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# Scalar curvature of self-shrinker

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**Abstract.** In this paper, we consider the scalar curvature of a self-shrinker and get the gap theorem of the scalar curvature. We get also a relationship between the upper bound of the square of the length of the second fundamental form and the Ricci mean value.

## 1. Introduction.

Let $x : M^n \to \mathbb{R}^{n+1}$ be an $n(n \geq 2)$-dimensional hypersurface in the $(n+1)$-dimensional Euclidean space. Let $x^T$ and $x^\perp$ denote the projection of the position vector $x$ onto tangent space and the normal space of $M^n$, respectively, then

$$x = x^T + x^\perp.$$

A hypersurface $M^n$ is called a self-shrinker if it satisfies the quasi-linear elliptic system:

$$H = -\langle x^\perp, e_{n+1}\rangle, \tag{1.1}$$

where $e_{n+1}$ an unit normal vector and $H$ is the mean curvature of $M^n$. Self-shrinkers play an important role in the study of the mean curvature flow. For example, Huisken's monotonicity formula for the mean curvature flow implies that any Type I blow-up limit is a self-similar shrinking solution (cf. [**5**] and [**6**]). In other words, not only self-shrinkers correspond to self-shrinking solutions to the mean curvature flow, but also they describe all possible Type I blow ups at a given singularity of the mean curvature flow. The simplest example of a self-shrinker in $\mathbb{R}^{n+1}$ is the round sphere of radius $\sqrt{n}$ centered at the origin. In a remarkable recent work, Colding and Minicozzi [**3**] proved that a self-shrinker which is a stable critical points of a certain entropy functional must be a sphere or cylinder. The round sphere is also known to minimize entropy among closed self-shrinkers.

To characterize the self-shrinkers by mean curvature $H$ and the square of the length of the second fundamental form $\|A\|^2$, some interesting gap theorems have been obtained. For examples, for a compact self-shrinker, we have $\max \|A\|^2 \geq 1$ and equality sign holds if and only if the self-shrinker is the round sphere of radius $\sqrt{n}$ centered at the origin [**8**]. For the generalization to the case of arbitrary codimension and complete self-shrinkers we refer readers to papers [**1**], [**2**] and [**4**]. Considering the mean curvature, by making use of Minkowski's formula [**7**], one can see that following gap theorem holds: for a

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compact self-shrinker we have $\min H^2 - n \leq 0 \leq \max H^2 - n$, where each equality sign holds if and only if the self-shrinker is $\mathbb{S}^n(\sqrt{n})$. For the scalar $R$, it is easy to prove that $\min R \leq n-1$ and the equality holds if and only if the self-shrinker is $\mathbb{S}^n(\sqrt{n})$. In fact, from Gauss equation $R = H^2 - \|A\|^2$ we have

$$R - \frac{n-1}{n}H^2 = \frac{1}{n}H^2 - \|A\|^2 \leq 0,$$

where the equality holds if and only if the hypersurface is totally umbilical. Hence we have

$$\begin{aligned}\min R - (n-1) &= \min R - \frac{n-1}{n}n \leq \min R - \frac{n-1}{n}\min H^2 \\ &= \min R + \max\Big(-\frac{n-1}{n}H^2\Big) \leq \max\Big(R - \frac{n-1}{n}H^2\Big) \leq 0,\end{aligned}$$

and the conclusion follows immediately. From this we see that $\min R \leq n-1$ is a necessary condition that a compact Riemannian manifold can be immersed in Euclidean space as a codimension 1 self-shrinker.

In this paper, we define Ricci mean value of a hypersurface as follows:

$$c = \frac{1}{nV}\int_M Ric(x^T, x^T)dM, \tag{1.2}$$

where $V$ is the volume of $M^n$ and $Ric(x^T, x^T)$ denotes the Ricci curvature in tangent vector $x^T$. The main purpose of this paper is to get the gap theorem for the scalar curvature. As a corollary of the main theorem we get also a relationship between the upper bound of the square of the length of the second fundamental form and the Ricci mean value. Explicitly, we prove following results:

THEOREM 1.1. *For a compact self-shrinker with scalar curvature $R$ if either condition $R \leq n-1+c$ or $R \geq n-1+c$ is satisfied, then $c = 0$, $R = n-1$ and the self-shrinker is isometrically homeomorphic to sphere $\mathbb{S}^n(\sqrt{n})$.*

In other words, we get, on a compact self-shrinker, it holds

$$\min R - (n-1) \leq c \leq \max R - (n-1),$$

where each of the equality signs holds if and only if $c = 0$ and the self-shrinker is isometrically homeomorphic to sphere $\mathbb{S}^n(\sqrt{n})$. In particular, if $R$ is constant then $x(M^n)$ is isometrically homeomorphic to sphere $\mathbb{S}^n(\sqrt{n})$.

THEOREM 1.2. *For a compact self-shrinker, we have*

$$1 - \max\|A\|^2 \leq c \leq (n-1)(\max\|A\|^2 - 1)$$

*where each of the equality signs holds if and only if $c = 0$ and the self-shrinker is isometrically homeomorphic to sphere $\mathbb{S}^n(\sqrt{n})$.*

The conclusion of Theorem 1.2 is closely related to an interesting problem, i.e. the second gap problem. The second gap problem means that dose exist a number $\delta(>1)$ such that if $\|A\|^2 \leq \delta$ then $\|A\|^2 = 1$ or $\|A\|^2 = \delta$, and there exists compact self-shrinker with $\|A\|^2 = \delta$? In other words, for compact self-shrinkers, dose exist a number $\delta$ such that $\max\|A\|^2 \geq \delta$ and identity holds if and only if $\|A\|^2 = \delta$ and there exists compact self-shrinker with $\|A\|^2 = \delta$? The interval $[1, \delta]$ is called the second gap of $\|A\|^2$.

From Theorem 1.2 we see that it holds that *if* $c > 0$, *then* $\max\|A\|^2 > 1+(c/(n-1))$; *if* $c < 0$, *then* $\max\|A\|^2 > 1 - c$. This shows that if there exists the second gap $[1, \delta]$ of $\|A\|^2$, then $c(x) \neq 0$ ($x(M)$ is self-shrinker with $\max\|A(x)\|^2 = \delta$) and, if $c(x) > 0$ then $\delta > 1 + (c(x)/(n-1))$; if $c(x) < 0$ then $\delta > 1 - c(x)$.

We establish first a new integral formula for a compact self-shrinker and then, by making use of the new formula, we prove above results.

## 2. An integral formula on a compact hypersurface.

Let $x : M^n \to \mathbb{R}^{n+1}$ be an $n$-dimensional hypersurface in the $(n+1)$-dimensional Euclidean space with inner product $<\cdot,\cdot>$. Define nonnegative function

$$u = \frac{1}{2}\|x\|^2.$$

Let $x^T$ denote the projection of the position vector $x$ and $Ric(x^T, x^T)$ denote the Ricci curvature in tangent vector $x^T$. We introduce the notation

$$c = \frac{1}{nV}\int_M Ric(x^T, x^T)dM, \tag{2.1}$$

where $dM$ is volume element and $V$ is the volume of $M^n$. Quantity $c$ relies on the metric of $M^n$ and immersion $x$ but it is a constant on $x(M^n)$ for fixed $x$. We have following integral formula:

Lemma 2.1. *For a compact hypersurface $M^n$ in $\mathbb{R}^{n+1}$, we have formula*

$$\int_M [R\|x^\perp\|^2 - n(n-1+c)]dM = 0. \tag{2.2}$$

For a self-shrinker, using condition $\|x^\perp\|^2 = H^2$ and Minkowski's formula, we have the following integral formula:

Corollary 2.2. *For a compact self-shrinker, it holds that*

$$\int_M [R - (n-1+c)]H^2 dM = 0. \tag{2.3}$$

Proof of Lemma 2.1. Let $f$ be a smooth function on $M^n$. Choosing a local field of orthonormal tangent frames $e_i$ $(1 \leq i \leq n)$ and normal vector $e_{n+1}$, we can denote the components of the co-derivative of $f_i = e_i(f)$ by $f_{i,j}$. Laplacian $\Delta$ is defined by $\Delta f := \sum_i f_{i,i}$. Noting that the following formula can be easily gotten by a direct calculation and making use of Ricci identity:

$$\Delta\left(\frac{1}{2}\|\nabla f\|^2\right) = \sum_{i,j} f_{i,j}^2 + \sum_k f_k(\Delta f)_k + \sum_{ij} R_{ij} f_i f_j,$$

we have

$$-\int_M \sum_{ij} R_{ij} f_i f_j dM = \int_M \left[\sum_{i,j} f_{i,j}^2 - (\Delta f)^2\right] dM. \tag{2.4}$$

In the fact, (2.4) holds on a compact Riemannian manifold because it dose not involve the structure of the hypersurface. Next step, we will apply (2.4) to the function $u$ which is determined by isometrically immersion $x$. As $u$ satisfies following equation

$$\begin{aligned} u_{i,j} &= \delta_{ij} + \langle x, e_{n+1}\rangle h_{ij}, \\ \Delta u &= n + \langle x, e_{n+1}\rangle H, \end{aligned}$$

we have

$$\begin{aligned} &\sum_{i,j} u_{i,j}^2 - (\Delta u)^2 \\ &= -n(n-1) - 2(n-1)\langle x, e_{n+1}\rangle H + \left(\sum_{ij} h_{ij}h_{ij} - H^2\right)\langle x, e_{n+1}\rangle^2. \end{aligned}$$

Making use of (2.4) and the first Minkowski's integral formula

$$\int_M (n + \langle x, e_{n+1}\rangle H) dM = 0,$$

we have

$$-\int_M \sum_{ij} R_{ij} u_i u_j dM = \int_M \left[n(n-1) + \left(\sum_{ij} h_{ij}h_{ij} - H^2\right)\langle x, e_{n+1}\rangle^2\right] dM,$$

which implies (2.2). This completes the proof of the lemma. □

## 3. Proofs of main theorems.

In the section, we will prove Theorem 1.1 and Theorem 1.2. We need the following proposition:

Proposition 3.1. *Let $M^n$ be a compact Riemannian manifold and $x : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion. Then there exists a point $q \in M^n$ such that scalar curvature $R(q)$ is positive.*

Proof. The function $u = (1/2)\langle x, x\rangle$ attains the maximum at a point $q \in M^n$ as $M^n$ is compact and $u$ is continuous. We have

$$du|_q = 0, \quad d^2u|_q \le 0.$$

Let $e_{n+1}$ be the unit normal vector at point $q$. The second fundamental form $A(q)$ is defined as follows:

$$A(q) = <d^2x|_q, e_{n+1}>.$$

From $<x, dx>|_q = du|_q = 0$ we see $x(q) \perp T_qM$. So, we have

$$x(q) = <x(q), e_{n+1}>e_{n+1}, \quad <x(q), e_{n+1}> \neq 0.$$

We have

$$0 \geq d^2u|_q = <dx, dx>|_q + <x, d^2x>|_q = <dx, dx>|_q + <x(q), e_{n+1}>A(q).$$

Since $<dx, dx>|_q$ is positive definite, we know that $<x(q), e_{n+1}>A(q)$ is negative definite. Hence we know that $A(q)$ must be definite. Let $\lambda_i$ $(1 \leq i \leq n)$ be the eigenvalue of $A(q)$. Then we have

$$\lambda_i \lambda_j > 0.$$

Hence we have

$$R(q) = (\|H\|^2 - \|A\|^2)(q) = \left(\sum_i \lambda_i\right)^2 - \sum_i \lambda_i^2 = \sum_{i \neq j} \lambda_i \lambda_j > 0.$$

This completes the proof of Proposition 3.1. □

Proof of Theorem 1.1. Firstly, we prove claim: suppose $R \leq n-1+c$ or $R \geq n-1+c$, then $R = n-1+c$ on $M$. From Corollary 2.2 we see that the assumption of the theorem implies

$$[R - (n-1+c)]H^2 = 0.$$

On open set $U = \{q \in M : H(q) \neq 0\}$, we have $R = n-1+c$ on $U$. We will prove that constant number $n-1+c$ is positive. In fact, on one hand, there exits a point $q_0$ such that $R$ is positive at $q_0$ (Proposition 3.1). On the other hand, we have $R = H^2 - \|A\|^2 \leq H^2 = 0$ on $M \setminus U$. Hence $q_0 \in U$ and so we have $n-1+c = R(q_0) > 0$ on $U$. Note that $R$ is a positive constant on $U$ and is non-positive on $M \setminus U$, we know that $M \setminus U$ is empty as $R$ is continuous on $M$. This completes the proof of the claim: $R = n-1+c$ on $M$.

Secondly, we prove claim: $R = n-1+c$ on $M^n$ implies $c = 0$. It is well known that on a compact hypersurface of $\mathbb{R}^{n+1}$ it holds the second Minkowski's integral formula

$$\int_M [(n-1)H + R<x, e_{n+1}>]dM = 0.$$

In particular, for self-shrinker we have

$$\int_M H[R - (n-1)]dM = 0. \tag{3.1}$$

From the first claim we have

$$c\int_M HdM = 0.$$

As $R$ is constant on $M$ and $M$ is compact we know that $R$ is a positive constant. Using Gaussian equation we have

$$0 < R = H^2 - \|A\|^2 \le H^2,$$

which means $H \neq 0$ everywhere. So we have $\int HdM \neq 0$. This completes the proof of the claim: $c = 0$.

Thirdly, we prove claim: $M^n$ is isometrically homeomorphic to sphere $\mathbb{S}^n(\sqrt{n})$. From inequality

$$0 \ge \frac{1}{n}H^2 - \|A\|^2 = -\frac{n-1}{n}H^2 + H^2 - \|A\|^2 = -\frac{n-1}{n}H^2 + R$$

and equality

$$\int_M H^2 dM = \int_M ndM,$$

we have

$$0 \ge \int_M \left(\frac{1}{n}H^2 - \|A\|^2\right)dM = \int_M (-(n-1) + R)dM.$$

From the second claim we know $R = n-1$. We get $(1/n)H^2 - \|A\|^2 = 0$, which means $M$ is totally umbilical. This completes proof of the claim and so completes the proof of Theorem 1.1. □

Corollary 3.2. *A compact self-shrinker with constant scalar curvature is isometrically homeomorphic to* $\mathbb{S}^n(\sqrt{n})$.

Proof of Theorem 1.2. From Corollary 2.2 and Gaussian equation we have

$$\int_M (H^2 - \|A\|^2 - n + 1 - c)H^2 dM = 0.$$

Noting

$$\int_M (H^2 - n)dM = 0,$$

we have

$$\int_M (H^2 - n)H^2 dM = \int_M (H^2 - n)^2 dM \ge 0.$$

Hence we have

$$\int_M [\|A\|^2 - (1-c)]H^2 dM \geq 0.$$

We see that if $\|A\|^2 \leq 1-c$ then $\|A\|^2 = 1-c$ and $H^2 = n$, which implies $c = 0$ and $x(M)$ is isometrically homeomorphic to sphere $\mathbb{S}^n(\sqrt{n})$. In other words, we have

$$\sup \|A\|^2 \geq 1-c, \tag{3.2}$$

where equality holds implies $c = 0$ and $x(M)$ is isometrically homeomorphic to sphere $\mathbb{S}^n(\sqrt{n})$.

On the other hand, as $H^2 \leq n\|A\|^2$, we have $R = H^2 - \|A\|^2 \leq (n-1)\|A\|^2$. From Theorem 1.1 we have

$$\sup \|A\|^2 \geq 1 + \frac{c}{n-1}. \tag{3.3}$$

From (3.2) and (3.3) we complete the proof of Theorem 1.2. □

Remark 3.3. For a given compact Riemannian manifold $(M^n, g)$, if it can be isometrically immersed in $\mathbb{R}^{n+1}$ as a self-shrinker, then we have a non-empty set

$$\mathbb{X} := \{x : M^n \to \mathbb{R}^{n+1} | \text{isometrically immersion as a self-shrinker}\}.$$

Functional $c : \mathbb{X} \to \mathbb{R}$ which is defined by (1.2) needs to satisfy

$$\min R(g) - (n-1) \leq c(x) \leq \max R(g) - (n-1), \quad x \in \mathbb{X}.$$

Hence we can define two numbers $\alpha$ and $\beta$ as follows:

$$\alpha = \inf_{x \in \mathbb{X}} c(x), \quad \beta = \sup_{x \in \mathbb{X}} c(x).$$

Our inequality can be written as follows:

$$\min R(g) - (n-1) \leq \alpha \leq \beta \leq \max R(g) - (n-1). \tag{3.4}$$

Theorem 1.1 shows that if there exists $x_0$ such that $c(x_0) = \alpha$, then $c(x_0) = 0$ and $M^n$ is isometric to sphere $\mathbb{S}^n(\sqrt{n})$.

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