J. Math. Soc. Japan
Vol. 70, No. 3 (2018) pp. 1047–1062
doi: 10.2969/jmsj/75337533

Examples of four dimensional cusp singularities

By Hiroyasu Tsuchihashi

(Received May 28, 2016)
(Revised Dec. 19, 2016)

Abstract. We give some examples of four dimensional cusp singularities which are not of Hilbert modular type. We construct them, using quadratic cones and subgroups of reflection groups.

Introduction.

In [**8**], we showed that an r-dimensional cusp singularity $\mathrm{Cusp}(C, \Gamma)$ is obtained from a pair (C, Γ) of an open cone C in $\mathbf{R}^r$ and a subgroup Γ of $GL(r, \mathbf{Z})$ satisfying the following three conditions, where r is an integer greater than 1.

1. C is strongly convex, i.e., $\overline{xy} \subset C$ for any $x, y \in C$ and $\overline{C} \cap \overline{-C} = \{0\}$.
2. C is Γ-invariant, i.e., $\gamma C = C$ for all $\gamma \in \Gamma$.
3. Γ acts on $D_C := C/\mathbf{R}_{>0}$ properly discontinuously, freely and D_C/Γ is compact.

$\mathrm{Cusp}(C, \Gamma)$ is obtained by adding a point to the quotient of the tube domain $\mathbf{R}^r + \sqrt{-1}C$ under the action of the semidirect product of $\mathbf{Z}^r$ and Γ. In the 2-dimensional case, $\mathrm{Cusp}(C, \Gamma)$ is nothing but a Hilbert modular cusp singularity. Hilbert modular cusp singularities exist in all dimensions greater than 1, where C is the interior of a simplicial cone and D_C/Γ is a real torus. It is also known that there exist other higher dimensional cusp singularities of arithmetic type (see [**6**] and [**7**, Section 3], for instance). We gave in [**8**] some 3-dimensional explicit examples of (C, Γ) such that D_C/Γ are not real tori. In 1991, Ishida [**3**] gave explicit 4-dimensional examples. Until quite recently no other 4-dimensional explicit examples seem to be found. On the other hand, Vinberg [**10**] gave a way to obtain a subgroup Γ of $GL(r, \mathbf{R})$ acting properly discontinuously on a strongly convex open cone C in $\mathbf{R}^r$. Here Γ is generated by reflections with respect to the hyperplanes containing the $(r-1)$-dimensional faces of a polyhedral cone satisfying certain conditions. Moreover, he gave a simple necessary and sufficient condition for the cone C to be quadratic, i.e., defined by a quadratic polynomial. In this paper, using the results in [**10**], we give some explicit examples of 4-dimensional pairs (C, Γ) such that Γ are subgroups of reflection groups.

In Section 1, we show that for any open strongly convex cone C in $\mathbf{R}^r$, any subgroup of $GL(r, \mathbf{Z})$ preserving C, acts on D_C properly discontinuously. In Section 2, we show that if a quadratic polynomial P defines a cone C in $\mathbf{R}^r$ and there exists a subgroup Γ of $GL(r, \mathbf{Z})$ satisfying the above conditions, then all coefficients of P may be assumed to be

2010 *Mathematics Subject Classification.* Primary 14J17; Secondary 32S05.
Key Words and Phrases. cusp singularity, quadratic cone, reflection.

integers and $P(x) \neq 0$ for any point x in $\mathbf{Z}^r \setminus \{0\}$. In Section 3, we show that if a quadratic cone C contains a rational polyhedral cone satisfying certain conditions, then there exists a reflection group Γ contained in $GL(r, \mathbf{Z})$ and acting on C with compact D_C/Γ. In Section 4, we study the structure of exceptional sets of resolutions of $\mathrm{Cusp}(C, \Gamma)$ for pairs (C, Γ) such that Γ is a subgroup of a reflection group. Finally, we give three 4-dimensional examples of pairs (C, Γ) with quadratic C, and one with non-quadratic C and a resolution of $\mathrm{Cusp}(C, \Gamma)$ whose exceptional set consists of 4 irreducible components.

1. Groups acting on cones.

Let N be a free $\mathbf{Z}$-module of rank $r > 1$, let $M = \mathrm{Hom}(N, \mathbf{Z})$ and let $\langle\ ,\ \rangle : M \times N \longrightarrow \mathbf{Z}$ be the natural pairing. For an open cone C in $N_{\mathbf{R}} = N \otimes \mathbf{R}$, let $D_C = C/\mathbf{R}_{>0}$ and let $p_C : C \longrightarrow D_C$ be the natural projection.

Definition. $\Gamma_C = \{\gamma \in GL(N) \mid \gamma C = C\}$ for an open cone C in $N_{\mathbf{R}}$.

Let $C^* = \{x \in M_{\mathbf{R}} \mid \langle x, y\rangle > 0 \text{ for } y \in \overline{C} \setminus \{0\}\}$. If C is an open strongly convex cone in $N_{\mathbf{R}}$, then $\Gamma_{C^*} = \{{}^t\gamma \mid \gamma \in \Gamma_C\}$, where ${}^t\gamma$ is the element in $GL(M)$ satisfying $\langle {}^t\gamma x, y\rangle = \langle x, \gamma y\rangle$ for any elements x and y in M and N, respectively.

Theorem 1. *If C is an open strongly convex cone in $N_{\mathbf{R}}$, then Γ_C acts on D_C properly discontinuously, i.e., $\{\gamma \in \Gamma \mid \gamma S \cap S \neq \emptyset\}$ is finite for every compact subset S of D_C.*

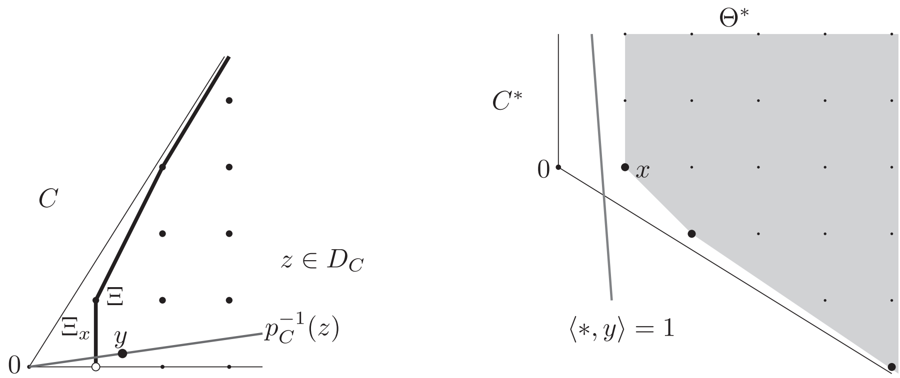

Figure 1.

Proof. Let Θ^* be the convex hull of $C^* \cap M$ and let Ξ be the boundary of $\{y \in C \mid \langle x, y\rangle \geq 1 \text{ for } x \in \Theta^*\}$. Then the restriction $p_{C|\Xi} : \Xi \longrightarrow D_C$ of p_C to Ξ is a homeomorphism (see Figure 1). Let $\Xi_x = \{y \in \Xi \mid \langle x, y\rangle = 1\}$ for each element x in $C^* \cap M$. Then Ξ_x is closed in Ξ. Let L be the set of vertices on Θ^*. Then L is contained in M and $\Xi = \bigcup_{x \in L} \Xi_x$. For any point y in Ξ, $\{x \in L \mid y \in \Xi_x\} \subset \{x \in C^* \cap M \mid \langle x, y\rangle = 1\}$ is finite.

Let S be a compact subset of D_C. Then $L_0 = \{x \in L \mid S \cap p_C(\Xi_x) \neq \emptyset\}$ is finite. If $\gamma S \cap S \neq \emptyset$ for an element γ in Γ_C, then there exist elements x_1, x_2 in L_0 with ${}^t\gamma x_1 = x_2$.

On the other hand, $K = \{y \in C \cap N \mid \langle x_1, y\rangle = c\}$ contains linearly independent r elements for a positive integer c. Then $\{\gamma \in \Gamma_C \mid {}^t\gamma x_1 = x_1\} \subset \{\gamma \in \Gamma_C \mid \gamma K = K\}$ is a finite set. Hence $\{\gamma \in \Gamma_C \mid {}^t\gamma x_1 = x_2\}$ is also finite for any elements x_1, x_2 in L_0. Therefore, $\{\gamma \in \Gamma_C \mid \gamma S \cap S \neq \emptyset\}$ is finite. □

For an open strongly convex cone C with compact D_C/Γ_C, there exists a normal subgroup Γ of Γ_C with a finite index acting on D_C freely. For example, we obtain such a group as the intersection with the kernel of $SL(N) \to SL(N/nN)$ for a suitable positive integer n.

2. Quadratic cones.

We fix a coordinate $(X_1, X_2, \ldots, X_r)$ of N throughout the rest of this paper. For a homogeneous polynomial $P(X_1, X_2, \ldots, X_r)$ of r variables, we denote by C_P the open cone defined by

$$\{(x_1, x_2, \ldots, x_r) \in N_{\mathbf{R}} \mid P(x_1, x_2, \ldots, x_r) > 0\}.$$

Definition. We call a cone C in $N_{\mathbf{R}}$ quadratic, if there exists a homogeneous quadratic polynomial $P(X_1, X_2, \ldots, X_r)$ such that C is a connected component of C_P.

If a quadratic cone C defined by a polynomial P is strongly convex, then the signature of P is $(1, r-1)$ and $C \cup (-C) = C_P$.

Theorem 2. *Let C be a quadratic strongly convex cone in $N_{\mathbf{R}}$ defined by a polynomial P. If D_C/Γ_C is compact, then there exists a positive real number c such that all coefficients of cP are integers and P has no isotropic elements in N, i.e., $P(x) \neq 0$ for all x in $N \setminus \{0\}$.*

Proof. First, we show that there exists a finite set K contained in $C \cap N$ such that the convex hull of $p_C(\Gamma_C K)$ is equal to D_C. Let Ξ be the boundary of the convex hull of $C \cap N$ and let $J = \Xi \cap N$. Then the convex hull of $p_C(J)$ is equal to D_C. On the other hand, J/Γ_C is finite, because D_C/Γ_C is compact. Hence there exists a finite set K such that $\Gamma_C K = J$.

Let x be an element in K. We may assume that $P(x) = 1$, multiplying P by a positive number. Then $P(\gamma x) = 1$ for any element γ in Γ_C. Hence all coefficients of P are rational, by the following lemma.

Lemma. *There exist $m = r(r+1)/2$ elements $\gamma_1, \gamma_2, \ldots, \gamma_m$ in Γ_C and an element x in K such that $f(\gamma_1 x), f(\gamma_2 x), \ldots, f(\gamma_m x)$ are linearly independent, where $f : N \longrightarrow \mathbf{Z}^m$ is the map sending $(x_1, x_2, \ldots, x_r)$ to $(x_1^2, \ldots, x_r^2, x_1x_2, \ldots, x_{r-1}x_r)$.*

Proof. Suppose that $f(\gamma_1 x), f(\gamma_2 x), \ldots, f(\gamma_m x)$ are linearly dependent for any element x in K and any m elements $\gamma_1, \gamma_2, \ldots, \gamma_m$ in Γ_C. Then $f(\Gamma_C x)$ is contained in an $(m-1)$-dimensional linear subspace of $\mathbf{R}^m$. It implies that there exists a homogeneous quadratic polynomial $Q_x(x_1, x_2, \ldots, x_r)$ such that $Q_x(\gamma x) = 0$ for all γ in Γ_C. Since K is finite, there exists a point x_0 on $\partial C \setminus \{0\}$ such that $Q_x(x_0) \neq 0$ for all x in K. Then there exists a non-zero element y_0 in $M_{\mathbf{R}}$ such that $\langle y_0, x_0\rangle < 0$ and that $\langle y_0, \gamma x\rangle > 0$ for all x in K and for all γ in Γ_C, because there exists a hyperplane H with $H \cap \partial C = \mathbf{R}_{\geq 0}x_0$. Hence D_C is not equal to the convex hull of $p_C(\Gamma_C K)$, a contradiction.

Next, suppose that $P(y_0) = 0$ for an element y_0 in $N \setminus \{0\}$. We may assume that y_0 is primitive and that $y_0 \in \partial C$. Let x_0 be a vertex on the boundary of the convex hull of $\{x \in C^* \cap M \mid \langle x, y_0\rangle = 1\}$, which is not empty. Then $x_0 \in M$ and $y_0 \in \overline{\Theta_{x_0}}$, where

$$\Theta_{x_0} = \{y \in C \mid \langle x_0, y\rangle = 1, \langle x, y\rangle \geq 1 \text{ for } x \in C^* \cap M\}.$$

Since $\overline{\Theta_{x_0}}$ is compact, $\Theta_{x_0} \cap N$ is a finite set. Hence $\Gamma_0 = \{\gamma \in \Gamma_C \mid \gamma\Theta_{x_0} = \Theta_{x_0}\}$ is a finite group. Therefore, $p_C(\Theta_{x_0})/\Gamma_0$ is not compact. However, $p_C(\Theta_{x_0})$ is closed in D_C. It implies that D_C/Γ_C is not compact. □

In the 2-dimensional case, the converse of the above theorem holds, because $C = \mathbf{R}_{\geq 0}v_1 + \mathbf{R}_{\geq 0}v_2$ for two eigenvectors v_1 and v_2 in $N_{\mathbf{R}} \setminus N_{\mathbf{Q}}$ of an element in $SL(N)$.

Proposition 3. *An open strongly convex cone C in $N_{\mathbf{R}}$ with compact D_C/Γ_C, is quadratic, if and only if there exists a homomorphism $f : N \to M$ such that $f_{\mathbf{R}}(C) = C^*$ and that $f \circ \gamma = {}^t\gamma^{-1} \circ f$ for any element γ in Γ_C.*

Proof. Assume that C is quadratic, i.e., there exists a regular symmetric matrix A of index $(1, r-1)$ such that C is a connected component of $\{x \in N_{\mathbf{R}} \mid {}^t xAx > 0\}$. We may assume that all entries of A are integers, by Theorem 2. Let $f : N \to M$ be the homomorphism satisfying $\langle f(y), x\rangle = {}^t yAx$. Since the index of A is $(1, r-1)$,

$$\{y \in N_{\mathbf{R}} \mid {}^t yAx > 0 \text{ for } x \in \overline{C} \setminus \{0\}\} = C.$$

Therefore, $f_{\mathbf{R}}(C) = C^*$. Let γ be any element in Γ_C. Then ${}^t\gamma A\gamma = A$. Hence

$$\langle f(\gamma y), x\rangle = {}^t(\gamma y)Ax = {}^t y {}^t\gamma Ax = {}^t yA\gamma^{-1}x = \langle f(y), \gamma^{-1}x\rangle = \langle {}^t\gamma^{-1}f(y), x\rangle.$$

Therefore, $f \circ \gamma = {}^t\gamma^{-1} \circ f$.

Conversely, assume that there exists a homomorphism $f : N \to M$ as in the proposition. We define a symmetric bilinear form on $N_{\mathbf{R}}$ by $x \cdot y = \langle f_{\mathbf{R}}(x), y\rangle + \langle f_{\mathbf{R}}(y), x\rangle$. Then there exists a symmetric and integer matrix A with $x \cdot y = {}^t xAy$. For any element γ in Γ_C, $\gamma x \cdot \gamma y = x \cdot y$, because $\langle f_{\mathbf{R}}(\gamma x), \gamma y\rangle = \langle {}^t\gamma^{-1} f_{\mathbf{R}}(x), \gamma y\rangle = \langle f_{\mathbf{R}}(x), y\rangle$. Since $f_{\mathbf{R}}(C) = C^*$, $x \cdot y > 0$ for any points x and y in C. Hence $x \cdot x \geq 0$ for any point x on ∂C, because the function $N_{\mathbf{R}} \ni x \mapsto x \cdot x \in \mathbf{R}$ is continuous. Let Θ be the convex hull of $C \cap N$. Since $\partial\Theta/\Gamma_C$ is compact, $\{x \cdot x \mid x \in \partial\Theta\}$ has the maximal value d. Let $S_d = \{x \in N_{\mathbf{R}} \mid x \cdot x = d\}$. Then $S_d \cap C \subset \Theta$. Since Θ is closed and $\Theta \cap \partial C = \emptyset$, $S_d \cap \partial C = \emptyset$. Hence $x \cdot x = 0$ for any point x on ∂C. Therefore, C is a connected component of $\{x \in N_{\mathbf{R}} \mid x \cdot x > 0\}$. □

The above proposition can be applied to decide whether the cone C is quadratic for a pair (C, Γ) satisfying the conditions 1, 2 and 3 in Introduction. We give an example. Let $r = 3$. Let S be the surface and Δ be its triangulation obtained from the hexagon in Figure 2, identifying the edges $\overline{v_1v_2}$, $\overline{v_3v_4}$ and $\overline{v_5v_6}$ with $\overline{v_2v_3}$, $\overline{v_4v_5}$ and $\overline{v_6v_1}$, respectively.

Then $\chi(S) = -1$ and the double $\mathbf{Z}$-weight on Δ as in Figure 2 satisfies the monodromy condition and the convexity condition (see [**8**, Definitions 1.3 and 1.5]). Hence we obtain a map $\sigma : \{\text{all vertices of } \widetilde{\Delta}\} \to N$ and a homomorphism $\rho : \pi_1(S) \to GL(N)$ such that $\sigma(\gamma v) = \rho(\gamma)\sigma(v)$ for all vertices v of $\widetilde{\Delta}$ and all elements γ in $\pi_1(S)$ by [**8**], where $\widetilde{\Delta}$ is the pull-back of Δ under the universal covering $\varpi : \widetilde{S} \to S$. Let $C = \mathbf{R}_{>0}\Theta$, where Θ is the convex hull of the image of σ, and let $\Gamma = \rho(\pi_1(S))$. Then the pair (C, Γ) satisfies the conditions 1, 2 and 3 in Introduction. There exist vertices $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_6$ of $\widetilde{\Delta}$ with $\varpi(\tilde{v}_i) = v_i$ such that $\overline{\tilde{v}_1\tilde{v}_2\tilde{v}_3}$, $\overline{\tilde{v}_3\tilde{v}_4\tilde{v}_5}$, $\overline{\tilde{v}_5\tilde{v}_6\tilde{v}_1}$ and $\overline{\tilde{v}_1\tilde{v}_3\tilde{v}_5}$ are triangles of $\widetilde{\Delta}$. Here we may assume that $\sigma(\tilde{v}_1) = \mathbf{e}_1$, $\sigma(\tilde{v}_3) = \mathbf{e}_2$ and $\sigma(\tilde{v}_5) = \mathbf{e}_3$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of N. Let $\mathbf{f}_i = \sigma(\tilde{v}_{2i}) (= 2\mathbf{e}_i + 2\mathbf{e}_{i+1} - \mathbf{e}_{i+2})$ for each i in $\mathbf{Z}/3\mathbf{Z}$. Let $\Sigma = \{\gamma\tau \mid \gamma \in \Gamma, \tau \prec \mu_i, i = 0, 1, 2, 3\}$, where $\mu_0 = \mathbf{R}_{\geq 0}\mathbf{e}_1 + \mathbf{R}_{\geq 0}\mathbf{e}_2 + \mathbf{R}_{\geq 0}\mathbf{e}_3$ and $\mu_i = \mathbf{R}_{\geq 0}\mathbf{e}_i + \mathbf{R}_{\geq 0}\mathbf{e}_{i+1} + \mathbf{R}_{\geq 0}\mathbf{f}_i$ for $i = 1, 2, 3$. Then Σ is a non-singular fan with $|\Sigma| \setminus \{0\} = C$ and Γ acts on the set of 1-dimensional cones in Σ transitively, because Δ has only one vertex. Hence we have a resolution of $\mathrm{Cusp}(C, \Gamma)$ whose exceptional set is irreducible.

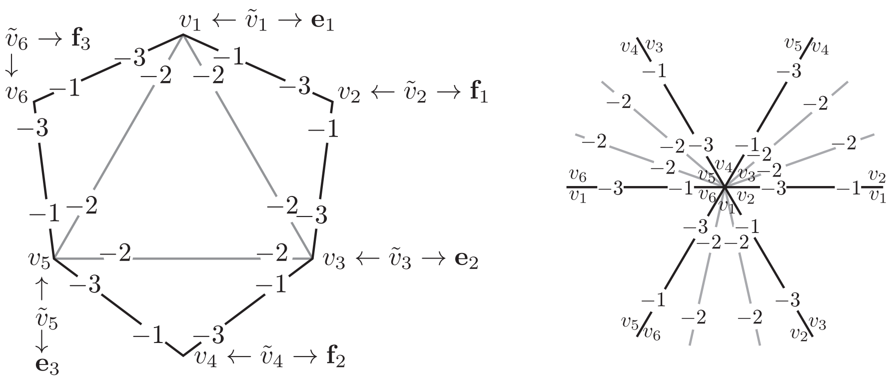

Figure 2.

Proposition 4. *The above cone C is not quadratic.*

Proof. Let γ_i be the elements in $GL(N)$ sending $\mathbf{e}_i$, $\mathbf{f}_i$ and $\mathbf{e}_{i+1}$ to $\mathbf{f}_i$, $\mathbf{e}_{i+1}$ and $\mathbf{f}_i + 3\mathbf{e}_{i+1} - \mathbf{e}_i$, respectively for all i in $\mathbf{Z}/3\mathbf{Z}$. Then γ_i are in Γ_C. We easily see that also is in Γ_C the element sending $\mathbf{e}_i$ to $\mathbf{e}_{i+1}$, which we denote by δ. Let $\mathbf{e}_0 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{e}_0^* = \mathbf{e}_1^* + \mathbf{e}_2^* + \mathbf{e}_3^*$, where $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$ is the basis of M dual to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then $\delta\mathbf{e}_0 = \mathbf{e}_0$ and ${}^t\delta\mathbf{e}_0^* = \mathbf{e}_0^*$. Suppose that there exists an injective homomorphism $f : N \to M$ satisfying $f \circ \gamma = {}^t\gamma^{-1} \circ f$ for any element γ in Γ_C. Then $f(\mathbf{e}_0) = c\mathbf{e}_0^*$ for a non-zero integer c, because any fixed point of ${}^t\delta^{-1}$ is on $\mathbf{R}\mathbf{e}_0^*$. We see by an easy calculation that $\gamma_i\mathbf{e}_0 = 9\mathbf{e}_i + 20\mathbf{e}_{i+1} - 6\mathbf{e}_{i+2}$ and ${}^t\gamma_i^{-1}\mathbf{e}_0^* = 9\mathbf{e}_i^* + 3\mathbf{e}_{i+1}^* + 23\mathbf{e}_{i+2}^*$. Hence $\gamma_1\mathbf{e}_0 + \gamma_2\mathbf{e}_0 + \gamma_3\mathbf{e}_0 = 23\mathbf{e}_0$ and ${}^t\gamma_1^{-1}\mathbf{e}_0^* + {}^t\gamma_2^{-1}\mathbf{e}_0^* + {}^t\gamma_3^{-1}\mathbf{e}_0^* = 35\mathbf{e}_0^*$. It implies $c = 0$. Hence C is not quadratic, by Proposition 3. $\square$

3. Reflections.

Let P be a quadratic homogeneous polynomial of r variables with the signature $(1, r-1)$, and let C be a connected component of C_P. Then C is strongly convex and $C_P = C \cup (-C)$. We assume that all coefficients of P are integers with no common divisors greater than 1, throughout this section. Let $B_P : N \times N \to \mathbf{Z}$ be the symmetric bilinear form with $B_P(x,x) = 2P(x)$.

Definition. $x \cdot y = B_P(x,y)$ for elements $x, y \in N_{\mathbf{R}}$.

We easily see that $\gamma x \cdot \gamma y = x \cdot y$ for any element γ in Γ_C. For an element v in $N_{\mathbf{R}}$ with $v \cdot v \neq 0$, we define a linear transformation γ_v and a hyperplane H_v of $N_{\mathbf{R}}$ as follows:

$$\gamma_v : x \mapsto x - 2\frac{x \cdot v}{v \cdot v}v, \qquad H_v = \{x \in N_{\mathbf{R}} \mid x \cdot v = 0\}.$$

We see by easy calculation that $\gamma_v^2 = \mathrm{id}$, $\gamma_v v = -v$, $\gamma_v x = x$ for any x in H_v and $\gamma_v x \cdot \gamma_v y = x \cdot y$ for any x, y in $N_{\mathbf{R}}$. Hence $\gamma_v C = C$ or $-C$. If $v \cdot v < 0$, then $\gamma_v C = C$, because $C \cap H_v \neq \emptyset$. Hence we have:

Proposition 5. *If v is an element in N with $v \cdot v < 0$ and $2(\mathbf{e}_i \cdot v / v \cdot v) \in \mathbf{Z}$ for each fundamental vector $\mathbf{e}_i$, then γ_v is in Γ_C.*

Any element v in N with $v \cdot v = -2$ satisfies the assumption of the above proposition. Let $F_\gamma = \{x \in C \mid \gamma x = x\}$ for an element γ in Γ_C.

Proposition 6. *Let γ be an element in Γ_C with $F_\gamma \neq \emptyset$ and $\dim F_\gamma = r-1$. Then there exists an element v in N with $\gamma = \gamma_v$.*

Proof. $r-1$ of the eigenvalues of γ are equal to 1. The other is equal to -1 and $\gamma^2 = 1$, by Theorem 1. Hence there exists a non-zero element v in N with $\gamma v = -v$. For any element x in $N_{\mathbf{R}}$, there exists a real number c_x with $x - \gamma x = c_x v$, because $\gamma(x - \gamma x) = -(x - \gamma x)$. On the other hand, $\gamma x \cdot v = x \cdot \gamma v$, because $\gamma^2 = 1$. Hence $(x - \gamma x) \cdot v = 2x \cdot v$. Therefore, $c_x = 2(x \cdot v / v \cdot v)$. □

Here we note that an eigenvector h of γ_v corresponding to the eigenvalue -1 and the linear function α on $N_{\mathbf{R}}$ with $\alpha(h) = 2$ and vanishing on H_v in [**10**], are nothing but v and the function $\alpha(x) = 2v \cdot x / v \cdot v$, respectively.

Proposition 7. *Let v and w be elements in N with $v \cdot v < 0$ and $w \cdot w < 0$. If $v \cdot w / (\sqrt{-v \cdot v}\sqrt{-w \cdot w}) = 0$, $1/2$, $1/\sqrt{2}$ or $\sqrt{3}/2$, then $|\gamma_v \gamma_w| = 2, 3, 4$ or 6, respectively, and $\lambda = \{y \in N_{\mathbf{R}} \mid v \cdot y \geq 0, w \cdot y \geq 0\}$ is a fundamental domain of the action of $\langle \gamma_v, \gamma_w \rangle$ on $N_{\mathbf{R}}$.*

Proof. We may assume that $v \cdot v = w \cdot w = -1$ replacing v and w with $v/\sqrt{-v \cdot v}$ and $w/\sqrt{-w \cdot w}$, respectively. Assume that $v \cdot w = \sqrt{3}/2$. Then $\gamma_v \gamma_w$ sends v and w to $2v + \sqrt{3}w$ and $-\sqrt{3}v - w$, respectively. Hence $|\gamma_v \gamma_w| = 6$. Moreover,

$$\lambda = \mathbf{R}_{\geq 0}(-2v - \sqrt{3}w) + \mathbf{R}_{\geq 0}(-\sqrt{3}v - 2w) + \{y \in N_{\mathbf{R}} \mid v \cdot y = w \cdot y = 0\}.$$

We see by easy calculation that $r-2 \leq \dim(\gamma\lambda \cap \lambda) \leq r-1$ for any γ in $\langle \gamma_v, \gamma_w \rangle \setminus \{1\}$. For the other cases, calculation is easier. □

If $v \cdot w/(\sqrt{-v \cdot v}\sqrt{-w \cdot w}) = -1/2, -1/\sqrt{2}$ or $-\sqrt{3}/2$, then $|\gamma_v\gamma_w| = 3, 4$ or 6, respectively, however, $\dim(\gamma_v\gamma_w\gamma_v\lambda \cap \lambda) = r$. Let σ be an r-dimensional rational polyhedral cone. For each $(r-1)$-dimensional face τ of σ, we denote by $v(\tau)$ the unique primitive element v in N determined by the condition that $v \cdot y = 0$ for all points y in τ and $v \cdot y \geq 0$ for all points y in σ.

Theorem 8. *If there exists an r-dimensional rational polyhedral cone σ satisfying the following three conditions, then $p_C(\sigma \setminus \{0\})$ is a fundamental domain of the action of Γ on D_C, $\Sigma = \{\gamma\lambda \mid \gamma \in \Gamma, \lambda \prec \sigma\}$ is a fan and $|\Sigma| = C \cup \{0\}$, where $\Gamma = \langle \gamma_{v(\tau)} \mid \tau \prec \sigma, \dim \tau = n-1 \rangle$.*

1. *$\sigma \setminus \{0\} \subset C$.*
2. *$v(\tau) \cdot v(\tau) < 0$ and $\gamma_{v(\tau)} \in \Gamma_C$ for any $(r-1)$-dimensional face τ of σ.*
3. *$v(\tau) \cdot v(\mu)/(\sqrt{-v(\tau) \cdot v(\tau)}\sqrt{-v(\mu) \cdot v(\mu)}) = 0,\ 1/2,\ 1/\sqrt{2}$ or $\sqrt{3}/2$ for any $(r-1)$-dimensional faces τ and μ of σ with $\dim(\tau \cap \mu) = r-2$.*

Proof. We can define distance $\overline{vw}$ on $S_C = \{v \in C \mid v \cdot v = 1\} \simeq D_C$ by $\cosh \overline{vw} = v \cdot w$ and angle $\angle H_v^C H_w^C$ of two hyperplanes $H_v^C = H_v \cap S_C$ and $H_w^C = H_w \cap S_C$ on S_C by $\cos \angle H_v^C H_w^C = v \cdot w/(\sqrt{-v \cdot v}\sqrt{-w \cdot w})$ for $v, w \in N_{\mathbf{R}}$ with $v \cdot v < 0$, $w \cdot w < 0$. Then we may regard D_C as a hyperbolic space and $(p_C)_{\mathbf{R}}(\sigma \setminus \{0\})$ as a Coxeter polyhedron, by the conditions 2, 3 and Proposition 7. Hence we see by [**4**, Theorem 7.1.3] that the assertions of the theorem hold. □

4. Structure of exceptional sets.

We keep the notations and the assumptions in the previous section. Let σ be an r-dimensional rational polyhedral cone satisfying the conditions of Theorem 8. Let $W = T_N \operatorname{emb}(\Sigma)$ be the toric variety associated to the fan Σ in Theorem 8. For a cone $\tau \neq \{0\}$ in Σ, we denote by $V(\tau)$ the closure of $\operatorname{orb}(\tau)$ in W, which is a compact toric variety (see [**5**, Corollary 1.7]). Let $\operatorname{ord} : T_N \to N_{\mathbf{R}}$ be the homomorphism induced by $-\log|\ | : \mathbf{C}^\times \to \mathbf{R}$. Let $\widetilde{U}$ be the interior of the closure of $\operatorname{ord}^{-1}(C)$ in W and let $\widetilde{X} = W \setminus T_N$. Then $\widetilde{U}$ is an open neighborhood of $\widetilde{X}$. Let Γ_0 be a subgroup of Γ with a finite index acting on D_C freely. Then Γ_0 acts on $\widetilde{U}$ freely. Let $U = \widetilde{U}/\Gamma_0$ and let $X = \widetilde{X}/\Gamma_0$. Then the cusp singularity $\operatorname{Cusp}(C, \Gamma_0)$ is obtained by contracting X to a point in U (see [**8**]).

Let λ be a face of σ with $1 \leq s := \dim \lambda \leq r-2$, and let $p_\lambda : N \to N/(\mathbf{R}\lambda \cap N)$ be the natural projection. Let $\mu_1, \mu_2, \ldots, \mu_l$ be the $(r-1)$-dimensional faces of σ with $\lambda \prec \mu_i$ and let $\Gamma_\lambda = \langle \gamma_{v(\mu_i)} \mid i = 1, \ldots, l \rangle$. Then Γ_λ acts on $N/(\mathbf{R}\lambda \cap N)$. Let $\Sigma_\lambda = \{(p_\lambda)_{\mathbf{R}}(\tau) \mid \tau \in \Sigma, \lambda \prec \tau\}$. Then Σ_λ is a Γ_λ-invariant fan in $N/(\mathbf{R}\lambda \cap N)$. Moreover, $V(\lambda) \simeq T_{N/(\mathbf{R}\lambda \cap N)} \operatorname{emb}(\Sigma_\lambda)$, by [**5**, Corollary 1.7]. Hence $V(\lambda)$ is non-singular, if and only if so is $(p_\lambda)_{\mathbf{R}}(\sigma)$.

Now, assume that $(p_\lambda)_{\mathbf{R}}(\sigma)$ is non-singular, i.e., $(p_\lambda)_{\mathbf{R}}(\sigma) = \mathbf{R}_{\geq 0}w_1 + \mathbf{R}_{\geq 0}w_2 + \cdots + \mathbf{R}_{\geq 0}w_{r-s}$ for a basis $\{w_1, w_2, \ldots, w_{r-s}\}$ of $N/(\mathbf{R}\lambda \cap N)$. Then there exist elements u_1, u_2, $\ldots$, u_{r-s} in $N \cap \sigma$ with $w_i = p_\lambda(u_i)$. Let $\{u_{r-s+1}, \ldots, u_r\}$ be a basis of $\mathbf{R}\lambda \cap N$. Then $\{u_1, u_2, \ldots, u_r\}$ is a basis of N. Moreover, so is $\{u_1, \ldots, u_{i-1}, \gamma_{v(\mu_i)}u_i, u_{i+1}, \ldots, u_r\}$, because $\gamma_{v(\mu_i)}$ is in $GL(N)$ and $\gamma_{v(\mu_i)}u_j = u_j$ if $i \neq j$. Hence there exist integers $c_{i,j}$ $(1 \leq i \leq r-s, 1 \leq j \leq r)$ with

$$u_i + \gamma_{v(\mu_i)}u_i + c_{i,1}u_1 + \cdots + c_{i,i-1}u_{i-1} + c_{i,i+1}u_{i+1} + \cdots + c_{i,r}u_r = 0.$$

Therefore,

$$w_i + \gamma_{v(\mu_i)}w_i + c_{i,1}w_1 + \cdots + c_{i,i-1}w_{i-1} + c_{i,i+1}w_{i+1} + \cdots + c_{i,r-s}w_{r-s} = 0.$$

These numbers $c_{i,j}$ determine the structure of $V(\lambda)$. Especially, when $s = r-3$, they are nothing but double $\mathbf{Z}$-weights in [**5**, 1.7]. We easily see that $c_{i,j} \leq 0$. Moreover, $|\gamma_{v(\mu_i)}\gamma_{v(\mu_j)}| = +\infty$, if $c_{i,j} \leq -2$ and $c_{j,i} \leq -2$, $c_{i,j} = -1$ and $c_{j,i} \leq -4$ or $c_{i,j} = 0$ and $c_{j,i} \neq 0$. Hence if $v(\mu_i) \cdot v(\mu_j)/(\sqrt{-v(\mu_i) \cdot v(\mu_i)}\sqrt{-v(\mu_j) \cdot v(\mu_j)}) = 0, 1/2, 1/\sqrt{2}$ or $\sqrt{3}/2$, then $\{c_{i,j}, c_{j,i}\} = \{0\}, \{-1\}, \{-1,-2\}$ or $\{-1,-3\}$, respectively, by Proposition 7.

We explain some examples of $V(\lambda)$ for the convenience of the next section. First, we consider the case $s = r-2$ and $(p_\lambda)_{\mathbf{R}}(\sigma)$ is non-singular. If $c_{1,2} = c_{2,1} = 0$, then $V(\lambda) \simeq \mathbf{P}^1 \times \mathbf{P}^1$. If $c_{1,2} = c_{2,1} = -1$, then $V(\lambda) \simeq S_6$. If $c_{1,2} = -1$ and $c_{2,1} = -2$ (resp. -3), then $V(\lambda) \simeq S_8$ (resp. S_{12}). Here S_i are toric surfaces obtained from Coxeter groups as follows (see [**2**, 5.1] for the definition of Coxeter group). For each $i = 6, 8, 12$, let G_i be a subgroup of $GL(2, \mathbf{Z})$ generated by two elements g_1 and $g_{2,i}$ defined by

$$g_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad g_{2,6} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad g_{2,8} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad g_{2,12} = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}.$$

Then G_i are Coxeter groups with $|G_i| = i$. Let $\Lambda_i = \{\text{faces of } g\mathbf{R}^2_{\geq 0} \mid g \in G_i\}$. Then Λ_i is a non-singular fan for each i. Let $S_i = T_{\mathbf{Z}^2}\operatorname{emb}(\Lambda_i)$ be the compact toric surface associated to the fan Λ_i. Then the complement of the algebraic torus in S_6, is a cycle of 6 rational curves with the self-intersection numbers all equal to -1. The complement of the algebraic torus in S_8 (resp. S_{12}), is a cycle of 8 (resp. 12) rational curves with the self-intersection numbers repeating -1, -2 (resp. -1, -3).

Next, we consider the case $s = r-3$ and assume that $(p_\lambda)_{\mathbf{R}}(\sigma)$ is non-singular except the case (7). We denote by V_i the toric variety $V(\lambda)$ in (i), which appears in the following sections as an irreducible component of the exceptional set of a resolution of 4-dimensional cusp singularities.

(1a) If $c_{1,2} = c_{2,1} = 0$, $c_{1,3} = c_{3,1} = c_{3,2} = -1$, $c_{2,3} = -2$, then the complement of the algebraic torus in V_{1a}, consists of 26 toric surfaces 6, 8 and 12 of which are biholomorphic to S_8, S_6 and $\mathbf{P}^1 \times \mathbf{P}^1$, respectively (see Figure 3). The self-intersection numbers $(E_{|V})^2$ in irreducible components $V \simeq S_8$ of rational curves $E = V \cdot W$, are equal to -2 and -1, if $W \simeq \mathbf{P}^1 \times \mathbf{P}^1$ and S_6, respectively.

(1b) If $c_{1,2} = c_{2,1} = 0$, $c_{1,3} = c_{3,1} = c_{2,3} = -1$, $c_{3,2} = -2$, then the complement of the algebraic torus in V_{1b}, consists of 26 toric surfaces 6, 8 and 12 of which

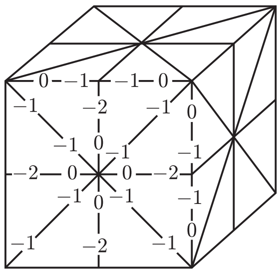

Figure 3.

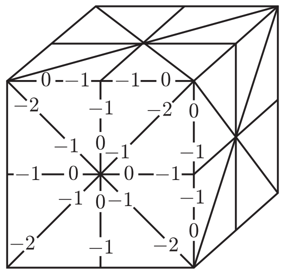

Figure 4.

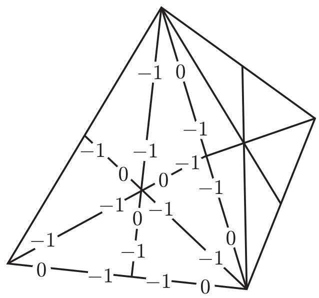

Figure 5.

are biholomorphic to S_8, S_6 and $\mathbf{P}^1 \times \mathbf{P}^1$, respectively (see Figure 4). The self-intersection numbers $(E_{|V})^2$ in irreducible components $V \simeq S_8$ of rational curves $E = V \cdot W$, are equal to -1 and -2, if $W \simeq \mathbf{P}^1 \times \mathbf{P}^1$ and S_6, respectively.

(2) If $c_{1,2} = c_{2,1} = 0$, $c_{1,3} = c_{3,1} = c_{3,2} = c_{2,3} = -1$, then the complement of the algebraic torus in V_2, consists of 14 toric surfaces 8 and 6 of which are biholomorphic to S_6 and $\mathbf{P}^1 \times \mathbf{P}^1$, respectively (see Figure 5).

(3) If $c_{1,2} = c_{2,1} = c_{1,3} = c_{3,1} = 0$, $c_{2,3} = c_{3,2} = -1$ then $V_3 \simeq \mathbf{P}^1 \times S_6$.

(4) If $c_{1,2} = c_{2,1} = c_{1,3} = c_{3,1} = 0$, $c_{2,3} = -1$, $c_{3,2} = -2$ then $V_4 \simeq \mathbf{P}^1 \times S_8$.

(5) If $c_{1,2} = c_{2,1} = c_{1,3} = c_{3,1} = 0$, $c_{2,3} = -1$, $c_{3,2} = -3$ then $V_5 \simeq \mathbf{P}^1 \times S_{12}$.

(6) If $c_{i,j} = 0$ for all i, j, then $V_6 \simeq \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.

(7) If $(p_\lambda)_{\mathbf{R}}(\sigma)$ is simplicial, $v(\mu_i) \cdot v(\mu_j) = 0$ for $1 \le i < j \le 3$ and $u_1 = \mathbf{f}_1$, $u_2 = \mathbf{f}_1 + 2\mathbf{f}_2$, $u_3 = \mathbf{f}_3$ for a basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_r\}$ of N, then $V_7 \simeq \mathbf{P}^1 \times \big(\mathbf{P}^1 \times \mathbf{P}^1/(-1,-1)\big)$.

5. Examples with quadratic C.

We fix $r = 4$, throughout the rest of this paper.

EXAMPLE 1. Let $P(x_1, x_2, x_3, x_4) = -x_1^2 - x_2^2 - x_3^2 + 7x_4^2$. Let σ be the cone generated by the following six elements in N.

$$u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ u_2 = \begin{bmatrix} 7 \\ 7 \\ 0 \\ 4 \end{bmatrix}, \ u_3 = \begin{bmatrix} 7 \\ 7 \\ 7 \\ 5 \end{bmatrix}, \ u_4 = \begin{bmatrix} 14 \\ 7 \\ 0 \\ 6 \end{bmatrix}, \ u_5 = \begin{bmatrix} 21 \\ 7 \\ 7 \\ 9 \end{bmatrix}, \ u_6 = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

Let C be the connected component of C_P containing u_1. Then $\sigma \setminus \{0\} \subset C$. Let

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \ v_4 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ v_5 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

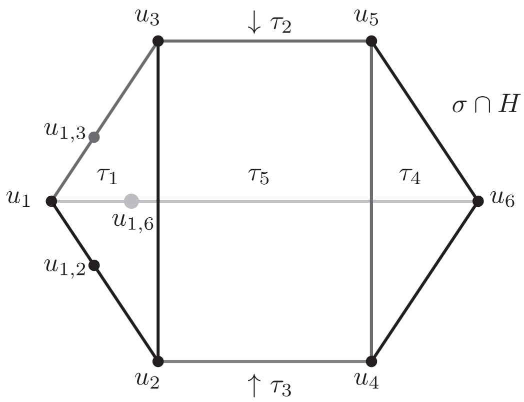

Figure 6.

Then $\tau_i := \sigma \cap H_{v_i}$ $(i = 1, \ldots, 5)$ are 3-dimensional faces of σ (see Figure 6 which shows the intersection with a hyperplane H). Moreover, we see by Proposition 5 and easy calculation that $v(\tau_i) = v_i$ satisfy the conditions 2, 3 of Theorem 8. Let Σ be the fan in Theorem 8 defined for this σ. Then $V(\lambda)$ are singularities in $T_N \operatorname{emb}(\Sigma)$ for all cones λ in Σ with $\dim \lambda \geq 2$. Noting that $\sigma^\vee$ is spanned by $i(v_1)$, $i(v_2)$, …, $i(v_5)$, where $i : N \to M$ is the homomorphism satisfying $\langle i(x), y\rangle = B_P(x, y)$, we see that all 3-dimensional faces of $\sigma^\vee$ are non-singular. Let $\lambda = \mathbf{R}_{\geq 0} u_1$ and let

$$u_{1,2} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad u_{1,3} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_{1,6} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then $\{u_1, u_{1,2}, u_{1,3}, u_{1,6}\}$ is a basis of N and $(p_\lambda)_{\mathbf{R}}(\sigma) = \mathbf{R}_{\geq 0} p_\lambda(u_{1,2}) + \mathbf{R}_{\geq 0} p_\lambda(u_{1,3}) + \mathbf{R}_{\geq 0} p_\lambda(u_{1,6})$. Moreover, we see by easy calculation that the relations $u_{1,2} + \gamma_{v_2} u_{1,2} - u_{1,3} - u_{1,6} = 0$, $u_{1,3} + \gamma_{v_3} u_{1,3} - 2u_{1,2} = 0$ and $u_{1,6} + \gamma_{v_1} u_{1,6} - u_1 - u_{1,2} = 0$ hold. Hence $V(\lambda)$ is biholomorphic to V_{1a} in the previous section. Since $v_1 \cdot v_3 = v_1 \cdot v_5 = 0$, $v_3 \cdot v_5 = 1$, $v_3 \cdot v_3 = -1$ and $v_5 \cdot v_5 = -2$, $V(\mathbf{R}_{\geq 0} u_2)$ is biholomorphic to V_4. We see by similar calculation that $V(\mathbf{R}_{\geq 0} u_i)$ are biholomorphic to V_2, V_{1a}, V_2 and V_4 for $i = 3, 4, 5$ and 6, respectively.

Example 2. Let $P(x_1, x_2, x_3, x_4) = -x_1^2 - x_2^2 - x_3^2 + 15x_4^2$. Then the cone σ defined by v_1, v_2, …, v_6, satisfies the conditions of Theorem 8, where

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

(see Figure 7). We can verify that the divisors corresponding to the vertices attached ⓘ are biholomorphic to V_i in the previous section. For example, $v_2 \cdot v_4 = v_2 \cdot v_6 = v_4 \cdot v_6 = 0$,

$(\mathbf{R}w_2 + \mathbf{R}w_i) \cap M = \mathbf{Z}w_2 + \mathbf{Z}w_i$ for $i = 4, 6$ and $[(\mathbf{R}w_4 + \mathbf{R}w_6) \cap M : \mathbf{Z}w_4 + \mathbf{Z}w_6] = 2$, where w_i $(i = 2, 4, 6)$ are the elements in M satisfying $\langle w_2, x\rangle = B_P(v_2, x)$, $\langle w_4, x\rangle = (1/5)B_P(v_4, x)$ and $\langle w_6, x\rangle = B_P(v_6, x)$. Hence $V(\tau_2 \cap \tau_4 \cap \tau_6)$ is biholomorphic to V_7, where $\tau_i = \sigma \cap H_{v_i}$.

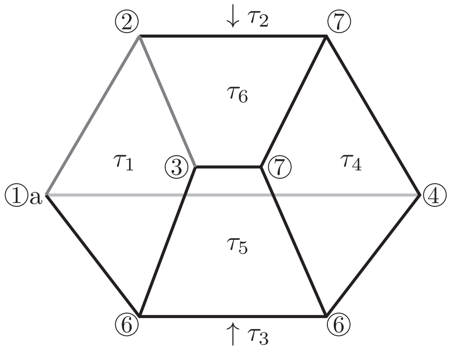

Figure 7.

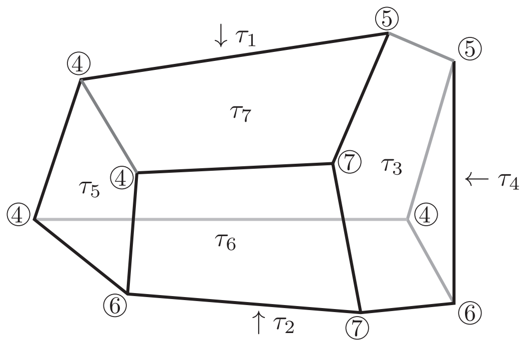

Figure 8.

Example 3. Let $P(x_1, x_2, x_3, x_4) = -3x_1^2 - 3x_2^2 - 5x_3^2 + x_4^2$. Then the cone σ defined by v_1, v_2, …, v_6, where

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix},$$

$$v_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 0 \\ 5 \\ 6 \\ 15 \end{bmatrix}, \quad v_7 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

(see Figure 8).

6. An example with non-quadratic C.

We fix a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ of N. Let γ_i be the elements in $GL(N)$ defined by the following relations for $i = 1, 2, 3, 4$. $\gamma_i \mathbf{e}_j = \mathbf{e}_j$ if $i \neq j$ and

$$\gamma_1 \mathbf{e}_1 = -\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3, \ \gamma_2 \mathbf{e}_2 = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_4, \ \gamma_3 \mathbf{e}_3 = \mathbf{e}_1 - \mathbf{e}_3 + \mathbf{e}_4, \ \gamma_4 \mathbf{e}_4 = 2\mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4.$$

Then $\Gamma_6 = \langle \gamma_i \mid i = 1, 2, 3, 4\rangle$ is a Coxeter group with the relations: $\gamma_i^2 = 1$ and

$$(*) \ \ (\gamma_1\gamma_2)^3 = (\gamma_3\gamma_4)^3 = (\gamma_1\gamma_3)^4 = (\gamma_2\gamma_4)^4 = (\gamma_1\gamma_4)^2 = (\gamma_2\gamma_3)^2 = 1.$$

Hence the Dynkin diagram of Γ_6 is Figure 9 (see [**2**, 2.3] for the definition of Dynkin diagram). Let $\sigma = \mathbf{R}_{\geq 0}\mathbf{e}_1 + \mathbf{R}_{\geq 0}\mathbf{e}_2 + \mathbf{R}_{\geq 0}\mathbf{e}_3 + \mathbf{R}_{\geq 0}\mathbf{e}_4$ and let τ_i be the 3-dimensional face of σ which does not contain $\mathbf{e}_i$ for each i. Then γ_i is a reflection with respect to the hyperplane containing τ_i. Moreover, the entries a_{ij} of the Cartan matrix in [**10**], are equal to $-c_{ji}$ if $i \neq j$, where c_{ji} are the coefficients in the above relations $\gamma_j \mathbf{e}_j = \sum c_{ji}\mathbf{e}_i$, because $2\mathbf{e}_j - \sum_{i \neq j} c_{ji}\mathbf{e}_i$ is an eigenvector of γ_j with the eigenvalue -1. Hence $a_{14} = a_{41} = a_{23} = a_{32} = 0$, $a_{12} \cdot a_{21} = a_{34} \cdot a_{43} = 1$, $a_{13} \cdot a_{31} = a_{24} \cdot a_{42} = 2$. Therefore, $C_6 = \bigcup_{\gamma \in \Gamma_6} \gamma\sigma \setminus \{0\}$ is an open strongly convex cone in $N_{\mathbf{R}}$ and $\Sigma_6 = \{\gamma\tau \mid \gamma \in \Gamma_6, \tau \prec \sigma\}$ is a Γ_6-invariant fan with $|\Sigma_6| = C_6 \cup \{0\}$, by [**10**, Theorem 1]. Moreover, C_6 is not quadratic, by [**10**, Theorem 6]. Since σ is non-singular, so is $T_N \operatorname{emb}(\Sigma_6)$. The 3-dimensional toric variety $V(\mathbf{R}_{\geq 0}\mathbf{e}_i)$ is biholomorphic to V_{1a} (resp. V_{1b}) in Section 4 for $i = 2, 3$ (resp. $1, 4$). The intersection $V(\mathbf{R}_{\geq 0}\mathbf{e}_i) \cap V(\mathbf{R}_{\geq 0}\mathbf{e}_j) = V(\mathbf{R}_{\geq 0}\mathbf{e}_i + \mathbf{R}_{\geq 0}\mathbf{e}_j)$ is the toric surface corresponding to the Coxeter group generated by $\{\gamma_k, \gamma_l\}$ for $\{k, l\} = \{1, 2, 3, 4\} \setminus \{i, j\}$. Hence it is biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ if $(i, j) = (2, 3), (1, 4)$, S_6 if $(i, j) = (3, 4), (1, 2)$ and S_8 if $(i, j) = (2, 4), (1, 3)$ by $(*)$. Note that $V(\mathbf{R}_{\geq 0}\mathbf{e}_i) \cap V(\mathbf{R}_{\geq 0}\mathbf{e}_j)$ is biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, if and only if $V(\mathbf{R}_{\geq 0}\mathbf{e}_i)$ and $V(\mathbf{R}_{\geq 0}\mathbf{e}_j)$ are biholomorphic.

Remark. Let Γ'_6, Σ'_6 and C'_6 be the subgroup of $GL(N)$, the fan and the cone in $N_{\mathbf{R}}$, respectively, obtained by transposing the coefficients $c_{2,4} = 1$ and $c_{4,2} = 2$ in the above relations $\gamma_i \mathbf{e}_i = \sum c_{ij}\mathbf{e}_j$. Then the irreducible components of $T_N \operatorname{emb}(\Sigma'_6) \setminus T_N$ are isomorphic to those of $T_N \operatorname{emb}(\Sigma_6) \setminus T_N$. However, they intersect to each other in a different way. $V(\mathbf{R}_{\geq 0}\mathbf{e}_i)$ are biholomorphic to V_{1a} (resp. V_{1b}) for $i = 1, 2$ (resp. $3, 4$). Hence $V(\mathbf{R}_{\geq 0}\mathbf{e}_i) \cap V(\mathbf{R}_{\geq 0}\mathbf{e}_j)$ is biholomorphic to S_6, if and only if $V(\mathbf{R}_{\geq 0}\mathbf{e}_i)$ and $V(\mathbf{R}_{\geq 0}\mathbf{e}_j)$ are biholomorphic. However, the following consideration for (C_6, Γ_6) holds also for (C'_6, Γ'_6), because the relations in $(*)$ do not change.

Hereafter, we simply write Γ, Σ and C for Γ_6, Σ_6 and C_6, respectively.

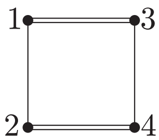

Figure 9.

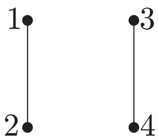

Figure 10.

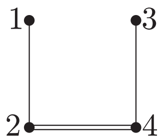

Figure 11.

Theorem 9. *There exists a subgroup Γ^0 of Γ of index 48 which acts on D_C freely. Conversely, if a subgroup Γ' of Γ acts on D_C freely, then Γ' is of index at least 48.*

Let $\Gamma^i = \langle \gamma_j \mid 1 \leq j \leq 4, j \neq i \rangle$ for each i. Then Γ^i is the stabilizer of $\mathbf{R}_{\geq 0}\mathbf{e}_i$ and $|\Gamma^i| = 48$. Hence the second assertion in the above theorem holds. Let $\Delta = \{p_C(\tau \setminus \{0\}) \mid \tau \in \Sigma, \tau \neq \{0\}\}$. Then Δ is a Γ-invariant tetrahedral decomposition of D_C. If we get Γ^0 in the above theorem, then Δ/Γ^0 is a tetrahedral decomposition of the 3-dimensional compact topological manifold D_C/Γ^0 consisting of 48 tetrahedra. Since Δ/Γ^0 has $48 \cdot 4/|\Gamma^i| = 4$ vertices, there exists a resolution of the cusp singularity $\operatorname{Cusp}(C, \Gamma^0)$ with an exceptional set consisting of 4 irreducible components. The rest of this section is devoted to the proof of the first assertion in the above theorem.

Let γ'_i be the elements in $GL(N)$ defined by the following relations for $i = 1, 2, 3, 4$. $\gamma'_i \mathbf{e}_j = \mathbf{e}_j$ if $i \neq j$ and

$$\gamma_1'\mathbf{e}_1 = -\mathbf{e}_1 + \mathbf{e}_2,\ \gamma_2'\mathbf{e}_2 = \mathbf{e}_1 - \mathbf{e}_2,\ \gamma_3'\mathbf{e}_3 = -\mathbf{e}_3 + \mathbf{e}_4,\ \gamma_4'\mathbf{e}_4 = \mathbf{e}_3 - \mathbf{e}_4.$$

Then $\Gamma' = \langle \gamma_i' \mid i = 1,2,3,4\rangle$ is a Coxeter group with the relations: $\gamma_i'^2 = 1$ and

$$(\gamma_1'\gamma_2')^3 = (\gamma_3'\gamma_4')^3 = (\gamma_1'\gamma_3')^2 = (\gamma_2'\gamma_4')^2 = (\gamma_1'\gamma_4')^2 = (\gamma_2'\gamma_3')^2 = 1.$$

Hence the Dynkin diagram of Γ' is Figure 10, $\Gamma' \simeq D_3 \times D_3$ and there exists a surjective homomorphism $q : \Gamma \to \Gamma'$ sending γ_i to γ_i'. Let $\Delta' = \{p(\gamma'\tau \setminus \{0\}) \mid \gamma' \in \Gamma', \tau \prec \sigma, \tau \neq \{0\}\}$, where $p : N_{\mathbf{R}} \setminus \{0\} \to S^3$ is the natural projection. Then Δ' is a tetrahedral decomposition of S^3 with 36 tetrahedra. Let $\tilde{f} : C \cup \{0\} \to N_{\mathbf{R}}$ be the piecewise linear map defined by $\tilde{f}(x) = q(\gamma)\gamma^{-1}x$, if x is in $\gamma\sigma$ for an element γ in Γ. Then $\tilde{f}$ induces a Galois covering $f : D_C \to S^3$ with $f(\gamma x) = q(\gamma)f(x)$ for any element γ in Γ, ramifying only along $\Xi_{13} \cup \Xi_{24}$, where $\Xi_{ij} = \bigcup_{\gamma' \in \Gamma'} p(\gamma'(\mathbf{R}_{\geq 0}\mathbf{e}_i + \mathbf{R}_{\geq 0}\mathbf{e}_j) \setminus \{0\})$, because $\langle \gamma_i, \gamma_j\rangle$ are the stabilizers of $\mathbf{R}_{\geq 0}\mathbf{e}_k + \mathbf{R}_{\geq 0}\mathbf{e}_l$, where $\{k,l\} = \{1,2,3,4\} \setminus \{i,j\}$, $q((\gamma_2\gamma_4)^2) = q((\gamma_1\gamma_3)^2) = 1$ and the restriction of q to $\langle \gamma_i, \gamma_j\rangle$ is an isomorphism if $(i,j) \neq (1,3), (2,4)$. Moreover, Δ is the pull-back of Δ' under f.

$$\begin{array}{ccc} D_C & \xrightarrow{f} & S^3 \\ & {\scriptstyle f_1}\searrow \quad \nearrow{\scriptstyle f_1'} & \\ & S^3 & \end{array} \qquad \begin{array}{ccc} \Delta & \xrightarrow{f} & \Delta' \\ & {\scriptstyle f_1}\searrow \quad \nearrow{\scriptstyle f_1'} & \\ & \Delta_1 & \end{array} \qquad \begin{array}{ccc} \Gamma & \xrightarrow{q} & \Gamma' \\ & {\scriptstyle q_1}\searrow \quad \nearrow{\scriptstyle q_1'} & \\ & \Gamma'' \supset \Gamma_1 & \end{array}$$

Let $\Gamma'' = \langle \gamma_i'' \mid i = 1,2,3,4\rangle$, where $\gamma_1'' = \gamma_1'$, $\gamma_2'' = \gamma_2$, $\gamma_3'' = \gamma_3'$, $\gamma_4'' = \gamma_4$. Then Γ'' is a Coxeter group whose Dynkin diagram is Figure 11 and there exist surjective homomorphisms $q_1 : \Gamma \to \Gamma''$ sending γ_i to γ_i'' and $q_1' : \Gamma'' \to \Gamma'$ sending γ_i'' to γ_i' with $q = q_1' \circ q_1$. We can define Galois coverings $f_1 : D_C \to S^3$ and $f_1' : S^3 \to S^3$ such that $f_1'(\gamma'' x) = q_1'(\gamma'')f_1'(x)$ for any element γ'' in Γ'' and that $f_1' \circ f_1 = f$, in a similar way as f. Then f_1' ramifies only along Ξ_{13}, $\mathrm{Gal}(f_1') = \ker(q_1')$ and $\Delta_1 = \{p(\gamma''\tau \setminus \{0\}) \mid \gamma'' \in \Gamma'', \tau \prec \sigma, \tau \neq \{0\}\}$ is the pull-back of Δ' under f_1'. Let $\gamma_0'' = \gamma_1''\gamma_2''\gamma_3''\gamma_4''$.

Lemma. *There exists a normal subgroup Γ_1 of $\ker(q_1')$ acting on S^3 freely with $\ker(q_1')/\Gamma_1 \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$, $\gamma_0''^3 \in \Gamma_1$ and $\gamma_0''\Gamma_1\gamma_0''^{-1} = \Gamma_1$.*

Proof. Let $\mathbb{P}$ be the convex hull of the 24 points

$$\begin{pmatrix} \pm 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pm 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \pm 2 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \\ \pm 1 \end{pmatrix}$$

in $\mathbf{R}^4$. Then the boundary $\partial\mathbb{P}$ of $\mathbb{P}$ consists of 24 octahedra which are on the hyperplanes defined by $\pm x_i \pm x_j = 2$ $(1 \leq i < j \leq 4)$, and is a regular polyhedron of type $(3,4,3)$ (see [**1**, 8.2]). For example, an octahedron has 6 vertices ${}^t(2,0,0,0)$, ${}^t(0,2,0,0)$, ${}^t(1,1,\pm 1,\pm 1)$. Let $\square$ be the barycentric subdivision of the octahedral decomposition $p(\partial\mathbb{P})$ of S^3 which is the image of $\partial\mathbb{P}$ under the projection $p : \mathbf{R}^4 \setminus \{0\} \to S^3$. Let $h : S^3 \to S^3$ be the homeomorphism induced by the linear transformation $\tilde{h}$ sending $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$ to ${}^t(1,1,0,0)$, ${}^t(2,1,1,0)$, ${}^t(1,1,1,1)$ and ${}^t(2,2,2,0)$, respectively. Then $h\Delta_1$ coincides

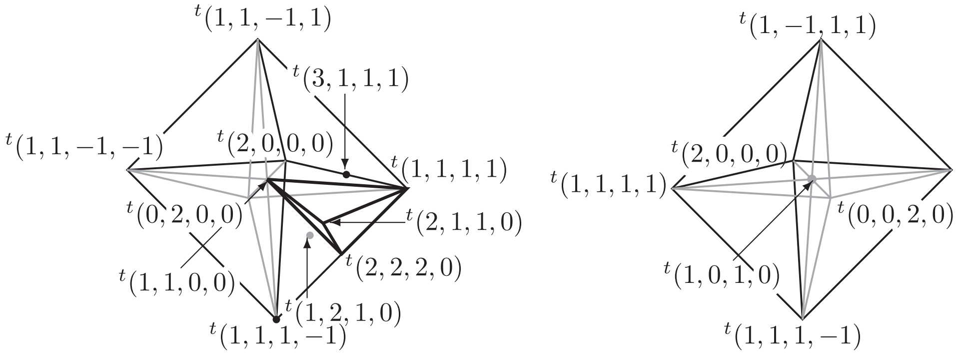

Figure 12.

with □, because $\tilde{h}(\gamma_1''\mathbf{e}_1) = {}^t(1,0,1,0)$, $\tilde{h}(\gamma_2''\mathbf{e}_2) = {}^t(1,2,1,0)$, $\tilde{h}(\gamma_3''\mathbf{e}_3) = {}^t(1,1,1,-1)$ and $\tilde{h}(\gamma_4''\mathbf{e}_4) = {}^t(3,1,1,1)$ (see Figure 12). Moreover, $h(f_1'^{-1}(\Xi_{13}))$ is the union of the diagonals of the octahedra on $p(\partial\mathbb{P})$. Since the barycentric subdivision of an octahedron has 48 tetrahedra, $|\Gamma''| = 24 \cdot 48 = 1152$. Since $\ker(q_1')$ is generated by the conjugates of $(\gamma_2''\gamma_4'')^2$, whose fixed points are contained in $f_1'^{-1}(\Xi_{13})$ and $|\ker(q_1')| = |\Gamma''|/|\Gamma'| = 1152/36 = 32$, $\tilde{h}\ker(q_1')\tilde{h}^{-1}$ consists of the following 32 matrices, where $\epsilon_i = \pm 1$ and $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$.

$$\begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & \epsilon_4 \end{pmatrix}, \quad \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ \epsilon_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_3 \\ 0 & 0 & \epsilon_4 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ \epsilon_3 & 0 & 0 & 0 \\ 0 & \epsilon_4 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & \epsilon_1 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & \epsilon_3 & 0 & 0 \\ \epsilon_4 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the fixed points of all matrices of order 2 in the above except $-I_4$, are contained in the diagonals of the octahedra and that any one of order 4 in the above is the product of two of order 2. The set consisting of $\pm I_4$, $\pm A$, $\pm B$ and $\pm C$ is a normal subgroup of $\tilde{h}\ker(q_1')\tilde{h}^{-1}$ acting on S^3 freely, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $J = \tilde{h}\gamma_0''\tilde{h}^{-1}$. Then

$$J = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}.$$

Hence $J^3 = -B$, $JAJ^{-1} = -A$, $JBJ^{-1} = B$ and $JCJ^{-1} = -C$. Since $|\ker(q_1')/\Gamma_1| = 4$ and $X^2 = -I_4$ for any element X of order 4 in $\tilde{h}\ker(q_1')\tilde{h}^{-1}$, $\ker(q_1')/\Gamma_1 \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$. □

Let $T_1 = S^3/\Gamma_1$ and let $g_1' : T_1 \to S^3$ be the Galois covering induced by f_1'. Then g_1' ramifies only along Ξ_{13}. Let $h_1 : D_C \to T_1$ be the composite of f_1 and the quotient map $S^3 \to T_1$ under Γ_1. Then h_1 ramifies only along $g_1'^{-1}(\Xi_{24})$ and $f = g_1' \circ h_1$. Moreover, γ_0'' induces an automorphism δ_1 on T_1 with $|\delta_1| = 3$, by the above lemma. Let $\gamma_0' = \gamma_1'\gamma_2'\gamma_3'\gamma_4'$. Then γ_0' has no fixed points on S^3 and $q_1'(\gamma_0'') = \gamma_0'$. Hence $g_1' \circ \delta_1 = \gamma_0' \circ g_1'$. In a similar way, we obtain Galois coverings $g_2' : T_2 \to S^3$ ramifying only along Ξ_{24}, $h_2 : D_C \to T_2$ ramifying only along $g_2'^{-1}(\Xi_{13})$ with $f = g_2' \circ h_2$ and an automorphism δ_2 on T_2 with $|\delta_2| = 3$ such that $g_2' \circ \delta_2 = \gamma_0' \circ g_2'$.

$$D_C \to T = T_1 \times_{S^3} T_2 \to T_0 = T/G_0 \to T_0/\langle\delta_0\rangle = D_C/\Gamma^0 \to S^3$$

Now, to show the existence of a subgroup Γ^0 in the theorem, we construct covering maps as above, where the left three arrows do not ramify and the right one ramifies along $\Xi_{13} \cup \Xi_{24}$. Let $T = T_1 \times_{S^3} T_2$ be the fiber product of g_1' and g_2'. Then T is a topological manifold, because $\Xi_{13} \cap \Xi_{24} = \emptyset$. Since $\mathrm{Gal}(g_i') \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$, any bijection between $\mathrm{Gal}(g_1') \setminus \{1\}$ and $\mathrm{Gal}(g_2') \setminus \{1\}$ induces an isomorphism. Hence there exists an isomorphism $\phi : \mathrm{Gal}(g_1') \simeq \mathrm{Gal}(g_2')$ such that $\phi(\delta_1\gamma\delta_1^{-1}) = \delta_2\phi(\gamma)\delta_2^{-1}$ for any element γ in $\mathrm{Gal}(g_1')$. Let $G_0 = \{(\gamma, \phi(\gamma)) \mid \gamma \in \mathrm{Gal}(g_1')\}$. Then G_0 has no fixed points on T, because $\Xi_{13} \cap \Xi_{24} = \emptyset$. Let $T_0 = T/G_0$ and let $g_0' : T_0 \to S^3$ be the covering induced by the natural projection $T \to S^3$. Then $\deg g_0' = 4$, because $\deg g_i' = 4$. Hence the pull-back of Δ' under g_0', consists of $36 \cdot 4 = 144$ tetrahedra. Let $h : D_C \to T_0$ be the composite of the map (h_1, h_2) and the quotient map $T \to T_0$. Then h is a surjective unramified covering, because it does not ramify along $g_0'^{-1}(\Xi_{13} \cup \Xi_{24})$ and T_0 is a topological manifold. Since $(\delta_1, \delta_2)G_0(\delta_1, \delta_2)^{-1} = G_0$, (δ_1, δ_2) induces an automorphism δ_0 on T_0 with $g_0' \circ \delta_0 = \gamma_0' \circ g_0'$. Since γ_0' has no fixed points on S^3, so does δ_0 on T_0. Hence the composite of h and the quotient map $T_0 \to T_0/\langle\delta_0\rangle$, is the quotient map under a subgroup of Γ with the index $144/3 = 48$ acting on D_C freely.

References

[1] H. S. M. Coxeter, Regular Polytopes, 3rd edn., Dover, New York, 1973.

[2] J. E. Humphreys, Reflection groups and Coxeter groups, Cambridge Studies in Advanced Mathematics, **29**, Cambridge.

[3] M. Ishida, Cusp singularities given by reflections of stellable cones, Internat. J. Math., **2** (1991), 635–657.

[4] J. G. Ratcliffe, Foundations of Hyperbolic Manifolds, 2nd edn., GTM, **149**, Springer.

[5] T. Oda, Convex Bodies and Algebraic Geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete 3, Folge-Band 15, Springer-Verlag, 1987.

[6] I. Satake, On numerical invariants of arithmetic varieties of **Q**-rank one, in Automorphic forms of several variables, Taniguchi Symposium, Katata, 1983 (I. Satake and Y. Morita, eds.), Progr. Math., **46**, Birkhauser, Basel-Boston-Stuttgart, 1984, 353–369.

[7] I. Satake and S. Ogata, Zeta functions associated to cones and their special values, Automorphic Forms and Geometry of Arithmetic Varieties, Adv. Stud. Pure Math., **15**, 1989, 1–27.

[8] H. Tsuchihashi, Higher dimensional analogues of periodic continued fractions and cusp singularities, Tohoku Math. J., **35** (1983), 607–639.

[9] H. Tsuchihashi, Three-Dimensional Cusp Singularities, Complex Analytic Singularities, Adv. Stud. Pure Math., **8**, 1986, 649–670.

[10] È. B. Vinberg, Discrete linear groups generated by reflections, Math. USSR Izvestija, **5** (1971), 1083–1119.

Hiroyasu Tsuchihashi
Miyagi University of Education
149 Aramaki Aza Aoba
Aobaku, Sendai
Miyagi 980-0845, Japan