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Classification of flat slant surfaces in complex Euclidean plane

In memory of Professor Seiichi Yamaguchi

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Abstract. It is well-known that the classification of flat surfaces in Euclidean 3-space is one of the most basic results in differential geometry. For surfaces in the complex Euclidean plane $\boldsymbol{C}^2$ endowed with almost complex structure J, flat surfaces are the simplest ones from intrinsic point of views. On the other hand, from J-action point of views, the most natural surfaces in $\boldsymbol{C}^2$ are slant surfaces, i.e., surfaces with constant Wintinger angle. In this paper the author completely classifies flat slant surfaces in $\boldsymbol{C}^2$. The main result states that, beside the totally geodesic ones, there are five large classes of flat slant surfaces in $\boldsymbol{C}^2$. Conversely, every non-totally geodesic flat slant surfaces in $\boldsymbol{C}^2$ is locally a surface given by these five classes.

1. Introduction.

Let M be an n-dimensional Riemannian manifold isometrically immersed in a Kählerian manifold $(\tilde{M}, g, J)$ endowed with Kähler metric g and almost complex structure J. For each vector X tangent to M, we put

$$JX = PX + FX, \tag{1.1}$$

where PX and FX are the tangential and normal components of JX. Then P is an endomorphism of the tangent bundle TM. For any nonzero vector X tangent to M at a point p, the angle $\theta(X)$, $0 \le \theta(X) \le \pi/2$, between JX and the tangent space T_pM is called the *Wirtinger angle* of X. The submanifold M is called *slant* if its Wirtinger angle θ is constant, *i.e.*, $\theta(X)$ is independent of the choice of the X in the tangent bundle TM. The Wirtinger angle θ of a slant immersion is called the *slant angle*. A slant submanifold with slant angle θ is simply called *θ-slant*. Slant submanifolds of a Kählerian manifold are characterized by the condition $P^2 = cI$ for some real number $c \in [-1, 0]$. Complex and totally real immersions are slant immersions with slant angle 0 and $\pi/2$, respectively. A slant immersion is called *proper slant* if it is neither complex nor totally real. A totally real immersion $f : M \to \tilde{M}$ is called Lagrangian if $\dim_{\boldsymbol{R}} M = \dim_{\boldsymbol{C}} \tilde{M}$.

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There exist ample examples of proper slant submanifolds in complex-space-forms (see, for examples, [**1**], [**4**]–[**6**], [**9**]).

When M is an oriented surface in a Kählerian manifold $\tilde{M}$, one also has the notion of *Kähler angle* α defined by $\alpha = \cos^{-1}(\langle JX, Y\rangle) \in [0, \pi]$, where $\{X, Y\}$ is a local positive orthonormal frame field on M. The Kähler angle α and the Wirtinger angle θ of an oriented surface M are related by $\theta(p) = \min\{\alpha(p), \pi - \alpha(a)\}$. In this sense, an oriented surface in a Kählerian manifold is slant if and only if it has constant Kähler angle.

From J-action point of views, slant submanifolds are the simplest and the most natural submanifolds of a Kählerian manifold. Slant submanifolds arise naturally and play some important roles in the studies of submanifolds of Kählerian manifolds. For example, K. Kenmotsu and D. Zhou proved in [**7**] that every surface in a complex space form $\tilde{M}^2(4c)$ is proper slant if it has constant curvature and nonzero parallel mean curvature vector.

Flat surfaces in Euclidean 3-space $\boldsymbol{E}^3$ are the simplest surfaces from intrinsic point of views. The classification theorem of flat surfaces in $\boldsymbol{E}^3$ is one of most basic results in differential geometry (see, for instance [**8**]). For surfaces in the complex Euclidean plane $\boldsymbol{C}^2$, flat surfaces are also the simplest ones from intrinsic point of views. On the other hand, from J-action point of views, the most simplest surfaces in $\boldsymbol{C}^2$ are slant surfaces.

The purpose of this paper is thus to classify flat slant surfaces in the complex Euclidean plane. In section 2 we recall some basic facts, lemmas, and the existence and uniqueness theorems of slant submanifolds. In section 3, we prove the main theorem which states that, beside the totally geodesic ones, there are five large classes of flat slant surfaces in $\boldsymbol{C}^2$. Conversely, every non-totally geodesic flat slant surfaces in $\boldsymbol{C}^2$ is locally a surface given by these five classes. Class V of flat slant surfaces is related with the solutions of certain wave equation and of certain second order ordinary differential equation with two prescribed conditions. In section 4, we prove that the second order differential equation with the prescribed conditions alway has solutions. This existence result implies that class V of flat slant surfaces is indeed very large. In this section we also prove that, for any nonzero function λ of one variable, there exists a θ-Legendre curve in S^3 whose curvature in S^3 is given by $\lambda \csc\theta$. In the last section we provide explicit examples of flat slant surfaces of class V.

2. Basic formulas and existence theorem.

Let $x : M \to \tilde{M}^m$ be an isometric immersion of a Riemannian n-manifold into a Kählerian m-manifold. Denote by R and $\tilde{R}$ the Riemann curvature tensors of M and $\tilde{M}^m$, respectively. Denote by h and A the second fundamental form and the shape operator of the immersion x; and by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}^m$, respectively. The second fundamental form h and the shape operator A are related by $\langle A_\xi X, Y\rangle = \langle h(X, Y), \xi\rangle$.

The well-known *equation of Gauss* is given by

$$\tag{2.1} \begin{aligned}\tilde{R}(X, Y; Z, W) = R(X, Y; Z, W) &+ \langle h(X, Z), h(Y, W)\rangle \\ &- \langle h(X, W), h(Y, Z)\rangle,\end{aligned}$$

for X, Y, Z, W tangent to M and ξ, η normal to M.

For the second fundamental form h, we define its covariant derivative $\overline{\nabla} h$ with respect to the connection on $TM \oplus T^\perp M$ by

$$\tag{2.2} (\overline{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The *equation of Codazzi* is

$$\tag{2.3} (\tilde{R}(X, Y)Z)^\perp = (\overline{\nabla}_X h)(Y, Z) - (\overline{\nabla}_Y h)(X, Z),$$

where $(\tilde{R}(X, Y)Z)^\perp$ denotes the normal component of $\tilde{R}(X, Y)Z$.

For an endomorphism Q on the tangent bundle of the submanifold, we define its covariant derivative ∇Q by

$$\tag{2.4} (\nabla_X Q)Y = \nabla_X(QY) - Q(\nabla_X Y).$$

For a θ-slant submanifold M in a Kählerian n-manifold $\tilde{M}^n$, we have [**1**]

$$\tag{2.5} P^2 = -(\cos^2\theta)I, \quad \langle PX, Y\rangle + \langle X, PY\rangle = 0,$$

$$\tag{2.6} (\nabla_X P)Y = th(X, Y) + A_{FY}X,$$

$$\tag{2.7} D_X(FY) - F(\nabla_X Y) = fh(X, Y) - h(X, PY),$$

where I denotes the identity map and $th(X, Y)$ and $fh(X, Y)$ are the tangential and normal components of $Jh(X, Y)$.

If we define a symmetric bilinear TM-valued form α on M by

$$\tag{2.8} \alpha(X, Y) = th(X, Y),$$

then we obtain

$$\tag{2.9} h(X, Y) = \csc^2\theta(P\alpha(X, Y) - J\alpha(X, Y)).$$

For an n-dimensional θ-slant submanifold in $\boldsymbol{C}^n$ with $\theta \neq 0$, the equations of Gauss and Codazzi become

$$\tag{2.10} R(X, Y; Z, W) = \csc^2\theta\{\langle\alpha(X, W), \alpha(Y, Z)\rangle - \langle\alpha(X, Z), \alpha(Y, W)\rangle\}$$

$$(2.11)\quad (\nabla_X \alpha)(Y,Z) + \csc^2\theta\{P\alpha(X,\alpha(Y,Z)) + \alpha(X,P\alpha(Y,Z))\}$$
$$= (\nabla_Y \alpha)(X,Z) + \csc^2\theta\{P\alpha(Y,\alpha(X,Z)) + \alpha(Y,P\alpha(X,Z))\}.$$

We recall the following Existence Theorem from [**6**] for later use.

Existence Theorem. *Let $\theta \in (0,\pi/2]$. Suppose there exist an endomorphism P on the tangent bundle TM and a symmetric bilinear TM-valued form α on M such that*

$$P^2 = -(\cos^2\theta)I, \tag{2.12}$$

$$\langle PX, Y\rangle + \langle X, PY\rangle = 0, \tag{2.13}$$

$$\langle(\nabla_X P)Y, Z\rangle = \langle\alpha(X,Y),Z\rangle - \langle\alpha(X,Z),Y\rangle, \tag{2.14}$$

$$R(X,Y;Z,W) = \csc^2\theta\{\langle\alpha(X,W),\alpha(Y,Z)\rangle - \langle\alpha(X,Z),\alpha(Y,W)\rangle\}, \tag{2.15}$$

for $X, Y, Z, W \in TM$, and

$$(\nabla_X \alpha)(Y,Z) + \csc^2\theta\{P\alpha(X,\alpha(Y,Z)) + \alpha(X,P\alpha(Y,Z))\} \tag{2.16}$$

is totally symmetric. Then there exists a θ-slant isometric immersion from M into $\boldsymbol{C}^n$ whose second fundamental form is given by

$$h(X,Y) = \csc^2\theta(P\alpha(X,Y) - J\alpha(X,Y)). \tag{2.17}$$

Let M be a proper θ-slant surface in a Kählerian surface $\tilde{M}^2$. For a given unit tangent vector field e_1 of M, we choose a canonical orthonormal frame $\{e_1,e_2,e_3,e_4\}$ defined by

$$e_2 = (\sec\theta)Pe_1, \quad e_3 = (\csc\theta)Fe_1, \quad e_4 = (\csc\theta)Fe_2. \tag{2.18}$$

We call such an orthonormal frame an *adapted frame*.

We need the following lemmas

Lemma 2.1. *Let M be a slant surface in a Kähler surface with slant angle $\theta \in (0,\pi/2]$. Then, with respect to an adapted frame, we have*

$$\omega_3^4 - \omega_1^2 = -\cot\theta\{(\mathrm{trace}\,A_3)\omega^1 + (\mathrm{trace}\,A_4)\omega^2\}, \tag{2.19}$$

where A_3, A_4 are the shape operators with respect to e_3, e_4 and $\{\omega^1,\omega^2\}$ is the dual frame of $\{e_1,e_2\}$.

This lemma can be found in [**1**, p. 29].

Lemma 2.2. *A proper slant surface in $\boldsymbol{C}^2$ is flat if and only if its normal connection is flat.*

Proof. Follows from the fact that every proper slant surface in $\boldsymbol{C}^2$ satisfies $A_{FX}Y = A_{FY}X$ (cf. [**1**, p. 24]). □

Lemma 2.3. *If M is a flat slant surface in $\boldsymbol{C}^2$ with slant angle $\theta \in (0, \pi/2]$, then there exists an adapted slant frame e_1, e_2, e_3, e_4 such that the second fundamental form of M in $\boldsymbol{C}^2$ takes the form*:

$$h(e_1, e_1) = \mu e_3, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = \varphi e_4, \tag{2.20}$$

for some functions μ, φ.

Proof. Since M is a flat slant surface in $\boldsymbol{C}^2$ with slant angle $\theta \in (0, \pi/2]$, Lemma 2.2 implies that M has flat normal connection. Thus the shape operators of M are simultaneous diagonalizable. Hence there exists an orthonormal basis e_1, e_2 such that A_{Fe_1}, A_{Fe_2} are diagonalized with respect to e_1, e_2. Therefore, by applying $A_{FX}Y = F_{FY}X$, we conclude that the second fundamental form takes the form of (2.20). □

We also need the following.

Definition 2.1. Let S^3 denote the unit hypersphere in $\boldsymbol{C}^2$ centered at the origin. Then S^3 admits a canonical Sasakian structure with structure vector field $\xi = iz$, $z \in S^3$. A unit speed curve $z : I \to S^3 \subset \boldsymbol{C}^2$ defined over an open interval I is called a *θ-Legendre curve* if

$$\langle iz(s), z'(s) \rangle = \cos\theta \tag{2.21}$$

for some constant angle θ. A θ-Legendre curve with $\theta = \pi/2$ is known as a *Legendre curve*. θ-Legendre curves in S^3 are also known as generalized helices in S^3 (cf. [**4**]).

Lemma 2.4. *A unit speed curve $z : I \to S^3 \subset \boldsymbol{C}^2$ is a θ-Legendre curve with nonzero curvature in S^3 if and only if z satisfies the second order differential equation*:

$$z''(s) - (\csc\theta)\lambda(s) i z'(s) + (1 - (\cot\theta)\lambda(s)) z(s) = 0 \tag{2.22}$$

for some nonzero function λ over I.

Proof. If z is a (unit speed) θ-Legendre curve in S^3, we have $\langle z, z \rangle = \langle z', z' \rangle = 1$ and $\langle iz, z' \rangle = \cos\theta$. By taking differentiation of these equations we find

$$\langle z, z' \rangle = \langle z', z'' \rangle = \langle iz, z'' \rangle = 0, \quad \langle z'', z \rangle = -1. \tag{2.23}$$

If we put $z'' = a_1 z + a_2 iz + a_3 z' + a_4 iz'$, then by taking the scalar product of z'' with iz and z', we obtain $a_2 + a_3\cos\theta = 0$ and $a_3 + a_2\cos\theta = 0$ respectively.

Therefore we obtain $a_2 = a_3 = 0$. Thus, $z'' = a_1 z + a_4 i z'$. Also, by taking the inner product of z'' with z and by applying (2.23), we obtain $a_1 = a_4 \cos\theta - 1$.

If $a_4 = 0$, then $a_1 = -1$. Hence $z'' + z = 0$. In this case, z is an open part of a great circle of S^3 which is impossible by our assumption. Therefore, if we put $a_4 = \lambda \csc\theta$, we obtain $a_1 = \lambda \cot\theta - 1$ which implies (2.22).

Conversely, if $z = z(s)$ is a unit speed curve in S^3 satisfying (2.22), then by taking the inner product of (2.22) with z and by applying $\langle z'', z\rangle = -1$, we obtain $\langle iz, z'\rangle = \cos\theta$. Thus, z is a θ-Legendre curve. Moreover, from (2.22) we see that the curvature of z in S^3 is nonzero. $\square$

EXAMPLES 2.1. For any $\theta \in (0, \pi/2]$, there exist many θ-Legendre curves with nonzero curvature in S^3 (see Theorem 4.2). For examples, for any constant $\lambda \neq 0$, the map

$$
\begin{aligned}
z(s) = e^{(i\lambda\csc\theta)s/2}\Bigg(&\cos\left(\frac{s}{2}\sqrt{\lambda^2 + (2-\lambda\cot\theta)^2}\right) \\
&+ i\frac{(2\cos\theta - \lambda\csc\theta)}{\sqrt{\lambda^2 + (2-\lambda\cot\theta)^2}}\sin\left(\frac{s}{2}\sqrt{\lambda^2 + (2-\lambda\cot\theta)^2}\right), \\
&\frac{2\sin\theta}{\sqrt{\lambda^2 + (2-\lambda\cot\theta)^2}}\sin\left(\frac{s}{2}\sqrt{\lambda^2 + (2-\lambda\cot\theta)^2}\right)\Bigg)
\end{aligned}
\tag{2.24}
$$

defines a unit speed θ-Legendre curve with nonzero curvature in S^3.

3. The main theorem.

The main result of this paper is the following classification theorem for flat slant surfaces in $\boldsymbol{C}^2$.

THEOREM 3.1. *We have the following.*

(I) *Let $\theta \in (0, \pi/2]$. Then, for any θ-Legendre curve $z : I \to S^3 \subset \boldsymbol{C}^2$ defined on an open interval I and any function β of one variable defined on I, the map*

$$
\boldsymbol{x}(x, y) = z(y)x + \int_0^y \beta(u) z'(u)\, du. \tag{3.1}
$$

defines a flat θ-slant surface in $\boldsymbol{C}^2$.

(II) *Let $\theta \in (0, \pi/2]$. Then, for any given nonzero function $\varphi = \varphi(y)$ of one variable defined on an open interval I containing 0, the map*

$$
\boldsymbol{x}(x, y) = \left(x + i\cos\theta \int_0^y \frac{dt}{\varphi(t)}, \sin\theta \int_0^y \frac{e^{i(\csc\theta)t}}{\varphi(t)}\, dt\right) \tag{3.2}
$$

defines a flat θ-slant surface in $\boldsymbol{C}^2$.

(III) *If* $\mu(x) \neq 0$ *and* $k(y)$ *are two functions of one variable defined on some open intervals containing* 0 *such that* $\psi = k(y) + \cot\theta \int_0^x dx/\mu \neq 0$ *for some* $\theta \in (0, \pi/2]$, *then the map*

$$x(x, y) = \left(i \int_0^y k(y) e^{iy\csc\theta} dy + e^{iy\csc\theta} \cos\theta \int_0^x \frac{dx}{\mu(x)}, \sin\theta \int_0^x \frac{e^{ix\csc\theta}}{\mu(x)} dx \right) \tag{3.3}$$

defines a flat θ*-slant surface in* $\boldsymbol{C}^2$.

(IV) *If* $\theta \in (0, \pi/2)$ *and* $u(x), v(y)$ *are two functions of one variable defined on some open intervals containing* 0 *with* $v(0) \neq 0$ *and* $v'(y) \neq 0$, *then the map*

$$\begin{aligned} x(x, y) = \Bigg(&\int_0^x u(x) e^{ix\csc\theta} dx - iv(y) e^{ix\csc\theta} \sin\theta + iv(0)\sin\theta, \\ &\sin\theta\tan\theta \int_0^y v'(y) e^{iy\csc\theta} dy \Bigg) \end{aligned} \tag{3.4}$$

defines a flat θ*-slant surface in* $\boldsymbol{C}^2$.

(V) *Suppose* f *is a nonzero function of one variable defined on an open interval containing* 0 *such that* $f \neq \cot\theta$ *for some* $\theta \in (0, \pi/2]$ *and suppose* $\rho = \rho(x, y)$ *with* $\partial\rho/\partial y \neq 0$ *is a solution of the wave equation*:

$$\rho_{xy} - \frac{f'(x-y)}{f(x-y)} \rho_y + \{(\cot\theta) f(x-y) - f^2(x-y)\}\rho = 0 \tag{3.5}$$

and K *is a* $\boldsymbol{C}^2$*-valued solution of the ordinary differential equation*:

$$K''(u) + \left(i\csc\theta - \frac{f'(u)}{f(u) - \cot\theta} \right) K'(u) + f(u)(f(u) - \cot\theta) K(u) = 0, \tag{3.6}$$

satisfying $|K|^2 = 1$ *and* $|K'|^2 = (f - \cot\theta)^2$, *then*

$$\begin{aligned} x(x, y) = &\int_0^y \left\{ \frac{\rho_y K'(x-y)}{(\cot\theta - f) f} e^{ix\csc\theta} - \int_0^x (\rho K(x-y))_y e^{ix\csc\theta} dx \right\} dy \\ &+ \int_0^x \rho K(x-y) e^{ix\csc\theta} dx, \quad f = f(x-y) \end{aligned} \tag{3.7}$$

defines a flat θ*-slant surface in* $\boldsymbol{C}^2$.

(VI) *Conversely, locally every flat slant surface in* $\boldsymbol{C}^2$ *is either an open part of a slant plane or, up to rigid motions of* $\boldsymbol{C}^2$, *a surface given by one of the five classes of slant immersions defined by* (3.1), (3.2), (3.3), (3.4) *and* (3.7).

Proof. (I) Let $z : I \to S^3 \subset \boldsymbol{C}^2$ be a (unit-speed) θ-Legendre curve and β a nonzero function of one variable defined on I. Consider the map defined by (3.1). Then

$$\boldsymbol{x}_x(x, y) = z(y), \quad \boldsymbol{x}_y(x, y) = (x + \beta(y))z'. \tag{3.8}$$

Hence, $\boldsymbol{x}$ defines an isometric immersion from the surface M with metric

$$g = dx^2 + (x + \beta(y))^2\, dy^2 \tag{3.9}$$

into $\boldsymbol{C}^2$. Then M is a flat surface. If we put $e_1 = z(y)$ and $e_2 = z'(y)$, we obtain $\langle ie_1, e_2\rangle = \cos\theta$ due to the fact that z is a θ-Legendre curve. Thus $\boldsymbol{x} : M \to \boldsymbol{C}^2$ is an isometric θ-slant immersion.

(II) For any given nonzero function $\varphi = \varphi(y)$ of one variable defined on an open interval I containing 0, consider the map $\boldsymbol{x}$ defined by (3.2). Then we have

$$\boldsymbol{x}_x = (1, 0), \quad \boldsymbol{x}_y = \frac{1}{\varphi}(i\cos\theta, e^{i(\csc\theta)y}\sin\theta). \tag{3.10}$$

Let M denote the surface endowed with metric

$$g = dx^2 + \frac{1}{\varphi^2(y)}\, dy^2. \tag{3.11}$$

Then M is a flat surface and $\boldsymbol{x}$ defines an isometric θ-slant immersion from M into $\boldsymbol{C}^2$.

(III) Let $\mu(x) \neq 0$ and $k(y)$ be two functions of one variable defined on open intervals containing 0 such that $\psi = k(y) + \cot\theta \int_0^x dx/\mu \neq 0$. Consider the map $\boldsymbol{x}$ defined by (3.3). Then we have

$$\boldsymbol{x}_x = \left(\frac{\cos\theta}{\mu(x)} e^{iy\csc\theta}, \frac{\sin\theta}{\mu(x)} e^{ix\csc\theta}\right), \quad \boldsymbol{x}_y = (ie^{iy\csc\theta}\psi, 0). \tag{3.12}$$

Let M denote the surface endowed with metric

$$g = \frac{dx^2}{\mu^2(x)} + \psi^2\, dy^2. \tag{3.13}$$

Then M is a flat surface and $\boldsymbol{x}$ is an isometric θ-slant immersion from M into $\boldsymbol{C}^2$.

(IV) Let $\theta \in (0, \pi/2)$ and $u(x), v(y)$ be two functions of one variable defined on open intervals containing 0 with $v(0) \neq 0$ and $v'(y) \neq 0$. Consider the map $\boldsymbol{x}$ defined by (3.4). Then

$$\begin{aligned} \boldsymbol{x}_x &= ((u(x) + v(y))e^{ix\csc\theta}, 0), \\ \boldsymbol{x}_y &= (-iv'(y)\sin\theta e^{ix\csc\theta}, v'(y)\sin\theta\tan\theta e^{iy\csc\theta}). \end{aligned} \tag{3.14}$$

Let M denote the surface endowed with metric

$$g = (u(x) + v(y))^2\, dx^2 + (v'(y)\tan\theta)^2\, dy^2, \tag{3.15}$$

Then M is a flat surface and $\boldsymbol{x}$ is an isometric θ-slant immersion from M into $\boldsymbol{C}^2$.

(V) Let f be a nonzero function of one variable defined on an open interval containing 0 such that $f \neq \cot\theta$, $\rho = \rho(x, y)$ be a solution of (3.5) with $\rho_y \neq 0$, and K is a $\boldsymbol{C}^2$-valued solution of the (3.6) satisfying $|K|^2 = 1$ and $|K'|^2 = (f - \cot\theta)^2$. Consider the map $\boldsymbol{x}$ defined by (3.7). Then we have

$$\boldsymbol{x}_x = K(x - y)\rho e^{ix\csc\theta}, \quad \boldsymbol{x}_y = \frac{\rho_y K'}{f(\cot\theta - f)} e^{ix\csc\theta}, \tag{3.16}$$

where $f = f(x - y)$.

From $|K| = 1$, we obtain $\langle K, K'\rangle = 0$, where $\langle\, , \rangle$ is the standard inner product of $\boldsymbol{C}^2$. Thus we have $\langle \boldsymbol{x}_x, \boldsymbol{x}_y\rangle = 0$. Therefore, the map (3.7) defines an isometric immersion from the surface M endowed with metric

$$g = \rho^2\, dx^2 + \frac{\rho_y^2}{f^2} dy^2 \tag{3.17}$$

into $\boldsymbol{C}^2$. Since ρ is a solution of the wave equation (3.5), a straightforward computation shows that M is a flat surface and $\boldsymbol{x}$ is a θ-slant immersion.

Conversely, if M is a flat slant surface in $\boldsymbol{C}^2$ with slant angle $\theta = 0$, then M is a complex surface in $\boldsymbol{C}^2$. Thus it is a minimal surface. Since M is flat, M is totally geodesic due to the equation of Gauss. Therefore, M is an open part of a holomorphic line, i.e., an open part of a 0-slant plane.

Now, suppose M is a flat slant surface with slant angle $\theta \in (0, \pi/2]$. Then, by Lemma 2.3, there exists an adapted frame e_1, e_2, e_3, e_4 such that the second fundamental form of M takes the form:

$$h(e_1, e_1) = \mu e_3, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = \varphi e_4, \tag{3.18}$$

for some nonzero functions μ, φ. If $\mu = \varphi = 0$ identically, then M is totally geodesic. In this case, M is an open portion of a θ-plane.

We divide our study of non-totally geodesic flat θ-slant surfaces into four cases: (a) $\mu = 0$ and $\varphi \neq 0$, (b) $\mu, \varphi \neq 0$ and $e_2\mu = 0$, (c) $\mu, \varphi \neq 0$, $e_2\mu \neq 0$ and $e_1\varphi = 0$, or (d) $\mu \neq 0$, $e_2\mu \neq 0$ and $e_1\varphi \neq 0$, for the functions μ, φ given in (3.18).

Case (a). $\mu = 0$ and $\varphi \neq 0$.

In this case, equation (2.2) of Codazzi and (3.18) imply

$$\omega_1^2(e_1) = 0, \quad e_1\varphi = \varphi\omega_2^1(e_2) \tag{3.19}$$

which implies $[e_1, \varphi^{-1}e_2] = 0$. Thus there exists a coordinate chart $\{x, y\}$ on M such that $\partial/\partial x = e_1$ and $\partial/\partial y = \varphi^{-1}e_2$. Therefore the metric tensor of M is given by

$$g = dx^2 + \frac{1}{\varphi^2} dy^2. \tag{3.20}$$

Using (3.20) we obtain

$$\nabla_{\partial/\partial x}\frac{\partial}{\partial x}=0,\quad \nabla_{\partial/\partial x}\frac{\partial}{\partial y}=-\frac{\varphi_x}{\varphi}\frac{\partial}{\partial y},\quad \nabla_{\partial/\partial y}\frac{\partial}{\partial y}=\frac{\varphi_x}{\varphi^3}\frac{\partial}{\partial x}-\frac{\varphi_y}{\varphi}\frac{\partial}{\partial y}. \tag{3.21}$$

Applying (3.21) we get

$$R\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x}=\left(\frac{2\varphi_x^2-\varphi\varphi_{xx}}{\varphi^2}\right)\frac{\partial}{\partial y}, \tag{3.22}$$

where R is the curvature tensor of M. Since M is flat, (3.22) implies $(\varphi^{-1})_{xx}=0$. Thus

$$\varphi^{-1}=\beta(y)+\alpha(y)x \tag{3.23}$$

for some functions α,β. From (3.21) and (3.23) we get

$$\begin{aligned}&\nabla_{\partial/\partial x}\frac{\partial}{\partial x}=0,\quad \nabla_{\partial/\partial x}\frac{\partial}{\partial y}=\alpha(y)\varphi\frac{\partial}{\partial y},\\ &\nabla_{\partial/\partial y}\frac{\partial}{\partial y}=-\frac{\alpha}{\varphi}\frac{\partial}{\partial x}+(\beta'+\alpha'x)\varphi\frac{\partial}{\partial y}.\end{aligned} \tag{3.24}$$

From (1.1), (2.18), (3.18) and (3.20) we have

$$\begin{aligned}&h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right)=h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)=0,\\ &h\left(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right)=i\csc\theta\frac{\partial}{\partial y}+\frac{1}{\varphi}\cot\theta\frac{\partial}{\partial x}.\end{aligned} \tag{3.25}$$

Combining (3.24), (3.25) and the formula of Gauss we know that the immersion $\boldsymbol{x}$ of M in $\boldsymbol{C}^2$ satisfies

$$\begin{aligned}&\boldsymbol{x}_{xx}=0,\\ &\boldsymbol{x}_{xy}=\varphi\alpha(y)\boldsymbol{x}_y,\\ &\boldsymbol{x}_{yy}=\left(\frac{1}{\varphi}\cot\theta-\frac{\alpha}{\varphi}\right)\boldsymbol{x}_x+((\alpha'x+\beta')\varphi+i\csc\theta)\boldsymbol{x}_y.\end{aligned} \tag{3.26}$$

Integrating the first equation in (3.26) yields

$$\boldsymbol{x}(x,y)=P(y)x+D(y), \tag{3.27}$$

for some $\boldsymbol{C}^2$-valued functions $P(y),D(y)$. Hence we have

$$\boldsymbol{x}_x=P(y),\quad \boldsymbol{x}_y=P'(y)x+D'(y),\quad \boldsymbol{x}_{xy}=P'(y). \tag{3.28}$$

Applying the second equation in (3.26) and (3.28) we obtain

$$\beta(y)P'(y) = \alpha(y)D'(y). \tag{3.29}$$

From (3.20) and (3.28) we get $|P(y)| = 1$.

Case (a.1). $\alpha = 0$.

In this case (3.29) implies that $P'(y) = 0$, since $\beta = \varphi^{-1} \neq 0$. Thus, $P(y)$ is a unit constant vector, say c, in $\boldsymbol{C}^2$. From (3.23) we get $\varphi = 1/\beta(y)$ which implies that φ is a function of y. Hence, (3.27) and the third equation in (3.26) reduce to

$$\boldsymbol{x}(x,s) = cx + D(y). \tag{3.30}$$

$$D''(y) + \left(\frac{\varphi'}{\varphi} - i\csc\theta\right)D'(y) = \frac{\cot\theta}{\varphi}c. \tag{3.31}$$

Solving (3.31) yields

$$D(y) = ic\cos\theta\int_0^y \frac{dt}{\varphi(t)} + a\int_0^y \frac{e^{i(\csc\theta)y}}{\varphi(t)}dt + E,$$

for some vectors $a, E \in \boldsymbol{C}^2$. Without loss of generality, we may choose $E = 0$ by applying a suitable translation on $\boldsymbol{C}^2$ if necessary. Thus we have

$$\boldsymbol{x}(x,s) = c\left(x + i\cos\theta\int_0^y \frac{dt}{\varphi(t)}\right) + a\int_0^y \frac{e^{i(\csc\theta)y}}{\varphi(t)}dt. \tag{3.32}$$

If we choose the initial conditions to be

$$\boldsymbol{x}_x(0,0) = (1,0), \quad \boldsymbol{x}_y(0,0) = \frac{1}{\varphi(0)}(i\cos\theta, \sin\theta),$$

then (3.32) implies $c = (1,0)$, $a = (0, \sin\theta)$. Therefore, we obtain (3.2).

Case (a.2). $\alpha \neq 0$.

In this case (3.29) implies

$$D(y) = \int_0^y \frac{\beta(t)}{\alpha(t)}P'(t)\,dt + C \tag{3.33}$$

for some constant vector C. We may choose $C = 0$ by applying a suitable translation on $\boldsymbol{C}^2$ if necessary. Hence (3.27) yields

$$\boldsymbol{x}(x,y) = P(y)x + \int_0^y \frac{\beta(t)}{\alpha(t)}P'(t)\,dt. \tag{3.34}$$

Substituting (3.34) into the third equation in (3.26) gives

$$P''(y) - \left(\frac{\alpha'(y)}{\alpha(y)} + i\csc\theta\right)P'(y) + (\alpha^2(y) - \alpha(y)\cot\theta)P(y) = 0. \tag{3.35}$$

On the other hand, from (3.34) we find $\boldsymbol{x}_y = (1/(\alpha\varphi))P'(y)$. Comparing this with (3.20) we get $|P'(y)|^2 = \alpha^2(y)$. Hence, if $z(s) = P(y(s))$ is the arc-length reparametrization of $P(y)$, then we find $ds/dy = \pm\alpha$. Without loss of generality, we may assume $ds/dy = \alpha$, by reversing the orientation of $z(s)$ if necessary. Thus, by applying the chain rule, we have

$$P'(y) = \alpha z'(s), \quad P''(y) = \alpha'(y)z'(s) + \alpha^2 z''(s). \tag{3.36}$$

Substituting (3.36) into (3.35) we obtain

$$\alpha^2 z'' - \alpha\csc\theta i z' + (\alpha^2 - \alpha\cot\theta)z = 0. \tag{3.37}$$

Therefore, $z = z(s)$ is a θ-Legendre curve in S^3 according to Lemma 2.4. Consequently, the flat slant surface M is given by

$$\boldsymbol{x}(x,s) = z(s)x + \int_0^s \beta(u)z'(u)\,du. \tag{3.38}$$

where $z(s)$ is a θ-Legendre curve in S^3. Thus, in this case, we obtain (3.1).

Case (b). $\mu, \varphi \neq 0$ and $e_2\mu \neq 0$.

In this case, the equation of Codazzi and (3.18) imply

$$\omega_1^2(e_1) = 0, \quad e_1\varphi = \varphi\omega_2^1(e_2) = -\varphi^2\cot\theta \tag{3.39}$$

which implies $[(1/\mu)e_1, (1/\varphi)e_2] = 0$. Thus, there exists a coordinate chart $\{x, y\}$ such that $e_1 = \mu(\partial/\partial x)$, $e_2 = \varphi(\partial/\partial y)$. By the assumption: $e_2\mu = 0$, we know that $\mu = \mu(x)$ must be a function of x. Therefore, the metric tensor of M is given by

$$g = \frac{dx^2}{\mu^2(x)} + \frac{dy^2}{\varphi^2}. \tag{3.40}$$

Applying (3.39) and (3.40) we obtain $(1/\varphi)_x = \cot\theta/\mu$. Therefore

$$\frac{1}{\varphi} = k(y) + \cot\theta\int_0^x \frac{dx}{\mu(x)}, \tag{3.41}$$

for some function $k = k(y)$.

Using (3.40) and (3.41) we obtain

$$\nabla_{\partial/\partial x}\frac{\partial}{\partial x} = -\frac{\mu_x}{\mu}\frac{\partial}{\partial x}, \quad \nabla_{\partial/\partial x}\frac{\partial}{\partial y} = -\frac{\varphi_x}{\varphi}\frac{\partial}{\partial y}, \quad \nabla_{\partial/\partial y}\frac{\partial}{\partial y} = \frac{\mu^2\varphi_x}{\varphi^3}\frac{\partial}{\partial x} - \frac{\varphi_y}{\varphi}\frac{\partial}{\partial y}. \tag{3.42}$$

On the other hand, from (1.1), (2.18) and (3.18) we have

$$
\begin{aligned}
& h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = (\csc\theta) J \frac{\partial}{\partial x} - (\cot\theta) \frac{\varphi}{\mu} \frac{\partial}{\partial y}, \\
& h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0, \\
& h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = (\cot\theta) \frac{\mu}{\varphi} \frac{\partial}{\partial x} + (\csc\theta) J \frac{\partial}{\partial y}.
\end{aligned} \tag{3.43}
$$

Gauss's formula together with (3.42) and (3.43) implies that the immersion $\boldsymbol{x}$ of M in $\boldsymbol{C}^2$ satisfies the following system of partial differential equations:

$$\boldsymbol{x}_{xx} = \left(-\frac{\mu_x}{\mu} + i\csc\theta\right)\boldsymbol{x}_x - \frac{\varphi}{\mu}\cot\theta \boldsymbol{x}_y, \tag{3.44}$$

$$\boldsymbol{x}_{xy} = -\frac{\varphi_x}{\varphi}\boldsymbol{x}_y, \tag{3.45}$$

$$\boldsymbol{x}_{yy} = \left(\frac{\mu}{\varphi}\cot\theta + \frac{\mu^2\varphi_x}{\varphi^3}\right)\boldsymbol{x}_x - \left(\frac{\varphi_y}{\varphi} - i\csc\theta\right)\boldsymbol{x}_y. \tag{3.46}$$

Applying (3.41) and (3.39), equation (3.46) becomes

$$\boldsymbol{x}_{yy} = \left(-\frac{\varphi_y}{\varphi} + i\csc\theta\right)\boldsymbol{x}_y. \tag{3.47}$$

Solving (3.47) for $\boldsymbol{x}_y$ yields

$$\boldsymbol{x}_y = \frac{e^{iy\csc\theta}}{\varphi} A(x), \tag{3.48}$$

for some $\boldsymbol{C}^2$-valued function $A = A(x)$. Substituting (3.48) into (3.45) implies that $A'(x) = 0$. Thus

$$\boldsymbol{x}_y = \frac{e^{iy\csc\theta}}{\varphi} C, \tag{3.49}$$

for some vector C in $\boldsymbol{C}^2$. Substituting (3.49) into (3.44) gives

$$\boldsymbol{x}_{xx} + \left(\frac{\mu_x}{\mu} - i\csc\theta\right)\boldsymbol{x}_x = -\frac{e^{iy\csc\theta}\cot\theta}{\mu} C. \tag{3.50}$$

By solving (3.50) for $\boldsymbol{x}_x$ we obtain

$$\boldsymbol{x}_x = \frac{1}{\mu}(B(y)e^{ix\csc\theta} - iCe^{iy\csc\theta}\cos\theta), \tag{3.51}$$

for some $\boldsymbol{C}^2$-valued functions $B(y)$. Therefore, by integrating (3.51) with respect to x, we obtain

$$\boldsymbol{x}(x,y) = D(y) + B(y)\int_0^x \frac{e^{ix\csc\theta}}{\mu}dx - iCe^{iy\csc\theta}\cos\theta\int_0^x \frac{dx}{\mu}, \tag{3.52}$$

for some $\boldsymbol{C}^2$-valued functions $B(y), D(y)$ and some vector C.

From (3.41), (3.49) and (3.52) we obtain

$$D'(y) + B'(y)\int_0^x \frac{e^{ix\csc\theta}}{\mu}dx = Ck(y)e^{iy\csc\theta}. \tag{3.53}$$

Differentiating (3.53) with respect to x yields $B'(y) = 0$ which implies that B is a constant vector. Thus (3.53) implies $D'(y) = Ck(y)e^{iy\csc\theta}$. Hence

$$D(y) = C\int_0^y k(y)e^{iy\csc\theta}\,dy + E,$$

for some vector E. Without loss of generality, we may choose $E = 0$. Combining this with (3.52) yields

$$\boldsymbol{x}(x,y) = C\left\{\int_0^y k(y)e^{iy\csc\theta}\,dy - ie^{iy\csc\theta}\cos\theta\int_0^x \frac{dx}{\mu}\right\} + B\int_0^x \frac{e^{ix\csc\theta}}{\mu}dx \tag{3.54}$$

where B, C are constant vectors.

If we choose the following initial conditions:

$$\boldsymbol{x}_x(0,0) = \frac{1}{\mu(0)}(\cos\theta, \sin\theta), \quad \boldsymbol{x}_y(0,0) = \frac{1}{\varphi(0)}(i,0),$$

then (3.54) gives $B = (0, \sin\theta), C = (i,0)$. Hence we obtain (3.3).

CASE (c). $\mu, \varphi \neq 0$, $e_2\mu \neq 0$ and $e_1\varphi = 0$.

In this case, the equation of Codazzi and (3.18) imply

$$\omega_1^2(e_1) = e_2(\ln\mu), \quad \omega_1^2(e_2) = 0, \tag{3.55}$$

$$\mu\omega_1^2(e_2) + \varphi\omega_1^2(e_1) = \mu\varphi\cot\theta, \tag{3.56}$$

Apply (3.55) and (3.56) we get $[(1/\mu)e_1, (1/\varphi)e_2] = 0$. Thus, there exists a coordinate chart $\{x,y\}$ such that $e_1 = \mu(\partial/\partial x)$, $e_2 = \varphi(\partial/\partial y)$. Hence, the metric tensor of M is given by

$$g = \frac{1}{\mu^2}dx^2 + \frac{1}{\varphi^2}dy^2. \tag{3.57}$$

Using (3.55), (3.56) and (3.57) we obtain

$$\frac{1}{\varphi} = \frac{\mu_y}{\mu^2}\tan\theta. \tag{3.58}$$

Hence, by using $e_1\varphi = 0$, we obtain $(1/\mu)_{xy} = 0$. Therefore

$$\frac{1}{\mu} = u(x) + v(y), \tag{3.59}$$

for some functions $u = u(x)$ and $v = v(y)$. It follows from $e_2\mu \neq 0$ that $v'(y) \neq 0$. Moreover, (3.58) and (3.59) imply $\varphi^{-1} = v'(y)$. These show that the metric tensor of M is given by

$$g = (u(x) + v(y))^2\, dx^2 + (v'(y)\tan\theta)^2\, dy^2, \tag{3.60}$$

Using (3.60) we obtain

$$\begin{aligned} \nabla_{\partial/\partial x}\frac{\partial}{\partial x} &= \frac{u'(x)}{u+v}\frac{\partial}{\partial x} - \frac{(u+v)}{v'(y)}\cot^2\theta\frac{\partial}{\partial y}, \\ \nabla_{\partial/\partial x}\frac{\partial}{\partial y} &= \frac{v'(y)}{u+v}\frac{\partial}{\partial x}, \\ \nabla_{\partial/\partial y}\frac{\partial}{\partial y} &= \frac{v''(y)}{v'(y)}\frac{\partial}{\partial y}. \end{aligned} \tag{3.61}$$

On the other hand, from (1.1), (2.18) and (3.18) we have

$$\begin{aligned} h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) &= (\csc\theta)J\frac{\partial}{\partial x} + \frac{u+v}{v'(y)}\cot^2\theta\frac{\partial}{\partial y}, \\ h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= 0, \\ h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) &= -\frac{v'(y)}{u+v}\frac{\partial}{\partial x} + (\csc\theta)J\frac{\partial}{\partial y}. \end{aligned} \tag{3.62}$$

From (3.61), (3.62) and Gauss' formula we know that the immersion $\boldsymbol{x}$ of M in $\boldsymbol{C}^2$ satisfies the following system of partial differential equations:

$$\boldsymbol{x}_{xx} = \left(\frac{u'(x)}{u+v} + i\csc\theta\right)\boldsymbol{x}_x, \tag{3.63}$$

$$\boldsymbol{x}_{xy} = \frac{v'(y)}{u+v}\boldsymbol{x}_x, \tag{3.64}$$

$$\boldsymbol{x}_{yy} = -\frac{v'(y)}{u+v}\boldsymbol{x}_x + \left(\frac{v''(y)}{v'(y)} + i\csc\theta\right)\boldsymbol{x}_y. \tag{3.65}$$

Solving (3.64) for $\boldsymbol{x}_x$ yields

$$\boldsymbol{x}_x = (u+v)A(x), \tag{3.66}$$

for some $\boldsymbol{C}^2$-valued function $A = A(x)$. Substituting (3.66) into (3.63) yields $A'(x) = i\csc\theta A(x)$. Thus, we get $A(x) = Ce^{ix\csc\theta}$. Therefore, we obtain from (3.66) that

$$\boldsymbol{x}_x = C(u+v)e^{ix\csc\theta}, \tag{3.67}$$

for some vector C in $\boldsymbol{C}^2$. Integrating (3.67) with respect to x yields

$$\boldsymbol{x} = C\left\{\int_0^x u(x)e^{ix\csc\theta}\,dx - iv(y)\sin\theta e^{ix\csc\theta}\right\} + B(y), \tag{3.68}$$

for some $\boldsymbol{C}^2$-valued function $B(y)$. Substituting (3.68) into (3.65) yields

$$B''(y) - \left(\frac{v''(y)}{v'(y)} + i\csc\theta\right)B'(y) = 0. \tag{3.69}$$

Solving (3.69) yields

$$B(y) = D\int_0^y v'(y)e^{iy\csc\theta}\,dy + E,$$

for some vectors D, E. Consequently, we get

$$\begin{aligned}\boldsymbol{x}(x,y) = C&\left\{\int_0^x u(x)e^{ix\csc\theta}\,dx - ie^{ix\csc\theta}v(y)\sin\theta\right\}\\ &+ D\int_0^y v'(y)e^{iy\csc\theta}\,dy + E.\end{aligned} \tag{3.70}$$

If we choose the following initial conditions:

$$\begin{aligned}\boldsymbol{x}(0,0) &= (0,0),\\ \boldsymbol{x}_x(0,0) &= (u(0)+v(0),0),\\ \boldsymbol{x}_y(0,0) &= (-iv'(0)\sin\theta, v'(0)\sin\theta\tan\theta),\end{aligned}$$

then (3.70) gives

$$C = (1,0), \quad D = (0, \sin\theta\tan\theta), \quad E = (iv(0)\sin\theta, 0).$$

Hence we obtain (3.4).

CASE (d). $\mu \neq 0$, $e_2\mu \neq 0$ and $e_1\varphi \neq 0$.

In this case, (3.18) and the equation of Codazzi imply

$$e_2\mu = \mu\omega_1^2(e_1), \tag{3.71}$$

$$\mu\omega_1^2(e_2) + \varphi\omega_1^2(e_1) = \mu\varphi\cot\theta, \tag{3.72}$$

$$e_1\varphi = -\varphi\omega_1^2(e_2). \tag{3.73}$$

From (3.71), (3.72) and (3.73) we get $[(1/\mu)e_1, (1/\varphi)e_2] = 0$. Thus, there exists a coordinate chart $\{x, y\}$ such that $e_1 = \mu(\partial/\partial x)$, $e_2 = \varphi(\partial/\partial y)$. Hence, the metric tensor of M is given by

$$g = \rho^2\, dx^2 + \frac{\rho_y^2}{f^2(x-y)}\, dy^2. \tag{3.74}$$

Using (3.74), the flatness of M, and Gauss's Theorema Egregium, we find

$$\left(\frac{\varphi\mu_y}{\mu^2}\right)_y + \left(\frac{\mu\varphi_x}{\varphi^2}\right)_x = 0. \tag{3.75}$$

On the other hand, (3.71), (3.72) and (3.73) imply

$$\frac{\varphi\mu_y}{\mu^2} - \frac{\mu\varphi_x}{\varphi^2} = \cot\theta. \tag{3.76}$$

Applying (3.75) and (3.76) we obtain

$$\left(\frac{\varphi\mu_y}{\mu^2}\right)_x + \left(\frac{\varphi\mu_y}{\mu^2}\right)_y = 0. \tag{3.77}$$

Put $u = x + y$, $v = x - y$. Then (3.77) becomes $\left(\varphi\mu_y/\mu^2\right)_u = 0$. Thus

$$\frac{\varphi\mu_y}{\mu^2} = f(x-y), \tag{3.78}$$

for some function f of one variable. Therefore, we obtain $\varphi = -f(x-y)/\rho_y$, where $\rho = \mu^{-1}$. Applying (3.75) and (3.78) we know that ρ satisfies the wave equation

$$\rho_{xy} - \frac{f'(x-y)}{f(x-y)}\rho_y + \{(\cot\theta)f(x-y) - f^2(x-y)\}\rho = 0. \tag{3.79}$$

Applying (3.74) we obtain

$$\begin{aligned}
&\nabla_{\partial/\partial x}\frac{\partial}{\partial x} = \frac{\rho_x}{\rho}\frac{\partial}{\partial x} - \frac{\rho f^2}{\rho_y}\frac{\partial}{\partial y},\\
&\nabla_{\partial/\partial x}\frac{\partial}{\partial y} = \frac{\rho_y}{\rho}\frac{\partial}{\partial x} + \left(\frac{\rho_{xy}}{\rho_y} - \frac{f'}{f}\right)\frac{\partial}{\partial y},\\
&\nabla_{\partial/\partial y}\frac{\partial}{\partial y} = \left(-\frac{\rho_y\rho_{xy}}{\rho^2 f^2} + \frac{\rho_y^2 f'}{\rho^2 f^3}\right)\frac{\partial}{\partial x} + \left(\frac{\rho_{yy}}{\rho_y} + \frac{f'}{f}\right)\frac{\partial}{\partial y},
\end{aligned} \tag{3.80}$$

where $f = f(x-y)$.

From (1.1), (2.18), (3.18) and (3.74) we find

$$h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right) = i\csc\theta\frac{\partial}{\partial x}+\frac{f\rho}{\rho_y}\cot\theta\frac{\partial}{\partial y},$$

$$h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) = 0, \tag{3.81}$$

$$h\left(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right) = -\frac{\rho_y}{\rho f}\cot\theta\frac{\partial}{\partial x}+i\csc\theta\frac{\partial}{\partial y}.$$

Using (3.79), (3.80), (3.81) and the formula of Gauss we know that the immersion of $M^{\theta}_{f,\rho}$ in $\boldsymbol{C}^2$ satisfies

$$\boldsymbol{x}_{xx} = \left(\frac{\rho_x}{\rho}+i\csc\theta\right)\boldsymbol{x}_x + f(\cot\theta - f)\frac{\rho}{\rho_y}\boldsymbol{x}_y,$$

$$\boldsymbol{x}_{xy} = \frac{\rho_y}{\rho}\boldsymbol{x}_x + f(f-\cot\theta)\frac{\rho}{\rho_y}\boldsymbol{x}_y, \tag{3.82}$$

$$\boldsymbol{x}_{yy} = -\frac{\rho_y}{\rho}\boldsymbol{x}_x + \left(\frac{\rho_{yy}}{\rho_y}+\frac{f'}{f}+i\csc\theta\right)\boldsymbol{x}_y.$$

Combining the first and the second equations in (3.82) we find

$$\boldsymbol{x}_{xx}+\boldsymbol{x}_{xy} = \left(\frac{\rho_x}{\rho}+\frac{\rho_y}{\rho}\right)\boldsymbol{x}_x + i\csc\theta\boldsymbol{x}_x. \tag{3.83}$$

Solving (3.83) yields

$$\boldsymbol{x}_x = G(x-y)\rho e^{(i/2)\csc\theta(x+y)}, \tag{3.84}$$

where G is a positive function of one variable. Thus, if we put

$$G(x-y) = K(x-y)e^{(i/2)\csc\theta(x-y)}, \tag{3.85}$$

we obtain

$$\boldsymbol{x}_x = K(x-y)\rho e^{ix\csc\theta}. \tag{3.86}$$

On the other hand, by the second and the third equations of (3.82) we have

$$\boldsymbol{x}_{xy}+\boldsymbol{x}_{yy} = \left(\frac{\rho_{xy}}{\rho_y}+\frac{\rho_{yy}}{\rho_y}\right)\boldsymbol{x}_y + i\csc\theta\boldsymbol{x}_y. \tag{3.87}$$

Solving (3.87) yields

$$\boldsymbol{x}_y = H(x-y)\rho_y e^{(i/2)\csc\theta(x+y)}, \tag{3.88}$$

where H is a positive function of one variable. Hence, by putting $H(x-y) = F(x-y)e^{(i/2)\csc\theta(x-y)}$, we obtain

$$\boldsymbol{x}_y = F(x-y)\rho_y e^{ix\csc\theta}. \tag{3.89}$$

By taking the derivative of (3.86) with respect to y and compare it with the second equation of (3.82) and (3.89), we find

$$K' = f(\cot\theta - f)F. \tag{3.90}$$

Therefore, (3.89) becomes

$$\boldsymbol{x}_y = \frac{\rho_y K'}{f(\cot\theta - f)} e^{ix\csc\theta}, \tag{3.91}$$

By taking the derivative of (3.91) with respect to x and comparing it with the second equation of (3.82), we obtain

$$K'' + \left(i\csc\theta - \frac{f'}{f-\cot\theta}\right)K' + f(f-\cot\theta)K = 0 \tag{3.92}$$

by virtue of (3.79), (3.86) and (3.91).

Integrating (3.86) with respect to x we obtain

$$\boldsymbol{x} = \int_0^x K(x-y)\rho e^{ix\csc\theta}\,dx + H(y), \tag{3.93}$$

for some $\boldsymbol{C}^2$-valued function $H(y)$. Taking the derivative of (3.93) with respect to y and comparing with (3.91) yields

$$H'(y) = \frac{\rho_y K'}{f(\cot\theta - f)} e^{ix\csc\theta} - \int_0^x (K(x-y)\rho)_y e^{ix\csc\theta}\,dx. \tag{3.94}$$

Since ρ satisfies (3.79) and K satisfies (3.92), a direct computation shows that the right hand side of (3.94) is a function of y only. Thus, we obtain

$$\begin{aligned} \boldsymbol{x}(x,y) = &\int_0^y \left\{ \frac{\rho_y K'(x-y)}{f(\cot\theta - f)} e^{ix\csc\theta} - \int_0^x (\rho K(x-y))_y e^{ix\csc\theta}\,dx \right\} dy \\ &+ \int_0^x \rho K(x-y) e^{ix\csc\theta}\,dx + C, \end{aligned} \tag{3.95}$$

where C is a constant vector. If we choose the initial conditions: $\boldsymbol{x}(0,0) = (0,0)$, we obtain $C = (0,0)$. Thus, the immersion of M in $\boldsymbol{C}^2$ is given by (3.7) for some $\boldsymbol{C}^2$-valued function $K(u)$ which satisfies the second order ordinary differential equation (3.92). By comparing the metric g given by (3.74) with (3.86) and (3.91) we obtain $|K|^2 = 1$ and $|K'|^2 = (f - \cot\theta)^2$. This completes the proof of the theorem. □

REMARK 3.1. For any nonzero function f of one variable and any $\theta \in (0, \pi/2]$, the wave equation (3.5) admits infinitely many solutions. For example, every linear combination of

$$\rho_1 = \sin\left(\int_0^{x-y} f(t)\,dt - x\cot\theta\right),$$

$$\rho_2 = \cos\left(\int_0^{x-y} f(t)\,dt - x\cot\theta\right),$$

$$\rho_3 = \left(\int_0^{x-y} f(u)\cos(u\cot\theta)\,du\right)\rho_1 + \left\{\int_0^{x-y} f(u)\sin(u\cot\theta)\,du - 2\int_0^{x-y}\left(\int_0^u f(t)\,dt - x\cot\theta\right)du\right\}\rho_2,$$

$$\rho_4 = \left(\int_0^{x-y} f(u)\cos(u\cot\theta)\,du\right)\rho_2 - \left\{\int_0^{x-y} f(u)\sin(u\cot\theta)\,du - 2\int_0^{x-y}\left(\int_0^u f(t)\,dt - x\cot\theta\right)du\right\}\rho_1,$$

is a solution of the wave equation (3.5).

4. Existence theorems.

The following existence theorem together with Theorem 3.1 and Remark 3.1 imply that the class V of flat slant surfaces in $\boldsymbol{C}^2$ is very large.

THEOREM 4.1. *For any given nonzero function f of one variable defined on an open interval I and for any $\theta \in (0, \pi/2]$ such that $f \neq \cot\theta$, there exists a $\boldsymbol{C}^2$-valued solution $K = K(u)$ of the second order ordinary differential equation*:

$$K''(u) + \left(i\csc\theta - \frac{f'(u)}{f(u) - \cot\theta}\right)K'(u) + f(u)(f(u) - \cot\theta)K(u) = 0, \tag{4.1}$$

that also satisfies the two conditions:

$$|K|^2 = 1, \quad |K'|^2 = (f - \cot\theta)^2. \tag{4.2}$$

PROOF. Suppose f is a nonzero function of one variable defined on an open interval I such that $f \neq \cot\theta$ for some $\theta \in (0, \pi/2]$. Let $\rho(x, y)$ be any given solution of the wave equation:

$$\rho_{xy} - \frac{f'(x-y)}{f(x-y)}\rho_y + \{(\cot\theta)f(x-y) - f^2(x-y)\}\rho = 0 \tag{4.3}$$

with $\rho_y \neq 0$ (such a solution always exists, cf. Remark 3.1). Let U be a simply-connected open subset of $\boldsymbol{R}^2$ on which $f(x-y)$ is defined. We define a metric g on U by

$$g = \rho^2\, dx^2 + \frac{\rho_y^2}{f^2(x-y)} dy^2. \tag{4.4}$$

Then $M = (U, g)$ is a flat surface.

Consider the orthonormal frame $\{e_1, e_2\}$ on M given by

$$e_1 = \frac{1}{\rho}\frac{\partial}{\partial x}, \quad e_2 = -\frac{f(x-y)}{\rho_y}\frac{\partial}{\partial y}. \tag{4.5}$$

We define an endomorphism P on the tangent bundle TM by

$$Pe_1 = \cos\theta e_2, \quad Pe_2 = -\cos\theta e_1 \tag{4.6}$$

and also define a symmetric bilinear form α by

$$\alpha(e_1, e_1) = -\frac{\sin\theta}{\rho} e_1, \quad \alpha(e_1, e_2) = 0, \quad \alpha(e_2, e_2) = \frac{f(x-y)\sin\theta}{\rho_y} e_2. \tag{4.7}$$

By a long straightforward computation, we may prove that (M, P, α, θ) satisfies all of the conditions mentioned in the existence theorem of slant immersions (section 2). Therefore, by applying the existence theorem of slant immersions, we conclude that there exists a θ-slant isometric immersion $\boldsymbol{x} : M \to \boldsymbol{C}^2$ from M into $\boldsymbol{C}^2$ whose second fundamental form h is given by

$$h(X, Y) = \csc^2\theta(P\alpha(X, Y) - J\alpha(X, Y)). \tag{4.8}$$

Applying (4.5), (4.6), (4.7) and (4.8) we have

$$\begin{aligned} h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) &= i\csc\theta\frac{\partial}{\partial x} + \frac{f\rho}{\rho_y}\cot\theta\frac{\partial}{\partial y}, \\ h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= 0, \\ h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) &= -\frac{\rho_y}{\rho f}\cot\theta\frac{\partial}{\partial x} + i\csc\theta\frac{\partial}{\partial y}. \end{aligned} \tag{4.9}$$

Using (4.4) we also have

$$\begin{aligned} \nabla_{\partial/\partial x}\frac{\partial}{\partial x} &= \frac{\rho_x}{\rho}\frac{\partial}{\partial x} - \frac{\rho f^2}{\rho_y}\frac{\partial}{\partial y}, \\ \nabla_{\partial/\partial x}\frac{\partial}{\partial y} &= \frac{\rho_y}{\rho}\frac{\partial}{\partial x} + \left(\frac{\rho_{xy}}{\rho_y} - \frac{f'}{f}\right)\frac{\partial}{\partial y}, \\ \nabla_{\partial/\partial y}\frac{\partial}{\partial y} &= \left(-\frac{\rho_y\rho_{xy}}{\rho^2 f^2} + \frac{\rho_y^2 f'}{\rho^2 f^3}\right)\frac{\partial}{\partial x} + \left(\frac{\rho_{yy}}{\rho_y} + \frac{f'}{f}\right)\frac{\partial}{\partial y}, \end{aligned} \tag{4.10}$$

where $f = f(x-y)$. Thus, by (4.3), (4.9) and (4.10), we know that the immersion $\boldsymbol{x}$ must satisfies the system

$$
\begin{aligned}
\boldsymbol{x}_{xx} &= \left(\frac{\rho_x}{\rho} + i\csc\theta\right)\boldsymbol{x}_x + f(\cot\theta - f)\frac{\rho}{\rho_y}\boldsymbol{x}_y, \\
\boldsymbol{x}_{xy} &= \frac{\rho_y}{\rho}\boldsymbol{x}_x + f(f - \cot\theta)\frac{\rho}{\rho_y}\boldsymbol{x}_y, \\
\boldsymbol{x}_{yy} &= -\frac{\rho_y}{\rho}\boldsymbol{x}_x + \left(\frac{\rho_{yy}}{\rho_y} + \frac{f'}{f} + i\csc\theta\right)\boldsymbol{x}_y.
\end{aligned}
\tag{4.11}
$$

Therefore, as in the proof of Theorem 3.1, we conclude that there is a $\boldsymbol{C}^2$-valued function K of one variable so that

$$
\boldsymbol{x}_x = K(x-y)\rho e^{ix\csc\theta}, \quad \boldsymbol{x}_y = \frac{\rho_y K'}{f(\cot\theta - f)}e^{ix\csc\theta}. \tag{4.12}
$$

From (4.11) and (4.12), we also know that the function K satisfies the differential equation (4.1). Furthermore, by applying (4.4) and (4.11) we have $|K|^2 = 1$ and $|K|^2 = (f - \cot\theta)^2$. Consequently, we conclude that, for any given nonzero function $f \neq \cot\theta$, the differential equation (4.1) admits a $\boldsymbol{C}^2$-valued solution K which satisfies condition (4.2). □

The next theorem shows the existence of ample θ-Legendre curves in S^3 which implies that class I of flat slant surfaces in $\boldsymbol{C}^2$ is quite large too.

THEOREM 4.2. *For any $\theta \in (0, \pi/2]$ and any nonzero function $\lambda(s)$ defined on an open interval I, there is a unit speed θ-Legendre curve in S^3 whose curvature in S^3 is given by $\kappa = \lambda\csc\theta$.*

PROOF. Let $\lambda = \lambda(y)$ be a nonzero function defined on an open interval I. Denote by U the open subset of $\boldsymbol{R}^2$ given by the product: $\boldsymbol{R} \times I$. On U we define a Riemannian metric g by

$$
g = dx^2 + (x+y)^2\,dy^2. \tag{4.13}
$$

From (4.13) we have

$$
\begin{aligned}
&\nabla_{\partial/\partial x}\frac{\partial}{\partial x} = 0, \quad \nabla_{\partial/\partial x}\frac{\partial}{\partial y} = \left(\frac{1}{x+y}\right)\frac{\partial}{\partial y}, \\
&\nabla_{\partial/\partial y}\frac{\partial}{\partial y} = -(x+y)\frac{\partial}{\partial x} + \left(\frac{1}{x+y}\right)\frac{\partial}{\partial y}.
\end{aligned}
\tag{4.14}
$$

On $M = (U, g)$ we define an endomorphism P on the tangent bundle of M by

$$P\left(\frac{\partial}{\partial x}\right) = \frac{\cos\theta}{x+y}\frac{\partial}{\partial y}, \quad P\left(\frac{\partial}{\partial y}\right) = -(x+y)\cos\theta\frac{\partial}{\partial x}. \tag{4.15}$$

We also define a symmetric bilinear TM-valued form α by

$$\alpha\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \alpha\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0, \quad \alpha\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = -\lambda(y)\sin\theta\frac{\partial}{\partial y}. \tag{4.16}$$

By a straightforward computation we know that (M, P, α) satisfies all of the conditions mentioned in existence theorem of section 2. Thus, there exists an isometric θ-slant immersion $\boldsymbol{x} : M \to \boldsymbol{C}^2$ whose second fundamental form is given by

$$h(X, Y) = \csc^2\theta(P\alpha(X, Y) - J\alpha(X, Y)). \tag{4.17}$$

From (4.15), (4.16) and (4.17) we find

$$\begin{aligned} h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) &= h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0, \\ h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) &= \lambda(y)(x+y)\cot\theta\boldsymbol{x}_x + i\lambda\csc\theta\boldsymbol{x}_y. \end{aligned} \tag{4.18}$$

Combining (4.14), (4.18) and Gauss's formula, we obtain

$$\begin{aligned} &\boldsymbol{x}_{xx} = 0, \\ &\boldsymbol{x}_{xy} = \frac{\boldsymbol{x}_y}{x+y}, \\ &\boldsymbol{x}_{yy} = (\lambda\cot\theta - 1)(x+y)\boldsymbol{x}_x + \left(\frac{1}{x+y} + i\lambda\csc\theta\right)\boldsymbol{x}_y. \end{aligned} \tag{4.19}$$

Solving the first equation of (4.19) yields

$$\boldsymbol{x}(x, y) = C(y)x + D(y), \tag{4.20}$$

for some $\boldsymbol{C}^2$-valued functions $C(y)$ and $D(y)$. From (4.20) we have

$$\boldsymbol{x}_x = C(y), \quad \boldsymbol{x}_y = C'(y)x + D'(y). \tag{4.21}$$

If we put $z(y) = C(y)$, then, (4.19) and (4.21) imply $D'(y) = yz'(y)$. Hence, $D(y) = \int^y uz'(u)\,du$. Therefore, we obtain from (4.20) that

$$\boldsymbol{x}(x,y) = z(y)x + \int^{y} uz'(u)\,du. \tag{4.22}$$

Differentiating (4.22) yields

$$\boldsymbol{x}_x = z(y), \quad \boldsymbol{x}_y = (x+y)z'(y), \tag{4.23}$$

$$\boldsymbol{x}_{yy} = \frac{\boldsymbol{x}_y}{x+y} + (x+y)z''(y). \tag{4.24}$$

Comparing (4.24) with the third equation of (4.19) gives

$$z''(y) = (\lambda\cot\theta - 1)z(y) + i\lambda\csc\theta z'(y), \tag{4.25}$$

by virtue of (4.23). Using (4.13) and (4.23) we get $\langle z, z\rangle = 1$. Thus z defines a curve in S^3. Moreover, from (4.13) and (4.23) we also know that z is a unit speed curve. Consequently, by applying Lemma 2.1, we conclude that z is a unit speed θ-Legendre curve in S^3. These imply that for any nonzero function λ defined on an open interval, there exists a θ-Legendre curve in S^3. Since at each point on a curve in S^3 the position vector is normal to S^3, (4.25) implies that iz' is the principal normal vector of the curve z in S^3 and the curvature of the curve in S^3 is given by $\lambda\csc\theta$. $\square$

5. Some explicit examples of flat slant surfaces of class V.

Here we provide some explicit solutions of the differential equation (4.1) which also satisfy the two prescribed conditions given in (4.2). By applying these solutions we construct explicit examples of flat slant surfaces belonging to class V.

Let

$$f = \frac{1}{2}(\cot\theta + c), \quad c \neq \cot\theta. \tag{5.1}$$

Then (3.6) becomes

$$K'' + i\csc\theta K' - \frac{1}{4}(\cot^2\theta - c^2)K = 0. \tag{5.2}$$

The general solution of (5.2) is given by

$$K(u) = e^{-(i/2)u\csc\theta}\left(c_1\cos\left(\frac{\sqrt{1+c^2}}{2}u\right) + c_2\sin\left(\frac{\sqrt{1+c^2}}{2}u\right)\right). \tag{5.3}$$

From (5.3) we find

$$K(0) = c_1, \quad K'(0) = -\frac{i}{2}\csc\theta c_1 + \frac{\sqrt{1+c^2}}{2}c_2. \tag{5.4}$$

Since K needs to satisfy the two conditions:

$$|K(u)|^2 = 1, \quad |K'(u)|^2 = \frac{1}{4}(c - \cot\theta)^2 \tag{5.5}$$

required by Theorem 3.1, so we choose the following initial condition:

$$K(0) = c_1 = (1, 0). \tag{5.6}$$

From the condition $|K(u)|^2 = 1$, we have $\langle K(u), K'(u)\rangle = 0$. Thus, (5.4) implies $\langle c_1, c_2\rangle = 0$. Hence, by applying (5.6) we know that c_2 takes the form: $c_2 = (ia, z)$ for some real number a.

Since $f = (1/2)(\cot\theta + c)$, the wave equation (3.5) reduces to

$$4\rho_{xy} + (\cot^2\theta - c^2)\rho = 0. \tag{5.7}$$

It is easy to verify that

$$\rho = e^{\{(x+y)c + (x-y)\cot\theta\}/2} \tag{5.8}$$

is a solution of (5.7). Suppose that $\boldsymbol{x}$ is the immersion of the flat slant surface associated with K and ρ mentioned in Theorem 3.1. Then from the proof of Theorem 3.1 we know that the metric of the slant surface is given by

$$g = \rho^2\,dx^2 + \frac{\rho_y^2}{f^2(x-y)}dy^2. \tag{5.9}$$

Moreover, we also have

$$\boldsymbol{x}_x = K(x-y)\rho e^{ix\csc\theta}, \quad \boldsymbol{x}_y = \frac{\rho_y K'}{f(\cot\theta - f)}e^{ix\csc\theta}. \tag{5.10}$$

Using (5.1), (5.5), (5.9), (5.10) and the θ-slantness of the surface, we have

$$\langle iK(0), K'(0)\rangle = \frac{1}{2}(c - \cot\theta)\langle ie_1, e_2\rangle = \frac{1}{2}(c - \cot\theta)\cos\theta. \tag{5.11}$$

Combining (5.4), (5.6) and (5.11) yields

$$\langle ic_1, c_2\rangle = \frac{\sin\theta + c\cos\theta}{\sqrt{1+c^2}}. \tag{5.12}$$

Therefore, (5.6) and (5.12) imply

$$c_2 = \frac{1}{\sqrt{1+c^2}}(i(\sin\theta + c\cos\theta), z) \tag{5.13}$$

for some z.

On the other hand, by applying (5.4), (5.5) and (5.12) we find $|c_2|^2 = 1$. Thus, from (5.13), we obtain $|z|^2 = (\cos\theta - c\sin\theta)^2$. In particular, if we choose $z = \cos\theta - c\sin\theta$, then we have

$$c_2 = \frac{1}{\sqrt{1+c^2}}(i(\sin\theta + c\cos\theta), \cos\theta - c\sin\theta). \tag{5.14}$$

By combining (5.3), (5.6) and (5.13) we have

(5.15)

$$\begin{aligned} K(u) = \frac{e^{-(i/2)u\csc\theta}}{\sqrt{1+c^2}}\Bigg(&\sqrt{1+c^2}\cos\left(\frac{\sqrt{1+c^2}}{2}u\right) \\ &+ i(\sin\theta + c\cos\theta)\sin\left(\frac{\sqrt{1+c^2}}{2}u\right), (\cos\theta - c\sin\theta)\sin\left(\frac{\sqrt{1+c^2}}{2}u\right)\Bigg). \end{aligned}$$

It is straightforward to verify that $K(u)$ satisfies conditions $|K(u)|^2 = 1$ and $|K'(u)|^2 = (1/4)(c - \cot\theta)^2$.

By a direct long computation, we find

$$\begin{aligned} \int_0^x \rho K(x-y)e^{ix\csc\theta}\,dx = {} & \frac{e^{\{(x+y)(c+i\csc\theta)+(x-y)\cot\theta\}/2}}{\sqrt{1+c^2}(c+i\csc\theta)(c\sin\theta+\cos\theta)} \\ & \cdot ((c^2\sin\theta(1+i\cos\theta) + \cos\theta(i\sin\theta - c) + ic)\sin w \\ & \quad + \sqrt{1+c^2}\{\cos\theta(1+i\cos\theta) + c\sin\theta(1-i\cos\theta)\}\cos w, \\ & \quad (\sin\theta\cos\theta - c\sin^2\theta)\{(c+\cot\theta+i\csc\theta)\sin w - \sqrt{1+c^2}\cos w\}), \end{aligned} \tag{5.16}$$

where $w = (1/2)\sqrt{1+c^2}(x-y)$.

On the other hand, by a straightforward but very long computation we obtain

$$\frac{4\rho_y K'(x-y)}{\cot^2\theta - c^2}e^{ix\csc\theta} - \int_0^x (\rho K(x-y))_y e^{ix\csc\theta}\,dx = 0. \tag{5.17}$$

Consequently, by combining (3.95), (5.1), (5.8), (5.16) and (5.17), we conclude that, up to rigid motions of $\boldsymbol{C}^2$, the flat slant surface is given by

$$(5.18)\quad \boldsymbol{x}(x,y) = \frac{e^{\{(x+y)(c+i\csc\theta)+(x-y)\cot\theta\}/2}}{\sqrt{1+c^2}(c+i\csc\theta)(c\sin\theta+\cos\theta)}$$
$$\times\Biggl(\biggl((c^2\sin\theta(1+i\cos\theta)+\cos\theta(i\sin\theta-c)+ic)\sin\biggl(\frac{\sqrt{1+c^2}}{2}(x-y)\biggr)$$
$$+\sqrt{1+c^2}\{\cos\theta(1+i\cos\theta)+c\sin\theta(1-i\cos\theta)\}$$
$$\times\cos\biggl(\frac{\sqrt{1+c^2}}{2}(x-y)\biggr),(\sin\theta\cos\theta-c\sin^2\theta)$$
$$\times\biggl\{(c+\cot\theta+i\csc\theta)\sin\biggl(\frac{\sqrt{1+c^2}}{2}(x-y)\biggr)$$
$$-\sqrt{1+c^2}\cos\biggl(\frac{\sqrt{1+c^2}}{2}(x-y)\biggr)\biggr\}\Biggr).$$

By a straightforward long computation one can verify that (5.18) defines a flat θ-slant surface belonging to class V.

When $c=0$, (5.18) reduces to

$$(5.19)\quad \boldsymbol{x}(x,y) = e^{\{i(x+y)\csc\theta+(x-y)\cot\theta\}/2}\sin\theta$$
$$\times\biggl(\sin\theta\sin\frac{x-y}{2}+(\cos\theta-i)\cos\frac{x-y}{2},(1-i\cos\theta)\sin\frac{x-y}{2}$$
$$+i\sin\theta\cos\frac{x-y}{2}\biggr).$$

Remark 5.1. The conditions $|K|=1$ and $|K'|=(f-\cot\theta)^2$ in statement V of Theorem 3.1 are necessary. For instance, although

$$K(u)=e^{-(i/2)u\csc\theta}\biggl(\cos\frac{u}{2}+i(\csc\theta+\cos\theta\cot\theta)\sin\frac{u}{2},\cos\theta\sin\frac{u}{2}\biggr).$$

is a $\boldsymbol{C}^2$-valued solution of (3.6) associated with $f=(1/2)\cot\theta$, the map (3.7) with $\rho=e^{(x-y)\cot\theta/2}$ does not define a flat θ-slant surface in $\boldsymbol{C}^2$.

Added in Proof. For the visualization of some flat slant surfaces in $\boldsymbol{C}^2$, including the one defined by (5.19), see [**10**].

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