complex space. If X has 1-EP, then X has n-EP for every $n \geq 1$.*

Proof. This lemma was proved in [25] with some gaps. We repeat the details here for correction and also for the reader's convenience.

(i) First observe that X is weakly disc-convex. By a result of Shiffman [20], X has (HEP).

(ii) Given $n \geq 2$. Since the problem is local, without lost of generality we may assume that $Z = \Delta^n$. By the theorem of Josefson (see [13, p. 170]), there exists $u \in PSH(\Delta^n)$ such that $S \subset \tilde{S} := u^{-1}(-\infty)$.

We put

$$S' = \{z \in \Delta^{n-1} : \{z\} \times \Delta \subset \tilde{S}\} \quad \text{and}$$

$$S'' = \{w \in \Delta : \Delta^{n-1} \times \{w\} \subset \tilde{S}\}.$$

Then S' and S'' are pluripolar in Δ^{n-1} and Δ respectively. By Lemma 4.2, without loss of generality we may assume that $S'' = \emptyset$.

We put $S^w = \{z \in \Delta^{n-1} : (z, w) \in S\}$ for each $w \in \Delta$ and $S_z = \{w \in \Delta : (z, w) \in S\}$ for each $z \in \Delta^{n-1}$.

Then S^w is closed pluripolar in Δ^{n-1} for each $w \in \Delta$ and S_z is closed polar in Δ for $z \notin S'$.

(iii) Now assume that f is a holomorphic mapping from $(\Delta^{n-1} \times \Delta)\backslash S$ into X.

For each $w \in \Delta$, consider the holomorphic mapping $f^w : \Delta^{n-1}\backslash S^w \to X$ given by $f^w(z) = f(z, w)$ for all $z \in \Delta^{n-1}\backslash S^w$. By the inductive hypothesis, f^w is extended to the mapping $\tilde{f}^w \in H(\Delta^{n-1}, X)$. Similarly, for each $z \notin S'$, the holomorphic mapping $f_z : \Delta\backslash S_z \to X$ given by $f_z(w) = f(z, w)$ for all $w \in \Delta\backslash S_z$, is extended to the mapping $\tilde{f}_z \in H(\Delta, X)$. Thus we can define the mappings

$$f_1 : (\Delta^{n-1}\backslash S') \times \Delta \to X \text{ by } f_1(z, w) = \tilde{f}_z(w) \quad \text{and}$$

$$f_2 : \Delta^{n-1} \times \Delta \to X \text{ by } f_2(z, w) = \tilde{f}^w(z).$$

We now prove that f_1 is continuous on $(\Delta^{n-1}\backslash S') \times \Delta$.

Indeed, assume that $\{(z_k, w_k)\} \subset (\Delta^{n-1}\backslash S') \times \Delta$, $\{(z_k, w_k)\} \to (z_0, w_0) \in (\Delta^{n-1}\backslash S') \times \Delta$.

Put $P = (\bigcup_{k=1}^{\infty} S_{z_k}) \cup S_{z_0}$. Then P is closed polar in Δ. Since the sequence $\{\tilde{f}_{z_k}\}$ converges uniformly to $\tilde{f}_{z_0}$ in $H(\Delta\backslash P, X)$, by the inductive hypothesis, we see that the sequence $\{\tilde{f}_{z_k}\}$ converges uniformly to $\tilde{f}_{z_0}$ in $H(\Delta, X)$. Hence $\tilde{f}_{z_k}(w_k) = f_1(z_k, w_k) \to f_{z_0}(w_0) = f_1(z_0, w_0)$. Thus f_1 is continuous on $(\Delta^{n-1}\backslash S') \times \Delta$.

Similarly, f_2 is continuous on $\Delta^{n-1} \times \Delta$.

Since $(\Delta^{n-1} \times \Delta)\backslash S$ is dense in $(\Delta^{n-1}\backslash S') \times \Delta$ and $f_1 = f_2$ on $(\Delta^{n-1} \times \Delta)\backslash S$, we have $f_1 = f_2$ on $(\Delta^{n-1}\backslash S') \times \Delta$.

This implies that the mapping f_2 satisfies the following: $(f_2)^w = \tilde{f}^w \in H(\Delta^{n-1}, X)$ for all $w \in \Delta$ and $(f_2)_z = \tilde{f}_z \in H(\Delta, X)$ for all $z \in \Delta^{n-1}\backslash S$, where $(f_2)^w$ and $(f_2)_z$ are given by $(f_2)^w(z) = (f_2)_z(w) = f_2(z, w)$. By a theorem of Shiffman [21], f_2 is holomorphic.

(iv) Now assume that the sequence $\{f_k\} \subset H((\Delta^{n-1} \times \Delta)\backslash S, X)$ converges uniformly to $f \in H((\Delta^{n-1} \times \Delta)\backslash S, X)$. We must prove that $\{\hat{f}_k\} \to \hat{f}$ in $H(\Delta^{n-1} \times \Delta, X)$, i.e., we must prove that if $\{(z_k, w_k)\} \subset \Delta^{n-1} \times \Delta$ such that $\{(z_k, w_k)\} \to (z_0, w_0) \in \Delta^{n-1} \times \Delta$ then $\{\hat{f}_k(z_k, w_k)\} \to \hat{f}(z_0, w_0)$.

As before, we consider the following holomorphic mappings $f^{k,w_k} : \Delta^{n-1}\backslash S^{w_k} \to X$, $z \mapsto f_k(z, w_k)$ and $f^{w_0} : \Delta^{n-1}\backslash S^{w_0} \to X$, $z \mapsto f(z, w_0)$ and $\hat{f}^{k,w_k} : \Delta^{n-1} \to X$, $z \mapsto \hat{f}_k(z, w_k)$ and $\hat{f}^{w_0} : \Delta^{n-1} \to X$, $z \mapsto \hat{f}(z, w_0)$.

Put $P = (\bigcup_{k=1}^{\infty} S^{w_k}) \cup S^{w_0}$. Then P is a closed pluripolar set in Δ^{n-1}.

Since $\{f^{k,w_k}\} \to f^{w_0}$ in $H(\Delta^{n-1}\backslash P, X)$, by the inductive hypothesis, we have $\{\hat{f}^{k,w_k}\} \to \hat{f}^{w_0}$ in $H(\Delta^{n-1}, X)$.

Thus $\{\hat{f}^{k,w_k}(z_k) = \hat{f}_k(z_k, w_k)\} \to \hat{f}^{w_0}(z_0) = \hat{f}(z_0, w_0)$. Q.E.D.

4.4. Proof of Theorem 4.

By Lemma 4.3, it suffices to show that X has 1-EP, i.e., the restriction $R : H(\Delta, X) \to H(\Delta\backslash S, X)$ is homeomorphic for every closed polar set S in Δ.

By the hypothesis, X has Δ^*-EP and hence, X contains no complex lines [24]. Then every compact subset of X has a hyperbolic neighbourhood in X (see [4], [29], [30]).

Let $\{f_j\} \subset H(\Delta\backslash S, X)$ be such that $\{f_j\} \to f \in H(\Delta\backslash S, X)$ in $H(\Delta\backslash S, X)$. By the hypothesis, there are unique holomorphic extensions $\bar{f}_j : \Delta \to X$ of f_j and $\bar{f} : \Delta \to X$ of f over Δ. We will show that $\{\bar{f}_j\} \to \bar{f}$ in $H(\Delta, X)$.

Given $z_0 \in S$. Since S is closed polar in Δ, there exists a neighbourhood U of z_0 in Δ such that $\partial U \cap S = \emptyset$. Then the set $K = \bigcup_{j \geq 1} \bar{f}_j(\partial U)$ is relatively compact in X. By the hypothesis and by the maximum principle, it follows that $(\bar{K})^{\wedge}_{PSH(X)}$ is compact and $\bigcup_{j \geq 1} \bar{f}_j(U) \subset (\bar{K})^{\wedge}_{PSH(X)}$. Take a hyperbolic neighbourhood W of $(\bar{K})^{\wedge}_{PSH(X)}$ in X. Then the family $\{\bar{f}_j|_U\}$ is equicontinuous. On the other hand, since $\{\bar{f}_j(\lambda)\}$ is relatively compact for each $\lambda \in U$, by the Ascoli theorem, the family $\{\bar{f}_j\}$ is relatively compact in $H(U, X)$. This implies that every subsequence $\{\bar{f}_{j_k}\}_{k=1}^{\infty}$ of the sequence $\{\bar{f}_j\}_{j=1}^{\infty}$ contains a subsequence $\{\bar{f}_{j_{k_l}}\}_{l=1}^{\infty}$ which converges, uniformly on compact subsets, to the mapping F in $H(U, X)$. The equality $F|_{U\backslash S} = f$ implies $F = \bar{f}$ on U. This follows that $\{\bar{f}_j|_U\} \to \bar{f}|_U$ in $H(U, X)$. Thus $\{\bar{f}_j\} \to \bar{f}$ in $H(\Delta, X)$. Q.E.D.

5. Generalization of Alexander theorem to complex spaces

First of all we give the following

5.1. Definition. Let M be a complex space.

i) An open subset A of M is said to be of type (S) if there exists a biholomorphic mapping from A onto an analytic subset of $\boldsymbol{C}^m$.

ii) The space M is said to be Stein-type if for each $p \in M$ there exist a neighbourhood W_p of p and $r_p > 0$ and a neighbourhood S_p of p being of type (S) such that, for each $f \in H(\Delta, M)$, if $f(0) \in W_p$ then $f(\Delta_{r_p}) \subset S_p$.

The class of complex spaces of Stein-type is rather large. It is easy to see that one contains Stein complex spaces and hyperbolic complex spaces.

We now give some notations

Put

$$z = (z', z_n) \in \boldsymbol{C}^{n-1} \times \boldsymbol{C} \quad \text{for each } z \in \boldsymbol{C}^n$$

$$B_r^n = \{z \in \boldsymbol{C}^n : \|z\| < r\}, \quad B_1^n = B^n$$

$$B(a, r) = \{z \in \boldsymbol{C}^n : \|z - a\| < r\} \quad \text{for each } a \in \boldsymbol{C}^n, r > 0$$

$$P(a, R) = \{z = (z_1, \dots, z_n) \in \boldsymbol{C}^n : |z_j - a_j| < R_j \text{ for each } 1 \le j \le n\}$$

for each $R = (R_1, \dots, R_n) \in \boldsymbol{R}_*^{+n}$ and $a = (a_1, \dots, a_n) \in \boldsymbol{C}^n$

Reasoning as in [13, p. 60], we have the following

5.2. Lemma. *Let $f : P(z, R) \to \boldsymbol{C}$ be a mapping satisfying the following: There exists $o < r_n < R_n$ such that f is holomorphic in $P(z', R') \times P(z_n, r_n)$ and $f(\tilde{z}', .)$ is holomorphic in $P(z_n, R_n)$ for each $\tilde{z}' \in P(z', R')$. Then f is holomorphic in $P(z, R)$.*

We now prove the following proposition which is a generalization of the theorem of Forelli to complex spaces (see [19, p. 49]).

5.3. Proposition. *Let M be a complex space of Stein type. Let $f : B^n \to M$ be such that the restriction of f to each complex line through the origin is holomorphic and f is a C^∞-mapping in an open neighbourhood of the origin. Then f is holomorphic in B^n.*

Proof. Step 1. Assume that f is holomorphic in $B_{1-\alpha}^n$ for some $0 < \alpha < 1$ and $f(B(z, \alpha))$ is included in a subset of type (S) of M for each $z \in B_{1-\alpha}^n$.

It is easy to see that $f(B(z, r_h))$ is also included in a subset of type (S) of M for each $0 < h < 1$, $z \in B_h^n$, where $r_h = \min(1 - h, \alpha)$.

Put $B_*^n = B^n \backslash \{z_n = 0\}$. Consider the holomorphic mapping $\varphi : B_*^n \to \boldsymbol{C}^n$ given by $\varphi(z_1, \dots, z_n) = (z_1/z_n, \dots, z_{n-1}/z_n, z_n)$. Put $\varphi(B_*^n) = T$ and $\varphi_1 : B_*^n \to T$ given by $\varphi_1(z) = \varphi(z)$ for each $z \in B_*^n$. Then φ_1 is biholomorphic. Put $g = f \circ \varphi_1^{-1} : T \to M$ and $T_{R,h} = \{t = (t', z_n) \in T : \|t'\| < R, \ 0 < |z_n|^2 < h/(1 + R^2)\}$ for each $R > 0$ and $0 < h < 1$.

It is easy to see that $\{T_{R,h}\}_h$ is a sequence of open sets which is increasing when h is increasing and $T = \bigcup\{T_{R,h} : R > 0, 0 < h < 1\}$.

Take $R > 0$, $0 < h < 1$ and we prove that g is holomorphic in $T_{R,h}$.

Indeed, we have the following assertions

- $\varphi_1^{-1}(T_{R,h}) \subset B_h^n \Subset B^n$
- $\forall 0 < \epsilon < h/(1 + R^2)$, $\exists \delta_h = \delta_h(\epsilon) > 0$, $\forall (t', z_n) \in T_{R,h}$ with $h/(1 + R^2) - \epsilon < |z_n|^2 < h/(1 + R^2) : g(P((t', z_n), \delta_h))$ is included in a set of type (S).

Indeed, put $D_\epsilon = \{(t', z_n) \in T : \|t'\| \le R, \ h/(1 + R^2) - \epsilon \le |z_n|^2 \le h/(1 + R^2)\}$. Then D_ϵ is compact and $D_\epsilon \subset T$ and $\varphi_1^{-1}(D_\epsilon) \subset B_h^n$. On the other hand, since D_ϵ is a compact subset of the open set T, it implies that there is $\mu > 0$ such that $P(D_\epsilon, \mu) = \bigcup\{P((t', z_n), \mu) : (t', z_n) \in D_\epsilon\} \Subset T$. This follows that $\varphi_1^{-1}(P(D_\epsilon, \mu)) \Subset B_*^n$. Hence, there is $0 < h_1 < 1$ such that $\varphi_1^{-1}(\overline{P(D_\epsilon, \mu)}) \Subset B_{h_1}^n$. Thus $\bigcup\{\varphi_1(B_*(z, r_{h_1})) : z \in B_{h_1}^n\} \supset \overline{P(D_\epsilon, \mu)}$, where $B_*(z, r_{h_1}) = B(z, r_{h_1}) \backslash \{z_n = 0\}$. By a theorem on the Lebesgue number, there is $v > 0$ such that, for each $z \in \overline{P(D_\epsilon, \mu)}$, $B(z, v) \cap \overline{P(D_\epsilon, \mu)} \subset B_*(\tilde{z}, r_{h_1})$ for some $\tilde{z} = \tilde{z}(z) \in B_{h_1}^n$.

Put $\delta_h = \min(v/\sqrt{n}, \mu)$.

Consider $(t', z_n) \in \operatorname{Int} D_\epsilon$. Then $\varphi_1^{-1}(P((t', z_n), \delta_h)) \subset B_*(\tilde{z}, r_{h_1})$ for some $\tilde{z} \in B_{h_1}^n$, and hence, $f \circ \varphi_1^{-1}(P((t', z_n), \delta_h)) \subset f(B_*(\tilde{z}, r_{h_1})) \subset$ a subset of type (S).

Since $\varphi_1^{-1}(T_{R,1-\alpha}) \subset B_{1-\alpha}^n$, it implies that g is holomorphic in $T_{R,1-\alpha}$.

- If $h \le 1 - \alpha$ then g is holomorphic in $T_{R,h}$
- If $1 - \alpha < h$ then $h_0 := \sup\{\tilde{h} \le h$: g is holomorphic in $T_{R,\tilde{h}}\} \le h$. We now prove that $h_0 = h$.

Suppose that $h_0 < h$. Choose $\epsilon = \min\left\{\dfrac{h_0}{2(1+R^2)}, \dfrac{h-h_0}{1+R^2}\right\}$. Put $\delta_1 = \min\left\{\delta_h, \dfrac{h_0}{2(1+R^2)}, \dfrac{h-h_0}{1+R^2}\right\} > 0$. Take (t', z_n) such that $\|t'\| < R$, $|z_n|^2 = \dfrac{h_0}{1+R^2} - \dfrac{\delta_1}{2}$. Consider the polydisc $P(t', \min_{j=\overline{1,n-1}}(R - |t_j|)) \times P(z_n, \delta_1)$. Note that g is holomorphic in $P(t', \min_{j=\overline{1,n-1}}(R - |t_j|)) \times P(z_n, \delta_1/2)$ and, for each $\tilde{t}' \in P(t', \min_{j=\overline{1,n-1}}(R - |t_j|))$, $g(\tilde{t}', \tilde{z}_n)$ is holomorphic in $P(z_n, \delta_1)$ since f is holomorphic on every complex line passing through the origin. By Lemma 5.2, we have g is holomorphic in $P(t', \min_{j=\overline{1,n-1}}(R - |t_j|)) \times P(z_n, \delta_1)$ for each (t', z_n) with $\|t'\| < R$, $|z_n| = \dfrac{h_0}{1+R^2} - \dfrac{\delta_1}{2}$. Thus g is holomorphic in $T_{R, h_0 + \delta_1(1+R^2)/2}$. This is a contradiction. Hence g is holomorphic in $T_{R,h}$, i.e., g is holomorphic on T. Thus g is holomorphic in B_*^n, i.e., f is holomorphic in B_*^n. Since $B^n = \bigcup_{j=1}^n (B^n \backslash \{z_j = 0\}) \cup B_{1-\alpha}^n$, it implies that f is holomorphic in B^n.

Step 2. Assume that there exists $r_1 \in (0, 1)$ such that f is holomorphic in $B_{r_1}^n$.

Take $p_0 \in \partial B_{r_1}^n$. For the point $f(p_0) \in M$ take $W_0 = W_{f(p_0)}$, $r_0 = r_{f(p_0)}$, $S_0 = S_{f(p_0)}$ as the definition of Stein-type, i.e., for each $\varphi \in H(\Delta, M)$, if $\varphi(0) \in W_0$ then $\varphi(\Delta_{r_0}) \subset S_0$.

Since $\lim_{\alpha \to 1^{-0}} \dfrac{r_1(1-\alpha)}{1 - \alpha.r_1^2} = 0 < r_0$, there exists $\alpha_0 \in (0, 1)$ such that $\dfrac{r_1(1-\alpha_0)}{1-\alpha_0.r_1^2} < r_0$ and $f(\alpha_0 p_0) \in W_0$.

Since $\lim_{p \to p_0} \dfrac{\|p\|.(1-\alpha_0)}{1-\alpha_0.\|p\|^2} = \dfrac{r_1(1-\alpha_0)}{1-\alpha_0.r_1^2} < r_0$, there exists $B(p_0, \delta) \subset B^n$ such that $\dfrac{\|p\|.(1-\alpha_0)}{1-\alpha_0.\|p\|^2} < r_0$ for each $p \in B(p_0, \delta)$ and $f(\alpha_0.B(p_0, \delta)) = f(B(\alpha_0 p_0, \alpha_0 \delta)) \subset W_0$.

We now prove that $f(B(p_0, \delta)) \subset S_0$. Indeed, take $p \in B(p_0, \delta)$. Consider the Mobius map $\psi : \Delta \to \Delta$ given by $\psi(z) = \dfrac{z - \|\alpha_0 p\|}{1 - \|\alpha_0 p\|.z}$. Put $\psi(\|p\|) = p'$.

Consider the map $\varphi : \Delta \to B^n$ given by $\varphi(z) = \dfrac{z.p}{\|p\|}$ and the composite map $\phi := f \circ \varphi \circ \psi^{-1} : \Delta \to M$. Then $\phi(0) = f(\alpha_0 p) \in W_0$, $\phi(p') = f(p)$. On the other hand, since $|p'| = \dfrac{\|p\|.(1-\alpha_0)}{1-\alpha_0.\|p\|^2} < r_0$, it implies that $p' \in \Delta_{r_0}$, and hence, $\phi(p') = f(p) \in S_0$.

For each $p \in \bar{B}^n_{r_1}$, put

$$\delta_p = \sup\{\delta : f(B(p,\delta)) \text{ is included in a subset of type (S)}\}.$$

Then $\delta_p > 0$. It is easy to see that $|\delta_{p_0} - \delta_{p_1}| \le \|p_0 - p_1\|$ for all $p_0, p_1 \in \bar{B}^n_{r_1}$. This implies that the function $\delta : \bar{B}^n_{r_1} \to \boldsymbol{R}^+_*$ is continuous. Thus there is $\min_{p \in \bar{B}^n_{r_1}} \delta(p) = \delta_{r_1} > 0$. Then $f(B(p, \delta_{r_1}/2))$ is included in a subset of type (S) for each $p \in B^n_{r_1}$.

Choose $t \in \boldsymbol{R}$ such that $t.r_1 + t.(\delta_{r_1}/2) = 1$. Consider the biholomorphic mapping $\chi_t : B^n \to B^n_{r_1+\delta_{r_1}/2}$ given by $z \mapsto z/t$. Then $\chi_t^{-1}(B^n_{r_1}) = B^n_{t.r_1}$, $\chi_t^{-1}(B^n_{\delta_{r_1}/2}) = B^n_{t.(\delta_{r_1}/2)}$.

Clearly, $f \circ (\chi_{t\,|\,B^n_{t.r_1}})$ is holomorphic and $f \circ \chi_t\left(B\left(p, \dfrac{t\delta_{r_1}}{2}\right)\right) = f\left(B\left(\dfrac{p}{t}, \dfrac{\delta_{r_1}}{2}\right)\right)$ is included in a subset of type (S) for all $p \in B^n_{tr_1}$. By Step 1, we have f is holomorphic in $B^n_{r_1+\delta_{r_1}/2}$.

STEP 3. By the theorem of Forelli [19, p. 49], there exists $r_0 > 0$ such that f is holomorphic in $B^n_{r_0}$. Put $r^* = \sup\{r \in (0,1)\colon f \text{ is holomorphic in } B^n_r\}$. Then f is holomorphic in $B^n_{r^*}$.

Suppose $r^* < 1$. By Step 2, there is $\delta_{r^*} > 0$ such that f is holomorphic in $B^n_{r^*+\delta_{r^*}/2}$. This is a contradiction. Q.E.D.

5.4. Proposition. *Let R be a positive real number. Assume that a family $\{f_j\} \subset H(P(0,R), \boldsymbol{C})$ satisfies the following*:

There exist $0 < r < R$ and $f \in H(P(0,R), \boldsymbol{C})$ such that $\{f_j\}$ converges uniformly on compact subsets to f in $P(0',R) \times P(0_n, r)$ and $\{f_j(z',.)\}$ converges uniformly on compact subsets to $f(z',.)$ in $P(0_n, R)$ for each $z' \in P(0',R)$.

Then $\{f_j\}$ converges uniformly on compact subsets to f in $P(0,R)$.

Proof. Put $g_j = f_j - f$. Then $\{g_j\}$ converges uniformly on compact subsets to 0 in $P(0',R) \times P(0_n, r)$ (1) and $\{g_j(z',.)\}$ converges uniformly on compact subsets to 0 in $P(0_n, R)$ for each $z' \in P(0',R)$ (2).

Take $0 < r_1 < r < R_1 < R$.

By (1), it follows that $\forall \epsilon > 0$, $\exists j_0(\epsilon)$, $\forall j \ge j_0(\epsilon)$, $\forall z \in P(0', R_1) \times P(0_n, r_1)$: $|g_j(z)| < \epsilon$ (3). Consider the Hartogs expansion of g_j:

$$g_j = \sum_{k=0}^{\infty} c_k^{(j)}(z').z_n^k$$

From (3) we have $|c_k^{(j)}(z')| \le \dfrac{\epsilon}{r_1^k}$, $\forall j \ge j_0(\epsilon)$, $\forall k \ge 0$, $\forall z' \in P(0',R)$ (4).

Take $r < R_2 < R_1$. We now prove that $\exists n_0 = n_0(R_2)$, $\forall k, j \geq n_0$, $\forall z' \in P(0', R_2) : |c_k^{(j)}(z').R_2^k| \leq 1$ (5).

Indeed, suppose this does not hold. Then there exist sequences $\{k_i\}, \{j_i\}$, $\{z'_i\} \subset P(0', R_2)$ such that $|c_{k_i}^{(j_i)}(z'_i).R_2^{k_i}| > 1$ for all $i \geq 1$ (6). Put $v_i(z') = \dfrac{1}{k_i}.\log|c_{k_i}^{(j_i)}(z')|$, $\forall z' \in P(0', R_1)$. From (4) we get $v_i \leq -\log r_1$ in $P(0', R_1)$ for all $i \geq j_0(\epsilon)$ (7). Fix $z' \in P(0', R_1)$. By (2), there is $j_1 = j_1(\epsilon, z')$ such that $|g_j(z', .)| < \epsilon$ in $P(0_n, R_1)$ for each $j \geq j_1$. Hence $|c_k^{(j)}(z')| \leq \dfrac{\epsilon}{R_1^k}$ for all $j \geq j_1$, $k \geq 0$. Thus, for each $i \geq j_1$, $v_i(z') \leq \dfrac{1}{k_i}.\log\left(\dfrac{\epsilon}{R_1^{k_i}}\right) = \dfrac{\log \epsilon}{k_i} - \log R_1 < -\log R_1$, and hence, $\limsup_{i \to \infty} v_i(z') \leq -\log R_1 < -\log R_2$ (8). From (7), (8) and by Hartogs Lemma, it follows that $\exists n_0$, $\forall i \geq n_0$, $\forall z' \in P(0', R_2) : v_i(z') < -\log R_2$. Hence $v_i(z'_i) < -\log R_2$ for all $i \geq n_0$, i.e., $\dfrac{1}{k_i}.\log|c_{k_i}^{(j_i)}(z'_i)| < -\log R_2$ for all $i \geq n_0$. This follows that $|c_{k_i}^{(j_i)}(z'_i).R_2^{k_i}| < 1$ for all $i \geq n_0$. This contradicts to (6).

Take $r < R_3 < R_2$. From (5) we get $|c_k^{(j)}(z').R_3^k| \leq \left(\dfrac{R_3}{R_2}\right)^j$, $\forall k, j \geq n_0$, $\forall z' \in P(0', R_2)$ (9). Let ϵ_0 be any positive number. Take $j_2 = j_2(R_2, R_3, \epsilon_0)$ large enough such that $\sum_{j=j_2}^{\infty} \left(\dfrac{R_3}{R_2}\right)^j < \dfrac{\epsilon}{2}$ (10). Choose $j^* = \max(n_0(\epsilon_0), j_2)$. Put

$$\epsilon_1 = \frac{\epsilon_0}{2(\sum_{j=0}^{j^*} (R_3/r_1)^j)} > 0$$

From (4) and by the above-mentioned argument, there exists $j_0 = j_0(\epsilon_1)$ such that $|c_k^{(j)}(z')| \leq \dfrac{\epsilon_1}{r_1^k}$, $\forall j \geq j_0$, $\forall k \geq 0$, $\forall z' \in P(0', R)$ (11). By (10) and (11), we have, for all $j \geq j_0$, $z \in P(0, R_3)$,

$$\begin{aligned} |g_j(z)| &\leq \sum_{k=0}^{j^*} |c_k^{(j)}(z')|.R_3^k + \sum_{k=j^*}^{\infty} |c_k^{(j)}(z')|.R_3^k \\ &\leq \sum_{k=0}^{j^*} |c_k^j(z').r_1^k|.\left(\frac{R_3}{r_1}\right)^k + \frac{\epsilon_0}{2} \\ &\leq \sum_{k=0}^{j^*} \epsilon_1.\left(\frac{R_3}{r_1}\right)^k + \frac{\epsilon_0}{2} = \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = \epsilon_0. \end{aligned}$$

Hence $|g_j(z)| < \epsilon_0$ $\forall j \geq j_0$, $\forall z \in P(0, R_3)$. Q.E.D.

5.5. Remark. It is easy to see that Proposition 5.4 holds for general polydiscs in $\boldsymbol{C}^n$. Namely, we have the following

Given $R = (R_1, \dots, R_n) \in \boldsymbol{R}_*^{+n}$. *Assume that a family* $\{f_j\} \subset H(P(z,R), \boldsymbol{C})$ *satisfies the following:*

There exist $0 < r_n < R_n$ *and* $f \in H(P(z,R), \boldsymbol{C})$ *such that* $\{f_j\}$ *converges uniformly on compact subsets to* f *in* $P(z', R') \times P(z_n, r_n)$ *and* $\{f_j(z', .)\}$ *converges uniformly on compact subsets to* $f(z', .)$ *in* $P(z_n, R_n)$ *for each* $z' \in P(z', R')$.

Then $\{f_j\}$ *converges uniformly on compact subsets to* f *in* $P(z,R)$.

5.6. Proof of Theorem 5.

Take any sequence $\{f_j\}_{j=1}^{\infty} \subset \mathscr{F}$.

Step 1. Without loss of generality we may assume that $\{f_j(0)\} \to p \in M$. Choose W_p, r_p, S_p as in the definition of Stein-type. Without loss of generality we may assume that $f_j(0) \in W_p$ for all $j \geq 1$. Then $f_j(B_{r_p}^n) \subset S_p$ for all $j \geq 1$. By Alexander theorem [1, Thm 6.2.], it follows that the sequence $\{f_j\}$ contains a subsequence which converges uniformly to $f \in H(B_{r_p}^n, M)$ in $H(B_{r_p}^n, M)$. Without loss of generality we may assume that the sequence $\{f_j\}$ converges uniformly to $f \in H(B_{r_p}^n, M)$ in $H(B_{r_p}^n, M)$.

For each $z_0 \in B^n$ consider the holomorphic mapping $\varphi_{z_0} : \Delta \to B^n$ given by $z \mapsto \dfrac{z.z_0}{\|z_0\|}$. Then $f_j \circ (\varphi_{z_0|\Delta_{r_p}})$ converges uniformly to $f \circ (\varphi_{z_0|\Delta_{r_p}})$. On the other hand, we have $\{f_j \circ \varphi_{z_0}\}$ converges uniformly to $F_{z_0} \in H(\Delta, M)$. This follows that $F_{z_0|\Delta_{r_p}} = f \circ (\varphi_{z_0|\Delta_{r_p}})$, and hence, $F_{z_0} = f \circ \varphi_{z_0}$. Define the holomorphic mapping $F : B^n \to M$, $z \mapsto F_z(\|z\|)$. Then $F_{|B_{r_p}^n} = f$ and F is holomorphic on each complex line through the origin. By Proposition 5.3, it implies that F is holomorphic in B^n.

Step 2. Assume that $\{f_j\}$ converges uniformly on compact subsets to F in $B_{1-\alpha}^n$ for some $0 < \alpha < 1$. Assume that, for each $z \in B_{1-\alpha}^n$, there exist $j(z) \geq 1$ and a subset S_z of type (S) such that $F(B(z,\alpha)) \subset S_z$ and $f_j(B(z,\alpha)) \subset S_z$ for each $j \geq j(z)$.

Then $\{f_j\}$ converges uniformly on compact subsets to F on every complex line passing through the origin (*).

From (*) and by the same argument as in Step 1 of the proof of Proposition 5.3, it implies that the sequence $\{f_j\}$ converges uniformly to F in B^n.

We now prove the assertion (*). Indeed, suppose that there is a complex line ℓ passing through the origin such that $\{f_j|\ell\}$ does not converge uniformly on compact subsets to $F|\ell$. Then there exist $\{z_j\} \subset \ell$, $z_0 \in \ell$, $\{z_j\} \to z_0$ such that the sequence $\{f_j(z_j)\}$ does not converge to $F(z_0)$. Hence there are a neighbourhood U_0 of $F(z_0)$ and an infinite subset N_0 of $\boldsymbol{N}$ such that $f_j(z_j) \notin U_0$ for all $j \in N_0$. Take an infinite subset N_1 of N_0 such that the sequence $\{f_j|\ell\}_{j \in N_1}$ converges uniformly on compact subsets to G, where G is a holomorphic mapping defined on ℓ. Then $G|_{\ell \cap B_{1-\alpha}^n} = F|_{\ell \cap B_{1-\alpha}^n}$, i.e., $G = F|\ell$. This follows that the sequence $\{f_j|\ell\}_{j \in N_1}$ converges uniformly on compact subsets to $F|\ell$, i.e., $\{f_j(z_j)\}_{j \in N_1}$ converges to $F(z_0)$. This is impossible.

Step 3. Assume that the sequence $\{f_j\}$ converges uniformly to F in $B_{r_1}^n$ for some $0 < r_1 < 1$. We now prove that there exist $r_1' \in (r_1, 1)$ such that $\{f_j\}$ converges uniformly to F in $B_{r_1'}^n$.

Indeed, take $p_0 \in \partial B^n_{r_1}$. For the point $F(p_0) \in M$ take $W_0 = W_{F(p_0)}$, $r_0 = r_{F(p_0)}$, $S_0 = S_{F(p_0)}$ as the definition of Stein-type. Repeating as in Step 2 of the proof of Proposition 5.3, it follows that there are $\delta_{p_0} > 0$ and $j(p_0) \geq 1$ such that $f_j(B(p_0, \delta_{p_0})) \subset S_0$ for all $j \geq j(p_0)$ and $F(B(p_0, \delta_{p_0})) \subset S_0$.

For each $p \in \bar{B}^n_{r_1}$ put $\delta_p = \sup\{\delta > 0\colon \exists$ a subset S_p of type (S), $\exists j(\delta) \geq 1$ such that $F(B(p,\delta)) \subset S_p$ and $f_j(B(p,\delta)) \subset S_p$ for all $j \geq j(\delta)\}$. Then δ is continuous on $\bar{B}^n_{r_1}$, and hence, there exists $\min_{p \in \bar{B}^n_{r_1}} \delta(p) = \delta_{r_1} > 0$. Thus $\forall z \in \bar{B}^n_{r_1}$, $\exists$ a subset S_z of type (S), $\exists j(z) \geq 1$ such that $F(B(z, \delta_{r_1}/2)) \subset S_z$ and $f_j(B(z, \delta_{r_1}/2)) \subset S_z$ for all $j \geq j(z)$.

Reasoning again as in Step 2 of the proof of Proposition 5.3, it follows that $\{f_j\}$ converges uniformly to F in $B^n_{r_1 + \delta_{r_1}/2}$.

Step 4. Put

$$r^* = \sup\{r \in (0,1)\colon \text{the sequence } \{f_j\} \text{ converges uniformly to } F \text{ in } B^n_r\}.$$

Then the sequence $\{f_j\}$ converges uniformly to F in $B^n_{r^*}$.

If $r^* < 1$ then, by Step 3, there is $r_0 \in (r^*, 1)$ such that the sequence $\{f_j\}$ converges uniformly to F in $B^n_{r_0}$. This is a contradiction. Q.E.D.

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