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Behavior of least-energy solutions to Matukuma type equations

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Abstract. Behavior of least-energy solutions to Matukuma type equations are discussed. Especially, how they vanish or blow up are investigated. In either case, the exact solution to a special equation plays a central role to analyze the behavior.

1. Introduction.

In this paper, we consider the behavior of least-energy solutions to

$$\Delta u + K(x)u^p = 0 \quad \text{in } \boldsymbol{R}^n. \tag{1.1}$$

By the terminology "least-energy" solution, we mean that a positive solution which is obtained by the minimization problem

$$S_p = \inf_{u \in \mathscr{D}, u \not\equiv 0} \frac{\displaystyle\int_{\boldsymbol{R}^n} |\nabla u|^2 \, dx}{\left(\displaystyle\int_{\boldsymbol{R}^n} K(x) u^{p+1} \, dx\right)^{2/(p+1)}}, \tag{1.2}$$

where $\mathscr{D}$ is the completion of $C_0^\infty(\boldsymbol{R}^n)$ with respect to the norm $\|\nabla \cdot\|_2$, $1 < p < (n+2)/(n-2)$ and $n \geq 3$. Note that $\mathscr{D}$ is a Hilbert space with its inner product $\langle \phi, \psi \rangle := \int_{\boldsymbol{R}^n} \nabla\phi \nabla\psi \, dx$ for $\phi, \psi \in \mathscr{D}$. The standing assumption on $K(x)$ is

$$K(x) > 0 \quad \text{in } \boldsymbol{R}^n, \quad K(x) \in C^1(\boldsymbol{R}^n), \quad \frac{x \cdot \nabla K(x)}{K(x)} \in L^\infty(\boldsymbol{R}^n). \tag{K.0}$$

We will show here that the asymptotic behavior of solutions as $p \uparrow (n+2)/(n-2)$ and $p \downarrow (n+2-2\ell)/(n-2)$, with $\ell \in (0,2)$, which will be determined by the asymptotic behavior of $K(x)$ at infinity.

In the bounded domain cases, the asymptotic behavior of least-energy solutions (not restricted to Matukuma type equations) as $p \uparrow (n+2)/(n-2)$ is studied

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by Atkinson and Peletier [**1**], Han [**9**] (Dirichlet problems), Brezis and Peletier [**4**], Rey [**19**] (small linear term with the critical exponent), Budd, Knaap and Peletier [**5**] (scalar-field equation with the Neumann problem) and others. For the scalar-field equation in the whole space, such asymptotics are investigated by Pan and Wang ([**18**]). However, the case $p \downarrow (n+2-2\ell)/(n-2)$ is not so well-investigated compared with the case $p \uparrow (n+2)/(n-2)$. Here, we give a unified view to these behaviors.

It would be worth reviewing the results by Yanagida and Yotsutani [**21**] on radial solutions with radial $K(x)$ (see also Yanagida [**20**]). Let us consider the problem

$$\begin{cases} \dfrac{1}{r^{n-1}}(r^{n-1}u_r)_r + K(r)u^p = 0, & r > 0, \\ u > 0, & r > 0, \\ \lim\limits_{r\to\infty} r^{n-2}u < \infty, & u(0) < \infty. \end{cases} \tag{1.3}$$

Suppose that $K(r) > 0$ in $(0,\infty)$, $K(r) \in C^1((0,\infty))$ and that rK_r/K is non-increasing and non-constant in $(0,\infty)$. Let us define

$$\sigma := -\lim_{r\downarrow 0}\frac{rK_r(r)}{K(r)}, \quad \ell := -\lim_{r\to\infty}\frac{rK_r(r)}{K(r)},$$

$$p_\sigma := \frac{n+2-2\sigma}{n-2}, \quad p_\ell := \max\left\{1, \frac{n+2-2\ell}{n-2}\right\}.$$

Theorem A ((i) of Theorem 2.1 of Yanagida and Yotsutani [**21**]). *Under the above assumptions, for any $p \in (p_\ell, p_\sigma)$, there exists a unique solution u_p to* (1.3).

Note that $u_p(0) = \|u\|_\infty$ in this case.

As for the behavior of $u_p(0)$, Yanagida and Yotsutani [**21**] obtained the following.

Theorem B (Theorems 2.3 and 2.4 of [**21**]).

(a) *Suppose that $\sigma < \infty$. Then $u_p(0) \to \infty$ as $p \uparrow p_\sigma$.*

(b) *Suppose that $0 < \ell < 2$. Then $u_p(0) \to 0$ as $p \downarrow p_\ell$.*

Remark 1.1. Under (K.0), we have $\sigma = 0$.

Here, we investigate more detailed properties of u_p as $p \downarrow p_\ell$ or $p \uparrow (n+2)/(n-2)$ than in [**21**] without the radial symmetry assumption on $K(x)$. For the purpose of $p \downarrow p_\ell$, we further assume that

$$0 < \ell < 2, \tag{L}$$

(K.1)
$$\begin{cases} x \cdot \nabla K(x) + \ell K(x) \geq 0, \not\equiv 0 & \text{in } \boldsymbol{R}^n, \\ \lim\limits_{|x| \to \infty} |x|^{\ell} K(x) = c_0, \end{cases}$$

where $c_0 > 0$.

In turn, for $p \uparrow (n+2)/(n-2)$, we assume

(K.2)
$$\begin{cases} x \cdot \nabla K(x) < 0 & \text{in } \boldsymbol{R}^n \backslash \{0\}, \\ K(x) \leq c_1 |x|^{-s} & \text{in } \boldsymbol{R}^n \backslash B_{R_1}, \end{cases}$$

where $c_1 > 0$, $R_1 > 0$, $s > 0$ are constants and $B_R := \{x \in \boldsymbol{R}^n \mid |x| \leq R\}$. We can take suitable radial $K(x)$ which satisfies the Yanagida and Yotsutani conditions and also (K.1) or (K.2).

Note that (1.1) has only $u \equiv 0$ in $\mathscr{D}$ due to the Pohozaev identity at $p = p_\ell$ under (K.0) and (K.1), and at $p = (n+2)/(n-2)$ under (K.0) and (K.2). Indeed, the Pohozaev identity, which will be proved in Lemma 2.2 in Section 2, yields

(1.4)
$$\int_{\boldsymbol{R}^n} \left\{ \left(\frac{n-2}{2} - \frac{n}{p+1} \right) - \frac{1}{p+1} \frac{x \cdot \nabla K(x)}{K(x)} \right\} K(x) |u|^{p+1} \, dx = 0.$$

At $p = p_\ell$, the integrand of (1.4) is nonpositive and integrable by (K.0), while that is nonnegative at $p = (n+2)/(n-2)$. With the aid of elliptic regularity theory and the maximum principle, we get $u \equiv 0$ (see also Theorems 1 and 3 of Naito [**17**]).

First, we consider the behavior of solutions as $p \downarrow p_\ell$. To this end, we introduce a constant

$$S(\ell) := \inf_{u \in \mathscr{D}, u \not\equiv 0} \frac{\displaystyle\int_{\boldsymbol{R}^n} |\nabla u|^2 \, dx}{\left(\displaystyle\int_{\boldsymbol{R}^n} |x|^{-\ell} |u|^{p_\ell + 1} \, dx \right)^{2/(p_\ell + 1)}}.$$

Note that $S(\ell) > 0$ is attained by a function

$$(1 + c|x|^{2-\ell})^{-(n-2)/(2-\ell)}$$

with any $c > 0$ (see Egnell [**7**] and Horiuchi [**10**], [**11**]). The value $S(\ell)$ is explicitly written as

(1.5)
$$S(\ell) = \frac{(n-2)^2 \omega_n \displaystyle\int_0^\infty r^{n+1-2\ell} (1 + r^{2-\ell})^{-2(n-\ell)/(2-\ell)} \, dr}{\left(\omega_n \displaystyle\int_0^\infty r^{n-1-\ell} (1 + r^{2-\ell})^{-2(n-\ell)/(2-\ell)} \, dr \right)^{(n-2)/(n-\ell)}},$$

where ω_n is the area of the unit sphere in $\boldsymbol{R}^n$. Thus $\mathscr{D}$ can be embedded in

$$L_\ell := \left\{ u \,\middle|\, \int_{\boldsymbol{R}^n} |x|^{-\ell} |u|^{p_\ell+1}\, dx < \infty \right\}$$

under (L). When $\ell = 0$, $S(0)$ is the usual best Sobolev constant and we denote $S(0)$ simply by S.

Theorem 1.1. *Suppose that* (L), (K.0) *and* (K.1) *hold. Then there holds* $S_p \to c_0^{-(n-2)/(n-\ell)} S(\ell)$ *as* $p \downarrow p_\ell$. *Moreover, any least-energy solution* u_p *to* (1.1) *satisfies* $u_p \rightharpoonup 0$ *weakly in* $\mathscr{D}$, *and* $\|u_p\|_\infty \to 0$ *as* $p \downarrow p_\ell$ *although* $\|\nabla u_p\|_2^2 = S_p^{(p+1)/(p-1)} \to c_0^{-(n-2)/(2-\ell)} S(\ell)^{(n-\ell)/(2-\ell)}$ *as* $p \downarrow p_\ell$.

Remark 1.2. Since $u_{p_\ell} \equiv 0$, u_p never converges strongly to 0 as $p \downarrow p_\ell$. Theorem 1.1 implies that u_p appears at "infinity" as p exceeds p_ℓ.

To see how u_p converges to 0, we rescale u_p. Let

$$\mu_p^{\alpha_p} = \|u_p\|_\infty \quad \text{with } \alpha_p = \frac{2-\ell}{p-1}.$$

Theorem 1.2. *For a solution* u_p *obtained in Theorem* 1.1, *let*

$$v_p(x) := \frac{1}{\mu_p^{\alpha_p}} u_p\left(\frac{x}{\mu_p}\right). \tag{1.6}$$

Then v_p *converges uniformly to*

$$U(x) := \left(1 + \frac{c_0}{(n-2)(n-\ell)} |x|^{2-\ell}\right)^{-(n-2)/(2-\ell)} \tag{1.7}$$

on any annulus in $\boldsymbol{R}^n$ *and strongly in* $\mathscr{D}$. *Moreover, the maximum point of* u_p *converges to* 0 *as* $p \downarrow p_\ell$.

Remark 1.3. v_p converges to a solution to

$$\Delta U + c_0 |x|^{-\ell} U^{(n+2-2\ell)/(n-2)} = 0 \quad \text{in } \boldsymbol{R}^n \tag{1.8}$$

with $U(0) = 1$. However, any positive solution to (1.8) must be radially symmetric (due to Theorem 2 of Bianchi [**2**]) and is of the form (1.7) (by Egnell [**7**] and Horiuchi [**10**], [**11**]).

If we add some concrete assumptions on $K(x)$, we have the vanishing order.

In the following, we mean

$$\nabla = \nabla_x = {}^t\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right), \quad \nabla_y = {}^t\left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right),$$

$$\Delta = \Delta_x = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_i^2}, \qquad \Delta_y = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}.$$

THEOREM 1.3. *Suppose that* (K.0) *and* (K.1) *hold. Moreover, assume that*

$$\text{(K.3)} \qquad \lim_{t \downarrow 0} \frac{1}{t^{\ell+q}} \left\{ \ell + \frac{x \cdot \nabla_x K(x/t)}{K(x/t)} \right\} K\left(\frac{x}{t}\right) = \frac{c_2}{|x|^{\ell+q}}$$

holds locally uniformly in $\boldsymbol{R}^n \backslash \{0\}$ *with* $q > 0$ *and* $c_2 > 0$. *In addition, if* $n - \ell \le q$, *assume further that*

$$\text{(K.4)} \qquad \begin{cases} K(x) = c_0|x|^{-\ell}(1 - c_3|x|^{-q}) & \text{in } \boldsymbol{R}^n \backslash B_{R_2}, \\ x \cdot \nabla K(x) + \ell K(x) > 0 & \text{in } B_{R_3}, \end{cases}$$

for some $R_3 \in (0, R_2)$. *Then, if* $n - \ell > q$, *there holds*

$$\lim_{p \downarrow p_\ell} \frac{(n-2)(p+1) - 2(n-\ell)}{\|u_p\|_\infty^{(p-1)(\ell+q)/(2-\ell)}} = \frac{2c_2 \int_{\boldsymbol{R}^n} |x|^{-\ell-q} U^{2(n-\ell)/(n-2)}\, dx}{c_0 \int_{\boldsymbol{R}^n} |x|^{-\ell} U^{2(n-\ell)/(n-2)}\, dx}.$$

If $n - \ell = q$, *there holds*

$$\lim_{p \downarrow p_\ell} \frac{(2-\ell)\{(n-2)(p+1) - 2(n-\ell)\}}{(p-1)\|u_p\|_\infty^{(p-1)(n-\ell)/(2-\ell)} |\log \|u_p\|_\infty|} = \frac{I}{c_0 \int_{\boldsymbol{R}^n} |x|^{-\ell} U^{2(n-\ell)/(n-2)}\, dx}$$

with

$$\lim_{p \downarrow p_\ell} \frac{2}{\mu_p^n |\log \mu_p|} \int_{\boldsymbol{R}^n} \left(\ell + \frac{x \cdot \nabla_x K(x/\mu_p)}{K(x/\mu_p)} \right) K\left(\frac{x}{\mu_p}\right) v_p^{p+1}\, dx =: I,$$

If $n - \ell < q$, *there holds*

$$\lim_{p \downarrow p_\ell} \frac{(n-2)(p+1) - 2(n-\ell)}{\|u_p\|_\infty^{(p-1)n/(2-\ell)}} = \frac{I_q}{c_0 \int_{\boldsymbol{R}^n} |x|^{-\ell} U^{2(n-\ell)/(n-2)}\, dx}$$

with

$$\lim_{p \downarrow p_\ell} \frac{2}{\mu_p^n} \int_{\boldsymbol{R}^n} \left(\ell + \frac{x \cdot \nabla_x K(x/\mu_p)}{K(x/\mu_p)} \right) K\left(\frac{x}{\mu_p}\right) v_p^{p+1}\, dx =: I_q,$$

where $v_p(x) = \mu_p^{-(2-\ell)/(p-1)} u_p(x/\mu_p)$.

Remark 1.4. In the case $\ell = 2$ or $\ell > 2$, the situation is much different from this case. Further arguments will be appear in Kabeya [**12**] or Kabeya and Yanagida [**13**].

Now in turn, we consider the behavior as $p \uparrow (n+2)/(n-2)$ in the spirit of Pan and Wang ([**18**]). Since they treated the scalar-field equation, solutions they obtained decay exponentially at infinity. However, the Matukuma type equation is not the case. Solutions decay at most $|x|^{-(n-2)}$-order at infinity. This requires careful treatments.

Theorem 1.4. *Suppose that* (K.0) *and* (K.2) *hold. Then there holds* $S_p \to K(0)^{-(n-2)/n} S$ *as* $p \uparrow (n+2)/(n-2)$. *Moreover, any least-energy solution* u_p *to* (1.1) *converges to* 0 *locally uniformly in* $\boldsymbol{R}^n \backslash \{0\}$ *and* $\|u_p\|_\infty \to \infty$ *as* $p \uparrow (n+2)/(n-2)$ *although* $\|\nabla u_p\|_2^2 = S_p^{(p+1)/(p-1)} \to K(0)^{-(n-2)/2} S^{n/2}$ *as* $p \uparrow (n+2)/(n-2)$.

Remark 1.5. Similar to Remark 1.2, u_p never converges strongly to 0 as $p \uparrow (n+2)/(n-2)$. As we see below, u_p blows up at the origin and vanishes as $p \uparrow (n+2)/(n-2)$.

Similar to Theorem 1.2, we introduce a scaling. Let

$$v_p^{-2/(p-1)} := \|u_p\|_\infty.$$

Theorem 1.5. *For a solution* u_p *obtained in Theorem* 1.4, *let*

$$w_p(x) := v_p^{2/(p-1)} u_p(v_p x).$$

Then w_p *converges to*

$$W(x) := \left(1 + \frac{K(0)}{n(n-2)}|x|^2\right)^{-(n-2)/2} \tag{1.9}$$

locally uniformly in $\boldsymbol{R}^n$ *and strongly in* $\mathscr{D}$ *as* $p \uparrow (n+2)/(n-2)$. *Moreover, the maximum point* P_p *converges to* 0 *as* $p \uparrow (n+2)/(n-2)$.

Remark 1.6. w_p converges to a solution to

$$\Delta W + K(0) W^{(n+2)/(n-2)} = 0 \quad \text{in } \boldsymbol{R}^n \tag{1.10}$$

with $W(0) = 1$. However, it is well-known that any positive solution to (1.10) must be radially symmetric and is of the form (1.9) by Caffarelli, Gidas and Spruck ([**6**]).

As for the blow-up rate of $\|u_p\|_\infty$, we get the following under some additional assumptions similar to Theorem 1.3.

Theorem 1.6. *Suppose that* (K.0) *and* (K.2) *with* $s > 2(n-2)/(n+2)$ *hold. Moreover, assume that*

$$\text{(K.5)} \qquad \lim_{t \downarrow 0} \frac{x \cdot \nabla_x K(tx)}{t^m} = -c_4 |x|^m$$

locally uniformly in $\boldsymbol{R}^n$ *with* $c_4 > 0$. *When* $m \geq n$, *suppose further that*

$$\text{(K.6)} \qquad x \cdot \nabla K(x) = -c_5 |x|^m \quad \text{in } B_{r_1}$$

with $r_1 > 0$, $c_5 > 0$. *Then, if* $0 < m < n$, *there holds*

$$\lim_{p \uparrow (n+2)/(n-2)} \{(n-2)(p+1) - 2n\} \|u_p\|_\infty^{(p-1)m/2} = -\frac{2c_4 \int_{\boldsymbol{R}^n} |x|^m W^{2n/(n-2)}\, dx}{K(0) \int_{\boldsymbol{R}^n} W^{2n/(n-2)}\, dx}.$$

If $m \geq n$, *then*

$$\lim_{p \uparrow (n+2)/(n-2)} \{(n-2)(p+1) - 2n\} \|u_p\|_\infty^{(p-1)n/2} = \frac{I_1}{K(0) \int_{\boldsymbol{R}^n} W^{2n/(n-2)}\, dx},$$

where

$$I_1 := 2 \lim_{p \uparrow (n+2)/(n-2)} v_p^{-n} \int_{\boldsymbol{R}^n} (x \cdot \nabla_x K(v_p x)) w_p(x)^{p+1}\, dx$$

with $w_p(x) = v_p^{2/(p-1)} u_p(v_p x)$.

Remark 1.7. The additional assumption on s is needed to ensure the decay order of u_p at infinity.

In view of Theorems 1.1–1.6, we can say a global features of a least-energy solution. A least-energy solution to (1.1) with $K(x)$ satisfying (K.0), (K.1) and (K.2) with (L) (take $K(x) = 1/(1+|x|)$ with $n = 3$ for instance) suddenly appears at "infinity" as p exceeds p_ℓ and the "energy" concentrates to the origin as $p \uparrow (n+2)/(n-2)$ like a δ-function and the solution vanishes at $p = (n+2)/(n-2)$.

This paper is organized as follows. Theorems concerning $p \downarrow p_\ell$ are proved in Section 2 and those on $p \uparrow (n+2)/(n-2)$ are done in Section 3.

2. Proofs of Theorems 1.1, 1.2 and 1.3.

In this section, we enumerate several lemmas and give proofs of Theorems 1.1, 1.2 and 1.3. The fundamental lemma is to assure the existence of a minimizer of (1.2), i.e., the existence of a least-energy solution to (1.1).

Lemma 2.1. *Under* (L), (K.0) *and* (K.1), *there exists a minimizer of* (1.2) *for any* $p \in (p_\ell, (n+2)/(n-2))$. *Similarly, under* (K.0) *and* (K.2), *there exists* $p_0 < (n+2)/(n-2)$ *such that a minimizer of* (1.2) *exists for any* $p \in (p_0, (n+2)/(n-2))$.

Proof. First we note that S_p is uniformly bounded. Indeed, take $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ $(\varphi \not\equiv 0)$. Then, from the definition of S_p, we see that

$$S_p \le \frac{\displaystyle\int_{\boldsymbol{R}^n} |\nabla \varphi|^2\,dx}{\left(\displaystyle\int_{\boldsymbol{R}^n} K(x)\varphi^{p+1}\,dx\right)^{2/(p+1)}}.$$

Hence the right-hand side is uniformly bounded in p.

Thus we may take a minimizing sequence $\{u_j\}$ such that

$$\int_{\boldsymbol{R}^n} K(x)u_j^{p+1}\,dx = 1 \quad \text{and} \quad \int_{\boldsymbol{R}^n} |\nabla u_j|^2\,dx \to S_p$$

as $j \to \infty$. We can choose a subsequence (still denoted by $\{u_j\}$) such that

$$\begin{cases} u_j \rightharpoonup u_\infty & \text{weakly in } \mathscr{D}, \\ u_j \to u_\infty & \text{locally strongly in } L^{p+1}(\boldsymbol{R}^n), \\ u_j \to u_\infty & \text{a.e. in } \boldsymbol{R}^n. \end{cases}$$

Moreover, by the Hölder and Sobolev inequalities, we have

$$\begin{aligned} (2.1) \qquad & \int_{\boldsymbol{R}^n\setminus B_R} K(x)|u_j|^{p+1}\,dx \\ & \le \left(\int_{\boldsymbol{R}^n\setminus B_R} K(x)^{2n/(2n-(p+1)(n-2))}\,dx\right)^{(2n-(p+1)(n-2))/2n} \\ & \quad \times \left(\int_{\boldsymbol{R}^n\setminus B_R} |u_j|^{2n/(n-2)}\,dx\right)^{(p+1)(n-2)/2n} \\ & \le S^{-(p+1)/2}\left(\int_{\boldsymbol{R}^n\setminus B_R} K(x)^{2n/(2n-(p+1)(n-2))}\,dx\right)^{(2n-(p+1)(n-2))/2n} \\ & \quad \times \left(\int_{\boldsymbol{R}^n} |\nabla u_j|^2\,dx\right)^{(p+1)/2} \end{aligned}$$

for any $R > 0$. From (K.1), we see that there exists $R_0 > 0$ such that

$$K(x) \le \frac{2c_0}{|x|^{\ell}} \quad \text{on } \boldsymbol{R}^n \backslash B_{R_0}.$$

Thus we have

$$\int_{\boldsymbol{R}^n \backslash B_{R_0}} K(x)^{2n/\{2n-(p+1)(n-2)\}}\, dx \le C_0 R_0^{n-2n\ell/\{2n-(p+1)(n-2)\}} < \infty$$

by $p > p_\ell$ with a constant $C_0 > 0$.

In the case of (K.2), we get

$$\int_{\boldsymbol{R}^n \backslash B_{R_1}} K(x)^{2n/\{2n-(p+1)(n-2)\}}\, dx \le C_1 R_1^{n-2ns/\{2n-(p+1)(n-2)\}} < \infty$$

if $2n - (p+1)(n-2) - 2s < 0$, i.e., $p_0 := \max\{1, (n+2-2s)/(n-2)\} < p < (n+2)/(n-2)$, with a constant $C_1 > 0$.

Hence, in either case, for arbitrarily given $\varepsilon > 0$, we can take $R > 0$ sufficiently large independent of j such that $\int_{\boldsymbol{R}^n \backslash B_R} K(x) u_j^{p+1}\, dx < \varepsilon$. Since u_j converges to u_∞ in $L^{p+1}(B_R)$, we see that

$$\int_{B_R} K(x) u_j^{p+1}\, dx - \varepsilon < \int_{\boldsymbol{R}^n} K(x) u_j^{p+1}\, dx < \int_{B_R} K(x) u_j^{p+1}\, dx + \varepsilon$$

and thus

$$\lim_{j \to \infty} \int_{\boldsymbol{R}^n} K(x) u_j^{p+1}\, dx = \int_{\boldsymbol{R}^n} K(x) u_\infty^{p+1}\, dx. \tag{2.2}$$

Finally, we prove

$$\lim_{j \to \infty} \int_{\boldsymbol{R}^n} |\nabla u_j|^2\, dx = \int_{\boldsymbol{R}^n} |\nabla u_\infty|^2\, dx.$$

Since u_j converges to u_∞ weakly in $\mathscr{D}$, we have

$$\int_{\boldsymbol{R}^n} |\nabla u_\infty|^2\, dx \le \int_{\boldsymbol{R}^n} |\nabla u_j|^2\, dx. \tag{2.3}$$

On the other hand, (2.2) and the fact that $\int_{\boldsymbol{R}^n} K(x) u_j^{p+1}\, dx = 1$ imply that $\int_{\boldsymbol{R}^n} K(x) u_\infty^{p+1}\, dx = 1$. Thus we get

$$S_p \le \int_{\boldsymbol{R}^n} |\nabla u_\infty|^2\, dx.$$

Taking $j \to \infty$ in (2.3) and using the Fatou lemma, we have

$$\int_{\boldsymbol{R}^n} |\nabla u_\infty|^2\, dx \le S_p \le \int_{\boldsymbol{R}^n} |\nabla u_\infty|^2\, dx,$$

i.e.,

$$\int_{\boldsymbol{R}^n} |\nabla u_\infty|^2 \, dx = S_p.$$

This implies the strong convergence of u_j in $\mathscr{D}$ and the existence of a minimizer of (1.2). □

REMARK 2.1. Any minimizer v of (1.2) must satisfy

$$\Delta v + \frac{S_p}{\left(\int_{\boldsymbol{R}^n} K(x)|v|^{p+1} \, dx\right)^{(p-1)/(p+1)}} K(x) v^p = 0. \tag{2.4}$$

Thus, setting

$$u := \frac{S_p^{1/(p-1)}}{\left(\int_{\boldsymbol{R}^n} K(x)|v|^{p+1} \, dx\right)^{1/(p+1)}} v,$$

we see that u is a solution to (1.1). Hence we obtain

$$\int_{\boldsymbol{R}^n} |\nabla u|^2 \, dx = \int_{\boldsymbol{R}^n} K(x) u^{p+1} \, dx = S_p^{(p+1)/(p-1)}. \tag{2.5}$$

Note that $\|\nabla u\|_2$ is uniformly bounded even as $p \downarrow p_\ell$ under (L).

Here we prove the Pohozaev identity (1.4). A proof is also given in Proposition 1 in Naito [**17**] under similar assumptions (instead of his decay condition, we assume $u \in \mathscr{D}$), we give a proof for the sake of self-containedness.

LEMMA 2.2. *For any nonnegative solution* $u \in \mathscr{D}$, *the Pohozaev identity*

$$\int_{\boldsymbol{R}^n} \left\{ \left(\frac{n-2}{2} - \frac{n}{p+1} \right) - \frac{1}{p+1} \frac{x \cdot \nabla K(x)}{K(x)} \right\} K(x) u^{p+1} \, dx = 0$$

holds under (K.0).

PROOF. First we prove that

$$\int_{\boldsymbol{R}^n} |\nabla u|^2 \, dx = \int_{\boldsymbol{R}^n} K(x) u^{p+1} \, dx \tag{2.6}$$

holds for any nonnegative solution $u \in \mathscr{D}$ to (1.1). Indeed, note that $\int_{\boldsymbol{R}^n} K(x) u^{p+1} \, dx < \infty$ in view of (2.1) on $\boldsymbol{R}^n \backslash B_R$ and the Sobolev inequality for B_R $(R > 0)$. Moreover, u is a classical solution under (K.0). Multiplying the both sides of (1.1) by u and integrating it over B_R, we have

$$\int_{B_R} |\nabla u|^2\,dx - \int_{\partial B_R} u\frac{\partial u}{\partial \nu}\,dS = \int_{B_R} K(x)u^{p+1}\,dx. \tag{2.7}$$

By the Schwarz inequality, we have

$$\left|\int_{\partial B_R} u\frac{\partial u}{\partial \nu}\,dS\right| \le \left(\int_{\partial B_R}\frac{u^2}{R}\,dS\right)^{1/2}\left(\int_{\partial B_R} R|\nabla u|^2\,dS\right)^{1/2}.$$

Using the Hardy inequality

$$\left(\frac{n-2}{2}\right)^2\int_{\boldsymbol{R}^n}\frac{u^2}{|x|^2}\,dx \le \int_{\boldsymbol{R}^n}|\nabla u|^2\,dx,$$

and

$$\int_{\boldsymbol{R}^n}\frac{u^2}{|x|^2}\,dx = \int_0^\infty\left(\int_{\partial B_R}\frac{u^2}{R^2}\,dS\right)dR$$

(note that $\boldsymbol{R}^n = \bigcup_{R\ge 0}\{(\theta,R)\,|\,\theta\in\partial B_R\}$), we can choose a sequence $\{R_j\}$ ($R_j\to\infty$ as $j\to\infty$) such that

$$\int_{\partial B_{R_j}}\frac{u^2}{R_j}\,dS\to 0,\quad \int_{\partial B_{R_j}} R_j|\nabla u|^2\,dS\to 0,$$

as $j\to\infty$. Choosing $R=R_j$ in (2.7) and letting $j\to\infty$, we obtain (2.6).

Next identical to the proof of Proposition 1 in [**17**], multiplying the both sides of (1.1) by $x\cdot\nabla u$, and integrating it over B_R, we have

$$\begin{aligned}\int_{\partial B_R}&\left\{R\left(\frac{\partial u}{\partial \nu}\right)^2 - \frac{R}{2}|\nabla u|^2 + \frac{R}{p+1}K(x)u^{p+1}\right\}dS\\ &+\int_{B_R}\left\{\frac{n-2}{2}|\nabla u|^2 - \frac{n}{p+1}K(x)u^{p+1} - \frac{x\cdot\nabla K(x)}{p+1}u^{p+1}\right\}dx = 0,\end{aligned}\tag{2.8}$$

where ν is the outward normal unit vector to ∂B_R. As above, since

$$\int_{\boldsymbol{R}^n}|\nabla u|^2\,dx = \int_0^\infty\left(\int_{\partial B_R}|\nabla u|^2\,dS\right)dR < \infty,$$

and since

$$\int_{\boldsymbol{R}^n}K(x)u^{p+1}\,dx = \int_0^\infty\left(\int_{\partial B_R}K(x)u^{p+1}\,dS\right)dR < \infty,$$

we can choose a sequence $\{R_j\}$ ($R_j\to\infty$ as $j\to\infty$) such that

$$\int_{\partial B_{R_j}} R_j |\nabla u|^2 \, dS \to 0, \quad \int_{\partial B_{R_j}} R_j K(x) u^{p+1} \, dS \to 0$$

as $j \to \infty$. Moreover, since $(\partial u/\partial \nu)^2 \le |\nabla u|^2$, choosing $R = R_j$ and letting $j \to \infty$ in (2.8), we have

$$\int_{\boldsymbol{R}^n} \left\{ \left(\frac{n-2}{2} - \frac{n}{p+1} \right) - \frac{1}{p+1} \frac{x \cdot \nabla K(x)}{K(x)} \right\} K(x) u^{p+1} \, dx = 0$$

by using (2.6) and $x \cdot \nabla K / K \in L^\infty(\boldsymbol{R}^n)$. □

From (2.5), we see that any least-energy solution u_p to (1.1) is uniformly bounded in $\mathscr{D}$. We will show an a priori estimate of $\|u\|_\infty$.

Lemma 2.3. *Suppose that* (K.0) *and* (K.1) *hold. For any least-energy solution u_p to* (1.1), *there exists $C_2 > 0$ independent of p near p_ℓ such that* $\|u_p\|_\infty \le C_2$.

Proof. We regard (1.1) as

$$\Delta u_p + L(x) u_p = 0 \quad \text{in } \boldsymbol{R}^n$$

with $L(x) = K(x) u_p^{p-1}$. Then there exists $\delta > 0$ independent of p such that $L(x) \in L^{n/2+\delta}_{\mathrm{loc}}(\boldsymbol{R}^n)$. Indeed, since $K(x) \in L^\infty(\boldsymbol{R}^n)$ by (K.0) and (K.1), we have only to consider u^{p-1} with $p = p_\ell + \varepsilon$. In view of

$$(p_\ell + \varepsilon - 1)\frac{n}{2} = \frac{(2-\ell)n}{n-2} + \frac{n}{2}\varepsilon < \frac{2n}{n-2}$$

if $\varepsilon > 0$ is sufficiently small, we can take $\delta > 0$ independent of ε such that $(p-1)(n/2+\delta) < 2n/(n-2)$. Moreover, we have

$$\int_{B(Q,r)} L(x)^{n/2+\delta} \, dx \le C_3 |B(Q,r)|^{(2n-(n-2)(p-1)(n/2+\delta))/2n}$$

with some constant $C_3 > 0$ independent of p since $\|\nabla u_p\|_2$ is bounded, where $B(Q,r)$ denotes the ball centered at Q with radius r.

Then, by Lemma 2.7 of Pan and Wang [**18**] (see also Lemma 7 of Han [**9**]), we have

$$\sup_{B(Q,r)} u_p \le C_4 \left(\frac{1}{r^n} \int_{B(Q,2r)} u_p^{2n/(n-2)} \, dx \right)^{(n-2)/2n} \tag{2.9}$$

for any $Q \in \boldsymbol{R}^n$ and $r > 0$, and the constant C_4 depends only on n, δ and $r^\delta \|L\|_{L^{n/2+\delta}(B(Q,2r))}$. The deduction of (2.9) is done by the Moser iteration method. Note that $r^\delta \|L\|_{L^{n/2+\delta}(B(Q,2r))}$ is independent of p as we have seen above. Moreover, by the Sobolev inequality and by the uniform boundedness of $\|\nabla u_p\|_2$, the right-hand side of (2.9) is uniformly bounded. Taking Q as the maximum point of u_p, we see that $\|u_p\|_\infty$ is uniformly bounded. $\square$

Next, we investigate the behavior of S_p.

Lemma 2.4. *$S_p \to c_0^{-(n-2)/(n-\ell)} S(\ell)$ as $p \downarrow p_\ell$.*

Proof. As in Pan and Wang [**18**], we prove

$$\limsup_{p \downarrow p_\ell} S_p \le c_0^{-(n-2)/(n-\ell)} S(\ell) \quad \text{and} \quad \liminf_{p \downarrow p_\ell} S_p \ge c_0^{-(n-2)/(n-\ell)} S(\ell).$$

First, we show $\limsup_{p \downarrow p_\ell} S_p \le c_0^{-(n-2)/(n-\ell)} S(\ell)$. To this end, let

$$\hat{u}_\varepsilon := \left(\frac{1}{\varepsilon} + |x|^{2-\ell}\right)^{-(n-2)/(2-\ell)}$$

with $\varepsilon = p - p_\ell$ and calculate the quotient

$$Q_\varepsilon(\hat{u}_\varepsilon) := \frac{\displaystyle\int_{\boldsymbol{R}^n} |\nabla \hat{u}_\varepsilon|^2 \, dx}{\left(\displaystyle\int_{\boldsymbol{R}^n} K(x) \hat{u}_\varepsilon^{p+1} \, dx\right)^{2/(p+1)}}.$$

Since $\hat{u}_\varepsilon$ is radial, we see that

$$\int_{\boldsymbol{R}^n} |\nabla \hat{u}_\varepsilon|^2 \, dx = \omega_n \int_0^\infty r^{n-1} \left(\frac{d\hat{u}_\varepsilon}{dr}\right)^2 dr$$

with $r = |x|$. From

$$\frac{d\hat{u}_\varepsilon}{dr} = -(n-2)\left(\frac{1}{\varepsilon} + r^{2-\ell}\right)^{-(n-2)/(2-\ell)-1} r^{1-\ell},$$

we have

$$\int_0^\infty r^{n-1} \left(\frac{d\hat{u}_\varepsilon}{dr}\right)^2 dr = (n-2)^2 \int_0^\infty r^{n+1-2\ell} \left(\frac{1}{\varepsilon} + r^{2-\ell}\right)^{-2(n-\ell)/(2-\ell)} dr.$$

Letting $r = \varepsilon^{-1/(2-\ell)} \rho$, we get

$$\int_0^\infty r^{n-1} \left(\frac{d\hat{u}_\varepsilon}{dr}\right)^2 dr = (n-2)^2 \varepsilon^{(n-2)/(2-\ell)} \int_0^\infty \rho^{n+1-2\ell} (1 + \rho^{2-\ell})^{-2(n-\ell)/(2-\ell)} \, d\rho.$$

Similarly, we have

$$\int_{\boldsymbol{R}^n} K(x)\hat{u}_\varepsilon^{p+1}\,dx$$

$$= \varepsilon^{(n-\ell+(n-2)\varepsilon)/(2-\ell)} \int_{\boldsymbol{R}^n} \varepsilon^{-\ell/(2-\ell)} K(\varepsilon^{-1/(2-\ell)}y)(1+|y|^{2-\ell})^{-(2(n-\ell)+(n-2)\varepsilon)/(2-\ell)}\,dy$$

with the change of variables $x = \varepsilon^{-1/(2-\ell)}y$. By (K.1), we have

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-\ell/(2-\ell)} K(\varepsilon^{-1/(2-\ell)}y) = \frac{c_0}{|y|^\ell}$$

locally uniformly in $\boldsymbol{R}^n\backslash\{0\}$. Thus we can apply the Lebesgue convergence theorem to get

$$\begin{aligned} Q_\varepsilon(\hat{u}_\varepsilon) &= ((n-2)^2\omega_n\varepsilon^{(n-2)/(2-\ell)}\int_0^\infty \rho^{n+1-2\ell}(1+\rho^{2-\ell})^{-2(n-\ell)/(2-\ell)}\,d\rho) \\ &\quad \bigg/\Bigg(\varepsilon^{2(n-2)\{n-\ell+(n-2)\varepsilon\}/(2-\ell)\{2(n-\ell)+(n-2)\varepsilon\}}\Bigg(\int_{\boldsymbol{R}^n}\varepsilon^{-\ell/(2-\ell)}K(y/\varepsilon^{1/(2-\ell)}) \\ &\quad \times(1+|y|^{2-\ell})^{-\{2(n-\ell)+(n-2)\varepsilon\}/(2-\ell)}\,dy\Bigg)^{2(n-2)/\{2(n-\ell)+(n-2)\varepsilon\}}\Bigg) \\ &\to \frac{(n-2)^2\omega_n\int_0^\infty \rho^{n+1-2\rho}(1+\rho^{2-\ell})^{-2(n-\ell)/(n-2)}\,d\rho}{\left(c_0\omega_n\int_0^\infty \rho^{n-1-\ell}(1+|\rho|^{2-\ell})^{-2(n-\ell)/(2-\ell)}\,d\rho\right)^{(n-2)/(n-\ell)}} \\ &= c_0^{-(n-2)/(n-\ell)}S(\ell) \end{aligned}$$

as $\varepsilon \downarrow 0$ since $(1+|x|^{2-\ell})^{-(n-2)/(2-\ell)}$ attains $S(\ell)$ by (1.5). This implies that

$$\limsup_{\varepsilon\downarrow 0} S_{p_\ell+\varepsilon} \le c_0^{-(n-2)/(n-\ell)}S(\ell).$$

Next, we prove $\liminf_{\varepsilon\downarrow 0} S_{p_\ell+\varepsilon} \ge c_0^{-(n-2)/(n-\ell)}S(\ell)$. For u_ε which attains $S_{p_\ell+\varepsilon}$ with

$$\int_{\boldsymbol{R}^n} K(x)u_\varepsilon^{2(n-\ell)/(n-2)+\varepsilon}\,dx = 1,$$

we set

$$v_\varepsilon(y) = u_\varepsilon(x) \quad \text{with } x = \frac{y}{\varepsilon}.$$

Then we have

$$\int_{\boldsymbol{R}^n} |\nabla_x u_\varepsilon(x)|^2\,dx = \frac{1}{\varepsilon^{n-2}}\int_{\boldsymbol{R}^n} |\nabla_y v_\varepsilon(y)|^2\,dy$$

and

$$(2.10)\qquad \int_{\boldsymbol{R}^n} K(x)u_\varepsilon^{2(n-\ell)/(n-2)+\varepsilon}\,dx = \frac{1}{\varepsilon^{n-\ell}}\int_{\boldsymbol{R}^n}\frac{1}{\varepsilon^\ell}K\left(\frac{y}{\varepsilon}\right)v_\varepsilon^{2(n-\ell)/(n-2)+\varepsilon}\,dy = 1.$$

By (K.1), we have

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-\ell}K\left(\frac{y}{\varepsilon}\right) = \frac{c_0}{|y|^\ell}$$

locally uniformly in $\boldsymbol{R}^n\backslash\{0\}$. Thus we get

$$\liminf_{\varepsilon\downarrow 0}\frac{\displaystyle\int_{\boldsymbol{R}^n}|\nabla_x u_\varepsilon|^2\,dx}{\left(\displaystyle\int_{\boldsymbol{R}^n}K(x)u_\varepsilon^{2(n-\ell)/(n-2)+\varepsilon}\,dx\right)^{2(n-2)/(2(n-\ell)+(n-2)\varepsilon)}}$$

$$= \liminf_{\varepsilon\downarrow 0}\frac{\varepsilon^{-(n-2)}\displaystyle\int_{\boldsymbol{R}^n}|\nabla_y v_\varepsilon(y)|^2\,dy}{\varepsilon^{-2(n-2)(n-\ell)/(2(n-\ell)+(n-2)\varepsilon)}\left(\displaystyle\int_{\boldsymbol{R}^n}\varepsilon^{-\ell}K\left(\frac{y}{\varepsilon}\right)v_\varepsilon^{2(n-\ell)/(n-2)+\varepsilon}\,dy\right)^{2(n-2)/(2(n-\ell)+(n-2)\varepsilon)}}$$

$$\geq \liminf_{\varepsilon\downarrow 0}\frac{\displaystyle\int_{\boldsymbol{R}^n}|\nabla_y v_\varepsilon|^2\,dy}{\left(c_0\displaystyle\int_{\boldsymbol{R}^n}|y|^{-\ell}v_\varepsilon^{2(n-\ell)/(n-2)}\,dy\right)^{(n-2)/(n-\ell)}\left(\displaystyle\max_{\boldsymbol{R}^n} v_\varepsilon\right)^{2(n-2)\varepsilon/(2(n-\ell)+(n-2)\varepsilon)}}$$

$$\geq c_0^{-(n-2)/(n-\ell)}S(\ell)$$

noting that $\limsup_{\varepsilon\downarrow 0}(\max_{\boldsymbol{R}^n} v_\varepsilon)^\varepsilon \leq 1$ by the a priori estimate (2.9) in Lemma 2.3, (2.5) and (2.10). Hence we see that $\lim_{p\downarrow p_\ell} S_p = c_0^{-(n-2)/(n-\ell)}S(\ell)$. □

Next, we show that $\|u_p\|_\infty \to 0$ as $p\downarrow p_\ell$.

LEMMA 2.5. *Suppose that* (K.0) *and* (K.1) *hold. Then, for any least-energy solution* u_p *to* (1.1) *satisfies* $\|u_p\|_\infty \to 0$ *as* $p\downarrow p_\ell$.

PROOF. Multiplying (1.4) by $2(p+1)$ and adding $2\ell\int_{\boldsymbol{R}^n} Ku_p^{p+1}\,dx$ to the both sides, we have

$$(2.11)\qquad (n-2)(p-p_\ell)\int_{\boldsymbol{R}^n}K(x)u_p^{p+1}\,dx = 2\int_{\boldsymbol{R}^n}\left(\ell + \frac{x\cdot\nabla K(x)}{K(x)}\right)K(x)u_p^{p+1}\,dx.$$

From (2.5), we see that

$$\lim_{p \downarrow p_\ell} \int_{\boldsymbol{R}^n} \left(\ell + \frac{x \cdot \nabla K(x)}{K(x)} \right) K(x) u_p^{p+1}\, dx = 0.$$

Since $\ell K(x) + x \cdot \nabla K(x) \geq 0$, $\not\equiv 0$, $u_p \to 0$ in any compact set in $\{x \in \boldsymbol{R}^n \mid \ell K(x) + x \cdot \nabla K(x) > 0\}$ as $p \downarrow p_\ell$. Suppose that there exists a sequence $\{p_j\}$ $(p_j \to p_\ell$ as $j \to \infty)$ such that $u_{p_j} \to u_*$ $(\not\equiv 0)$ locally uniformly as $j \to \infty$. Hence u_* must be a solution to $\Delta u + K(x) u^{p_\ell} = 0$ in $\boldsymbol{R}^n$. By the Pohozaev identity, $u_* \equiv 0$ is the only nonnegative solution, a contradiction. Thus $u_p \to 0$ locally uniformly in $\boldsymbol{R}^n$ as $p \downarrow p_\ell$.

Suppose that $\|u_p\|_\infty \geq C_5 > 0$ for any p $(>p_\ell)$ sufficiently close to p_ℓ. Then the maximum point Q_p of u_p goes to ∞ as $p \downarrow p_\ell$. Let $\tilde{u}_p(x) = u(x + Q_p)$. Then $\tilde{u}_p$ is a solution to

$$\Delta \tilde{u}_p + K(x + Q_p) \tilde{u}_p^p = 0.$$

Since $\tilde{u}_p$ is bounded, choosing a subsequence $\{\tilde{p}_j\}$ $(\tilde{p}_j \to p_\ell$ as $j \to \infty)$, we see that $\tilde{u}_{\tilde{p}_j}$ converges locally uniformly to a classical nonnegative solution of $\Delta \tilde{u}_* = 0$. Note that $\tilde{u}_* \in \mathscr{D}$ in view of the boundedness of $\tilde{u}_p$ in $\mathscr{D}$. However, since $\tilde{u}_*(0) > 0$, there exists no nonnegative solution to $\Delta \tilde{u}_* = 0$ in $\mathscr{D}$. This is a contradiction. Thus $\|u_p\|_\infty \to 0$ as $p \downarrow p_\ell$. $\square$

Lemma 2.6. *For a least-energy solution u_p to* (1.1) *with x replaced by y, let*

$$v_p(x) := \frac{1}{\mu_p^{\alpha_p}} u_p(y), \quad \alpha_p = \frac{2-\ell}{p-1}, \quad y = \frac{x}{\mu_p}, \tag{2.12}$$

with $\mu_p^{\alpha_p} = \|u_p\|_\infty$. Then v_p converges uniformly to

$$U(x) := \left(1 + \frac{c_0}{(n-2)(n-\ell)} |x|^{2-\ell} \right)^{-(n-2)/(2-\ell)} \tag{2.13}$$

on any annulus in $\boldsymbol{R}^n$. Moreover, the maximum point of u_p converges to 0 *as $p \downarrow p_\ell$.*

Proof. We use the scaling (2.12). Since $u(y)$ satisfies

$$\Delta_y u(y) + K(y) u^p = 0 \quad \text{in } \boldsymbol{R}^n,$$

we have

$$\mu_p^{\alpha_p + 2} \Delta_x v_p(x) + \mu_p^{\alpha_p p} K\left(\frac{x}{\mu_p} \right) v_p^p = 0.$$

Since $\alpha_p = (2-\ell)/(p-1)$, we see that $v_p(x)$ is a solution to

$$\Delta_x v_p + \mu_p^{-\ell} K\left(\frac{x}{\mu_p}\right) v_p^p = 0$$

and $\|v_p\|_\infty = 1$. By (K.1), we get

$$\lim_{p \downarrow p_\ell} \mu_p^{-\ell} K\left(\frac{x}{\mu_p}\right) = \frac{c_0}{|x|^\ell}$$

locally uniformly in $\boldsymbol{R}^n \backslash \{0\}$. Thus, choosing a subsequence $\{p_j\}$ ($p_j \to p_\ell$ as $j \to \infty$) such that $v_{p_j} \to U$ in locally uniformly in $\boldsymbol{R}^n \backslash \{0\}$. Note that U is a solution to (1.8). As is commented in Remark 1.3, it is known that

$$U(x) = \left(1 + \frac{c_0}{(n-2)(n-\ell)} |x|^{2-\ell}\right)^{-(n-2)/(2-\ell)}$$

is the only solution to the limiting equation. Thus we see that $v_p \to U$ locally uniformly in $\boldsymbol{R}^n \backslash \{0\}$ without choosing a subsequence (by the uniqueness, we have $\limsup_{p \downarrow p_\ell} v_p(x) = \liminf_{p \downarrow p_\ell} v_p(x) = U(x)$).

Since the maximum point of U is the origin and since U is a strictly decreasing function, we see that $Q_p \to 0$ as $p \downarrow p_\ell$. □

Remark 2.2. Note that $U(x)$ attains $S(\ell)$ and satisfies

$$\int_{\boldsymbol{R}^n} |\nabla U|^2 \, dx = c_0 \int_{\boldsymbol{R}^n} |x|^{-\ell} U^{p_\ell + 1} \, dx$$

since U is a solution to (1.8). Thus we have

$$S(\ell) = \frac{\int_{\boldsymbol{R}^n} |\nabla U|^2 \, dx}{\left(\int_{\boldsymbol{R}^n} |x|^{-\ell} U^{p_\ell+1} \, dx\right)^{2/(p_\ell+1)}} = c_0^{(n-2)/(n-\ell)} \left(\int_{\boldsymbol{R}^n} |\nabla U|^2 \, dx\right)^{(2-\ell)/(n-\ell)}$$

i.e.,

$$(c_0^{-(n-2)/(n-\ell)} S(\ell))^{(n-\ell)/(2-\ell)} = \int_{\boldsymbol{R}^n} |\nabla U|^2 \, dx.$$

Lemma 2.7. *For any least-energy solution u_p to* (1.1), *u_p converges weakly to* 0.

Proof. We will prove

$$\int_{\boldsymbol{R}^n} \nabla u_p \nabla \varphi \, dx \to 0 \quad \text{as } p \downarrow p_\ell$$

for any $\varphi \in \mathscr{D}$. Since u_p is a classical solution to (1.1) and since $u_p \to 0$ locally uniformly in $\boldsymbol{R}^n$, we see that $|\nabla u_p| \to 0$ locally uniformly in $\boldsymbol{R}^n$. For any $\varepsilon > 0$, we can choose $R > 0$ such that

$$\left(\int_{\boldsymbol{R}^n \setminus B_R} |\nabla \varphi|^2 \, dx\right)^{1/2} < \varepsilon.$$

Decomposing the integral as

$$\int_{\boldsymbol{R}^n} \nabla u_p \nabla \varphi \, dx = \int_{B_R} \nabla u_p \nabla \varphi \, dx + \int_{\boldsymbol{R}^n \setminus B_R} \nabla u_p \nabla \varphi \, dx,$$

we see that the first term converges to 0 as $p \downarrow p_\ell$ and that

$$\left| \int_{\boldsymbol{R}^n \setminus B_R} \nabla u_p \nabla \varphi \, dx \right| \leq C_6 \varepsilon$$

with $C_6 > 0$ independent of p since $\|\nabla u_p\|_2$ is uniformly bounded. This implies that $u_p \rightharpoonup 0$ weakly in $\mathscr{D}$ as $p \downarrow p_\ell$. □

Using the scaling defined in (2.12), we see some properties of the rescaled function.

Lemma 2.8. *The rescaled solution $v_p(x)$ defined in* (2.12) *satisfies $v_p \to U$ in $\mathscr{D}$, $\|\nabla v_p\|_2^2 \to c_0^{-(n-2)/(2-\ell)} S(\ell)^{(n-\ell)/(2-\ell)}$,*

$$\int_{\boldsymbol{R}^n} \mu_p^{-\ell} K(\mu_p^{-1} x) v_p^{p+1} \, dx \to c_0 \int_{\boldsymbol{R}^n} |x|^{-\ell} U^{2(n-\ell)/(n-2)} \, dx$$

and $\mu_p^{2(2-\ell)/(p-1)-(n-2)} \to 1$ as $p \downarrow p_\ell$.

Proof. First note that $v_p \to U$ locally uniformly in $\boldsymbol{R}^n \setminus \{0\}$ as $p \downarrow p_\ell$ as in the proof of Lemma 2.6.

Moreover, v_p also converges to U weakly in $\mathscr{D}$ by the local uniform convergence of v_p. This can be proved similarly as in the proof of Lemma 2.7. Thus we can apply the Fatou lemma to v_p.

Since

$$\int_{\boldsymbol{R}^n} |\nabla_y u_p(y)|^2 \, dy = \mu_p^{2(2-\ell)/(p-1)-(n-2)} \int_{\boldsymbol{R}^n} |\nabla_x v_p|^2 \, dx$$

and by Remark 2.2, we have

$$
\begin{aligned}
&(c_0^{-(n-2)/(n-\ell)} S(\ell))^{(n-\ell)/(2-\ell)} \\
&\quad = \|\nabla U\|_2^2 \le \liminf_{p \downarrow p_\ell} \int_{\boldsymbol{R}^n} |\nabla_x v_p(x)|^2\, dx \le \limsup_{p \downarrow p_\ell} \int_{\boldsymbol{R}^n} |\nabla_x v_p(x)|^2\, dx \\
&\quad = \limsup_{p \downarrow p_\ell} \mu_p^{-2(2-\ell)/(p-1)+(n-2)} \int_{\boldsymbol{R}^n} |\nabla_y u_p|^2\, dy \le \limsup_{p \downarrow p_\ell} \int_{\boldsymbol{R}^n} |\nabla_y u_p|^2\, dy
\end{aligned}
$$

by $\mu_p < 1$ in view of Lemma 2.5 for p sufficiently close to p_ℓ and $(n-2) - 2(2-\ell)/(p-1) > 0$. Moreover, since $u_p(y)$ is an unscaled least-energy solution to

$$
\Delta_y u_p + K(y) u_p^p = 0 \quad \text{in } \boldsymbol{R}^n,
$$

we see that

$$
\lim_{p \downarrow p_\ell} \int_{\boldsymbol{R}^n} |\nabla_y u_p|^2\, dy = c_0^{-(n-2)/(2-\ell)} S(\ell)^{(n-\ell)/(2-\ell)}
$$

by (2.5) and Lemma 2.4. Thus we obtain $\|\nabla v_p\|_2^2 \to c_0^{-(n-2)/(2-\ell)} S(\ell)^{(n-\ell)/(2-\ell)}$ as $p \downarrow p_\ell$. Thus, we have $\mu_p^{2(2-\ell)/(p-1)-(n-2)} \to 1$ as $p \downarrow p_\ell$.

Since the weak convergence of v_p together with the convergence of the corresponding norm implies the strong convergence, we see that v_p converges strongly to U in $\mathscr{D}$.

Moreover, since

$$
\int_{\boldsymbol{R}^n} |\nabla v_p|^2\, dx = \int_{\boldsymbol{R}^n} \frac{1}{\mu_p^\ell} K\left(\frac{x}{\mu_p}\right) v_p^{p+1}\, dx,
$$

since $v_p \to U$ in $\mathscr{D}$, and since

$$
\int_{\boldsymbol{R}^n} |\nabla U|^2\, dx = c_0 \int_{\boldsymbol{R}^n} |x|^{-\ell} U^{2(n-\ell)/(n-2)}\, dx,
$$

we see that

$$
\lim_{p \downarrow p_\ell} \int_{\boldsymbol{R}^n} \frac{1}{\mu_p^\ell} K\left(\frac{x}{\mu_p}\right) v_p^{p+1}\, dx = c_0 \int_{\boldsymbol{R}^n} |x|^{-\ell} U^{2(n-\ell)/(n-2)}\, dx. \qquad \square
$$

Proof of Theorem 1.1. Theorem 1.1 is immediately from Lemmas 2.4, 2.5, 2.7 and 2.8. □

Proof of Theorem 1.2. Theorem 1.2 is readily seen by Lemmas 2.6 and 2.8. □

Proof of Theorem 1.3. As in Lemma 2.6, we use the scaled solution

$$v_p(x) = \mu_p^{-(2-\ell)/(p-1)} u_p(y) \quad \text{with } y = \frac{x}{\mu_p}$$

where $\|u_p\|_\infty = \mu_p^{(2-\ell)/(p-1)}$.

The identity (2.11)

$$\{(n-2)(p+1) - 2(n-\ell)\} \int_{\boldsymbol{R}^n} K(y) u_p^{p+1}(y)\, dy$$
$$= 2 \int_{\boldsymbol{R}^n} \left(\ell + \frac{y \cdot \nabla_y K(y)}{K(y)} \right) K(y) u_p^{p+1}(y)\, dy$$

yields

$$\{(n-2)(p+1) - 2(n-\ell)\} \int_{\boldsymbol{R}^n} K\left(\frac{x}{\mu_p}\right) v_p(x)^{p+1}\, dx$$
$$= 2 \int_{\boldsymbol{R}^n} \left(\ell + \frac{x \cdot \nabla_x K(x/\mu_p)}{K(x/\mu_p)} \right) K\left(\frac{x}{\mu_p}\right) v_p(x)^{p+1}\, dx.$$

Under the assumption (K.1), we see that

$$\lim_{p \downarrow p_\ell} \frac{1}{\mu_p^\ell} K\left(\frac{x}{\mu_p}\right) = c_0 |x|^{-\ell}$$

uniformly in $\boldsymbol{R}^n \backslash B_r$, with any fixed $r > 0$. By Lemma 2.8, we have

$$\lim_{p \downarrow p_\ell} \int_{\boldsymbol{R}^n} K(y) u(y)^{p+1}\, dy$$
$$= \lim_{p \downarrow p_\ell} \mu_p^{(2-\ell)(p+1)/(p-1)-(n-\ell)} \int_{\boldsymbol{R}^n} \frac{1}{\mu_p^\ell} K\left(\frac{x}{\mu_p}\right) (v_p(x))^{p+1}\, dx$$
$$= c_0 \int_{\boldsymbol{R}^n} |x|^{-\ell} U(x)^{2(n-\ell)/(n-2)}\, dx.$$

Since the right-hand side tends to 0 as $R \to \infty$ independent of p, we get

$$\lim_{p \downarrow p_\ell} \frac{1}{\mu_p^{\ell+q}} \int_{\boldsymbol{R}^n} \left(\ell + \frac{x \cdot \nabla_x K\left(\dfrac{x}{\mu_p}\right)}{K\left(\dfrac{x}{\mu_p}\right)} \right) K\left(\frac{x}{\mu_p}\right) v_p^{p+1}\, dx$$
$$= c_2 \int_{\boldsymbol{R}^n} |x|^{-\ell-q} U^{2(n-\ell)/(n-2)}\, dx$$

in view of (K.3). Here we note that if $0 < q < n - \ell$, then $|x|^{-\ell-q} \in L^1(B_1)$ and

$$\int_{\mathbf{R}^n \setminus B_R} |x|^{-\ell-q} v_p^{p+1}\, dx \le C_7 R^{\{(n+2)-(n-2)p-2(\ell+q)\}/2} \left(\int_{\mathbf{R}^n \setminus B_R} v_p^{2n/(n-2)}\, dx \right)^{(n-2)(p+1)/2n} \tag{2.14}$$

by the Hölder inequality with $C_7 > 0$ independent of $R > 0$.

Hence we obtain

$$\lim_{p \downarrow p_\ell} \frac{(n-2)(p+1) - 2(n-\ell)}{\mu_p^{\ell+q}} = \frac{2c_2 \int_{\mathbf{R}^n} |x|^{-\ell-q} U^{2(n-\ell)/(n-2)}\, dx}{c_0 \int_{\mathbf{R}^n} |x|^{-\ell} U^{2(n-\ell)/(n-2)}\, dx},$$

i.e.,

$$\lim_{p \downarrow p_\ell} \frac{(n-2)(p+1) - 2(n-\ell)}{\|u_p\|_\infty^{(p-1)(\ell+q)/(2-\ell)}} = \frac{2qc_2 \int_{\mathbf{R}^n} |x|^{-\ell-q} U^{2(n-\ell)/(n-2)}\, dx}{\int_{\mathbf{R}^n} |x|^{-\ell} U^{2(n-\ell)/(n-2)}\, dx}$$

by $\mu_p = \|u_p\|_\infty^{(p-1)/(2-\ell)}$ and Lemma 2.8.

If $q \ge n - \ell$, then $|x|^{-\ell-q} \notin L^1(B_1)$. We decompose

$$\int_{\mathbf{R}^n} \left(\ell + \frac{x \cdot \nabla_x K\left(\frac{x}{\mu_p}\right)}{K\left(\frac{x}{\mu_p}\right)} \right) K\left(\frac{x}{\mu_p}\right) v_p(x)^{p+1}\, dx$$

$$= \left(\int_{B_{\mu_p R_2}} + \int_{\mathbf{R}^n \setminus B_{\mu_p R_2}} \right) \left(\ell + \frac{x \cdot \nabla_x K\left(\frac{x}{\mu_p}\right)}{K\left(\frac{x}{\mu_p}\right)} \right) K\left(\frac{x}{\mu_p}\right) v_p(x)^{p+1}\, dx.$$

The first term yields

$$C_8 \mu_p^n \le \int_{B_{\mu_p R_2}} \left(\ell + \frac{x \cdot \nabla_x K\left(\frac{x}{\mu_p}\right)}{K\left(\frac{x}{\mu_p}\right)} \right) K\left(\frac{x}{\mu_p}\right) v_p(x)^{p+1}\, dx \le C_9 \mu_p^n$$

with $0 < C_8 < C_9$ since the integrand is bounded and never converges to 0 as $p \downarrow p_\ell$ in $B_{\mu_p R_3} \subset B_{\mu_p R_2}$ by (K.4). Moreover, we have

$$\lim_{p \downarrow p_\ell} \mu_p^{-n} \int_{B_{\mu_p R_2}} \left(\ell + \frac{x \cdot \nabla_x K\left(\dfrac{x}{\mu_p}\right)}{K\left(\dfrac{x}{\mu_p}\right)} \right) K\left(\frac{x}{\mu_p}\right) v_p(x)^{p+1}\, dx = C_{10} > 0.$$

As for the second term, we have

$$\int_{\boldsymbol{R}^n \setminus B_{\mu_p R_2}} \left(\ell + \frac{x \cdot \nabla_x K\left(\dfrac{x}{\mu_p}\right)}{K\left(\dfrac{x}{\mu_p}\right)} \right) K\left(\frac{x}{\mu_p}\right) (v_p(x))^{p+1}\, dx \tag{2.15}$$

$$= c_3 \mu_p^{\ell+q} \int_{\boldsymbol{R}^n \setminus B_{\mu_p R_2}} |x|^{-\ell-q} (v_p(x))^{p+1}\, dx,$$

by the order assumption in (K.4). We note that

$$\int_{B_1 \setminus B_{\mu_p R_2}} |x|^{-\ell-q}\, dx = \frac{\omega_n}{\ell+q-n} \{(\mu_p R_2)^{n-\ell-q} - 1\}$$

if $\ell + q > n$ and that

$$\int_{B_1 \setminus B_{\mu_p R_2}} |x|^{-\ell-q}\, dx = \omega_n \{-\log(\mu_p R_2)\}$$

if $\ell + q = n$.

Since $v_p(x) \le 1$ in $\boldsymbol{R}^n$, $v_p \in \mathscr{D}$ and since (2.14), we see that

$$\lim_{p \downarrow p_\ell} \mu_p^{-n} \int_{\boldsymbol{R}^n \setminus B_{\mu_p R_2}} \left(\ell + \frac{x \cdot \nabla_x K\left(\dfrac{x}{\mu_p}\right)}{K\left(\dfrac{x}{\mu_p}\right)} \right) K\left(\frac{x}{\mu_p}\right) v_p^{p+1}\, dx$$

exists and so does

$$\lim_{p \downarrow p_\ell} \mu_p^{-n} \int_{\boldsymbol{R}^n} \left(\ell + \frac{x \cdot \nabla_x K\left(\dfrac{x}{\mu_p}\right)}{K\left(\dfrac{x}{\mu_p}\right)} \right) K\left(\frac{x}{\mu_p}\right) v_p^{p+1}\, dx = I_q > 0$$

if $q > n - \ell$. Similarly, we also see that

$$\lim_{p \downarrow p_\ell} \mu_p^{-n} |\log \mu_p|^{-1} \int_{\boldsymbol{R}^n \setminus B_{\mu_p R_2}} \left(\ell + \frac{x \cdot \nabla_x K\left(\frac{x}{\mu_p}\right)}{K\left(\frac{x}{\mu_p}\right)} \right) K\left(\frac{x}{\mu_p}\right) v_p^{p+1}\, dx$$

exists and so does

$$\lim_{p \downarrow p_\ell} \mu_p^{-n} |\log \mu_p|^{-1} \int_{\boldsymbol{R}^n} \left(\ell + \frac{x \cdot \nabla_x K\left(\frac{x}{\mu_p}\right)}{K\left(\frac{x}{\mu_p}\right)} \right) K\left(\frac{x}{\mu_p}\right) v_p^{p+1}\, dx =: I > 0$$

if $q = n - \ell$.

Hence, if $q > n - \ell$, we get

$$\lim_{p \downarrow p_\ell} \frac{(n-2)(p+1) - 2(n-\ell)}{\|u_p\|_\infty^{(p-1)n/(2-\ell)}} = \frac{I_q}{c_0 \int_{\boldsymbol{R}^n} |x|^{-\ell} U^{2(n-\ell)/(n-2)}\, dx}$$

and if $q = n - \ell$, we have

$$\lim_{p \downarrow p_\ell} \frac{(2-\ell)\{(n-2)(p+1) - 2(n-\ell)\}}{(p-1)\|u_p\|_\infty^{(p-1)n/(2-\ell)} |\log \|u_p\|_\infty|} = \frac{\tilde{I}}{c_0 \int_{\boldsymbol{R}^n} |x|^{-\ell} U^{2(n-\ell)/(n-2)}\, dx}. \qquad \square$$

Remark 2.3. The difference of convergence rate is due to the integrability of $|x|^{-\ell-q}(v_p(x))^{p+1}$ in the right-hand side of (2.15) in $\boldsymbol{R}^n$.

3. Proofs of Theorems 1.4, 1.5 and 1.6.

In the spirit of Section 2, we consider the behavior of a least-energy solution as $p \uparrow (n+2)/(n-2)$. By Lemma 2.1, there exists a minimizer of (1.2) for any p sufficiently close to $(n+2)/(n-2)$. First we show the behavior of S_p as $p \uparrow (n+2)/(n-2)$.

Lemma 3.1. *$S_p \to K(0)^{-(n-2)/n} S$ as $p \uparrow (n+2)/(n-2)$.*

Proof. First of all, note that $\|K\|_\infty = K(0)$ by $x \cdot \nabla K(x) \le 0$ in $\boldsymbol{R}^n$. As in Lemma 2.4, we are going to prove

$$\limsup_{p \uparrow (n+2)/(n-2)} S_p \le K(0)^{-(n-2)/n} S \quad \text{and} \quad \liminf_{p \uparrow (n+2)/(n-2)} S_p \ge K(0)^{-(n-2)/n} S.$$

We show $\limsup_{p\uparrow(n+2)/(n-2)} S_p \le K(0)^{-(n-2)/n} S$ first. Let $\varepsilon := (n+2)/(n-2) - p$ and $\hat{w}_\varepsilon(x) := \varphi(|x|)(\varepsilon + |x|^2)^{-(n-2)/2}$ with $\varphi \in C_0^\infty(\boldsymbol{R}^n)$, $\varphi(0) = 1$, $\max_{\boldsymbol{R}^n} \varphi = 1$, $\operatorname{supp}\varphi \subset B_1$ and $\operatorname{supp}\varphi_r \subset B_1 \backslash B_{1/2}$. As in the proof of Lemma 2.5 of Pan and Wang [**18**] or Lemma 1.1 of Brezis and Nirenberg [**3**], we have

$$\|\nabla \hat{w}_\varepsilon\|_2^2 = M_1 \varepsilon^{-(n-2)/2} + O(1) \quad \text{with } M_1 := (n-2)^2 \int_{\boldsymbol{R}^n} \frac{|x|^2}{(1+|x|^2)^n} dx$$

as $\varepsilon \downarrow 0$.

As for $\int_{\boldsymbol{R}^n} K(x) \hat{w}_\varepsilon^{2n/(n-2)-\varepsilon}\, dx$, we get

$$\begin{aligned}
&\int_{\boldsymbol{R}^n} K(x) \hat{w}_\varepsilon^{2n/(n-2)-\varepsilon}\, dx \\
&\quad = \int_{\boldsymbol{R}^n} K(x) \varphi^{2n/(n-2)-\varepsilon} (\varepsilon + |x|^2)^{-n+(n-2)\varepsilon/2}\, dx \\
&\quad = \int_{\boldsymbol{R}^n} K(x) (\varepsilon + |x|^2)^{-n+(n-2)\varepsilon/2}\, dx \\
&\qquad + \int_{\boldsymbol{R}^n} K(x) (\varphi^{2n/(n-2)-\varepsilon} - 1)(\varepsilon + |x|^2)^{-n+(n-2)\varepsilon/2}\, dx \\
&\quad = \varepsilon^{-n/2+(n-2)\varepsilon/2} \int_{\boldsymbol{R}^n} K(\sqrt{\varepsilon} y)(1+|y|^2)^{-n+(n-2)\varepsilon/2}\, dy + O(1)
\end{aligned}$$

with $x = \sqrt{\varepsilon} y$ as $\varepsilon \downarrow 0$. Since $\max_{\boldsymbol{R}^n} K(x) = K(0)$ and since $(1+|y|^2)^{-n} \in L^1(\boldsymbol{R}^n)$, we see that

$$\int_{\boldsymbol{R}^n} K(0) \hat{w}_\varepsilon^{2n/(n-2)-\varepsilon}\, dx = M_2 K(0) \varepsilon^{-n/2} + O(1)$$

as $\varepsilon \downarrow 0$ in view of the Lebesgue dominant convergence theorem, where $M_2 = \int_{\boldsymbol{R}^n} (1+|y|^2)^{-n}\, dy$. Thus we have

$$\begin{aligned}
\tilde{Q}_\varepsilon(\hat{w}_\varepsilon) &= \frac{\displaystyle\int_{\boldsymbol{R}^n} |\nabla \hat{w}_\varepsilon|^2\, dx}{\left(\displaystyle\int_{\boldsymbol{R}^n} K(x) \hat{w}_\varepsilon^{2n/(n-2)-\varepsilon}\, dx\right)^{2(n-2)/\{2n-(n-2)\varepsilon\}}} \\
&= \frac{M_1 \varepsilon^{-(n-2)/2} + O(1)}{(M_2 K(0) \varepsilon^{-n/2} + O(1))^{2(n-2)/\{2n-(n-2)\varepsilon\}}} \\
&= \frac{M_1}{K(0)^{(n-2)/n} M_2^{(n-2)/n}} + o(1).
\end{aligned}$$

Hence we obtain

$$\limsup_{p\uparrow(n+2)/(n-2)} S_p \le K(0)^{-(n-2)/n} S$$

by $M_1 M_2^{-(n-2)/n} = S$ since the function $(1+|y|^2)^{-(n-2)/2}$ attains the best Sobolev constant S.

Next, we show $K(0)^{-(n-2)/n} S \le \liminf_{p\uparrow(n+2)/(n-2)} S_p$. Take $\tilde{w}_p \in \mathscr{D}$ which attains S_p. Then we have

$$\int_{\boldsymbol{R}^n} K(x)\tilde{w}_p^{p+1}\,dx$$

$$\le \left(\int_{\boldsymbol{R}^n} K(x)^{2n/(2n-(n-2)(p+1))}\,dx\right)^{(2n-(n-2)(p+1))/2n} \left(\int_{\boldsymbol{R}^n} \tilde{w}_p^{2n/(n-2)}\,dx\right)^{(n-2)(p+1)/2n}$$

by the Hölder inequality for $p \in (\max\{1,(n+2-2s)/(n-2)\},(n+2)/(n-2))$ (see the proof of Lemma 2.1). Hence we see that

$$(3.1)\qquad S \le \frac{\displaystyle\int_{\boldsymbol{R}^n} |\nabla \tilde{w}_p|^2\,dx}{\left(\displaystyle\int_{\boldsymbol{R}^n} \tilde{w}_p^{2n/(n-2)}\,dx\right)^{(n-2)/n}}$$

$$\le \frac{\left(\displaystyle\int_{\boldsymbol{R}^n} |\nabla \tilde{w}_p|^2\,dx\right)\left(\displaystyle\int_{\boldsymbol{R}^n} K(x)^{2n/(2n-(n-2)(p+1))}\,dx\right)^{(2n-(n-2)(p+1))/n(p+1)}}{\left(\displaystyle\int_{\boldsymbol{R}^n} K(x)\tilde{w}_p^{p+1}\,dx\right)^{2/(p+1)}}$$

$$= \left(\int_{\boldsymbol{R}^n} K(x)^{2n/(2n-(n-2)(p+1))}\,dx\right)^{(2n-(n-2)(p+1))/n(p+1)} S_p.$$

Fix $R>0$ sufficiently large so that $K(x) \le c_1|x|^{-s}$ on $\boldsymbol{R}^n \backslash B_R$. Then we get

$$\int_{\boldsymbol{R}^n} K(x)^{2n/\{2n-(n-2)(p+1)\}}\,dx$$

$$= \left(\int_{B_R} + \int_{\boldsymbol{R}^n\backslash B_R}\right) K(x)^{2n/\{2n-(n-2)(p+1)\}}\,dx$$

$$\le \frac{1}{n}(K(0))^{2n/(2n-(n-2)(p+1))}\omega_n R^n$$

$$+ \frac{\omega_n\{2n-(n-2)(p+1)\}}{n\{(n-2)(p+1)-2(n-s)\}}\left(\frac{c_1^2}{R^{(n-2)(p+1)-2(n-s)}}\right)^{n/(2n-(n-2)(p+1))}$$

since $K(0)$ is the maximum of $K(x)$. Thus we have

$$\left(\int_{\boldsymbol{R}^n} K(x)^{2n/\{2n-(n-2)(p+1)\}}\,dx\right)^{\{2n-(n-2)(p+1)\}/n(p+1)}$$

$$\le \left\{\frac{1}{n}(K(0))^{2n/\{2n-(n-2)(p+1)\}}\omega_n R^n + \frac{\omega_n\{2n-(n-2)(p+1)\}}{n\{(n-2)(p+1)-2(n-s)\}}\right.$$

$$\left.\times\left(\frac{c_1^2}{R^{(n-2)(p+1)-2(n-s)}}\right)^{n/\{2n-(n-2)(p+1)\}}\right\}^{\{2n-(n-2)(p+1)\}/n(p+1)}$$

$$\to K(0)^{(n-2)/n} \quad \text{as } p\uparrow (n+2)/(n-2)$$

since R can be taken large enough so that $c_1^2 < R^{(n-2)(p+1)-2(n-s)}$ for any p near $(n+2)/(n-2)$. Taking the limit in (3.1), we get

$$K(0)^{-(n-2)/n}S \le \liminf_{p\uparrow(n+2)/(n-2)} S_p.$$

Thus we obtain the desired equality. □

Next, we consider the behavior of $\|u_p\|_\infty$ as $p\uparrow(n+2)/(n-2)$. Unlike $p\downarrow p_\ell$, we have $\|u_p\|_\infty\to\infty$ as $p\uparrow(n+2)/(n-2)$.

Lemma 3.2. *Let u_p be a least-energy solution to* (1.1). *Then $\|u_p\|_\infty\to\infty$ and $u_p\to 0$ locally uniformly in $\boldsymbol{R}^n\backslash\{0\}$ as $p\uparrow(n+2)/(n-2)$.*

Proof. Suppose to the contrary that we can choose a subsequence $\{p_j\}$ ($p_j\uparrow(n+2)/(n-2)$ as $j\to\infty$) so that $\|u_{p_j}\|_\infty$ is uniformly bounded. Since $\|\nabla u_{p_j}\|_2$ is uniformly bounded, we can choose further a subsequence (still denoted by $\{p_j\}$) such that $u_{p_j}\rightharpoonup u_\infty$ weakly in $\mathscr{D}$ and $u_{p_j}\to u_\infty$ locally uniformly in $C^2(\boldsymbol{R}^n)$. By the Pohozaev identity (1.4), we have

$$\text{(3.2)}\qquad \frac{(n-2)(p_j+1)-2n}{2(p_j+1)}\int_{\boldsymbol{R}^n}|\nabla u_{p_j}|^2\,dx = \frac{1}{p_j+1}\int_{\boldsymbol{R}^n}(x\cdot\nabla K)u_{p_j}^{p_j+1}\,dx$$

in view of

$$\int_{\boldsymbol{R}^n}|\nabla u_{p_j}|^2\,dx = \int_{\boldsymbol{R}^n}K(x)u_{p_j}^{p_j+1}\,dx.$$

Since $\|\nabla u_{p_j}\|_2$ is uniformly bounded, we can take the limit as $j\to\infty$ in (3.2) to get

$$\text{(3.3)}\qquad \lim_{j\to\infty}\int_{\boldsymbol{R}^n}(x\cdot\nabla K)u_{p_j}^{p_j+1}\,dx = 0.$$

In view of $x \cdot \nabla K < 0$ in $\boldsymbol{R}^n \backslash \{0\}$, we see that $u_\infty \equiv 0$. Thus $u_{p_j} \to 0$ locally uniformly in $\boldsymbol{R}^n$.

By the Hölder inequality, we have

$$\begin{aligned}
&\int_{\boldsymbol{R}^n \backslash B_R} K(x) u_{p_j}^{p_j+1}\,dx \\
&\quad \le \left(\int_{\boldsymbol{R}^n \backslash B_R} K(x)^{2n/(2n-(n-2)(p_j+1))}\,dx \right)^{(2n-(n-2)(p_j+1))/2n} \\
&\quad\quad \times \left(\int_{\boldsymbol{R}^n \backslash B_R} u_{p_j}^{2n/(n-2)}\,dx \right)^{(n-2)(p_j+1)/2n} \\
&\quad \le \left[\frac{\omega_n \{2n-(n-2)(p_j+1)\}}{n\{(n-2)(p_j+1)-2(n-s)\}} \right]^{(2n-(n-2)(p_j+1))/2n} \left(\frac{c_1^2}{R^{(n-2)(p_j+1)-2(n-s)}} \right)^{1/2} \\
&\quad\quad \times \left(\int_{\boldsymbol{R}^n} u_{p_j}^{2n/(n-2)}\,dx \right)^{(n-2)(p_j+1)/2n} \\
&\quad \le 2c_1 R^{-s} \left(\int_{\boldsymbol{R}^n} u_{p_j}^{2n/(n-2)}\,dx \right)^{(n-2)(p_j+1)/2n}
\end{aligned}$$

for any sufficiently large j. Since the right-hand side of the above inequality is uniformly bounded by the Sobolev inequality, we can take $R > 0$ sufficiently large and $j_0 > 0$ for any given $\varepsilon > 0$ so that $\int_{\boldsymbol{R}^n \backslash B_R} K(x) u_{p_j}^{p_j+1}\,dx < \varepsilon$ holds for any $j > j_0$. As in (2.5), we get

$$S_{p_j}^{(p_j+1)/(p_j-1)} = \int_{\boldsymbol{R}^n} K(x) u_{p_j}^{p_j+1}\,dx = \left(\int_{B_R} + \int_{\boldsymbol{R}^n \backslash B_R} \right) K(x) u_{p_j}^{p_j+1}\,dx. \tag{3.4}$$

Since $u_j \to 0$ uniformly on B_R, letting $j \to \infty$ in (3.4), we obtain

$$K(0)^{-(n-2)/2} S^{n/2} < \varepsilon,$$

a contradiction. Thus $\|u_p\|_\infty \to \infty$ as $j \to \infty$. □

In view of $x \cdot \nabla K < 0$ in $\boldsymbol{R}^n \backslash \{0\}$ and (2.2), u_p blows up only at the origin. Next, as in the proof of Theorem 1.2, we rescale u_p.

Lemma 3.3. *Let u_p be a least-energy solution and define*

$$\nu_p^{-2/(p-1)} := \|u_p\|_\infty, \quad w_p(x) = \nu_p^{2/(p-1)} u_p(y), \quad y := \nu_p x.$$

Then $w_p(x)$ converges to

$$W(x) = \left(1 + \frac{K(0)}{n(n-2)}|x|^2\right)^{-(n-2)/2}$$

locally uniformly in $\boldsymbol{R}^n$ *as* $p \uparrow (n+2)/(n-2)$. *Moreover, the maximum point of* u_p *converges to* 0 *as* $p \uparrow (n+2)/(n-2)$.

Remark 3.1. As commented in Remark 1.3 (with $\ell = 0$), $W(x)$ is a unique solution to

$$\begin{cases} \Delta W + K(0) W^{(n+2)/(n-2)} = 0 & \text{in } \boldsymbol{R}^n, \\ W > 0 \quad \text{in } \boldsymbol{R}^n, W \to 0 & (|x| \to \infty), \\ W(0) = 1. \end{cases} \tag{3.5}$$

As in Remark 2.2 (with $\ell = 0$), we have $\|\nabla W\|_2^2 = K(0)^{-(n-2)/2} S^{n/2}$. Note that $v_p \to 0$ as $p \uparrow (n+2)/(n-2)$ by Lemma 3.2.

Proof. First note that u_p can blow up only at the origin. Thus the maximum point P_p tends to 0 as $p \uparrow (n+2)/(n-2)$. Since $u(y)$ with $y = v_p x$ is a solution to

$$\Delta_y u(y) + K(y) u(y)^p = 0,$$

we have

$$v_p^{-2-2/(p-1)} \Delta_x w_p(x) + v_p^{-2p/(p-1)} K(v_p x) w_p^p = 0,$$

i.e.,

$$\Delta_x w + K(v_p x) w^p = 0. \tag{3.6}$$

Since $\|w_p(0)\|_\infty$ is bounded, there exists a subsequence $\{p_j\}$ ($p_j \uparrow (n+2)/(n-2)$ as $j \to \infty$) such that w_{p_j} converges locally uniformly to $W(x)$, which is a solution to (3.5). However, as mentioned in the proof of Lemma 2.6, the uniqueness of solutions to (3.5) implies that w_p converges to W without extracting a subsequence. □

How does $\|u_p\|_\infty$ blow up? We answer this question.

Lemma 3.4. *Let* w_p *and* v_p *be defined as in Lemma* 3.3. *Then* w_p *satisfies* $w_p \to W$ *in* $\mathscr{D}$, $\|\nabla w_p\|_2^2 \to (K(0))^{-(n-2)/2} S^{n/2}$, $v_p^{-(n-2)+4/(p-1)} \to 1$ *and*

$$\int_{\boldsymbol{R}^n} K(v_p x) w_p^{p+1}\, dx \to K(0) \int_{\boldsymbol{R}^n} W^{2n/(n-2)}\, dx$$

as $p \uparrow (n+2)/(n-2)$.

Proof. As before, let $y = v_p x$. Then we have

$$\int_{\boldsymbol{R}^n} |\nabla_y u_p(y)|^2 \, dy = v_p^{n-2-4/(p-1)} \int_{\boldsymbol{R}^n} |\nabla_x w_p(x)|^2 \, dx.$$

As in the proof of Lemma 2.8, we get

$$\begin{aligned}
&(K(0))^{-(n-2)/2} S^{n/2} \\
&\quad = \|\nabla W\|_2^2 \le \liminf_{p \uparrow (n+2)/(n-2)} \int_{\boldsymbol{R}^n} |\nabla_x w_p(x)|^2 \, dx \\
&\quad \le \limsup_{p \uparrow (n+2)/(n-2)} \int_{\boldsymbol{R}^n} |\nabla_x w_p(x)|^2 \, dx \\
&\quad = \limsup_{p \uparrow (n+2)/(n-2)} v_p^{-(n-2)+4/(p-1)} \int_{\boldsymbol{R}^n} |\nabla_y u_p|^2 \, dy \le \limsup_{p \uparrow (n+2)/(n-2)} \int_{\boldsymbol{R}^n} |\nabla_y u_p|^2 \, dy
\end{aligned}$$

by noting Remark 2.2, $v_p \to 0$ as $p \uparrow (n+2)/(n-2)$ and $-(n-2) + 4/(p-1) > 0$. Since $u_p(y)$ is an unscaled least-energy solution to

$$\Delta_y u_p(y) + K(y) u_p^p(y) = 0 \quad \text{in } \boldsymbol{R}^n,$$

Lemma 3.1 implies that

$$\lim_{p \uparrow (n+2)/(n-2)} \int_{\boldsymbol{R}^n} |\nabla_y w_p(y)|^2 \, dy = (K(0))^{-(n-2)/2} S^{n/2}.$$

Thus, we have

$$\|\nabla w_p\|_2^2 \to (K(0))^{-(n-2)/2} S^{n/2}, \quad v_p^{-(n-2)+4/(p-1)} \to 1 \quad \text{as } p \uparrow (n+2)/(n-2).$$

In view of Lemma 3.3, w_p converges to W locally uniformly in $\boldsymbol{R}^n$, thus we see that w_p converges weakly to W in $\mathscr{D}$. Since the weak convergence together with the convergence of the corresponding norm implies the strong convergence, we see that w_p converges strongly to W.

Moreover, since

$$\int_{\boldsymbol{R}^n} |\nabla w_p|^2 \, dx = \int_{\boldsymbol{R}^n} K(v_p x) w_p^{p+1} \, dx,$$

since $w_p \to W$ in $\mathscr{D}$ and since

$$\int_{\boldsymbol{R}^n} |\nabla W|^2 \, dx = K(0) \int_{\boldsymbol{R}^n} W^{2n/(n-2)} \, dx,$$

we see that

$$\lim_{p \uparrow (n+2)/(n-2)} \int_{\boldsymbol{R}^n} K(v_p x) w_p^{p+1} \, dx = K(0) \int_{\boldsymbol{R}^n} W^{2n/(n-2)} \, dx. \qquad \square$$

PROOF OF THEOREM 1.4. Theorem 1.4 is obtained by Lemmas 3.1, 3.2 and 3.4. □

PROOF OF THEOREM 1.5. Theorem 1.5 is readily seen from Lemmas 3.3 and 3.4. □

To prove Theorem 1.6, we need to know the decay order of u_p.

LEMMA 3.5. *Suppose that $u_p \in \mathscr{D}$ is a positive solution to* (1.1) *with* (K.0), (K.2) *and p is sufficiently close to $(n+2)/(n-2)$. If $s > 2(n-2)/(n+2)$, then u_p decays at the rate $|x|^{-(n-2)}$ at infinity.*

PROOF. First we consider the case $s > 2$. Then by Theorem 2.9 of Li and Ni [**14**], we see that u_p decays at the rate $|x|^{-(n-2)}$ at infinity.

In the following, we consider the case $2(n-2)/(n+2) < s \le 2$. In this proof, various constants independent of x are denoted only by C. We use a modification of Lemma 2.3 of Li and Ni [**14**] (see also Theorem 2.25 of [**14**], Theorems 2.4, 2.8, 2.16 and 3.2 of Li and Ni [**15**], [**16**]). Using the Green function of $-\Delta$ on $\boldsymbol{R}^n$, we have

$$u_p(x) = C\int_{\boldsymbol{R}^n} \frac{K(y)u_p(y)^p}{|x-y|^{n-2}}\,dy.$$

By the Hölder inequality, we see that

$$\begin{aligned}
&\int_{\boldsymbol{R}^n} \frac{K(y)u_p(y)^p}{|x-y|^{n-2}}\,dy \\
&\quad \le \int_{|x-y|\le 1} \frac{K(y)u_p(y)^p}{|x-y|^{n-2}}\,dy \\
&\qquad + \left\{\int_{|x-y|\ge 1} \left(\frac{K(y)}{|x-y|^{n-2}}\right)^{2n/(2n-(n-2)p)} dy\right\}^{(2n-(n-2)p)/2n} \\
&\qquad \times \left(\int_{\boldsymbol{R}^n} u_p^{2n/(n-2)}\,dy\right)^{(n-2)p/2n} \\
&\quad < \infty
\end{aligned}$$

in view of $2n(n-2)/\{2n-(n-2)p\} > n$ for p sufficiently close to $(n+2)/(n-2)$. Since $\|\nabla u_p\|_2$ is finite in view of $u_p \in \mathscr{D}$, so is $\|u_p\|_{2n/(n-2)}$ due to the Sobolev inequality. Using this inequality, we derive the decay order of u_p using the technique in the proof of Lemma 2.3 of [**14**].

For sufficiently large $|x|$, since u_p is uniformly bounded for large $|x|$ by the elliptic regularity estimate, we see that

$$\int_{|x-y|\le 1} \frac{K(y)u_p(y)^p}{|x-y|^{n-2}}\,dy \le C|x|^{-s}$$

by assumption (K.2). As for the second term, we decompose as

$$\begin{aligned}
&\int_{|x-y|\ge 1} \left(\frac{K(y)}{|x-y|^{n-2}}\right)^{2n/(2n-(n-2)p)} dy \\
&= \left(\int_{1\le|x-y|\le|x|/2} + \int_{|x|/2\le|x-y|\le 2|x|} + \int_{|x-y|\ge 2|x|}\right)\left(\frac{K(y)}{|x-y|^{n-2}}\right)^{2n/(2n-(n-2)p)} dy \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

The first term yields

$$\begin{aligned}
I_1 &\le C|x|^{-2ns/(2n-(n-2)p)} \int_{1\le|x-y|\le|x|/2} |x-y|^{-2n(n-2)/(2n-(n-2)p)}\,dy \\
&\le C|x|^{-2ns/(2n-(n-2)p)} \int_1^{|x|/2} r^{n-1-2n(n-2)/(2n-(n-2)p)}\,dr \\
&\le C(|x|^{n\{2(2-s)-(n-2)p\}/(2n-(n-2)p)} + |x|^{-2ns/(2n-(n-2)p)})
\end{aligned}$$

in view of $|x|-|y| \le |x-y| \le |x|/2$ and the decay rate of $K(x)$.

In the similar fashion, we have

$$\begin{aligned}
I_3 &\le C|x|^{-2ns/(2n-(n-2)p)} \int_{|x-y|\ge 2|x|} \frac{1}{|x-y|^{2n(n-2)/(2n-(n-2)p)}}\,dy \\
&\le C|x|^{-2ns/(2n-(n-2)p)} \int_{2|x|}^{\infty} r^{n-1-2n(n-2)/(2n-(n-2)p)}\,dr \\
&= C|x|^{n\{2(2-s)-(n-2)p\}/(2n-(n-2)p)}
\end{aligned}$$

since the integrand is integrable at infinity as commented in the above.

Finally as for I_2, noting that $K(x) \in L^\infty(\boldsymbol{R}^n)$ by (K.2), we have

$$\begin{aligned}
I_2 &\le C|x|^{-2ns/(2n-(n-2)p)}\left(\int_{|y|\le 1} + \int_{1\le|y|\le 3|x|}\right)(K(y))^{2n/(2n-(n-2)p)}\,dy \\
&\le C|x|^{-2n(n-2)/(2n-(n-2)p)}(1 + |x|^{n-2ns/(2n-(n-2)p)}).
\end{aligned}$$

Combining these estimates, we obtain

$$u_p(x) \le C|x|^{-s} + C(I_1 + I_2 + I_3)^{(2n-(n-2)p)/2n}$$

i.e.,

$$u_p(x) \le C(|x|^{-s} + |x|^{-s+2-(n-2)p/2} + |x|^{-(n-2)}),$$

for sufficiently large $|x|$. Since we are concerned with the process $p \uparrow (n+2)/(n-2)$, we see that $2-(n-2)p/2 < 0$. If $s > (n-2)$ (this is possible for $n = 3$), we are done. Thus we assume $s < n-2$. Then we have

$$u_p(x) \le C|x|^{-s} \tag{3.7}$$

for sufficiently large $|x|$.

Again decomposing as

$$\begin{aligned} u(x) &= C\int_{\boldsymbol{R}^n} \frac{K(y)u(y)^p}{|x-y|^{n-2}}\,dy \\ &= C\left(\int_{|x-y|\le|x|/2} + \int_{|x|/2\le|x-y|\le 2|x|} + \int_{|x-y|\ge 2|x|}\right)\frac{K(y)u(y)^p}{|x-y|^{n-2}}\,dy, \end{aligned}$$

and using (3.7), we have

$$\int_{|x-y|\le|x|/2} \frac{K(y)u(y)^p}{|x-y|^{n-2}}\,dy \le C|x|^{2-(p+1)s}$$

and

$$\int_{|x|/2\le|x-y|\le 2|x|} \frac{K(y)u(y)^p}{|x-y|^{n-2}}\,dy \le C(|x|^{-(n-2)} + |x|^{2-(p+1)s})$$

in the similar way as above.

Since $|x-y|^{-(n-2)}$ is not integrable at infinity, the previous method is not applicable directly to the rest integral. We decompose

$$\int_{|x-y|\ge 2|x|} \frac{K(y)u_p(y)^p}{|x-y|^{n-2}}\,dy = \left(\int_{\substack{|x-y|\ge 2|x| \\ 2|x|\ge|y|\ge|x|}} + \int_{\substack{|x-y|\ge 2|x| \\ |y|\ge 2|x|}}\right)\frac{K(y)u_p(y)^p}{|x-y|^{n-2}}\,dy = I_4 + I_5.$$

Note that $|x-y| \ge 2|x|$ implies $|y| \ge |x|$. For sufficiently large $|x|$, $K(y)u_p(y)^p \le C|y|^{-(p+1)s}$ on $|x| \le |y| \le 2|x|$ by (3.7) and (K.2). Thus we have

$$\begin{aligned} I_4 &\le C\int_{\substack{|x-y|\ge 2|x| \\ 2|x|\ge|y|\ge|x|}} \frac{K(y)u_p(y)^p}{|x-y|^{n-2}}\,dy \\ &\le C|x|^{-(n-2)-(p+1)s}\int_{|x|\le|y|\le 2|x|} dy \le C|x|^{2-(p+1)s}. \end{aligned}$$

For I_5, we use the elementary inequality $|x-y| \geq ||y|-|x||$. For $|y| \geq 2|x|$, we have $|x-y| \geq |y|/2$. Thus we have

$$I_5 \leq C \int_{\substack{|x-y| \geq 2|x| \\ |y| \geq 2|x|}} \frac{|y|^{-(p+1)s}}{|y|^{n-2}} dy \leq C|x|^{2-(p+1)s},$$

in view of $1-(p+1)s < -1$ if p is sufficiently close to $(n+2)/(n-2)$ with $2(n-2)/(n+2) < s$. Hence we obtain

$$u_p(x) \leq C(|x|^{-(n-2)} + |x|^{2-(p+1)s})$$

for sufficiently large $|x|$.

If $2-(p+1)s \leq -(n-2)$, then we are done. If $2-(p+1)s > -(n-2)$, then we need to check that this process gains the decay rate. If $2-(p+1)s < -s$ i.e., $2/p < s$, then we obtain a better decay rate. Since our assumption is $2(n-2)/(n+2) < s$, $2/p < s$ is assured for any p sufficiently close to $(n+2)/(n-2)$. Thus we obtain

$$u_p(x) \leq C|x|^{2-(p+1)s}.$$

Repeating this arguments in finitely many times, we have the desired decay rate. Indeed, let $a_1 = 2-(p+1)s$ and $a_{n+1} = pa_n + 2 - s$ (this relation represents the decay order by this deduction). Then we get

$$a_n = p^{n-1}\left(a_1 + \frac{2-s}{p-1}\right) - \frac{2-s}{p-1}.$$

Since $a_1 + (2-s)/(p-1) = p(2-ps)/(p-1) < 0$ by $2/p < s$, we see $a_n \to -\infty$ as $n \to \infty$. Thus the repeat of this process is ensured. □

Now we are in a position to prove Theorem 1.6.

Proof of Theorem 1.6. Let v_p, w_p and y be defined in Lemma 3.3. Since $u_p(y)$ is a least-energy solution to (1.1), we have

$$\int_{\mathbf{R}^n} |\nabla u_p|^2 \, dy = \int_{\mathbf{R}^n} K(y) u_p^{p+1} \, dy.$$

Thus, the Pohozaev identity (1.4) yields

$$\{(n-2)(p+1) - 2n\} \int_{\mathbf{R}^n} K(y) u_p^{p+1} \, dy = 2 \int_{\mathbf{R}^n} (y \cdot \nabla_y K(y)) u_p^{p+1} \, dy.$$

Expressing the equality in terms of w_p, we get

$$\{(n-2)(p+1)-2n\}v_p^{n-2(p+1)/(p-1)}\int_{\boldsymbol{R}^n} K(v_p x)w_p^{p+1}\,dx$$
$$= 2v_p^{n-2(p+1)/(p-1)}\int_{\boldsymbol{R}^n}(x\cdot\nabla_x K(v_p x))w_p^{p+1}\,dx.$$

i.e.,

$$\{(n-2)(p+1)-2n\}\int_{\boldsymbol{R}^n} K(v_p x)w_p^{p+1}\,dx = 2\int_{\boldsymbol{R}^n}(x\cdot\nabla_x K(v_p x))w_p^{p+1}\,dx. \tag{3.8}$$

As we have seen in the proof of Lemma 3.4, we have

$$\lim_{p\uparrow(n+2)/(n-2)}\int_{\boldsymbol{R}^n} K(v_p x)w_p^{p+1}\,dx = K(0)\int_{\boldsymbol{R}^n} W^{2n/(n-2)}\,dx.$$

We also note that

$$\lim_{p\uparrow(n+2)/(n-2)}\frac{x\cdot\nabla_x K(v_p x)}{v_p^m} = -c_4|x|^m$$

locally uniformly in $\boldsymbol{R}^n$ by (K.5).

If $0<m<n$, then $|x|^m W^{2n/(n-2)}\in L^1(\boldsymbol{R}^n)$. Thus we can take an $L^1(\boldsymbol{R}^n)$ function (decaying at the rate $|x|^{m-2n+\varepsilon}$ with $\varepsilon>0$ sufficiently small so that $m-2n+\varepsilon<-n$) which is uniformly bigger than $v_p^{-m}x\cdot\nabla K(v_p x)w_p^{p+1}$, since w_p decays like $|x|^{-(n-2)}$ at infinity by applying Lemma 3.5 to (3.6) and the convergence of w_p to W in $\mathscr{D}$ by Lemma 3.4. Thus, from the Lebesgue dominant convergence theorem, we see that

$$\lim_{p\uparrow(n+2)/(n-2)}\frac{1}{v_p^m}\int_{\boldsymbol{R}^n}(x\cdot\nabla K(v_p x))w_p^{p+1}\,dx = -c_4\int_{\boldsymbol{R}^n}|x|^m W^{2n/(n-2)}\,dx.$$

Hence, by (3.8), we get

$$\lim_{p\uparrow(n+2)/(n-2)}\{(n-2)(p+1)-2n\}\,v_p^{-m} = -\frac{2c_4\int_{\boldsymbol{R}^n}|x|^m W^{2n/(n-2)}\,dx}{K(0)\int_{\boldsymbol{R}^n} W^{2n/(n-2)}\,dx}.$$

Since $v_p=\|u_p\|_\infty^{-(p-1)/2}$, we obtain

$$\lim_{p\uparrow(n+2)/(n-2)}\{(n-2)(p+1)-2n\}\|u_p\|_\infty^{(p-1)m/2} = -\frac{2c_4\int_{\boldsymbol{R}^n}|x|^m W^{2n/(n-2)}\,dx}{K(0)\int_{\boldsymbol{R}^n} W^{2n/(n-2)}\,dx}.$$

If $m \geq n$, then we see $|x|^m W^{2n/(n-2)} \notin L^1(\boldsymbol{R}^n)$. So we decompose

$$\int_{\boldsymbol{R}^n} (x \cdot \nabla_x K(v_p x)) w_p^{p+1}\, dx \tag{3.9}$$
$$= \left(\int_{B_{v_p^{-1} r_1}} + \int_{\boldsymbol{R}^n \setminus B_{v_p^{-1} r_1}} \right) (x \cdot \nabla_x K(v_p x)) w_p^{p+1}\, dx$$

as in the proof of Theorem 1.3. As for the first term of (3.9), we have

$$\int_{B_{v_p^{-1} r_1}} (x \cdot \nabla_x K(v_p x)) w_p(x)^{p+1}\, dx = -c_5 v_p^m \int_{B_{v_p^{-1} r_1}} |x|^m w_p(x)^{p+1}\, dx,$$

by (K.6). Note that

$$\int_{B_{v_p^{-1} r_1}} |x|^m w_p(x)^{p+1}\, dx = v_p^{-m-n} \int_{B_{r_1}} |y|^m w_p \left(\frac{y}{v_p} \right)^{p+1} dy$$

with $y = v_p x$. In view of the decay order of w_p ($\sim |x|^{-(n-2)}$ at infinity by Lemma 3.5) and the convergence property of w_p to W, we see that

$$\lim_{p \uparrow (n+2)/(n-2)} v_p^{-(n-2)(p+1)} \int_{B_{r_1}} |y|^m w_p \left(\frac{y}{v_p} \right)^{p+1} dy > 0$$

exists. Thus

$$\lim_{p \uparrow (n+2)/(n-2)} v_p^{m+2-(n-2)p} \int_{B_{v_p^{-1} r_1}} |x|^m w_p(x)^{p+1}\, dx$$

exists, i.e.,

$$\lim_{p \uparrow (n+2)/(n-2)} v_p^{2-(n-2)p} \int_{B_{v_p^{-1} r_1}} (x \cdot \nabla_x K(v_p x)) w_p(x)^{p+1}\, dx \tag{3.10}$$

exists.

For the second term of (3.9), we again set $y = v_p x$. Hence we get

$$\int_{\boldsymbol{R}^n \setminus B_{v_p^{-1} r_1}} (x \cdot \nabla_x K(v_p x)) w_p(x)^{p+1}\, dx = v_p^{-n} \int_{\boldsymbol{R}^n \setminus B_{r_1}} (y \cdot \nabla_y K(y)) \left(w_p \left(\frac{y}{v_p} \right) \right)^{p+1} dy.$$

As in the same reasoning in the above (the decay property of w_p), in view of y $(= v_p x) \in \boldsymbol{R}^n \setminus B_{r_1}$, we see

$$\lim_{p \uparrow (n+2)/(n-2)} v_p^{-(n-2)(p+1)} \int_{\boldsymbol{R}^n \setminus B_{r_1}} (y \cdot \nabla_y K(y)) w_p^{p+1} \left(\frac{y}{v_p} \right) dy$$

exists, i.e.,

$$\lim_{p\uparrow(n+2)/(n-2)} v_p^{2-(n-2)p} \int_{\boldsymbol{R}^n\setminus B_{v_p^{-1}r_1}} (x\cdot\nabla_x K(v_p x)) w_p(x)^{p+1}\,dx \tag{3.11}$$

exists.

By Lemma 3.4, $v_p^{-(n-2)p+(n+2)} \to 1$ as $p\uparrow(n+2)/(n-2)$. Hence, combining (3.10) and (3.11), we see that

$$I_1 := 2\lim_{p\uparrow(n+2)/(n-2)} v_p^{-n}\int_{\boldsymbol{R}^n}(x\cdot\nabla_x K(v_p x))w_p^{p+1}\,dx$$

exists and negative. Thus we obtain

$$\lim_{p\uparrow(n+2)/(n-2)}\{(n-2)(p+1)-2n\}\|u_p\|_\infty^{(p-1)n/2} = \frac{I_1}{K(0)\int_{\boldsymbol{R}^n} W^{2n/(n-2)}\,dx}$$

by (3.8). □

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