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# On the theory of $KM_2O$-Langevin equations for non-stationary and degenerate flows

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**Abstract.** We have developed the theory of $KM_2O$-Langevin equations for stationary and non-degenerate flow in an inner product space. As its generalization and refinement of the results in [**14**], [**15**], [**16**], we shall treat in this paper a general flow in an inner product space without both the stationarity property and the non-degeneracy property. At first, we shall derive the $KM_2O$-Langevin equation describing the time evolution of the flow and prove the fluctuation-dissipation theorem which states that there exists a relation between the fluctuation part and the dissipation part of the above $KM_2O$-Langevin equation. Next we shall prove the characterization theorem of stationarity property, the construction theorem of a flow with any given nonnegative definite matrix function as its two-point covariance matrix function and the extension theorem of a stationary flow without losing stationarity property.

## 1. Introduction.

With the aim of understanding the mathematical structure of the fluctuation-dissipation theorem in non-equilibrium statistical physics and then constructing certain experimental and mathematical principle in the modeling problem for time series analysis, we have developed the theory of $KM_2O$-Langevin equations for discrete time weakly stationary processes [**5**], [**7**], [**8**], [**10**]. In particular, we have constructed the fluctuation-dissipation principle based on the mathematical fluctuation-dissipation theorem and applied it to propose the tests for stationarity, causality, determinacy and chaotic property of time series [**6**], [**9**], [**13**], [**18**].

As a generalization and refinement of the paper [**5**], we have developed in the series of papers [**14**], [**15**], [**16**] the theory of $KM_2O$-Langevin equations for non-degenerate flows in an inner product space and proved three kinds of theorems; one is the characterization theorem for stationarity property of flows; the second is the construction theorem of $KM_2O$-Langevin matrices and stationary flows; the third is the extension theorem of stationary flows and positive definite functions.

However, there are various kinds of non-stationary time series data that are important to be treated from a practical point of view. Although some non-stationary flows can be transformed into stationary flows by non-linear transformations, we need the method to treat non-stationarity property directly. On the other hand, the theory for degenerate flows are also important in the analysis of deterministic time series such as those generated by dynamical systems.

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Lev-Ari and Kailath have obtained Schur and Levinson algorithms for non-degenerate non-stationary processes [**1**]. These algorithms, however, cannot be applied to degenerate processes. On the other hand, Sakai treated degenerate (singular) stationary processes [**21**]. He first analyzed periodic autoregressive processes [**20**], [**21**] based on Pagano's work [**19**]. Then, as its application, he has obtained an estimation method for the parameters of singular stationary processes by transforming multi-dimensional stationary processes into one-dimensional periodic stationary processes. This argument is essentially based on stationary and periodic stationary properties.

As stated in [**14**], the theory of $KM_2O$-Langevin equations is applicable to the analysis of not only non-degenerate stationary flows, but also degenerate (or) non-stationary flows, that is, general flows. In fact, by developping the non-linear information analysis for local stochastic processes and then the method of weight transformation by which degenerate flows can be transformed into non-degenerate flows, we have solved in [**4**], [**17**] the non-linear prediction problems for local stochastic processes which remained to be solved after Masani-Wiener's work [**2**]. In particular, we have proved in [**4**] the fluctuation-dissipation theorem for degenerate stationary flows and then given an algorithm for calculating $KM_2O$-Langevin matrix from the covariance matrix function associated with the degenerate stationary flows.

In this paper, we extend the results on non-degenerate stationary flows to general flows, that is, degenerate and non-stationary flows. We shall explain the contents of this paper. In Section 2, we review the fundamental facts about the $KM_2O$-Langevin equations for any pair $[\boldsymbol{X}, \boldsymbol{Y}]$ of two $d$-dimensional flows $\boldsymbol{X} = (X(n); 0 \le n \le N_+)$ and $\boldsymbol{Y} = (Y(l); -N_- \le l \le 0)$ in an real inner product space $W$ with an inner product $(*, \star)$. In particular, there exist various kinds of coefficients of the dissipation term in $KM_2O$-Langevin equations for degenerate flows. For this reason, we introduce two sets $\mathscr{LMD}_+(\boldsymbol{X})$ and $\mathscr{LMD}_-(\boldsymbol{Y})$ of $KM_2O$-Langevin dissipation matrix functions associated with the flows $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. The main purpose of this paper is to obtain an algorithm for constructing all the elements of these sets.

In Section 3, we introduce the covariance matrix functions $R(\boldsymbol{X}) = (R(\boldsymbol{X})(m,n); 0 \le m,n \le N_+)$ and $R(\boldsymbol{Y}) = (R(\boldsymbol{Y})(k,l); -N_- \le k,l \le 0)$ of the flows $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, by

$$R(\boldsymbol{X})(m,n) \equiv (X(m), {}^tX(n)) \quad (0 \le m,n \le N_+), \tag{1.1}$$

$$R(\boldsymbol{Y})(-m,-n) \equiv (Y(-m), {}^tY(-n)) \quad (0 \le m,n \le N_-). \tag{1.2}$$

These functions also reflect characteristic properties of the pair $[\boldsymbol{X}, \boldsymbol{Y}]$ of flows. We will obtain a certain relation between the sets $\mathscr{LMD}_+(\boldsymbol{X}), \mathscr{LMD}_-(\boldsymbol{Y})$ and the covariance matrix functions $R(\boldsymbol{X}), R(\boldsymbol{Y})$ in terms of a Cholesky factorization.

In Section 4, for a given $d$-dimensional flow $\boldsymbol{X} = (X(n); 0 \le n \le N)$ in the real inner product space $W$ and any integer $s$ $(0 \le s \le N)$, we define other $d$-dimensional flows $\boldsymbol{X}_+^{(s)} = (X_+^{(s)}(n); 0 \le n \le N-s), \boldsymbol{X}_-^{(s)} = (X_-^{(s)}(l); -s \le l \le 0)$ by

$$X_+^{(s)}(n) \equiv X(n+s) \quad (0 \le n \le N-s), \tag{1.3}$$

$$X_-^{(s)}(l) \equiv X(l+s) \quad (-s \le l \le 0). \tag{1.4}$$

We investigate the relations between $\mathscr{LMD}_+(\boldsymbol{X}_+^{(s)}), \mathscr{LMD}_-(\boldsymbol{X}_-^{(s)})$ and the covariance matrix function of the flow $\boldsymbol{X}$ by running $s$ from $s=0$ to $s=N$. As its result, we prove the fluctuation-dissipation theorem for degenerate and non-stationary flows which can be regarded as a generalization and a refinement of the fluctuation-dissipation theorem for non-degenerate stationary flows in [**14**].

In Section 5, we start with any nonnegative definite matrix function $R=(R(m,n); 0\le m,n\le N)$ with its value in $M(d;\boldsymbol{R})$. We introduce the system $\mathscr{LM}(R)$ of $\mathrm{KM_2O}$-Langevin matrix functions associated with the function $R$ and give an algorithm for obtaining all the elements of the set $\mathscr{LM}(R)$ by using the fluctuation-dissipation theorem in Section 4. Moreover, we construct a flow $\boldsymbol{X}=(X(n); 0\le n\le N)$ such that the matrix function $R$ becomes its covariance matrix function. These results can be considered as a generalization and a refinement of the construction theorem for non-degenerate stationary flows in [**15**].

Section 6 consists of four subsections and is devoted to the analysis of stationary flows. We shall see that most results about non-degenerate stationary flows in [**16**] hold also for degenerate stationary flows.

In Section 7, we treat periodic stationary flows as an example of non-stationary flows (though any periodic stationary flow of period one is a stationary flow), and then prove a characterization theorem for periodic stationarity of flows.

Finally in Section 8, we propose the test for the models of covariance matrix functions for non-stationary time series. This is based on essentially the same fluctuation-dissipation principle as in the test for stationarity which has been proposed in [**6**].

## 2. Notation and basic facts.

In this section, we shall review the basic facts about the theory of $\mathrm{KM_2O}$-Langevin equations ([**14**], [**15**], [**18**], [**4**]) and follow the notations and the terminologies in the papers above.

Let $(W,(\star,*))$ be any real inner product space with an inner product $(\star,*)$. For a $d$-dimensional flow $\boldsymbol{Z}=(Z(n); l\le n\le r)$ and two integers $n_1,n_2$ $(l\le n_1\le n_2\le r)$, we denote by $\mathrm{M}_{n_1}^{n_2}(\boldsymbol{Z})$ the subspace of $W$ spanned by $\{Z_j(m); 1\le j\le d, n_1\le m\le n_2\}$, where $Z_j(m)$ is the $j$th component of $Z(m)$ $(1\le j\le d, n_1\le m\le n_2)$.

Given a $d$-dimensional flow $\boldsymbol{X}=(X(n); 0\le n\le N_+)$ in $W$, we derive a new $d$-dimensional flow $v_+(\boldsymbol{X})=(v_+(\boldsymbol{X})(n); 0\le n\le N_+)$, to be called a forward $\mathrm{KM_2O}$-Langevin fluctuation flow associated with the flow $\boldsymbol{X}$, by projecting each component of $X(n)$ on the space $\mathrm{M}_0^{n-1}(\boldsymbol{X})$, i.e.,

$$v_+(\boldsymbol{X})(0)\equiv X(0), \tag{2.1}$$

$$v_+(\boldsymbol{X})(n)\equiv X(n)-P_{\mathrm{M}_0^{n-1}(\boldsymbol{X})}X(n) \quad (1\le n\le N_+). \tag{2.2}$$

The matrix function $V_+(\boldsymbol{X})=(V_+(\boldsymbol{X})(n); 0\le n\le N_+)$ obtained by

$$V_+(\boldsymbol{X})(n)\equiv(v_+(\boldsymbol{X})(n), {}^t v_+(\boldsymbol{X})(n)) \quad (0\le n\le N_+) \tag{2.3}$$

is called a forward $\mathrm{KM_2O}$-Langevin fluctuation matrix function associated with the flow $\boldsymbol{X}$, where $(v_+(\boldsymbol{X})(n), {}^t v_+(\boldsymbol{X})(n))$ denotes the inner product matrix of the vector $v_+(\boldsymbol{X})(n)$ and itself.

Given another $d$-dimensional flow $\boldsymbol{Y} = (Y(l); -N_- \le l \le 0)$ in $W$, we derive a backward KM$_2$O-Langevin fluctuation flow $\nu_-(\boldsymbol{Y}) = (\nu_-(\boldsymbol{Y})(l); -N_- \le l \le 0)$ and a backward KM$_2$O-Langevin fluctuation matrix function $V_-(\boldsymbol{Y}) = (V_-(\boldsymbol{Y})(n); 0 \le n \le N_-)$ in a similar way:

$$\nu_-(\boldsymbol{Y})(0) \equiv Y(0), \tag{2.4}$$

$$\nu_-(\boldsymbol{Y})(l) \equiv Y(l) - P_{\mathrm{M}_{l+1}^0(\boldsymbol{Y})} Y(l) \quad (-N_- \le l \le -1), \tag{2.5}$$

$$V_-(\boldsymbol{Y})(n) \equiv (\nu_-(\boldsymbol{Y})(-n), {}^t\nu_-(\boldsymbol{Y})(-n)) \quad (0 \le n \le N_-). \tag{2.6}$$

We have then the following fundamental theorems.

Theorem 2.1 ([**14**]).

(i) $\mathrm{M}_0^n(\boldsymbol{X}) = \mathrm{M}_0^n(\nu_+(\boldsymbol{X}))$ $(0 \le n \le N_+)$,

(ii) $(X(m), {}^t\nu_+(\boldsymbol{X})(n)) = 0$ $(0 \le m < n \le N_+)$,

(iii) $(\nu_+(\boldsymbol{X})(m), {}^t\nu_+(\boldsymbol{X})(n)) = \delta_{mn} V_+(\boldsymbol{X})(n)$ $(0 \le m, n \le N_+)$.

Theorem 2.2 ([**14**]).

(i) $\mathrm{M}_{-n}^0(\boldsymbol{Y}) = \mathrm{M}_{-n}^0(\nu_-(\boldsymbol{Y}))$ $(0 \le n \le N_-)$,

(ii) $(Y(-m), {}^t\nu_-(\boldsymbol{Y})(-n)) = 0$ $(0 \le m < n \le N_-)$,

(iii) $(\nu_-(\boldsymbol{Y})(-m), {}^t\nu_-(\boldsymbol{Y})(-n)) = \delta_{mn} V_-(\boldsymbol{Y})(n)$ $(0 \le m, n \le N_-)$.

Let $\boldsymbol{X} = (X(n); 0 \le n \le N_+)$ and $\boldsymbol{Y} = (Y(l); -N_- \le l \le 0)$ be two $d$-dimensional flows in $W$. Then there exist two matrix functions $\gamma_+ = (\gamma_+(n,k); 0 \le k < n \le N_+)$ and $\gamma_- = (\gamma_-(n,k); 0 \le k < n \le N_-)$ that satisfy

$$P_{\mathrm{M}_0^{n-1}(\boldsymbol{X})} X(n) = -\sum_{k=0}^{n-1} \gamma_+(n,k) X(k) \quad (1 \le n \le N_+), \tag{2.7}$$

$$P_{\mathrm{M}_{-n+1}^0(\boldsymbol{Y})} Y(-n) = -\sum_{k=0}^{n-1} \gamma_-(n,k) Y(-k) \quad (1 \le n \le N_-). \tag{2.8}$$

We call the function $\gamma_+$ (resp. $\gamma_-$) a forward (resp. backward) KM$_2$O-Langevin dissipation matrix function associated with the flow $\boldsymbol{X}$ (resp. $\boldsymbol{Y}$). In particular, we put

$$\delta_+(n) \equiv \gamma_+(n,0) \quad (1 \le n \le N_+), \tag{2.9}$$

$$\delta_-(n) \equiv \gamma_-(n,0) \quad (1 \le n \le N_-) \tag{2.10}$$

and call the function $\delta_+ = (\delta_+(n); 1 \le n \le N_+)$ (resp. $\delta_- = (\delta_-(n); 1 \le n \le N_-)$) a forward (resp. backward) KM$_2$O-Langevin partial correlation matrix function associated with the flow $\boldsymbol{X}$ (resp. $\boldsymbol{Y}$).

From (2.1), (2.2), (2.4) and (2.5), we can now derive the following form:

$$X(0) = \nu_{+}(\boldsymbol{X})(0), \tag{2.11}$$

$$X(n) = -\sum_{k=0}^{n-1} \gamma_{+}(n,k)X(k) + \nu_{+}(\boldsymbol{X})(n) \quad (1 \le n \le N_{+}), \tag{2.12}$$

$$Y(0) = \nu_{-}(\boldsymbol{Y})(0), \tag{2.13}$$

$$Y(-n) = -\sum_{k=0}^{n-1} \gamma_{-}(n,k)Y(-k) + \nu_{-}(\boldsymbol{Y})(-n) \quad (1 \le n \le N_{-}). \tag{2.14}$$

We call equation (2.12) (resp. (2.14)) a forward (resp. backward) $KM_2O$-Langevin equation with initial condition (2.11) (resp. (2.13)) which describes the time evolution of the $d$-dimensional flow $\boldsymbol{X}$ (resp. $\boldsymbol{Y}$) in $W$.

In general, $KM_2O$-Langevin dissipation matrix functions $\gamma_+$ and $\gamma_-$ are not uniquely determined. For this reason, we define two kinds of sets $\mathscr{LMD}_+(\boldsymbol{X})$, $\mathscr{LMD}_-(\boldsymbol{Y})$ by

$$\begin{aligned}\mathscr{LMD}_+(\boldsymbol{X}) \equiv \{\gamma_+ = (&\gamma_+(n,k); 0 \le k < n \le N_+);\\ &\gamma_+ \text{ is a forward } KM_2O\text{-Langevin dissipation}\\ &\text{matrix function associated with the flow } \boldsymbol{X}\},\end{aligned} \tag{2.15}$$

$$\begin{aligned}\mathscr{LMD}_-(\boldsymbol{Y}) \equiv \{\gamma_- = (&\gamma_-(n,k); 0 \le k < n \le N_-);\\ &\gamma_- \text{ is a backward } KM_2O\text{-Langevin dissipation}\\ &\text{matrix function associated with the flow } \boldsymbol{Y}\}.\end{aligned} \tag{2.16}$$

Remark 2.1. By using a weight transformation to be able to transform any degenerate flow into a non-degenerate flow, we have obtained in [**4**] a constructive method for finding the elements $\gamma_+^0, \gamma_-^0$ of $\mathscr{LMD}_+(\boldsymbol{X}), \mathscr{LMD}_-(\boldsymbol{Y})$ which have the smallest norms in the sets $\mathscr{LMD}_+(\boldsymbol{X}), \mathscr{LMD}_-(\boldsymbol{Y})$, respectively.

Definition 2.1. A flow $\boldsymbol{X} = (X(n); 0 \le n \le N_+)$ is said to be non-degenerate if $\boldsymbol{X}$ satisfies the following condition (2.17):

$$\{X_j(n); 1 \le j \le d, 0 \le n \le N_+ - 1\} \text{ is linearly independent in } W. \tag{2.17}$$

Otherwise, $\boldsymbol{X}$ is said to be degenerate. Similarly, a flow $\boldsymbol{Y} = (Y(l); -N_- \le l \le 0)$ is said to be non-degenerate if $\boldsymbol{Y}$ satisfies the following condition (2.18):

$$\{Y_j(l); 1 \le j \le d, -N_- + 1 \le l \le 0\} \text{ is linearly independent in } W. \tag{2.18}$$

Otherwise, $\boldsymbol{Y}$ is said to be degenerate.

Suppose we are given a single $d$-dimensional flow $\boldsymbol{X} = (X(n); 0 \le n \le N)$ in the real inner product space $W$. We introduce a new flow $\boldsymbol{X}^{(rev)} = (X^{(rev)}(l); -N \le l \le 0)$ by inverting the time evolution of the flow $\boldsymbol{X}$ by

$$X^{(rev)}(l) \equiv X(N+l) \quad (-N \le l \le 0). \tag{2.19}$$

By applying the results above to the natural pair $[\boldsymbol{X}, \boldsymbol{X}^{(rev)}]$ of flows to any fixed element $\gamma_+$ of $\mathscr{LMD}_+(\boldsymbol{X})$ and element $\gamma_-$ of $\mathscr{LMD}_-(\boldsymbol{X}^{(rev)})$, we can derive the following equations:

$$X(0) = \nu_+(\boldsymbol{X})(0), \tag{2.20}$$

$$X(n) = -\sum_{k=0}^{n-1} \gamma_+(n,k)X(k) + \nu_+(\boldsymbol{X})(n) \quad (1 \le n \le N), \tag{2.21}$$

$$X(N) = \nu_-(\boldsymbol{X}^{(rev)})(0), \tag{2.22}$$

$$X(N-n) = -\sum_{k=0}^{n-1} \gamma_-(n,k)X(N-k) + \nu_-(\boldsymbol{X}^{(rev)})(-n) \quad (1 \le n \le N). \tag{2.23}$$

We call equation (2.21) (resp. (2.23)) a forward (resp. backward) $\mathrm{KM_2O}$-Langevin equation with initial condition (2.20) (resp. (2.22)) which describes the time evolution of the $d$-dimensional flow $\boldsymbol{X}$ in $W$.

We define the set $\mathscr{LM}(\boldsymbol{X})$ by

$$\begin{aligned} \mathscr{LM}(\boldsymbol{X}) \equiv \{(\gamma_+, \gamma_-, V_+(\boldsymbol{X}), V_-(\boldsymbol{X}^{(rev)})); \\ \gamma_+ = (\gamma_+(n,k); 0 \le k < n \le N) \in \mathscr{LMD}_+(\boldsymbol{X}), \\ \gamma_- = (\gamma_-(n,k); 0 \le k < n \le N) \in \mathscr{LMD}_-(\boldsymbol{X}^{(rev)})\}. \end{aligned} \tag{2.24}$$

and call it the system of $\mathrm{KM_2O}$-Langevin matrix functions associated with the flow $\boldsymbol{X}$.

## 3. Covariance matrix functions.

Let $\boldsymbol{X} = (X(n); l \le n \le r)$ be any $d$-dimensional flow in an real inner product space $W$. For any integers $m, n$ $(l \le m, n \le r)$, we denote by $R(\boldsymbol{X})(m,n)$ the inner product matrix of $X(m)$ and $X(n)$, i.e.,

$$R(\boldsymbol{X})(m,n) \equiv (X(m), {}^tX(n)) \quad (l \le m, n \le r). \tag{3.1}$$

We call the matrix function $R(\boldsymbol{X}) = (R(\boldsymbol{X})(m,n); l \le m, n \le r)$ the covariance matrix function of the flow $\boldsymbol{X}$. From definition, $R(\boldsymbol{X})$ satisfies

$${}^tR(\boldsymbol{X})(m,n) = R(\boldsymbol{X})(n,m) \quad (l \le m, n \le r). \tag{3.2}$$

Let $\boldsymbol{X} = (X(n); 0 \le n \le N_+)$ and $\boldsymbol{Y} = (Y(l); -N_- \le l \le 0)$ be two $d$-dimensional flows in $W$. For each natural numbers $n, m$ $(1 \le n \le N_+ + 1, 1 \le m \le N_- + 1)$, we define two kinds of $nd \times nd$ symmetric matrix $T(\boldsymbol{X})(n)$ and $md \times md$ symmetric matrix $T(\boldsymbol{Y})(m)$ by

$$T(\boldsymbol{X})(n) \equiv \begin{pmatrix} R(\boldsymbol{X})(0,0) & R(\boldsymbol{X})(0,1) & \cdots & R(\boldsymbol{X})(0,n-1) \\ R(\boldsymbol{X})(1,0) & R(\boldsymbol{X})(1,1) & \cdots & R(\boldsymbol{X})(1,n-1) \\ \vdots & \vdots & \ddots & \vdots \\ R(\boldsymbol{X})(n-1,0) & R(\boldsymbol{X})(n-1,1) & \cdots & R(\boldsymbol{X})(n-1,n-1) \end{pmatrix} \tag{3.3}$$

and

(3.4)

$$T(\boldsymbol{Y})(m) \equiv \begin{pmatrix} R(\boldsymbol{Y})(0,0) & R(\boldsymbol{Y})(0,-1) & \cdots & R(\boldsymbol{Y})(0,-m+1) \\ R(\boldsymbol{Y})(-1,0) & R(\boldsymbol{Y})(-1,-1) & \cdots & R(\boldsymbol{Y})(-1,-m+1) \\ \vdots & \vdots & \ddots & \vdots \\ R(\boldsymbol{Y})(-m+1,0) & R(\boldsymbol{Y})(-m+1,-1) & \cdots & R(\boldsymbol{Y})(-m+1,-m+1) \end{pmatrix}.$$

Proposition 3.1.

(i) $T(\boldsymbol{X})(n) \geq 0 \ (1 \leq n \leq N_+ + 1)$,

(ii) $T(\boldsymbol{Y})(n) \geq 0 \ (1 \leq n \leq N_- + 1)$.

Proof. Property (i) follows from the fact

$$T(\boldsymbol{X})(n) = \left( \begin{pmatrix} X(0) \\ X(1) \\ \vdots \\ X(n-1) \end{pmatrix}, {}^t\begin{pmatrix} X(0) \\ X(1) \\ \vdots \\ X(n-1) \end{pmatrix} \right). \tag{3.5}$$

Property (ii) is proved in a similar way. □

Throughout this section, we will fix any elements $\gamma_+ = (\gamma_+(n,k); 0 \leq k < n \leq N_+)$, $\gamma_- = (\gamma_-(n,k); 0 \leq k < n \leq N_-)$ of $\mathscr{LMD}_+(\boldsymbol{X}), \mathscr{LMD}_-(\boldsymbol{Y})$, respectively, except for Lemma 3.6. We will show some lemmas for later discussion.

Lemma 3.1.

(i) $R(\boldsymbol{X})(n,l) = -\sum_{k=0}^{n-1} \gamma_+(n,k) R(\boldsymbol{X})(k,l) \ (1 \leq n \leq N_+, 0 \leq l \leq n-1)$,

(ii) $R(\boldsymbol{Y})(-n,-l) = -\sum_{k=0}^{n-1} \gamma_-(n,k) R(\boldsymbol{Y})(-k,-l) \ (1 \leq n \leq N_-, 0 \leq l \leq n-1)$.

Lemma 3.2.

(i) $V_+(\boldsymbol{X})(n) = \sum_{k=0}^{n-1} \gamma_+(n,k) R(\boldsymbol{X})(k,n) + R(\boldsymbol{X})(n,n) \ (1 \leq n \leq N_+)$,

(ii) $V_-(\boldsymbol{Y})(n) = \sum_{k=0}^{n-1} \gamma_-(n,k) R(\boldsymbol{Y})(-k,-n) + R(\boldsymbol{Y})(-n,-n) \ (1 \leq n \leq N_-)$.

Proof. By taking the inner product of both-hand sides in the forward $\mathrm{KM_2O}$-Langevin equation (2.12) and the vector $X(l)$ $(0 \leq l \leq n)$, we have Lemmas 3.1(i) and 3.2(i). Lemmas 3.1(ii) and 3.2(ii) are similarly proved. □

For any natural numbers $n, m$ $(1 \le n \le N_+ + 1, 1 \le m \le N_- + 1)$, we define other $nd \times nd$ matrix $G(\boldsymbol{X})(n)$ and $md \times md$ matrix $G(\boldsymbol{Y})(m)$ in the following way.

$$G(\boldsymbol{X})(n) \equiv \begin{pmatrix} I_d & & & & \\ \gamma_+(1,0) & I_d & & 0 & \\ \gamma_+(2,0) & \gamma_+(2,1) & I_d & & \\ \vdots & \vdots & & \ddots & \\ \gamma_+(n-1,0) & \gamma_+(n-1,1) & \cdots & \gamma_+(n-1,n-2) & I_d \end{pmatrix}, \tag{3.6}$$

$$G(\boldsymbol{Y})(m) \equiv \begin{pmatrix} I_d & & & & \\ \gamma_-(1,0) & I_d & & 0 & \\ \gamma_-(2,0) & \gamma_-(2,1) & I_d & & \\ \vdots & \vdots & & \ddots & \\ \gamma_-(m-1,0) & \gamma_-(m-1,1) & \cdots & \gamma_-(m-1,m-2) & I_d \end{pmatrix}, \tag{3.7}$$

where $I_d$ stands for the $d \times d$ identity matrix. Immediately from Lemmas 3.1 and 3.2, we have

Lemma 3.3. *For each natural numbers* $n, m$ $(1 \le n \le N_+ + 1, 1 \le m \le N_- + 1)$,

(i) $$G(\boldsymbol{X})(n)T(\boldsymbol{X})(n)^tG(\boldsymbol{X})(n) = \begin{pmatrix} V_+(\boldsymbol{X})(0) & & & 0 \\ & V_+(\boldsymbol{X})(1) & & \\ & & \ddots & \\ 0 & & & V_+(\boldsymbol{X})(n-1) \end{pmatrix},$$

(ii) $$G(\boldsymbol{Y})(m)T(\boldsymbol{Y})(m)^tG(\boldsymbol{Y})(m) = \begin{pmatrix} V_-(\boldsymbol{Y})(0) & & & 0 \\ & V_-(\boldsymbol{Y})(1) & & \\ & & \ddots & \\ 0 & & & V_-(\boldsymbol{Y})(m-1) \end{pmatrix}.$$

Lemma 3.4. *For each natural number* $n$ $(1 \le n \le N_+ + 1)$, *the following three conditions are equivalent with each other*:

(a-i) $T(\boldsymbol{X})(n) > 0$;

(a-ii) $V_+(\boldsymbol{X})(k) > 0$ $(0 \le k \le n-1)$;

(a-iii) $\{X_j(l); 1 \le j \le d, 0 \le l \le n-1\}$ *is linearly independent in* $W$.

*Similarly, for each natural number* $n$ $(1 \le n \le N_- + 1)$, *the following three conditions are equivalent with each other*:

(b-i) $T(\boldsymbol{Y})(n) > 0$;

(b-ii) $V_-(\boldsymbol{Y})(k) > 0$ $(0 \le k \le n-1)$;

(b-iii) $\{Y_j(l); 1 \le j \le d, -n+1 \le l \le 0\}$ *is linearly independent in* $W$.

Proof. By virtue of Lemma 3.3, we see that (a-i) and (a-ii) are equivalent. Moreover, (a-i) and (a-iii) are equivalent, because of (3.5). The same proof can be applied to the flow $\boldsymbol{Y}$. $\square$

By noting that $G(\boldsymbol{X})(n)$ and $G(\boldsymbol{Y})(m)$ are invertible, we see from Lemma 3.3 that

Lemma 3.5. *For each natural numbers* $n, m$ $(1 \leq n \leq N_+ + 1, 1 \leq m \leq N_- + 1)$,

$$\text{(i)} \quad T(\boldsymbol{X})(n) = G^{-1}(\boldsymbol{X})(n) \begin{pmatrix} V_+(\boldsymbol{X})(0) & & & 0 \\ & V_+(\boldsymbol{X})(1) & & \\ & & \ddots & \\ 0 & & & V_+(\boldsymbol{X})(n-1) \end{pmatrix} {}^t G^{-1}(\boldsymbol{X})(n),$$

$$\text{(ii)} \quad T(\boldsymbol{Y})(m) = G^{-1}(\boldsymbol{Y})(m) \begin{pmatrix} V_-(\boldsymbol{Y})(0) & & & 0 \\ & V_-(\boldsymbol{Y})(1) & & \\ & & \ddots & \\ 0 & & & V_-(\boldsymbol{Y})(m-1) \end{pmatrix} {}^t G^{-1}(\boldsymbol{Y})(m).$$

In Lemma 3.5, both $G^{-1}(\boldsymbol{X})(n)$ and $G^{-1}(\boldsymbol{Y})(n)$ have the form

$$\begin{pmatrix} I_d & & & \\ & I_d & 0 & \\ & * & \ddots & \\ & & & I_d \end{pmatrix}.$$

In general, any $ld \times ld$ nonnegative definite matrix $T$ can be factorized as follows:

$$T = LD{}^tL, \tag{3.8}$$

where $L$ is an $ld \times ld$ lower triangular matrix and $D$ is an $ld \times ld$ block diagonal matrix of the form such that

$$L = \begin{pmatrix} I_d & & & \\ & I_d & 0 & \\ & * & \ddots & \\ & & & I_d \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} V_1 & & & 0 \\ & V_2 & & \\ & & \ddots & \\ 0 & & & V_n \end{pmatrix},$$

where $V_i$ $(1 \leq i \leq n)$ are $d \times d$ matrices. We call this factorization a $d$-dimensional block Cholesky factorization. We should note that this factorization is not unique when $T$ is singular. Lemma 3.5 gives a $d$-dimensional block Cholesky factorizations of $T(\boldsymbol{X})(n)$ and $T(\boldsymbol{Y})(m)$. The converse is also true in the following meaning.

Lemma 3.6. *Let* $L_+ D_+ {}^t L_+$ *and* $L_- D_- {}^t L_-$ *be any d-dimensional Cholesky factorizations of* $T(\boldsymbol{X})(N_+ + 1)$ *and* $T(\boldsymbol{Y})(N_- + 1)$, *respectively. We divide* $L_\pm^{-1}$ *and* $D_\pm$ *into submatrices as follows*:

$$L_{\pm}^{-1} = \begin{pmatrix} I_d & & & & \\ \gamma_{\pm}(1,0) & I_d & & 0 & \\ \gamma_{\pm}(2,0) & \gamma_{\pm}(2,1) & I_d & & \\ \vdots & \vdots & & \ddots & \\ \gamma_{\pm}(N_{\pm},0) & \gamma_{\pm}(N_{\pm},1) & \cdots & \gamma_{\pm}(N_{\pm},N_{\pm}-1) & I_d \end{pmatrix},$$

$$D_{\pm} = \begin{pmatrix} V_{\pm}(0) & & & 0 \\ & V_{\pm}(1) & & \\ & & \ddots & \\ 0 & & & V_{\pm}(N_{\pm}) \end{pmatrix},$$

*where* $\gamma_+(n,k), \gamma_-(m,l), V_+(p), V_-(q)$ $(0 \le k < n \le N_+, 0 \le l < m \le N_-, 0 \le p \le N_+, 0 \le q \le N_-)$ *are* $d \times d$ *matrices. Then the matrix functions* $\gamma_+ = (\gamma_+(n,k);\ 0 \le k < n \le N_+), \gamma_- = (\gamma_-(m,l); 0 \le l < m \le N_-)$ *belong to the sets* $\mathscr{LMD}_+(\boldsymbol{X})$, $\mathscr{LMD}_-(\boldsymbol{Y})$, *respectively. Moreover, the matrix functions* $V_+ = (V_+(n); 0 \le n \le N_+)$ (*resp.* $V_- = (V_-(n); 0 \le n \le N_-)$) *is a forward* (*resp. backward*) $\mathrm{KM_2O}$-*Langevin fluctuation matrix function associated with the flow* $\boldsymbol{X}$ (*resp.* $\boldsymbol{Y}$).

Proof. First we note

$$D_+ = L_+^{-1} T(\boldsymbol{X})(N_+ + 1)\,{}^t L_+^{-1}. \tag{3.9}$$

We define a $d$-dimensional flow $v_+ = (v_+(n); 0 \le n \le N_+)$ by

$$v_+(0) \equiv X(0), \tag{3.10}$$

$$v_+(n) \equiv X(n) + \sum_{k=0}^{n-1} \gamma_+(n,k) X(k) \quad (1 \le n \le N_+). \tag{3.11}$$

Definitions (3.10), (3.11) can be rewritten as

$$\begin{pmatrix} v_+(0) \\ v_+(1) \\ \vdots \\ v_+(N_+) \end{pmatrix} = L_+^{-1} \begin{pmatrix} X(0) \\ X(1) \\ \vdots \\ X(N_+) \end{pmatrix}. \tag{3.12}$$

Then by virtue of (3.9), we have

$$\left( \begin{pmatrix} v_+(0) \\ v_+(1) \\ \vdots \\ v_+(N_+) \end{pmatrix}, {}^t \begin{pmatrix} v_+(0) \\ v_+(1) \\ \vdots \\ v_+(N_+) \end{pmatrix} \right) = D_+, \tag{3.13}$$

which shows

$$(v_+(m), {}^t v_+(n)) = 0 \quad (0 \le m, n \le N_+, m \ne n), \tag{3.14}$$

$$(v_+(n), {}^t v_+(n)) = V_+(n) \quad (0 \le n \le N_+). \tag{3.15}$$

Hence, by (3.10), (3.11), and (3.14), the flow $v_+ = (v_+(n); 0 \le n \le N_+)$ is a forward $KM_2O$-Langevin fluctuation flow associated with the flow $\boldsymbol{X}$ and the function $\gamma_+ = (\gamma_+(n,k); 0 \le k < n \le N_+)$ is a forward $KM_2O$-Langevin dissipation matrix function associated with the flow $\boldsymbol{X}$. Moreover, from (3.15), we see that the matrix function $V_+ = (V_+(n); 0 \le n \le N_+)$ is a forward $KM_2O$-Langevin fluctuation matrix function associated with the flow $\boldsymbol{X}$. The latter part of the lemma is also proved in the same way. □

## 4. Relations between the system of $KM_2O$-Langevin matrix functions and the covariance matrix function.

Let $(W, (\star, *))$ be any real inner product space with an inner product $(\star, *)$ and let $\boldsymbol{X} = (X(n); 0 \le n \le N)$ be any $d$-dimensional flow in the space $W$. In the previous section, we saw that the $KM_2O$-Langevin matrices and the covariance matrix functions are connected by Cholesky factorizations.

For further insight into their relations, for each integer $s$ $(0 \le s \le N)$, we define new flows $\boldsymbol{X}_+^{(s)} = (X_+^{(s)}(n); 0 \le n \le N-s)$ and $\boldsymbol{X}_-^{(s)} = (X_-^{(s)}(l); -s \le l \le 0)$ by

$$X_+^{(s)}(n) \equiv X(n+s) \quad (0 \le n \le N-s), \tag{4.1}$$

$$X_-^{(s)}(l) \equiv X(l+s) \quad (-s \le l \le 0). \tag{4.2}$$

We define the set $\widetilde{\mathscr{LM}}(\boldsymbol{X})$ by

$$\begin{aligned}
\widetilde{\mathscr{LM}}(\boldsymbol{X}) \equiv \{ & (\tilde{\gamma}_+, \tilde{\gamma}_-, \tilde{V}_+(\boldsymbol{X}), \tilde{V}_-(\boldsymbol{X})); \\
& \tilde{\gamma}_+ = (\tilde{\gamma}_+(s,n,k) \equiv \gamma_+^{(s)}(n,k); 0 \le s \le N-1, 1 \le n \le N-s, 0 \le k < n), \\
& \tilde{\gamma}_- = (\tilde{\gamma}_-(s,n,k) \equiv \gamma_-^{(s)}(n,k); 1 \le s \le N, 1 \le n \le s, 0 \le k < n), \\
& \tilde{V}_+(\boldsymbol{X}) = (\tilde{V}_+(\boldsymbol{X})(s,n) \equiv V_+(\boldsymbol{X}_+^{(s)})(n); 0 \le s \le N, 0 \le n \le N-s), \\
& \tilde{V}_-(\boldsymbol{X}) = (\tilde{V}_-(\boldsymbol{X})(s,n) \equiv V_-(\boldsymbol{X}_-^{(s)})(n); 0 \le s \le N, 0 \le n \le s), \\
& \gamma_+^{(s)} \equiv (\gamma_+^{(s)}(n,k); 1 \le n \le N-s, 0 \le k < n) \in \mathscr{LMD}_+(\boldsymbol{X}_+^{(s)}) \quad (0 \le s \le N-1), \\
& \gamma_-^{(s)} \equiv (\gamma_-^{(s)}(n,k); 1 \le n \le s, 0 \le k < n) \in \mathscr{LMD}_-(\boldsymbol{X}_-^{(s)}) \quad (1 \le s \le N)\}.
\end{aligned} \tag{4.3}$$

The aim of this section is to study the structure of the set $\widetilde{\mathscr{LM}}(\boldsymbol{X})$. For that purpose, we investigate the relations between $\mathscr{LMD}_+(\boldsymbol{X}_+^{(s)}), \mathscr{LMD}_-(\boldsymbol{X}_-^{(s)})$ and the covariance matrix function of the flow $\boldsymbol{X}$ by running $s$ from $s = 0$ to $s = N$ and then to prove the fluctuation-dissipation theorem which can be regarded as a generalization and a refinement of the fluctuation-dissipation theorem for non-degenerate stationary flows in [**14**].

Throughout this section, for each integer $s$ $(0 \le s \le N)$, we will fix any elements $\gamma_+^{(s)} = (\gamma_+^{(s)}(n,k); 0 \le k < n \le N-s), \gamma_-^{(s)} = (\gamma_-^{(s)}(n,k); 0 \le k < n \le s)$ of $\mathscr{LMD}_+(\boldsymbol{X}_+^{(s)})$, $\mathscr{LMD}_-(\boldsymbol{X}_-^{(s)})$, respectively. First of all, we have from (3.1) that

PROPOSITION 4.1. *For each integer $s$ $(0 \le s \le N)$,*

(i) $R(\boldsymbol{X}_+^{(s)})(m,n) = R(\boldsymbol{X})(s+m, s+n)$ $(0 \le m, n \le N-s)$,

(ii) $R(\boldsymbol{X}_-^{(s)})(-m,-n) = R(\boldsymbol{X})(s-m, s-n)$ $(0 \le m, n \le s)$.

Putting $n = m = 0$ in Proposition 4.1, we have

PROPOSITION 4.2. *For each integer $s$ $(0 \le s \le N)$,*

$$V_\pm(\boldsymbol{X}_\pm^{(s)})(0) = R(\boldsymbol{X})(s,s).$$

Furthermore, by virtue of Lemmas 3.1, 3.2, and Proposition 4.1, we have the following Lemma 4.1, Corollary 4.1 and Lemma 4.2.

LEMMA 4.1. *For each integer $s$ $(0 \le s \le N)$,*

(i) $$R(\boldsymbol{X})(s+n, s+l) = -\sum_{k=0}^{n-1} \gamma_+^{(s)}(n,k) R(\boldsymbol{X})(s+k, s+l) \quad (0 \le l < n \le N-s),$$

(ii) $$R(\boldsymbol{X})(s-n, s-l) = -\sum_{k=0}^{n-1} \gamma_-^{(s)}(n,k) R(\boldsymbol{X})(s-k, s-l) \quad (0 \le l < n \le s).$$

COROLLARY 4.1.

(i) $R(\boldsymbol{X})(s+1, s) = -\delta_+^{(s)}(1) V_-(\boldsymbol{X}_-^{(s)})(0)$ $(0 \le s \le N-1)$,

(ii) $R(\boldsymbol{X})(s-1, s) = -\delta_-^{(s)}(1) V_+(\boldsymbol{X}_+^{(s)})(0)$ $(1 \le s \le N)$.

LEMMA 4.2. *For each integer $s$ $(0 \le s \le N)$,*

(i) $$V_+(\boldsymbol{X}_+^{(s)})(n) = \sum_{k=0}^{n-1} \gamma_+^{(s)}(n,k) R(\boldsymbol{X})(s+k, s+n) + R(\boldsymbol{X})(s+n, s+n) \quad (1 \le n \le N-s),$$

(ii) $$V_-(\boldsymbol{X}_-^{(s)})(n) = \sum_{k=0}^{n-1} \gamma_-^{(s)}(n,k) R(\boldsymbol{X})(s-k, s-n) + R(\boldsymbol{X})(s-n, s-n) \quad (1 \le n \le s).$$

LEMMA 4.3.

(i) *For any integers $n, s$ $(2 \le n \le N, 0 \le s \le N-n)$,*

$$\sum_{k=1}^{n-1} \gamma_+^{(s)}(n,k) X_+^{(s)}(k) = \sum_{k=1}^{n-1} (\gamma_+^{(s+1)}(n-1, k-1) + \delta_+^{(s)}(n) \gamma_-^{(s+n-1)}(n-1, n-k-1)) X_+^{(s)}(k).$$

(ii) *For any integers $n, s$ $(2 \le n \le N, n \le s \le N)$,*

$$\sum_{k=1}^{n-1} \gamma_-^{(s)}(n,k) X_-^{(s)}(-k) = \sum_{k=1}^{n-1} (\gamma_-^{(s-1)}(n-1, k-1) + \delta_-^{(s)}(n) \gamma_+^{(s-n+1)}(n-1, n-k-1)) X_-^{(s)}(-k).$$

PROOF. Let $l, n$ and $s$ be integers such that $1 \le l < n \le N$, $0 \le s \le N-n$. From Lemma 4.1(i), we have

$$(4.4)\quad R(\boldsymbol{X})(s+n,s+l) = -\delta_+^{(s)}(n)R(\boldsymbol{X})(s,s+l) - \sum_{k=1}^{n-1}\gamma_+^{(s)}(n,k)R(\boldsymbol{X})(s+k,s+l).$$

Replacing $l,n$ and $s$ in Lemma 4.1(i) by $l-1,n-1$, and $s+1$, respectively, we have

$$(4.5)\qquad R(\boldsymbol{X})(s+n,s+l) = -\sum_{k=0}^{n-2}\gamma_+^{(s+1)}(n-1,k)R(\boldsymbol{X})(s+k+1,s+l).$$

Furthermore, replacing $l,n$ and $s$ in Lemma 4.1(ii) by $n-l-1,n-1$, and $s+n-1$, respectively, we have

$$(4.6)\qquad R(\boldsymbol{X})(s,s+l) = -\sum_{k=0}^{n-2}\gamma_-^{(s+n-1)}(n-1,k)R(\boldsymbol{X})(s+n-k-1,s+l).$$

Hence, by substituting (4.5) and (4.6) into the left-hand side of (4.4) and the term $R(\boldsymbol{X})(s,s+l)$ in the right-hand side of (4.4), respectively, we obtain

$$(4.7)\qquad \sum_{k=1}^{n-1}C_k^s R(\boldsymbol{X})(s+k,s+l) = 0,$$

where $C_k^s$ $(1\le k\le n-1,\ 0\le s\le N-n)$ are $d\times d$ matrices defined by

$$(4.8)\qquad C_k^s \equiv \gamma_+^{(s)}(n,k) - \gamma_+^{(s+1)}(n-1,k-1) - \delta_+^{(s)}(n)\gamma_-^{(s+n-1)}(n-1,n-k-1).$$

Here we define an element $Z = {}^t(Z_1,Z_2,\dots,Z_d)$ of $W^d$ by $Z\equiv\sum_{k=1}^{n-1}C_k^s X_+^{(s)}(k)$, which implies that $Z_i\in \mathrm{M}_1^{n-1}(\boldsymbol{X}_+^{(s)})$ $(1\le i\le d)$. On the other hand, by (4.7), we see that $(Z, {}^tX_+^{(s)}(l)) = 0$ $(1\le l\le n-1)$. Hence we find that $Z_i\in(\mathrm{M}_1^{n-1}(\boldsymbol{X}_+^{(s)}))^\perp$ $(1\le i\le d)$, which shows $Z=0$, as required. □

We shall prove the following dissipation-dissipation theorem.

Theorem 4.1 (Dissipation-Dissipation Theorem). *For any given elements $\gamma_+^{(s)}$ of $\mathscr{LMD}_+(\boldsymbol{X}_+^{(s)})$ and $\gamma_-^{(s)}$ of $\mathscr{LMD}_-(\boldsymbol{X}_-^{(s)})$ $(0\le s\le N)$, we can transform other matrices besides $\delta_+^{(s)}(n)$ $(1\le n\le N, 0\le s\le N-n)$, $\delta_-^{(s)}(m)$ $(1\le m\le N, m\le s\le N)$ among them to construct other elements $\gamma_+^{\prime(s)}$ of $\mathscr{LMD}_+(\boldsymbol{X}_+^{(s)})$ and $\gamma_-^{\prime(s)}$ of $\mathscr{LMD}_-(\boldsymbol{X}_-^{(s)})$ such that the following Dissipation-Dissipation Theorem ((DDT)) holds:*

(i) *For any integers $n,k,s$ $(1\le k<n\le N, 0\le s\le N-n)$,*

$$\gamma_+^{\prime(s)}(n,k) = \gamma_+^{\prime(s+1)}(n-1,k-1) + \delta_+^{\prime(s)}(n)\gamma_-^{\prime(s+n-1)}(n-1,n-k-1).$$

(ii) *For any integers $n,k,s$ $(1\le k<n\le N, n\le s\le N)$,*

$$\gamma_-^{\prime(s)}(n,k) = \gamma_-^{\prime(s-1)}(n-1,k-1) + \delta_-^{\prime(s)}(n)\gamma_+^{\prime(s-n+1)}(n-1,n-k-1).$$

Proof. DDT demands nothing for $n=1$. Assume that we do not change matrices $\delta_+^{(s)}(n)$ $(1\le n\le n_0-1, 0\le s\le N-n)$, $\delta_-^{(s)}(m)$ $(1\le m\le n_0-1, m\le s\le N)$ and can construct two systems of $\{\gamma_+^{\prime(s)}(n,k); 0\le k<n\le n_0-1, 0\le s\le N-n\}$ and

$\{\gamma'^{(s)}_{-}(n,k); 0 \le k < n \le n_0 - 1, n \le s \le N\}$ that satisfy DDT. Then we define $d \times d$ matrices $\gamma'^{(s)}_{+}(n_0,k), \gamma'^{(s)}_{-}(n_0,k)$ by

$$\gamma'^{(s)}_{+}(n_0,k) \equiv \gamma'^{(s+1)}_{+}(n_0-1,k-1) + \delta^{(s)}_{+}(n_0)\gamma'^{(s+n_0-1)}_{-}(n_0-1,n_0-k-1) \quad (1 \le k \le n_0-1,\ 0 \le s \le N-n_0), \tag{4.9}$$

$$\gamma'^{(s)}_{-}(n_0,k) \equiv \gamma'^{(s-1)}_{-}(n_0-1,k-1) + \delta^{(s)}_{-}(n_0)\gamma'^{(s-n_0+1)}_{+}(n_0-1,n_0-k-1) \quad (1 \le k \le n_0-1, n_0 \le s \le N), \tag{4.10}$$

$$\delta'^{(s)}_{+}(n_0) \equiv \delta^{(s)}_{+}(n_0) \quad (0 \le s \le N-n_0), \tag{4.11}$$

$$\delta'^{(s)}_{-}(n_0) \equiv \delta^{(s)}_{-}(n_0) \quad (n_0 \le s \le N). \tag{4.12}$$

Then from (4.9), (4.11) we obtain

$$-\sum_{k=0}^{n_0-1} \gamma'^{(s)}_{+}(n_0,k)X^{(s)}_{+}(k) = -\delta^{(s)}_{+}(n_0)X^{(s)}_{+}(0) - \sum_{k=1}^{n_0-1}\{\gamma'^{(s+1)}_{+}(n_0-1,k-1) + \delta^{(s)}_{+}(n_0)\gamma'^{(s+n_0-1)}_{-}(n_0-1,n_0-k-1)\}X^{(s)}_{+}(k).$$

Applying Lemma 4.3(i) to the second term of the right-hand side above, we have

$$\begin{aligned}-\sum_{k=0}^{n_0-1} \gamma'^{(s)}_{+}(n_0,k)X^{(s)}_{+}(k) &= -\delta^{(s)}_{+}(n_0)X^{(s)}_{+}(0) - \sum_{k=1}^{n_0-1} \gamma^{(s)}_{+}(n_0,k)X^{(s)}_{+}(k) \\ &= -\sum_{k=0}^{n_0-1} \gamma^{(s)}_{+}(n_0,k)X^{(s)}_{+}(k),\end{aligned}$$

which gives

$$-\sum_{k=0}^{n_0-1} \gamma'^{(s)}_{+}(n_0,k)X^{(s)}_{+}(k) = P_{\mathrm{M}_0^{n_0-1}(\boldsymbol{X}^{(s)}_{+})}X^{(s)}_{+}(n_0).$$

Similary we have

$$-\sum_{k=0}^{n_0-1} \gamma'^{(s)}_{-}(n_0,k)X^{(s)}_{-}(-k) = P_{\mathrm{M}^0_{-n_0+1}(\boldsymbol{X}^{(s)}_{-})}X^{(s)}_{-}(-n_0).$$

Therefore, we see from (4.9), (4.10), and the assumption of the induction that two systems of $\{\gamma'^{(s)}_{+}(n,k), \gamma'^{(u)}_{+}(n_0,j); 0 \le k < n \le n_0-1, 0 \le j \le n_0-1, 0 \le s \le N-n, 0 \le u \le N-n_0\}$, and $\{\gamma'^{(s)}_{-}(n,k), \gamma'^{(u)}_{-}(n_0,j); 0 \le k < n \le n_0-1, 0 \le j \le n_0-1, n \le s \le N, n_0 \le u \le N\}$ become $\mathrm{KM_2O}$-Langevin dissipation matrix functions that satisfy DDT for $n \le n_0$.

Therefore, we have proved Theorem 4.1. □

REMARK 4.1. By using the weight transformation, we can show from Lemma 4.3 that DDT holds for the minimum $\mathrm{KM_2O}$-Langevin dissipation matrix function $\gamma^0_{+}(\boldsymbol{X}^{(s)}_{+})$ of the flow $\boldsymbol{X}^{(s)}_{+}$ $(0 \le s \le N)$ and the minimum $\mathrm{KM_2O}$-Langevin dissipation matrix function $\gamma^0_{-}(\boldsymbol{X}^{(s)}_{-})$ of the flow $\boldsymbol{X}^{(s)}_{-}$ $(0 \le s \le N)$.

Next, we shall prove the fluctuation-dissipation theorem. For that purpose, we shall prepare some lemmas.

Lemma 4.4.

(i) *For any integers* $n, l, s$ $(2 \le n \le N, 0 \le l \le N, 0 \le s \le N-n)$,

$$\sum_{k=1}^{n-1} \gamma_+^{(s)}(n,k) R(\boldsymbol{X})(s+k,l)$$

$$= \sum_{k=1}^{n-1} (\gamma_+^{(s+1)}(n-1,k-1) + \delta_+^{(s)}(n)\gamma_-^{(s+n-1)}(n-1,n-k-1)) R(\boldsymbol{X})(s+k,l).$$

(ii) *For any integers* $n, l, s$ $(2 \le n \le N, 0 \le l \le N, n \le s \le N)$,

$$\sum_{k=1}^{n-1} \gamma_-^{(s)}(n,k) R(\boldsymbol{X})(s-k,l)$$

$$= \sum_{k=1}^{n-1} (\gamma_-^{(s-1)}(n-1,k-1) + \delta_-^{(s)}(n)\gamma_+^{(s-n+1)}(n-1,n-k-1)) R(\boldsymbol{X})(s-k,l).$$

Proof. By taking the inner product of the both-hand sides in Lemma 4.3(i) and the vector $X(l)$, we have (i). Statement (ii) is similarly proved. □

Lemma 4.5.

(i) *For any integers* $n, s$ $(1 \le n \le N-1, 0 \le s \le N-n-1)$,

$$R(\boldsymbol{X})(s+n+1,s) = -\sum_{k=0}^{n-1} \gamma_+^{(s+1)}(n,k) R(\boldsymbol{X})(s+k+1,s) - \delta_+^{(s)}(n+1) V_-(\boldsymbol{X}_-^{(s+n)})(n).$$

(ii) *For any integers* $n, s$ $(1 \le n \le N-1, n+1 \le s \le N)$,

$$R(\boldsymbol{X})(s-n-1,s) = -\sum_{k=0}^{n-1} \gamma_-^{(s-1)}(n,k) R(\boldsymbol{X})(s-k-1,s) - \delta_-^{(s)}(n+1) V_+(\boldsymbol{X}_+^{(s-n)})(n).$$

Proof. Replacing $n$ and $l$ in Lemma 4.1(i) by $n+1$ and 0, we have

$$R(\boldsymbol{X})(s+n+1,s) = -\sum_{k=1}^{n} \gamma_+^{(s)}(n+1,k) R(\boldsymbol{X})(s+k,s) - \delta_+^{(s)}(n+1) R(\boldsymbol{X})(s,s).$$

Furthermore, by replacing $n$ and $l$ in Lemma 4.4(i) by $n+1$ and $s$, and substituting it into the first term of the right-hand side in the above equation, we have

$$R(\boldsymbol{X})(s+n+1,s) = -\sum_{k=1}^{n} \gamma_+^{(s+1)}(n,k-1) R(\boldsymbol{X})(s+k,s)$$

$$- \delta_+^{(s)}(n+1) \left\{ \sum_{k=1}^{n} \gamma_-^{(s+n)}(n,n-k) R(\boldsymbol{X})(s+k,s) + R(\boldsymbol{X})(s,s) \right\}.$$

By Lemma 4.2(ii), the second term of the right-hand side in the above equation is equal to $-\delta_+^{(s)}(n+1)V_-(\boldsymbol{X}_-^{(s+n)})(n)$, which shows that Lemma 4.5(i) holds. We can prove (ii) in a similar fashion. □

Theorem 4.2 (Fluctuation-Dissipation Theorem I).
(i) *For any integers* $n, s$ $(1 \le n \le N, 0 \le s \le N-s)$,

$$V_+(\boldsymbol{X}_+^{(s)})(n) = (I - \delta_+^{(s)}(n)\delta_-^{(s+n)}(n))V_+(\boldsymbol{X}_+^{(s+1)})(n-1).$$

(ii) *For any integers* $n, s$ $(1 \le n \le N, n \le s \le N)$,

$$V_-(\boldsymbol{X}_-^{(s)})(n) = (I - \delta_-^{(s)}(n)\delta_+^{(s-n)}(n))V_-(\boldsymbol{X}_-^{(s-1)})(n-1).$$

Proof. Applying Proposition 4.2 and Corollary 4.1(ii) to the right-hand side in Lemma 4.2(i), we see that Theorem 4.2(i) holds for $n = 1$. For each $n$ $(2 \le n \le N)$, by virtue of Lemma 4.2(i), we have

$$V_+(\boldsymbol{X}_+^{(s)})(n) = \sum_{k=1}^{n-1} \gamma_+^{(s)}(n,k)R(\boldsymbol{X})(s+k, s+n) + \delta_+^{(s)}(n)R(\boldsymbol{X})(s, s+n) + R(\boldsymbol{X})(s+n, s+n).$$

Replacing $l$ in Lemma 4.4(i) by $s+n$, and substituting it into the first term of the right-hand side above, we obtain

$$\begin{aligned} V_+(\boldsymbol{X}_+^{(s)})(n) &= \sum_{k=0}^{n-2} \gamma_+^{(s+1)}(n-1,k)R(\boldsymbol{X})(s+1+k, s+n) + R(\boldsymbol{X})(s+n, s+n) \\ &\quad + \delta_+^{(s)}(n)\left\{ R(\boldsymbol{X})(s, s+n) + \sum_{k=0}^{n-2} \gamma_-^{(s+n-1)}(n-1,k)R(\boldsymbol{X})(s+n-k-1, s+n) \right\}. \end{aligned}$$

By Lemma 4.2(i), the sum of the first and second term of the right-hand side above is equal to $V_+(\boldsymbol{X}_+^{(s+1)})(n-1)$. By Lemma 4.5(ii), the third term of the right-hand side above equals $-\delta_+^{(s)}(n)\delta_-^{(s+n)}(n)V_+(\boldsymbol{X}_+^{(s+1)})(n-1)$. Thus we have (i). Statement (ii) is similarly proved. □

Theorem 4.3 (Burg's relation). *For any integers* $n, s$ $(1 \le n \le N-1, 1 \le s \le N-n)$,

$$\sum_{k=0}^{n-1} R(\boldsymbol{X})(s+n, s+n-1-k)\,{}^t\gamma_-^{(s+n-1)}(n,k) = \sum_{k=0}^{n-1} \gamma_+^{(s)}(n,k)R(\boldsymbol{X})(s+k, s-1).$$

Proof. Let $n$ and $s$ be integers such that $1 \le n \le N-1$, $1 \le s \le N-n$. We define a $2d \times (n+2)d$ matrix $F(n+2)$ by

$$F(n+2) \equiv \begin{pmatrix} 0 & \gamma_+^{(s)}(n,0) & \gamma_+^{(s)}(n,1) & \cdots & \gamma_+^{(s)}(n,n-1) & I \\ I & \gamma_-^{(s+n-1)}(n,n-1) & \gamma_-^{(s+n-1)}(n,n-2) & \cdots & \gamma_-^{(s+n-1)}(n,0) & 0 \end{pmatrix}.$$

We recall

$$T(\boldsymbol{X}_+^{(s-1)})(n+2) = \begin{pmatrix} R(\boldsymbol{X})(s-1,s-1) & R(\boldsymbol{X})(s-1,s) & \cdots & R(\boldsymbol{X})(s-1,s+n) \\ R(\boldsymbol{X})(s,s-1) & R(\boldsymbol{X})(s,s) & \cdots & R(\boldsymbol{X})(s,s+n) \\ \vdots & \vdots & \ddots & \vdots \\ R(\boldsymbol{X})(s+n,s-1) & R(\boldsymbol{X})(s+n,s) & \cdots & R(\boldsymbol{X})(s+n,s+n) \end{pmatrix}.$$

By Lemmas 4.1 and 4.2, we obtain

$$F(n+2)T(\boldsymbol{X}_+^{(s-1)})(n+2) = \begin{pmatrix} A(n,s) & 0 & \cdots & 0 & V_+(\boldsymbol{X}_+^{(s)})(n) \\ V_-(\boldsymbol{X}_-^{(s+n-1)})(n) & 0 & \cdots & 0 & B(n,s) \end{pmatrix}, \tag{4.13}$$

where $A(n,s)$ and $B(n,s)$ are $d \times d$ matrices defined by

$$A(n,s) \equiv R(\boldsymbol{X})(s+n,s-1) + \sum_{k=0}^{n-1} \gamma_+^{(s)}(n,k)R(\boldsymbol{X})(s+k,s-1), \tag{4.14}$$

$$B(n,s) \equiv R(\boldsymbol{X})(s-1,s+n) + \sum_{k=0}^{n-1} \gamma_-^{(s+n-1)}(n,k)R(\boldsymbol{X})(s+n-1-k,s+n). \tag{4.15}$$

Hence we have

$$F(n+2)T(\boldsymbol{X}_+^{(s-1)})(n+2)\,{}^tF(n+2) = \begin{pmatrix} V_+(\boldsymbol{X}_+^{(s)})(n) & A(n,s) \\ B(n,s) & V_-(\boldsymbol{X}_-^{(s+n-1)})(n) \end{pmatrix}. \tag{4.16}$$

By noting that $F(n+2)T(\boldsymbol{X}_+^{(s-1)})(n+2)\,{}^tF(n+2)$ is a symmetric matrix, we find

$${}^tB(n,s) = A(n,s), \tag{4.17}$$

which gives the proof of the theorem. □

Immediately from (4.16), we have

COROLLARY 4.2. *For two integers $n,s$ $(1 \le n \le N-1, 1 \le s \le N-n)$,*

$$\begin{pmatrix} V_+(\boldsymbol{X}_+^{(s)})(n) & A(n,s) \\ B(n,s) & V_-(\boldsymbol{X}_-^{(s+n-1)})(n) \end{pmatrix} \ge 0,$$

*where $A(n,s)$ and $B(n,s)$ are defined by* (4.14) *and* (4.15), *respectively.*

THEOREM 4.4 (Fluctuation-Dissipation Theorem II). *For two integers $n,s$ $(1 \le n \le N, 0 \le s \le N-n)$,*

$$\delta_+^{(s)}(n)V_-(\boldsymbol{X}_-^{(s+n-1)})(n-1) = V_+(\boldsymbol{X}_+^{(s+1)})(n-1)\,{}^t\delta_-^{(s+n)}(n).$$

PROOF. By Corollary 4.1, the statement is true for $n = 1$. Further, by Lemma 4.5 and Theorem 4.3, we have the proof for $n \ge 2$. □

## 5. Construction theorem for general flows.

Let us given any $M(d;\boldsymbol{R})$-valued nonnegative definite function $R = (R(m,n); 0 \le m, n \le N)$ of two variables. The aim of this section is to investigate the structure of the set of all the systems of KM$_2$O-Langevin matrices associated with the matrix function $R$

and then to obtain an algorithm of constructing all $d$-dimensional flows $\boldsymbol{X} = (X(n); 0 \le n \le N)$ whose covariance matrix function are equal to the matrix function $R$. Our result here can be regarded as a generalization of the construction theorem for non-degenerate stationary flows, which was obtained in [**15**]. We follow the same notation in [**15**].

For any natural number $n$ $(1 \le n \le N)$, we define the set $\widetilde{\mathscr{LM}}(R;n)$ of four kinds of matrix functions $\tilde{\gamma}_+, \tilde{\gamma}_-, \tilde{V}_+, \tilde{V}_-$ by

$$(5.1)\quad \widetilde{\mathscr{LM}}(R;n) \equiv \{(\tilde{\gamma}_+, \tilde{\gamma}_-, \tilde{V}_+, \tilde{V}_-);$$

$$\tilde{\gamma}_+ = (\tilde{\gamma}_+(s,m,k) \equiv \gamma_+^{(s)}(m,k); 0 \le s \le N-1,$$
$$1 \le m \le n, m \le N-s, 0 \le k < m),$$

$$\tilde{\gamma}_- = (\tilde{\gamma}_-(s,m,k) \equiv \gamma_-^{(s)}(m,k); 1 \le s \le N, 1 \le m \le n, m \le s, 0 \le k < m),$$

$$\tilde{V}_+ = (\tilde{V}_+(s,m) \equiv V_+^{(s)}(m); 0 \le s \le N, 0 \le m \le n, m \le N-s),$$

$$\tilde{V}_- = (\tilde{V}_-(s,m) \equiv V_-^{(s)}(m); 0 \le s \le N, 0 \le m \le n, m \le s) \text{ satisfy}$$

the following (PAC), (DDT), (FDT).

$$\delta_+^{(s)}(m) V_-^{(s+m-1)}(m-1) = -\left\{ R(s+m,s) + \sum_{k=0}^{m-2} \gamma_+^{(s+1)}(m-1,k) R(s+k+1,s) \right\}$$
$$(1 \le m \le n, 0 \le s \le N-m),$$

$$\delta_-^{(s)}(m) V_+^{(s-m+1)}(m-1) = -\left\{ R(s-m,s) + \sum_{k=0}^{m-2} \gamma_-^{(s-1)}(m-1,k) R(s-k-1,s) \right\}$$
$$(1 \le m \le n, m \le s \le N),$$

$$\gamma_+^{(s)}(m,k) = \gamma_+^{(s+1)}(m-1,k-1) + \delta_+^{(s)}(m) \gamma_-^{(s+m-1)}(m-1,m-k-1)$$
$$(1 \le k < m \le n, 0 \le s \le N-m),$$

$$\gamma_-^{(s)}(m,k) = \gamma_-^{(s-1)}(m-1,k-1) + \delta_-^{(s)}(m) \gamma_+^{(s-m+1)}(m-1,m-k-1)$$
$$(1 \le k < m \le n, m \le s \le N),$$

$$V_+^{(s)}(0) = V_-^{(s)}(0) = R(s,s) \quad (0 \le s \le N),$$
$$V_+^{(s)}(m) = (I - \delta_+^{(s)}(m)\delta_-^{(s+m)}(m)) V_+^{(s+1)}(m-1) \quad (1 \le m \le n, 0 \le s \le N-m),$$
$$V_-^{(s)}(m) = (I - \delta_-^{(s)}(m)\delta_+^{(s-m)}(m)) V_-^{(s-1)}(m-1) \quad (1 \le m \le n, m \le s \le N)\},$$

where $\delta_+^{(s)}(m) \equiv \gamma_+^{(s)}(m,0)$ $(1 \le m \le n, m \le N-s)$, $\delta_-^{(s)}(m) \equiv \gamma_-^{(s)}(m,0)$ $(1 \le m \le n, m \le s)$.

The aim of this section is to give an algorithm for obtaining all the elements of the set $\widetilde{\mathscr{LM}}(R;N)$ by running $n$ from $n=1$ to $n=N$. We note from (DDT) and (FDT) that all the elements of the set $\widetilde{\mathscr{LM}}(R;n)$ can be determined by two kinds of matrix functions $\tilde{\delta}_+, \tilde{\delta}_-$ defined by

(5.2) $$\tilde{\delta}_+ = (\tilde{\delta}_+(s,m) \equiv \delta_+^{(s)}(m); 0 \le s \le N-1, 1 \le m \le n, m \le N-s),$$

(5.3) $$\tilde{\delta}_- = (\tilde{\delta}_-(s,m) \equiv \delta_-^{(s)}(m); 1 \le s \le N, 1 \le m \le n, m \le s).$$

Therefore, for our purpose, we have only to give an algorithm for obtaining all the matrix functions $\tilde{\delta}_+, \tilde{\delta}_-$. We shall give it in the following.

Algorithm 5.1.

[Step 0] *We define* $d \times d$ *matrices* $V_\pm^{(s)}(0)$ *by*

(5.4) $$V_\pm^{(s)}(0) \equiv R(s,s) \quad (0 \le s \le N).$$

[Step 1] We shall prove

Lemma 5.1.

(i) *For any* $s$ $(0 \le s \le N-1)$, *the set of solutions* $Y_+$ $(\in M(d;\boldsymbol{R}))$ *of the following equation*

$(p_1^+)$ $$Y_+ V_-^{(s)}(0) = -R(s+1,s)$$

*is equal to the set of all* $d \times d$ *matrices* $\delta_+^{(s)}(1)$ *defined by*

(5.5) $$\delta_+^{(s)}(1) = -R(s+1,s)V_-^{(s)}(0)^+ + A_+(1)(I - V_-^{(s)}(0)V_-^{(s)}(0)^+),$$

*with an element* $A_+(1)$ *of* $M(d;\boldsymbol{R})$, *where the matrix* $V_-^{(s)}(0)^+$ *is the Moore-Penrose generalized inverse of the matrix* $V_-^{(s)}(0)$.

(ii) *For any* $s$ $(1 \le s \le N)$, *the set of solutions* $Y_-$ $(\in M(d;\boldsymbol{R}))$ *of the following equation*

$(p_1^-)$ $$Y_- V_+^{(s)}(0) = -R(s-1,s)$$

*is equal to the set of all* $\delta_-^{(s)}(1)$ *defined by*

(5.6) $$\delta_-^{(s)}(1) = -R(s-1,s)V_+^{(s)}(0)^+ + A_-(1)(I - V_+^{(s)}(0)V_+^{(s)}(0)^+),$$

*with an element* $A_+(1)$ *of* $M(d;\boldsymbol{R})$, *where the matrix* $V_+^{(s)}(0)^+$ *is the Moore-Penrose generalized inverse of the matrix* $V_+^{(s)}(0)$.

Proof. Since the matrix function $R$ has a nonnegative definite property, we obtain

$$\begin{pmatrix} R(s,s) & R(s,s+1) \\ R(s+1,s) & R(s+1,s+1) \end{pmatrix} \ge 0.$$

With the help of (5.4), we rewrite the above relation as

$$\begin{pmatrix} V_-^{(s)}(0) & R(s,s+1) \\ R(s+1,s) & R(s+1,s+1) \end{pmatrix} \ge 0,$$

which shows by Lemma A.1 in Appendix that there exists a $d \times d$ matrix $Y_+$ such that $Y_+ V_-^{(s)}(0) = -R(s+1,s)$. Hence, there exists a solution of the equation to be solved. Thus, it follows from the theory of generalized inverse matrix that (i) holds. Statement (ii) is similarly proved. □

Therefore, we see that

$$(5.7)\quad \widetilde{\mathscr{LM}}(R;1) = \{(\tilde{\gamma}_+, \tilde{\gamma}_-, \tilde{V}_+, \tilde{V}_-);$$

$$\tilde{\gamma}_+ = (\tilde{\gamma}_+(s,1,0) = \gamma_+^{(s)}(1,0); 0 \le s \le N-1),$$

$$\tilde{\gamma}_- = (\tilde{\gamma}_-(s,1,0) = \gamma_-^{(s)}(1,0); 1 \le s \le N),$$

$$\tilde{V}_+ = (\tilde{V}_+(s,n) = V_+^{(s)}(n); 0 \le s \le N, 0 \le n \le N-s, n \le 1),$$

$$\tilde{V}_- = (\tilde{V}_-(s,n) = V_-^{(s)}(n); 0 \le s \le N, 0 \le n \le s, n \le 1),$$

$$\gamma_+^{(s)}(1,0) = \delta_+^{(s)}(1),$$

$$\gamma_-^{(s)}(1,0) = \delta_-^{(s)}(1),$$

$$V_+^{(s)}(0) = V_-^{(s)}(0) = R(s,s) \quad (0 \le s \le N),$$

$$V_+^{(s)}(1) = (I - \delta_+^{(s)}(1)\delta_-^{(s+1)}(1))V_+^{(s+1)}(0) \quad (0 \le s \le N-1),$$

$$V_-^{(s)}(1) = (I - \delta_-^{(s)}(1)\delta_+^{(s-1)}(1))V_-^{(s-1)}(0) \quad (1 \le s \le N),$$

$$\delta_+^{(s)}(1), \delta_-^{(s)}(1) \text{ are given by (5.5) and (5.6), respectively}\}.$$

Moreover, by Lemma 5.1 and a direct calculation, we can prove

LEMMA 5.2.

(i) $R(s+1,s) = -\delta_+^{(s)}(1)R(s,s)$ $(0 \le s \le N-1)$,

(ii) $R(s-1,s) = -\delta_-^{(s)}(1)R(s,s)$ $(1 \le s \le N)$,

(iii) $V_+^{(s)}(1) = \delta_+^{(s)}(1)R(s,s+1) + R(s+1,s+1)$ $(0 \le s \le N-1)$,

(iv) $V_-^{(s)}(1) = \delta_-^{(s)}(1)R(s,s-1) + R(s-1,s-1)$ $(1 \le s \le N)$.

[Step 2] Let us fix any natural number $n$ $(1 \le n \le N)$ and any element $(\tilde{\gamma}_+, \tilde{\gamma}_-, \tilde{V}_+, \tilde{V}_-)$ of the set $\widetilde{\mathscr{LM}}(R;n)$. Then we shall consider the following equations $(p_n^\pm)$ for $Y_\pm(l,s)$ $(\in M(d;\boldsymbol{R}))$:

$(p_n^+)$ $\quad R(s+l,s) = -\sum_{k=0}^{l-2}\gamma_+^{(s+1)}(l-1,k)R(s+k+1,s) - Y_+(l,s)V_-^{(s+l-1)}(l-1)$ $(2 \le \forall l \le n, 0 \le \forall s \le N-l)$,

$(p_n^-)$ $\quad R(s-l,s) = -\sum_{k=0}^{l-2}\gamma_-^{(s-1)}(l-1,k)R(s-k-1,s) - Y_-(l,s)V_+^{(s-l+1)}(l-1)$ $(2 \le \forall l \le n, l \le \forall s \le N)$.

Concerning these equations, we shall consider the following statements $(b_n^\pm)$:
$(b_n^+)$ The set of solutions $Y_+(l,s)$ of equation $(p_n^+)$ is equal to the set of all $d \times d$ matrices $\delta_+^{(s)}(l)$ defined by

$$(5.8)\quad \delta_+^{(s)}(l) = -\left\{R(s+l,s) + \sum_{k=0}^{l-2}\gamma_+^{(s+1)}(l-1,k)R(s+k+1,s)\right\}V_-^{(s+l-1)}(l-1)^+$$
$$+ A_+(l)(I - V_-^{(s+l-1)}(l-1)V_-^{(s+l-1)}(l-1)^+) \quad (0 \le s \le N-l)$$

with a certain element $A_+(l)$ of $M(d;\boldsymbol{R})$, where $V_-^{(s+l-1)}(l-1)^+$ is the Moore-Penrose generalized inverse of the matrix $V_-^{(s+l-1)}(l-1)$.

$(b_n^-)$ The set of solutions $Y_-(l,s)$ of equation $(p_n^-)$ is equal to the set of all $d\times d$ matrices $\delta_-^{(s)}(l)$ defined by

$$(5.9)\quad \delta_-^{(s)}(l) = -\left\{R(s-l,s) + \sum_{k=0}^{l-2}\gamma_-^{(s-1)}(l-1,k)R(s-k-1,s)\right\}V_+^{(s-l+1)}(l-1)^+ \\ + A_-(l)(I - V_+^{(s-l+1)}(l-1)V_+^{(s-l+1)}(l-1)^+) \quad (l\le s\le N)$$

with a certain element $A_-(l)$ of $M(d;\boldsymbol{R})$, where $V_+^{(s-l+1)}(l-1)^+$ is the Moore-Penrose generalized inverse of the matrix $V_+^{(s-l+1)}(l-1)$.

Moreover, for proving the statements $(b_n^\pm)$, we shall consider the following statements $(e_n^\pm)$ and $(f_n^\pm)$:

$$(e_n^+)\quad V_+^{(s)}(m) = \sum_{k=0}^{m-1}\gamma_+^{(s)}(m,k)R(s+k,s+m) + R(s+m,s+m) \\ (1\le m\le n, 0\le s\le N-m),$$

$$(e_n^-)\quad V_-^{(s)}(m) = \sum_{k=0}^{m-1}\gamma_-^{(s)}(m,k)R(s-k,s-m) + R(s-m,s-m) \\ (1\le m\le n, m\le s\le N),$$

$$(f_n^+)\quad R(s+m,s+l) = -\sum_{k=0}^{m-1}\gamma_+^{(s)}(m,k)R(s+k,s+l) \quad (0\le l<m\le n, 0\le s\le N-m),$$

$$(f_n^-)\quad R(s-m,s-l) = -\sum_{k=0}^{m-1}\gamma_-^{(s)}(m,k)R(s-k,s-l) \quad (0\le l<m\le n, m\le s\le N).$$

Theorem 5.1. *The statements $(b_N^\pm),(e_N^\pm),(f_N^\pm)$ hold.*

Proof. We shall prove the statements $(b_n^\pm),(e_n^\pm),(f_n^\pm)$ by induction with respect to $n$. It follows from Lemmas 5.1 and 5.2 that the statements $(b_1^\pm),(e_1^\pm),(f_1^\pm)$ hold. Let us fix any natural number $n_0$ $(2\le n_0\le N)$ and assume that the statements $(b_n^\pm),(e_n^\pm),(f_n^\pm)$ hold for any $n$ $(1\le n\le n_0-1)$.

First, we shall prove $(b_{n_0}^+)$. By using $(e_{n_0-1}^\pm)$ and $(f_{n_0-1}^\pm)$, we can see from the same calculation as we had in the proof of Theorem 4.3 (Corollary 4.2) that

$$\begin{pmatrix} V_+^{(s+1)}(n_0-1) & A(R)(n_0-1,s+1) \\ B(R)(n_0-1,s+1) & V_-^{(s+n_0-1)}(n_0-1)\end{pmatrix} \ge 0,$$

where

$$A(R)(n_0-1,s+1) \equiv R(s+n_0,s) + \sum_{k=0}^{n_0-2}\gamma_+^{(s+1)}(n_0-1,k)R(s+k+1,s),$$

$$B(R)(n_0-1,s+1) \equiv R(s,s+n_0) + \sum_{k=0}^{n_0-2}\gamma_-^{(s+n_0-1)}(n_0-1,k)R(s+n_0-1-k,s+n_0).$$

So by Lemma A.1 in Appendix, we see that there exists a $d \times d$ matrix $Y$ such that

$$YV_{-}^{(s+n_0-1)}(n_0-1) = -R(s+n_0,s) - \sum_{k=0}^{n_0-2} \gamma_{+}^{(s+1)}(n_0-1,k)R(s+k+1,s).$$

Hence, we see from the theory of generalized inverse matrix that $(b_{n_0}^{+})$ is true. In a similar way, we can prove $(b_{n_0}^{-})$.

Next, we shall prove $(e_{n_0}^{+})$. By (DDT), we have

$$\begin{aligned}
&\sum_{k=0}^{n_0-1} \gamma_{+}^{(s)}(n_0,k)R(s+k,s+n_0) + R(s+n_0,s+n_0) \\
&\quad = \sum_{k=1}^{n_0-1} \gamma_{+}^{(s+1)}(n_0-1,k-1)R(s+k,s+n_0) + R(s+n_0,s+n_0) \\
&\qquad + \delta_{+}^{(s)}(n_0)\left\{ R(s,s+n_0) + \sum_{k=1}^{n_0-1} \gamma_{-}^{(s+n_0-1)}(n_0-1,n_0-k-1)R(s+k,s+n_0) \right\}.
\end{aligned}$$

Since $(e_{n_0-1}^{+})$ holds, we see that the sum of the first term and the second term of the right-hand side in the above equation is equal to $V_{+}^{(s+1)}(n_0-1)$. Moreover, the third term of the right-hand side above equals $-\delta_{+}^{(s)}(n_0)\delta_{-}^{(s+n_0)}(n_0)V_{+}^{(s+1)}(n_0-1)$, because $(b_{n_0}^{-})$ holds. Thus we obtain

$$\sum_{k=0}^{n_0-1} \gamma_{+}^{(s)}(n_0,k)R(s+k,s+n_0) + R(s+n_0,s+n_0) = (I - \delta_{+}^{(s)}(n_0)\delta_{-}^{(s+n_0)}(n_0))V_{+}^{(s+1)}(n_0-1).$$

It follows from (FDT) that the right-hand side above is equal to $V_{+}^{(s)}(n_0)$. Thus $(e_{n_0}^{+})$ is proved. In a similar way, we can prove $(e_{n_0}^{-})$.

Finally, we shall prove that $(f_{n_0}^{+})$ holds for $1 \le l \le n_0-1$, $m = n_0$. Let $l$ be any integer such that $1 \le l \le n_0-1$. From (DDT), we have

$$\begin{aligned}
&-\sum_{k=0}^{n_0-1} \gamma_{+}^{(s)}(n_0,k)R(s+k,s+l) \\
&\quad = -\sum_{k=1}^{n_0-1} \gamma_{+}^{(s+1)}(n_0-1,k-1)R(s+k,s+l) \\
&\qquad - \delta_{+}^{(s)}(n_0)\left\{ R(s,s+l) + \sum_{k=1}^{n_0-1} \gamma_{-}^{(s+n_0-1)}(n_0-1,n_0-k-1)R(s+k,s+l) \right\}.
\end{aligned}$$

By the assumption that $(f_{n_0-1}^{+})$ holds, we see that the first term of the right-hand side in the above equation is equal to $R(s+n_0,s+l)$. Furthermore, the second term vanishes, because $(f_{n_0-1}^{-})$ holds. Thus, $(f_{n_0}^{+})$ holds for $1 \le l \le n_0-1$, $m = n_0$.

We shall now prove that $(f_{n_0}^+)$ is true for $l=0$, $m=n_0$. It follows from (DDT) that

$$
\begin{aligned}
&-\sum_{k=0}^{n_0-1}\gamma_+^{(s)}(n_0,k)R(s+k,s)\\
&\quad=-\sum_{k=1}^{n_0-1}\gamma_+^{(s+1)}(n_0-1,k-1)R(s+k,s)\\
&\qquad-\delta_+^{(s)}(n_0)\left\{\sum_{k=1}^{n_0-1}\gamma_-^{(s+n_0-1)}(n_0-1,n_0-k-1)R(s+k,s)+R(s,s)\right\}.
\end{aligned}
$$

By $(e_{n_0-1}^-)$, the second term of the right-hand side above is equal to $-\delta_+^{(s)}(n_0)V_-^{(s+n_0-1)}(n_0-1)$. Hence, we have

$$
\begin{aligned}
-\sum_{k=0}^{n_0-1}\gamma_+^{(s)}(n_0,k)R(s+k,s)&=-\sum_{k=0}^{n_0-2}\gamma_+^{(s+1)}(n_0-1,k)R(s+k+1,s)\\
&\quad-\delta_+^{(s)}(n_0)V_-^{(s+n_0-1)}(n_0-1).
\end{aligned}
$$

The right-hand side above equals $R(s+n_0,s)$ because $(b_{n_0}^+)$ holds. Thus $(f_{n_0}^+)$ holds for $l=0$, $n=n_0$. It can be seen in a similar way that $(f_{n_0}^-)$ holds.

Consequently, we have completed the proof of Theorem 5.1 by mathematical induction. □

[Last step] *For any element* $(\tilde\gamma_+,\tilde\gamma_-,\tilde V_+,\tilde V_-)$ *of the set* $\widetilde{\mathscr{LM}}(R;N)$, *we can define* $d\times d$ *matrices* $\delta_\pm(n),\gamma_\pm(n,k),V_\pm(n)$ *by*

(5.10) $$\delta_+(n)\equiv\delta_+^{(0)}(n)\quad(0\le n\le N),$$

(5.11) $$\delta_-(n)\equiv\delta_-^{(N)}(n)\quad(0\le n\le N),$$

(5.12) $$\gamma_+(n,k)\equiv\gamma_+^{(0)}(n,k)\quad(0\le k<n\le N),$$

(5.13) $$\gamma_-(n,k)\equiv\gamma_-^{(N)}(n,k)\quad(0\le k<n\le N),$$

(5.14) $$V_+(n)\equiv V_+^{(0)}(n)\quad(0\le n\le N),$$

(5.15) $$V_-(n)\equiv V_-^{(N)}(n)\quad(0\le n\le N).$$

We shall prove

Theorem 5.2. *For each integer* $n$ $(0\le n\le N)$, $V_\pm(n)\ge 0$.

Proof. We define $(N+1)d\times(N+1)d$ matrices $T(R)(N+1)$ and $G(N+1)$ by

$$
T(R)(N+1)\equiv\begin{pmatrix}R(0,0)&R(0,1)&\cdots&R(0,N)\\R(1,0)&R(1,1)&\cdots&R(1,N)\\\vdots&\vdots&\ddots&\vdots\\R(N,0)&R(N,1)&\cdots&R(N,N)\end{pmatrix},
$$

$$G(N+1) \equiv \begin{pmatrix} I_d & & & & \\ \gamma_+(1,0) & I_d & & 0 & \\ \gamma_+(2,0) & \gamma_+(2,1) & I_d & & \\ \vdots & \vdots & & \ddots & \\ \gamma_+(N,0) & \gamma_+(N,1) & \cdots & \gamma_+(N,N-1) & I_d \end{pmatrix}.$$

From $(e_N^+), (f_N^+)$ in Theorem 5.1, we have

$$G(N+1)T(R)(N+1)^t G(N+1) = \begin{pmatrix} V_+(0) & & & 0 \\ & V_+(1) & & \\ & & \ddots & \\ 0 & & & V_+(N) \end{pmatrix}. \tag{5.16}$$

Since $T(R)(N+1)$ is nonnegative definite, so is $G(N+1)T(R)(N+1)^t G(N+1)$. Hence, $V_+(n) \geq 0$ $(0 \leq n \leq N)$. In the same way, we can prove $V_-(n) \geq 0$ $(0 \leq n \leq N)$. □

Thus, by using four kinds of matrix functions $\gamma_+, \gamma_-, V_+, V_-$ defined in [Last step] of Algorithm 5.1, we define the set $\mathscr{LM}(R)$ by

(5.17) $\mathscr{LM}(R) \equiv \{(\gamma_+, \gamma_-, V_+, V_-)$; there exists an element $(\tilde{\gamma}_+, \tilde{\gamma}_-, \tilde{V}_+, \tilde{V}_-)$ of $\widetilde{\mathscr{LM}}(R;N)$ such that $\gamma_\pm = (\gamma_\pm(n,k); 0 \leq k < n \leq N)$ and $V_\pm = (V_\pm(n); 0 \leq n \leq N)$ are defined by (5.12), (5.13), (5.14), (5.15), respectively$\}$.

DEFINITION 5.1. For a given $M(d;\boldsymbol{R})$-valued non-negative definite function $R = (R(m,n); 0 \leq m,n \leq N)$, we call the set $\mathscr{LM}(R)$ defined by (5.17) the system of $KM_2O$-Langevin matrix functions associated with the function $R$.

It is to be noted that any component $V_+$ (resp. $V_-$) of elements of the set $\mathscr{LM}(R)$ is the same matrix function which is uniquely determined by the matrix function $R$.

Let $W$ be any real inner product space the dimension of which is equal to or greater than $(N+1)d$. For a given $M(d;\boldsymbol{R})$-valued nonnegative definite function $R = (R(m,n); 0 \leq m,n \leq N)$, we construct a $d$-dimensional flow in $W$ as follows.

ALGORITHM 5.2.

[Step 1] *Let* $(\gamma_+, \gamma_-, V_+, V_-)$ *be any element of the set* $\mathscr{LM}(R)$ *defined by* (5.17). *Let* $\{\xi_{jn}; 1 \leq j \leq d, 0 \leq n \leq N\}$ *be any orthonormal system of* $W$. *We construct* $d$-*dimensional vectors* $\xi_+(n)$ $(0 \leq n \leq N)$ *by*

$$\xi_+(n) \equiv {}^t(\xi_{1n}, \xi_{2n}, \ldots, \xi_{dn}) \quad (0 \leq n \leq N). \tag{5.18}$$

[Step 2] *Let* $W_+(n)$ $(0 \leq n \leq N)$ *be any* $d \times d$ *matrices such that*

$$V_+(n) = W_+(n)^t W_+(n). \tag{5.19}$$

*We define a* $d$-*dimensional flow* $v_+ = (v_+(n); 0 \leq n \leq N)$ *in* $W$ *by*

$$v_+(n) \equiv W_+(n)\xi_+(n) \quad (0 \leq n \leq N). \tag{5.20}$$

*Then we have*

$$(v_+(m), {}^t v_+(n)) = \delta_{mn} V_+(n) \quad (0 \le n \le N). \tag{5.21}$$

[Step 3] *We shall inductively construct a d-dimensional flow* $\boldsymbol{X} = (X(n); 0 \le n \le N)$ *in* $W$ *by*

$$X(0) \equiv v_+(0), \tag{5.22}$$

$$X(n) \equiv -\sum_{k=0}^{n-1} \gamma_+(n,k) X(k) + v_+(n) \quad (1 \le n \le N). \tag{5.23}$$

For a $d$-dimensional flow $\boldsymbol{X} = (X(n); 0 \le n \le N)$ defined by Algorithm 5.2, we obtain the following construction theorem.

Theorem 5.3 (Construction Theorem for general flows). *The d-dimensional flow* $\boldsymbol{X}$ *does not depend upon the choice of an element* $(\gamma_+, \gamma_-, V_+, V_-)$ *of the set* $\mathscr{LM}(R)$ *and* $R$ *is the covariance matrix function of the flow* $\boldsymbol{X}$, *i.e.*, $R(\boldsymbol{X}) = R$.

Proof. From (5.21)–(5.23), it is easily verified that the function $\gamma_+$ is a forward $KM_2O$-Langevin dissipation matrix associated with the flow $\boldsymbol{X}$. Hence, comparing (5.16) with Lemma 3.5(i), we see that $R$ is the covariance matrix function of the flow $\boldsymbol{X}$. □

## 6. Stationary flows.

In this section, we shall treat degenerate stationary flows and extend the results on non-degenerate stationary flows in [**14**], [**15**], [**16**]. To begin with, we recall the definition of stationarity property. Let $W$ be any real inner product space.

Definition 6.1. Let $\boldsymbol{X} = (X(n); l \le n \le r)$ be any $d$-dimensional flow in the space $W$. We say that $\boldsymbol{X}$ has stationarity property if there exists an $M(d; \boldsymbol{R})$-valued function $R = (R(n); |n| \le r - l)$ such that

$$R(\boldsymbol{X})(m,n) = R(m-n) \quad (l \le m, n \le r). \tag{6.1}$$

The matrix function $R$ is called the covariance matrix function of the stationary flow $\boldsymbol{X}$.

Definition 6.2. Let $[\boldsymbol{X}, \boldsymbol{Y}]$ be any pair of two $d$-dimensional flows $\boldsymbol{X} = (X(n); 0 \le n \le N)$ and $\boldsymbol{Y} = (Y(l); -N \le l \le 0)$ in the space $W$. We say that the pair $[\boldsymbol{X}, \boldsymbol{Y}]$ has stationarity property if there exists an $M(d; \boldsymbol{R})$-valued function $R = (R(n); |n| \le N)$ such that

$$R(\boldsymbol{X})(m,n) = R(m-n) \quad (0 \le m, n \le N), \tag{6.2}$$

$$R(\boldsymbol{Y})(-m,-n) = R(-m+n) \quad (0 \le m, n \le N). \tag{6.3}$$

The matrix function $R$ is called the covariance matrix function of the stationary pair $[\boldsymbol{X}, \boldsymbol{Y}]$ of flows.

### 6.1. Fluctuation-Dissipation Theorem.

Througout this subsection, we assume that the pair $[\boldsymbol{X}, \boldsymbol{Y}]$ of two $d$-dimensional flows $\boldsymbol{X} = (X(n); 0 \le n \le N)$ and $\boldsymbol{Y} = (Y(l); -N \le l \le 0)$ in the space $W$ has stationarity property. Let $R$ be its covariance matrix function.

We shall fix any element $\gamma_+ = (\gamma_+(n,k); 0 \le k < n \le N)$ of $\mathscr{LMD}_+(\boldsymbol{X})$ and any element $\gamma_- = (\gamma_-(n,k); 0 \le k < n \le N)$ of $\mathscr{LMD}_-(\boldsymbol{Y})$.

Since the pair $[\boldsymbol{X}_+^{(0)}, \boldsymbol{X}_-^{(N)}]$ of flows is a stationary pair of flows whose covariance matrix function is $R$, we can see from the proof of Lemmas 4.1–4.5 that the following Lemmas 6.1–6.5 hold.

Lemma 6.1. *For any integers $n, l$ $(1 \le n \le N, 0 \le l \le n-1)$,*

$$\text{(i)} \quad R(n-l) = -\sum_{k=0}^{n-1} \gamma_+(n,k) R(k-l),$$

$$\text{(ii)} \quad {}^tR(n-l) = -\sum_{k=0}^{n-1} \gamma_-(n,k)\, {}^tR(k-l).$$

Lemma 6.2. *For each natural number $n$ $(1 \le n \le N)$,*

$$\text{(i)} \quad V_+(\boldsymbol{X})(n) = \sum_{k=0}^{n-1} \gamma_+(n,k)\, {}^tR(n-k) + R(0),$$

$$\text{(ii)} \quad V_-(\boldsymbol{Y})(n) = \sum_{k=0}^{n-1} \gamma_-(n,k) R(n-k) + R(0).$$

Lemma 6.3. *For each natural number $n$ $(2 \le n \le N)$,*

$$\text{(i)} \quad \sum_{k=1}^{n-1} \gamma_+(n,k) X(k) = \sum_{k=1}^{n-1} (\gamma_+(n-1,k-1) + \delta_+(n)\gamma_-(n-1,n-k-1)) X(k),$$

$$\text{(ii)} \quad \sum_{k=1}^{n-1} \gamma_-(n,k) Y(-k) = \sum_{k=1}^{n-1} (\gamma_-(n-1,k-1) + \delta_-(n)\gamma_+(n-1,n-k-1)) Y(-k).$$

Lemma 6.4. *For any integers $n, l$ $(2 \le n \le N, 0 \le l \le N)$,*

$$\text{(i)} \quad \sum_{k=1}^{n-1} \gamma_+(n,k) R(k-l) = \sum_{k=1}^{n-1} (\gamma_+(n-1,k-1) + \delta_+(n)\gamma_-(n-1,n-k-1)) R(k-l),$$

$$\text{(ii)} \quad \sum_{k=1}^{n-1} \gamma_-(n,k)\, {}^tR(k-l) = \sum_{k=1}^{n-1} (\gamma_-(n-1,k-1) + \delta_-(n)\gamma_+(n-1,n-k-1))\, {}^tR(k-l).$$

Lemma 6.5. *For each natural number $n$ $(1 \leq n \leq N-1)$,*

(i) $R(n+1) = -\sum_{k=0}^{n-1} \gamma_+(n,k)R(k+1) - \delta_+(n+1)V_-(\boldsymbol{Y})(n)$,

(ii) ${}^tR(n+1) = -\sum_{k=0}^{n-1} \gamma_-(n,k){}^tR(k+1) - \delta_-(n+1)V_+(\boldsymbol{X})(n)$.

By virtue of Lemma 6.3, we can apply the same method as in Theorem 4.1 to show the following Theorem 6.1.

Theorem 6.1 (Dissipation-Dissipation Theorem). *For any elements $\gamma_+$ of $\mathscr{LMD}_+(\boldsymbol{X})$ and $\gamma_-$ of $\mathscr{LMD}_-(\boldsymbol{Y})$, we can transform them to construct other elements $\gamma'_+$ of $\mathscr{LMD}_+(\boldsymbol{X})$ and $\gamma'_-$ of $\mathscr{LMD}_-(\boldsymbol{Y})$ such that the following Dissipation-Dissipation Theorem holds*:

(i) *For any integers $n,k$ $(1 \leq k < n \leq N)$,*

$$\gamma'_+(n,k) = \gamma'_+(n-1,k-1) + \delta'_+(n)\gamma'_-(n-1,n-k-1).$$

(ii) *For any integers $n,k$ $(1 \leq k < n \leq N)$,*

$$\gamma'_-(n,k) = \gamma'_-(n-1,k-1) + \delta'_-(n)\gamma'_+(n-1,n-k-1).$$

Remark 6.1. By using the weight transformation, we can show that DDT holds for the minimum $\mathrm{KM_2O}$-Langevin dissipation matrix function $\gamma_+^0$ of the flow $\boldsymbol{X}$ and the minimum $\mathrm{KM_2O}$-Langevin dissipation matrix function $\gamma_-^0$ of the flow $\boldsymbol{Y}$.

By virtue of Lemmas 6.2, 6.4 and 6.5, we can apply the same method as in Theorem 4.2 to show the following Theorem 6.2.

Theorem 6.2 (Fluctuation-Dissipation Theorem I). *For each natural number $n$ $(1 \leq n \leq N)$,*

(i) $V_+(\boldsymbol{X})(n) = (I - \delta_+(n)\delta_-(n))V_+(\boldsymbol{X})(n-1)$,

(ii) $V_-(\boldsymbol{Y})(n) = (I - \delta_-(n)\delta_+(n))V_-(\boldsymbol{Y})(n-1)$.

By using Lemmas 6.1 and 6.2, we can apply the same method as in Theorem 4.3 to show the following Theorem 6.3 which is stronger than Theorem 4.3.

Theorem 6.3 (Burg's relation). *For each integer $n$ $(1 \leq n \leq N)$,*

$$\sum_{k=0}^{n-1} R(k+1)\,{}^t\gamma_-(n,k) = \sum_{k=0}^{n-1} \gamma_+(n,k)R(k+1).$$

Proof. We have only to show the case for $n = N$. For that purpose, we define $2d \times (N+2)d$ matrix $F(N+2)$ and $(N+2)d \times (N+2)d$ matrix $\tilde{T}(n+2)$ by

$$F(N+2) \equiv \begin{pmatrix} 0 & \gamma_+(N,0) & \gamma_+(N,1) & \cdots & \gamma_+(N,N-1) & I \\ I & \gamma_-(N,N-1) & \gamma_-(N,N-2) & \cdots & \gamma_-(N,0) & 0 \end{pmatrix},$$

$$\tilde{T}(n+2) \equiv \begin{pmatrix} R(0) & {}^tR(1) & \cdots & {}^tR(N) & 0 \\ R(1) & R(0) & \cdots & {}^tR(N-1) & {}^tR(N) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R(N) & R(N-1) & \cdots & R(0) & {}^tR(1) \\ 0 & R(N) & \cdots & R(1) & R(0) \end{pmatrix}.$$

Then, by Lemmas 6.1 and 6.2, we have

$$F(N+2)\tilde{T}(N+2){}^tF(N+2) = \begin{pmatrix} V_+(\boldsymbol{X})(N) & \sum_{k=0}^{N-1}\gamma_+(N,k)R(k+1) \\ \sum_{k=0}^{N-1}\gamma_-(N,k){}^tR(k+1) & V_-(\boldsymbol{Y})(N) \end{pmatrix}.$$

Since $F(N+2)\tilde{T}(N+2){}^tF(N+2)$ is symmetric, we find that Theorem 6.3 holds for $n = N$. □

By using Lemma 6.5 and Theorem 6.3, we can apply the same method as in Theorem 4.4 to show the following Theorem 6.4.

Theorem 6.4 (Fluctuation-Dissipation Theorem II). *For each natural number* $n$ $(1 \le n \le N)$

$$\delta_+(n)V_-(\boldsymbol{Y})(n-1) = V_+(\boldsymbol{X})(n-1){}^t\delta_-(n).$$

**6.2. Characterization Theorem.**

Let $[\boldsymbol{X}, \boldsymbol{Y}]$ be any pair of two $d$-dimensional flows $\boldsymbol{X} = (X(n); 0 \le n \le N)$ and $\boldsymbol{Y} = (Y(l); -N \le l \le 0)$ in the inner product space $W$. Then we have the following theorem, that characterizes stationarity property in terms of $\mathrm{KM_2O}$-Langevin matrices. This theorem has been proved in [**14**] for non-degenerate stationary pair of flows.

Theorem 6.5 (Characterization Theorem). *The pair* $[\boldsymbol{X}, \boldsymbol{Y}]$ *of flows has stationary property if and only if there exist a* $\mathrm{KM_2O}$*-Langevin dissipation matrix function* $\gamma_+$ *of the flow* $\boldsymbol{X}$ *and a* $\mathrm{KM_2O}$*-Langevin dissipation matrix function* $\gamma_-$ *of the flow* $\boldsymbol{Y}$ *such that*

(DDT) *For any integers* $k, n$ $(1 \le k < n \le N)$,

$$\text{(i)} \quad \gamma_+(n,k) = \gamma_+(n-1,k-1) + \delta_+(n)\gamma_-(n-1,n-k-1),$$

$$\text{(ii)} \quad \gamma_-(n,k) = \gamma_-(n-1,k-1) + \delta_-(n)\gamma_+(n-1,n-k-1).$$

(FDT) *For each integer* $n$ $(1 \le n \le N)$,

$$\text{(i)} \quad V_+(\boldsymbol{X})(n) = (I - \delta_+(n)\delta_-(n))V_+(\boldsymbol{X})(n-1),$$

$$\text{(ii)} \quad V_-(\boldsymbol{Y})(n) = (I - \delta_-(n)\delta_+(n))V_-(\boldsymbol{Y})(n-1),$$

$$\text{(iii)} \quad \delta_+(n)V_-(\boldsymbol{Y})(n-1) = V_+(\boldsymbol{X})(n-1){}^t\delta_-(n),$$

$$\text{(iv)} \quad V_+(\boldsymbol{X})(0) = V_-(\boldsymbol{Y})(0).$$

Proof. If $[\boldsymbol{X}, \boldsymbol{Y}]$ has stationarity property, we have already seen that (DDT) and (FDD) hold. Therefore it suffices to show that (DDT) and (FDT) imply stationarity property of the pair $[\boldsymbol{X}, \boldsymbol{Y}]$. First, we note that the pair $[\boldsymbol{X}, \boldsymbol{Y}]$ has stationarity property if the following [Claim-$p$] is true for $p = N$.

Claim-$p$. *There exists a matrix function* $R = (R(n); |n| \le p)$ *such that*

$$R(\boldsymbol{X})(m,n) = R(m-n) \quad (0 \le m,n \le p), \tag{6.4}$$

$$R(\boldsymbol{Y})(-m,-n) = R(-m+n) \quad (0 \le m,n \le p). \tag{6.5}$$

We will now show two lemmas under the condition that (DDT) and (FDT) hold.

Step 1. [Claim-1] holds.

Proof of Step 1. We put

$$R(0) \equiv V_+(\boldsymbol{X})(0). \tag{6.6}$$

It follows from (2.1), (2.3) and (FDT)(iv) that

$$R(\boldsymbol{X})(0,0) = R(\boldsymbol{Y})(0,0) = V_-(\boldsymbol{Y})(0) = R(0). \tag{6.7}$$

Next we put

$$\begin{cases} R(1) \equiv -\delta_+(1)V_-(\boldsymbol{Y})(0), \\ R(-1) \equiv {}^tR(1). \end{cases} \tag{6.8}$$

We see from Lemma 3.1(i) and (6.7) that

$$R(\boldsymbol{X})(1,0) = -\delta_+(1)V_-(\boldsymbol{Y})(0), \tag{6.9}$$

$$R(\boldsymbol{Y})(-1,0) = -\delta_-(1)V_+(\boldsymbol{X})(0) \tag{6.10}$$

and so by (FDT)(iii), we obtain

$$\begin{cases} R(\boldsymbol{X})(1,0) = R(\boldsymbol{Y})(0,-1) = R(1), \\ R(\boldsymbol{X})(0,1) = R(\boldsymbol{Y})(-1,0) = R(-1). \end{cases} \tag{6.11}$$

Furthermore, Lemma 3.2(i) implies that $R(\boldsymbol{X})(1,1) = V_+(\boldsymbol{X})(1) - \delta_+(1)R(\boldsymbol{X})(0,1)$. Applying (6.11) and (6.10) to the second term of the right-hand side above, we see that $R(\boldsymbol{X})(1,1) = V_+(\boldsymbol{X})(1) - \delta_+(1)R(\boldsymbol{Y})(-1,0) = V_+(\boldsymbol{X})(1) + \delta_+(1)\delta_-(1)V_+(\boldsymbol{X})(0)$. Therefore, by (FDT)(i) and (6.6), $R(\boldsymbol{X})(1,1) = V_+(\boldsymbol{X})(0) = R(0)$. Similarly, $R(\boldsymbol{Y})(-1,-1) = R(0)$. Thus, by noting (6.7) and (6.11), we can prove Step 1. (Step 1) □

Step 2. Let $p_0$ be an arbitrary natural number such that $1 \le p_0 \le N-1$. We assume that [Claim-$p_0$] holds. Then, [Claim-$(p_0+1)$] also holds.

Proof of Step 2. From assumption, there exists a matrix function $R = (R(n); |n| \le p_0)$ such that

$$R(\boldsymbol{X})(m,n) = R(m-n) \quad (0 \le m,n \le p_0), \tag{6.12}$$

$$R(\boldsymbol{Y})(-m,-n) = R(-m+n) \quad (0 \le m,n \le p_0). \tag{6.13}$$

Therefore we have the following relations as in Lemmas 6.1, 6.2 and Theorem 6.3.

$$R(p_0 - l) = -\sum_{k=0}^{p_0-1} \gamma_+(p_0,k)R(k-l) \quad (0 \le l \le p_0 - 1), \tag{6.14}$$

$${}^tR(p_0 - l) = -\sum_{k=0}^{p_0-1} \gamma_-(p_0,k)\,{}^tR(k-l) \quad (0 \le l \le p_0 - 1), \tag{6.15}$$

$$V_{+}(\boldsymbol{X})(p_0) = \sum_{k=0}^{p_0-1} \gamma_{+}(p_0,k)\,{}^{t}R(p_0-k) + R(0), \tag{6.16}$$

$$V_{-}(\boldsymbol{Y})(p_0) = \sum_{k=0}^{p_0-1} \gamma_{-}(p_0,k)R(p_0-k) + R(0), \tag{6.17}$$

$$\sum_{k=0}^{p_0-1} R(k+1)\,{}^{t}\gamma_{-}(p_0,k) = \sum_{k=0}^{p_0-1} \gamma_{+}(p_0,k)R(k+1). \tag{6.18}$$

To prove Step 2, it suffices to show that

$$R(\boldsymbol{X})(p_0+1,l) = R(p_0+1-l) \quad (1 \le l \le p_0), \tag{6.19}$$

$$R(\boldsymbol{Y})(-p_0-1,-l) = R(-p_0-1+l) \quad (1 \le l \le p_0), \tag{6.20}$$

$$R(\boldsymbol{X})(p_0+1,0) = {}^{t}R(\boldsymbol{Y})(-p_0-1,0), \tag{6.21}$$

$$R(\boldsymbol{X})(p_0+1,p_0+1) = R(0), \tag{6.22}$$

$$R(\boldsymbol{Y})(-p_0-1,-p_0-1) = R(0). \tag{6.23}$$

Let $1 \le l \le p_0$. By Lemma 3.1 and (6.12), we have

$$R(\boldsymbol{X})(p_0+1,l) = -\sum_{k=0}^{p_0} \gamma_{+}(p_0+1,k)R(k-l).$$

Applying (DDT)(i) into the first term of the right-hand side above, we obtain

$$\begin{aligned} R(\boldsymbol{X})(p_0+1,l) = &-\sum_{k=0}^{p_0-1} \gamma_{+}(p_0,k)R(k+1-l) \\ &- \delta_{+}(p_0+1)\left\{ {}^{t}R(l) + \sum_{k=0}^{p_0-1} \gamma_{-}(p_0,k)\,{}^{t}R(k-p_0+l) \right\}. \end{aligned} \tag{6.24}$$

From (6.14), we see that the first term in the right-hand side above is equal to $R(p_0+1-l)$. Furthermore, replacing $l$ by $p_0-l$ in (6.15), we know that the second term of the right-hand side in (6.24) vanishes. This gives (6.19). Relation (6.20) is similarly proved.

By Lemma 3.1, (6.12) and (DDT)(i), we have as in (6.24),

$$\begin{aligned} &R(\boldsymbol{X})(p_0+1,0) \\ &\quad = -\sum_{k=0}^{p_0-1} \gamma_{+}(p_0,k)R(k+1) - \delta_{+}(p_0+1)\left\{ R(0) + \sum_{k=0}^{p_0-1} \gamma_{-}(p_0,k)R(p_0-k) \right\}. \end{aligned}$$

Substituting (6.17) into the above equation, we see that

$$R(\boldsymbol{X})(p_0+1,0) = -\sum_{k=0}^{p_0-1} \gamma_{+}(p_0,k)R(k+1) - \delta_{+}(p_0+1)V_{-}(\boldsymbol{Y})(p_0). \tag{6.25}$$

Similarly,

$$R(\boldsymbol{Y})(-p_0-1,0) = -\sum_{k=0}^{p_0-1} \gamma_-(p_0,k)R(-k-1) - \delta_-(p_0+1)V_+(\boldsymbol{X})(p_0). \tag{6.26}$$

Therefore, by (FDT)(iii) and (6.18), we have (6.21).

By using Lemma 3.2,

$$R(\boldsymbol{X})(p_0+1,p_0+1) = -\sum_{k=0}^{p_0} \gamma_+(p_0+1,k)R(\boldsymbol{X})(k,p_0+1) + V_+(\boldsymbol{X})(p_0+1).$$

Applying (DDT)(i) and (6.19) into the first term of the right-hand side above, we observe

$$\begin{aligned} R(\boldsymbol{X})(p_0+1,p_0+1) = & -\sum_{k=0}^{p_0-1} \gamma_+(p_0,k)\,{}^tR(p_0-k) + V_+(\boldsymbol{X})(p_0+1) \\ & - \delta_+(p_0+1)\left\{ R(\boldsymbol{X})(0,p_0+1) + \sum_{k=0}^{p_0-1} \gamma_-(p_0,k)R(-k-1) \right\}. \end{aligned}$$

From (6.16), the first term of the right-hand side above equals $R(0) - V_+(\boldsymbol{X})(p_0)$. Moreover, (6.26) and (6.21) imply that the third term of the right-hand side above is equal to $\delta_+(p_0+1)\delta_-(p_0+1)V_+(\boldsymbol{X})(p_0)$, so that

$$R(\boldsymbol{X})(p_0+1,p_0+1) = R(0) + V_+(\boldsymbol{X})(p_0+1) - (I - \delta_+(p_0+1)\delta_-(p_0+1))V_+(\boldsymbol{X})(p_0).$$

Substituting (FDT)(i) into the above equation, we have (6.22). We can prove (6.23) in the same fashion. Thus we have completed the proof. (Step 2) □

Therefore Theorem 6.5 follows from Steps 1 and 2. □

REMARK 6.2. We can show that the necessary part of Theorem 6.5 holds for the minimum KM$_2$O-Langevin dissipation matrix functions $\gamma_+^0, \gamma_-^0$.

**6.3. Construction Theorem.**

Let $R = (R(n); |n| \le N)$ be any $M(d;\boldsymbol{R})$-valued nonnegative definite function of one variable. Then $R$ is regarded as a nonnegative definite function of two variables by putting

$$R(m,n) \equiv R(m-n) \quad (0 \le m,n \le N). \tag{6.27}$$

Therefore we can construct a $d$-dimensional stationary flow $\boldsymbol{X} = (X(n); 0 \le n \le N)$ such that

$$R(\boldsymbol{X}) = R, \tag{6.28}$$

by using Algorithms 5.1 and 5.2. But because of stationarity property, the system $\mathscr{LM}(R)$ of KM$_2$O-Langevin matrix functions associated with the function $R$ can be obtained more effectively by the following algorithm.

Algorithm 6.1.

[Step 0] *We define* $d \times d$ *matrices* $V_\pm(0)$ *by*

$$V_\pm(0) \equiv R(0). \tag{6.29}$$

[Step 1] *We define* $d \times d$ *matrices* $\delta_\pm(1), \gamma_\pm(1,0), V_\pm(1)$ *by*

$$\delta_+(1) \equiv -R(1)V_-(0)^+ + A_+(1)(I - V_-(0)V_-(0)^+), \tag{6.30}$$

$$\delta_-(1) \equiv -{}^tR(1)V_+(0)^+ + A_-(1)(I - V_+(0)V_+(0)^+), \tag{6.31}$$

$$\gamma_+(1,0) \equiv \delta_+(1), \tag{6.32}$$

$$\gamma_-(1,0) \equiv \delta_-(1), \tag{6.33}$$

$$V_+(1) \equiv (I - \delta_+(1)\delta_-(1))V_+(0), \tag{6.34}$$

$$V_-(1) \equiv (I - \delta_-(1)\delta_+(1))V_-(0), \tag{6.35}$$

*where* $A_\pm(1)$ *are any elements of* $M(d;\boldsymbol{R})$.

[Step $m$] *Inductively as in* Algorithm 5.1, *we define* $d \times d$ *matrix functions* $\delta_\pm = (\delta_\pm(n); 0 \le n \le N), \gamma_\pm = (\gamma_\pm(n,k); 0 \le k < n \le N), V_\pm = (V_\pm(n); 0 \le n \le N)$ *as follows: for any* $2 \le m \le N$,

$$\delta_+(m) \equiv -\left\{R(m) + \sum_{k=0}^{m-2}\gamma_+(m-1,k)R(k+1)\right\}V_-(m-1)^+ \\ + A_+(m)(I - V_-(m-1)V_-(m-1)^+), \tag{6.36}$$

$$\delta_-(m) \equiv -\left\{{}^tR(m) + \sum_{k=0}^{m-2}\gamma_-(m-1,k)\,{}^tR(k+1)\right\}V_+(m-1)^- \\ + A_-(m)(I - V_+(m-1)V_+(m-1)^+), \tag{6.37}$$

$$\gamma_+(m,k) \equiv \gamma_+(m-1,k-1) + \delta_+(m)\gamma_-(m-1,m-k-1) \quad (1 \le k \le m-1), \tag{6.38}$$

$$\gamma_-(m,k) \equiv \gamma_-(m-1,k-1) + \delta_-(m)\gamma_+(m-1,m-k-1) \quad (1 \le k \le m-1), \tag{6.39}$$

$$\gamma_+(m,0) \equiv \delta_+(m), \tag{6.40}$$

$$\gamma_-(m,0) \equiv \delta_-(m), \tag{6.41}$$

$$V_+(m) \equiv (I - \delta_+(m)\delta_-(m))V_+(m-1), \tag{6.42}$$

$$V_-(m) \equiv (I - \delta_-(m)\delta_+(m))V_-(m-1), \tag{6.43}$$

*where* $A_\pm(m)$ *are any elements of* $M(d;\boldsymbol{R})$.

As a generalization of the construction theorem proved in [**15**] for non-degenerate case to degenerate case, we have

Theorem 6.6. *The set of all quadruplets* $(\gamma_+, \gamma_-, V_+, V_-)$ *of* $d \times d$ *matrix functions constructed by* Algorithm 6.1 *is equal to the system* $\mathscr{LM}(R)$ *of* $\mathrm{KM_2O}$-*Langevin matrix functions associated with the matrix function* $R$.

Proof. Since $R(s,s) = R(0)$ $(0 \le s \le N)$, we see at [Step 0] in Algorithm 5.1 that $V_{\pm}^{(s)}(0)$ are independent of $s$, i.e.,

$$V_{\pm}^{(s)}(0) = V_{\pm}(R)(0) \quad (0 \le s \le N). \tag{6.44}$$

Moreover, at [Step 1] in Algorithm 5.1, we can choose $\delta_{\pm}^{(s)}(1), \gamma_{\pm}^{(s)}(1,0), V_{\pm}^{(s)}(1)$ independently of $s$, because of (6.44) and the relations

$$R(s+1,s) = R(1) \quad (0 \le s \le N-1), \tag{6.45}$$

$$R(s-1,s) = R(-1) \quad (1 \le s \le N). \tag{6.46}$$

In general, at [Step 2], we can choose $\delta_{\pm}^{(s)}(m), \gamma_{\pm}^{(s)}(m,k), V_{\pm}^{(s)}(m)$ independently of $s$. Thus we have the proof. □

**6.4. Extension Theorem.**

As a generalization of the extension theorem proved in [**16**] for non-degenerate case to degenerate case, for an arbitrarily given stationary flow, we would like to extend it without losing stationarity property.

Let $\boldsymbol{X} = (X(n); 0 \le n \le N)$ be any stationary flow in $W$ with covariance matrix function $R = (R(n); |n| \le N)$. We take an arbitrary element $X(N+1) \in W^d$ and extend the flow $\boldsymbol{X}$ to the flow $\boldsymbol{X}^{(N+1)} = (X(n); 0 \le n \le N+1)$. To find a necessary and sufficient condition for the extended flow $\boldsymbol{X}^{(N+1)}$ to have stationarity property, we define a $d$-dimensional flow $\tilde{\boldsymbol{X}} = (\tilde{X}(n); 0 \le n \le N)$ by

$$\tilde{X}(n) \equiv X(n+1) \quad (0 \le n \le N). \tag{6.47}$$

The following lemma can be easily verified.

Lemma 6.6. *The flow $\boldsymbol{X}^{(N+1)} = (X(n); 0 \le n \le N+1)$ has stationarity property if and only if $\boldsymbol{X} = (X(n); 0 \le n \le N)$ and $\tilde{\boldsymbol{X}} = (\tilde{X}(n); 0 \le n \le N)$ has the same covariance matrix function, i.e., $R(\boldsymbol{X}) = R(\tilde{\boldsymbol{X}})$.*

Let us fix any KM$_2$O-Langevin dissipation matrix function $\gamma_+$ of the flow $\boldsymbol{X}$. By using Lemma 6.6 and the fact that $\mathrm{M}_1^N(\boldsymbol{X}) = \mathrm{M}_0^{N-1}(\tilde{\boldsymbol{X}})$, we have

Lemma 6.7. *The extended flow $\boldsymbol{X}^{(N+1)}$ has stationarity property if and only if*

$$P_{\mathrm{M}_1^N(\boldsymbol{X})}\tilde{X}(N) = -\sum_{k=0}^{N-1} \gamma_+(N,k)X(k+1), \tag{6.48}$$

$$(\tilde{X}(N) - P_{\mathrm{M}_1^N(\boldsymbol{X})}\tilde{X}(N), {}^t(\tilde{X}(N) - P_{\mathrm{M}_1^N(\boldsymbol{X})}\tilde{X}(N))) = V_+(\boldsymbol{X})(N). \tag{6.49}$$

Thus, we have the following extension theorem, which has been proved in [**16**] for non-degenerate and stationary flows.

Theorem 6.7. *Let $\boldsymbol{X} = (X(n); 0 \le n \le N)$ be any stationary flow in $W$. We define $X(N+1) \in W^d$ by*

$$X(N+1) \equiv -\sum_{k=0}^{N-1} \gamma_+(N,k)X(k+1) + \eta_+, \tag{6.50}$$

*where $\eta_+$ is some element of $W^d$. Then the extended flow $\boldsymbol{X}^{(N+1)} = (X(n); 0 \le n \le N+1)$ has stationarity property if and only if*

$$(X(n), {}^t\eta_+) = 0 \quad (1 \le n \le N), \tag{6.51}$$

$$(\eta_+, {}^t\eta_+) = V_+(\boldsymbol{X})(N). \tag{6.52}$$

More generally, let $\boldsymbol{X}^{(N_1,N_2)} = (X(n); N_1 \le n \le N_2)$ be any $d$-dimensional stationary flow in the space $W$. Let $R^{(N_2-N_1)} = (R(n); |n| \le N_2 - N_1)$ be the covariance matrix function of the flow $\boldsymbol{X}^{(N_1,N_2)}$. We fix any element $(\gamma_+, \gamma_-, V_+, V_-)$ of the system $\mathscr{LM}(R^{(N_2-N_1)})$ of KM$_2$O-Langevin matrix functions associated with the matrix function $R^{(N_2-N_1)}$.

Theorem 6.7 can be restated as follows.

Theorem 6.8. *We define a d-dimensional vector $X(N_2+1)$ by*

$$X(N_2+1) \equiv -\sum_{k=0}^{N_2-N_1-1} \gamma_+(N_2-N_1,k)X(N_1+k+1) + \eta_+, \tag{6.53}$$

*where $\eta_+$ is some d-dimensional vector in $W^d$.*

*Then the flow $\boldsymbol{X}^{(N_1,N_2+1)} = (X(n); N_1 \le n \le N_2+1)$ has stationarity property if and only if $\eta_+$ satisfies*

$$\begin{cases} (X(n), {}^t\eta_+) = 0 \quad (N_1+1 \le n \le N_2), \\ (\eta_+, {}^t\eta_+) = V_+(R)(N_2-N_1). \end{cases} \tag{6.54}$$

In a similar way, we have the following theorem for the backward extension.

Theorem 6.9. *We define a d-dimensional vector $X(N_1-1)$ by*

$$X(N_1-1) \equiv -\sum_{k=0}^{N_2-N_1-1} \gamma_-(N_2-N_1,k)X(N_2-k-1) + \eta_-, \tag{6.55}$$

*where $\eta_-$ is some d-dimensional vector in $W^d$.*

*Then the flow $X^{(N_1-1,N_2)} = (X(n); N_1-1 \le n \le N_2)$ has stationarity property if and only if $\eta_-$ satisfies*

$$\begin{cases} (X(n), {}^t\eta_-) = 0 \quad (N_1 \le n \le N_2-1), \\ (\eta_-, {}^t\eta_-) = V_-(R)(N_2-N_1). \end{cases} \tag{6.56}$$

By a repeated use of Theorems 6.8 and 6.9, we have the following.

Theorem 6.10. *Let $M_1$ and $M_2$ be two integers such that $M_1 \le N_1$ and $N_2 \le M_2$. Then, the stationary flow $\boldsymbol{X}^{(N_1,N_2)} = (X(n); N_1 \le n \le N_2)$ can be extended to a stationary flow defined on the set $\{M_1, M_1+1, \ldots, M_2-1, M_2\}$, i.e., there exists a stationary flow $\boldsymbol{X}^{(M_1,M_2)} = (X(n); M_1 \le n \le M_2)$.*

Any non-negative definite function of one variable can be extended by extending a corresponding stationary flow. Thus we have the following theorem, which can be proved just in the same way as in Theorem 4.1 in [**16**].

THEOREM 6.11. *Let* $R^{(N)} = (R(n); |n| \le N)$ *be any* $M(d; \boldsymbol{R})$*-valued nonnegative definite function such that*

$$ {}^tR(n) = R(-n) \quad (0 \le n \le N). \tag{6.57} $$

*For any fixed element* $(\gamma_+, \gamma_-, V_+, V_-)$ *of the system* $\mathscr{LM}(R^{(N)})$ *of* KM$_2$O*-Langevin matrix functions associated with the matrix function* $R^{(N)}$*, we define* $R(\pm(N+1)) \in M(d; \boldsymbol{R})$ *by*

$$ R(N+1) \equiv -\sum_{k=0}^{N-1} \gamma_+(N,k)R(k+1) + Q, \tag{6.58} $$

$$ R(-N-1) \equiv {}^tR(N+1), \tag{6.59} $$

*where* $Q$ *is some* $d \times d$ *matrix.*

*Then* $R^{(N+1)} = (R(n); |n| \le N+1)$ *has nonnegative definite property if and only if there exist two d-dimensional vectors* $\zeta$ *and* $\eta$ *in some real inner product space* $W$ *such that*

$$ \begin{cases} (\zeta, {}^t\eta) = Q, \\ (\zeta, {}^t\zeta) = V_+(N), \\ (\eta, {}^t\eta) = V_-(N). \end{cases} \tag{6.60} $$

## 7. Periodic stationary flows.

In [**19**], [**20**], [**21**], periodic autoregressive models are discussed and some algorithms to estimate the parameters of the models are proposed. In this section, we introduce the notion of periodic stationarity property for the pair of flows and characterize it in terms of KM$_2$O-Langevin matrix functions. Let $\boldsymbol{X} = (X(n); 0 \le n \le (N+1)p-1)$ be any $d$-dimensional flow in an real inner product space $W$.

DEFINITION 7.1. We say that the flow $\boldsymbol{X}$ has periodic stationarity property of period $p$ if its covariance matrix function satisfies

$$ R(\boldsymbol{X})(m+p, n+p) = R(\boldsymbol{X})(m,n) \quad (0 \le m, n \le Np-1). \tag{7.1} $$

Furthermore, we introduce the notion of periodic stationarity property for a pair of flows. Let $[\boldsymbol{X}, \boldsymbol{Y}]$ be any pair of two $d$-dimensional flows $\boldsymbol{X} = (X(n); 0 \le n \le (N+1)p-1)$, and $\boldsymbol{Y} = (Y(l); -(N+1)p+1 \le l \le 0)$ in $W$.

DEFINITION 7.2. We say that the pair $[\boldsymbol{X}, \boldsymbol{Y}]$ has periodic stationarity property of period $p$ if its covariance matrix functions satisfy

$$ R(\boldsymbol{X})(m+p, n+p) = R(\boldsymbol{X})(m,n) \quad (0 \le m, n \le Np-1), \tag{7.2} $$

$$ R(\boldsymbol{Y})(k-p, l-p) = R(\boldsymbol{Y})(k,l) \quad (-Np+1 \le k, l \le 0), \tag{7.3} $$

$$ R(\boldsymbol{X})(m,n) = R(\boldsymbol{Y})(m-(N+1)p+1, n-(N+1)p+1) \tag{7.4} $$

$$ (0 \le m \le (N+1)p-1, 0 \le n \le p-1). $$

Let us given any pair $[\boldsymbol{X}, \boldsymbol{Y}]$ of two $d$-dimensional flows $\boldsymbol{X} = (X(n); 0 \le n \le$

$(N+1)p-1)$, and $\boldsymbol{Y} = (Y(l); -(N+1)p+1 \le l \le 0)$ in $W$. For any element $\gamma_+ = (\gamma_+(n,k); 0 \le k < n \le (N+1)p-1)$ of $\mathscr{LMD}_+(\boldsymbol{X})$ and any element $\gamma_- = (\gamma_-(n,k); 0 \le k < n \le (N+1)p-1)$ of $\mathscr{LMD}_-(\boldsymbol{Y})$, we define $pd \times pd$ matrices $\Gamma_+(n,k), \Gamma_-(n,k)$ $(0 \le k \le n \le N)$, $\Lambda_+(\boldsymbol{X})(n), \Lambda_-(\boldsymbol{Y})(n)$ $(0 \le n \le N)$ formed by elements of a KM$_2$O-Langevin matrix as follows.

$$(7.5)\quad \Gamma_+(n,k) \equiv \begin{pmatrix} \gamma_+(np,kp) & \cdots & \gamma_+(np,(k+1)p-1) \\ \vdots & \ddots & \vdots \\ \gamma_+((n+1)p-1,kp) & \cdots & \gamma_+((n+1)p-1,(k+1)p-1) \end{pmatrix}$$
$$(0 \le k < n \le N),$$

$$(7.6)\quad \Gamma_+(n,n) \equiv \begin{pmatrix} I_d & & & 0 \\ \gamma_+(np+1,np) & I_d & & \\ \vdots & \vdots & \ddots & \\ \gamma_+((n+1)p-1,np) & \gamma_+((n+1)p-1,np+1) & \cdots & I_d \end{pmatrix}$$
$$(0 \le n \le N),$$

$$(7.7)\quad \Lambda_+(\boldsymbol{X})(n) \equiv \begin{pmatrix} V_+(\boldsymbol{X})(np) & & 0 \\ & \ddots & \\ 0 & & V_+(\boldsymbol{X})((n+1)p-1) \end{pmatrix} \quad (0 \le n \le N),$$

$$(7.8)\quad \Gamma_-(n,k) \equiv \begin{pmatrix} \gamma_-((n+1)p-1,(k+1)p-1) & \cdots & \gamma_-((n+1)p-1,kp) \\ \vdots & \ddots & \vdots \\ \gamma_-(np,(k+1)p-1) & \cdots & \gamma_-(np,kp) \end{pmatrix}$$
$$(0 \le k < n \le N),$$

$$(7.9)\quad \Gamma_-(n,n) \equiv \begin{pmatrix} I_d & \gamma_-((n+1)p-1,(n+1)p-2) & \cdots & \gamma_-((n+1)p-1,np) \\ & I_d & & \gamma_-((n+1)p-2,np) \\ & & \ddots & \vdots \\ & 0 & & I_d \end{pmatrix}$$
$$(0 \le n \le N),$$

$$(7.10)\quad \Lambda_-(\boldsymbol{Y})(n) \equiv \begin{pmatrix} V_-(\boldsymbol{Y})((n+1)p-1) & & 0 \\ & \ddots & \\ 0 & & V_-(\boldsymbol{Y})(np) \end{pmatrix} \quad (0 \le n \le N).$$

Furthermore, we define $pd \times pd$ matrices $\Delta_{++}(n), \Delta_{--}(n), \Delta_{+-}(n), \Delta_{-+}(n)$ $(1 \le n \le N)$ by

$$(7.11)\qquad \Delta_{++}(n) \equiv \Gamma_+(n,n)\Gamma_+^{-1}(n-1,n-1),$$

$$(7.12)\qquad \Delta_{--}(n) \equiv \Gamma_-(n,n)\Gamma_-^{-1}(n-1,n-1),$$

$$\Delta_{+-}(n) \equiv \Gamma_+(n,0)\Gamma_-^{-1}(n-1,n-1), \tag{7.13}$$

$$\Delta_{-+}(n) \equiv \Gamma_-(n,0)\Gamma_+^{-1}(n-1,n-1). \tag{7.14}$$

Then the following theorem characterizes the periodic stationarity property.

Theorem 7.1 (Characterization Theorem). *The pair* $[\boldsymbol{X},\boldsymbol{Y}]$ *of flows has periodic stationarity property of period* $p$ *if and only if there exist a* KM$_2$O*-Langevin dissipation matrix function* $\gamma_+$ *of the flow* $\boldsymbol{X}$ *and a* KM$_2$O*-Langevin dissipation matrix function* $\gamma_-$ *of the flow* $\boldsymbol{Y}$ *such that*

(DDT$_\mathrm{p}$) *For two integers* $k,n$ $(1 \le k < n \le N)$,

(i) $\Gamma_+(n,k) = \Delta_{++}(n)\Gamma_+(n-1,k-1) + \Delta_{+-}(n)\Gamma_-(n-1,n-k-1)$,

(ii) $\Gamma_-(n,k) = \Delta_{--}(n)\Gamma_-(n-1,k-1) + \Delta_{-+}(n)\Gamma_+(n-1,n-k-1)$,

(FDT$_\mathrm{p}$) *For each integer* $n$ $(1 \le n \le N)$,

(i) $\Lambda_+(\boldsymbol{X})(n) = \Delta_{++}(n)\Lambda_+(\boldsymbol{X})(n-1)\,{}^t\Delta_{++}(n) - \Delta_{+-}(n)\Lambda_-(\boldsymbol{Y})(n-1)\,{}^t\Delta_{+-}(n)$,

(ii) $\Lambda_-(\boldsymbol{Y})(n) = \Delta_{--}(n)\Lambda_-(\boldsymbol{Y})(n-1)\,{}^t\Delta_{--}(n) - \Delta_{-+}(n)\Lambda_+(\boldsymbol{X})(n-1)\,{}^t\Delta_{-+}(n)$,

(iii) $\Delta_{++}^{-1}(n)\Delta_{+-}(n)\Lambda_-(\boldsymbol{Y})(n-1) = \Lambda_+(\boldsymbol{X})(n-1)\,{}^t\Delta_{-+}(n)\,{}^t\Delta_{--}^{-1}(n)$,

(iv) $\Gamma_+^{-1}(0,0)\Lambda_+(\boldsymbol{X})(0)\,{}^t\Gamma_+^{-1}(0,0) = \Gamma_-^{-1}(0,0)\Lambda_-(\boldsymbol{Y})(0)\,{}^t\Gamma_-^{-1}(0,0)$.

To prove Theorem 7.1, we need some preparations. For the pair $[\boldsymbol{X},\boldsymbol{Y}]$ of $d$-dimensional flows, we define $pd$-dimensional flows $\boldsymbol{X}^p = (X^p(n); 0 \le n \le N)$, $\boldsymbol{Y}^p = (Y^p(l); -N \le l \le 0)$ by

$$X^p(n) \equiv \begin{pmatrix} X(np) \\ X(np+1) \\ \vdots \\ X((n+1)p-1) \end{pmatrix} \quad (0 \le n \le N), \tag{7.15}$$

$$Y^p(l) \equiv \begin{pmatrix} Y((l-1)p+1) \\ Y((l-1)p+2) \\ \vdots \\ Y(lp) \end{pmatrix} \quad (-N \le l \le 0). \tag{7.16}$$

Then the following lemma is easily verified.

Lemma 7.1. *The pair* $[\boldsymbol{X},\boldsymbol{Y}]$ *of flows has periodic stationarity property of period* $p$ *if and only if the pair* $[\boldsymbol{X}^p,\boldsymbol{Y}^p]$ *of flows has stationarity property.*

Lemma 7.2.

(i) *For any element* $\gamma_+$ *of* $\mathscr{LMD}_+(\boldsymbol{X})$ *and any element* $\gamma_-$ *of* $\mathscr{LMD}_-(\boldsymbol{Y})$, *there exist an element* $\gamma_+^p$ *of* $\mathscr{LMD}_+(\boldsymbol{X}^p)$ *and an element* $\gamma_-^p$ *of* $\mathscr{LMD}_-(\boldsymbol{Y}^p)$ *such that* (7.17)–(7.20) *hold.*

(ii) *Conversely, for any element* $\gamma_+^p$ *of* $\mathscr{LMD}_+(\boldsymbol{X}^p)$ *and an element* $\gamma_-^p$ *of* $\mathscr{LMD}_-(\boldsymbol{Y}^p)$, *there exist an element* $\gamma_+$ *of* $\mathscr{LMD}_+(\boldsymbol{X})$ *and an element* $\gamma_-$ *of* $\mathscr{LMD}_-(\boldsymbol{Y})$ *such that* (7.17)–(7.20) *hold*:

$$\gamma_+^p(n,k) = \Gamma_+^{-1}(n,n)\Gamma_+(n,k) \quad (0 \le k < n \le N), \tag{7.17}$$

$$\gamma_-^p(n,k) = \Gamma_-^{-1}(n,n)\Gamma_-(n,k) \quad (0 \le k < n \le N), \tag{7.18}$$

$$V_+(\boldsymbol{X}^p)(n) = \Gamma_+^{-1}(n,n)\Lambda_+(\boldsymbol{X})(n)^t\Gamma_+^{-1}(n,n) \quad (0 \le n \le N), \tag{7.19}$$

$$V_-(\boldsymbol{Y}^p)(n) = \Gamma_-^{-1}(n,n)\Lambda_-(\boldsymbol{Y})(n)^t\Gamma_-^{-1}(n,n) \quad (0 \le n \le N). \tag{7.20}$$

Proof. Let $\gamma_+, \gamma_-$ be any elements of $\mathscr{LMD}_+(\boldsymbol{X}), \mathscr{LMD}_-(\boldsymbol{Y})$, respectively. It follows from definition that

$$G((N+1)p) = \begin{pmatrix} \Gamma_+(0,0) & & & 0 \\ \Gamma_+(1,0) & \Gamma_+(1,1) & & \\ \vdots & \vdots & \ddots & \\ \Gamma_+(N,0) & \Gamma_+(N,1) & \cdots & \Gamma_+(N,N) \end{pmatrix}, \tag{7.21}$$

$$G((N+1)p)T(\boldsymbol{X})((N+1)p)^tG((N+1)p) = \begin{pmatrix} \Lambda_+(\boldsymbol{X})(0) & & 0 \\ & \ddots & \\ 0 & & \Lambda_+(\boldsymbol{X})(N) \end{pmatrix}, \tag{7.22}$$

where $G(\star)$ is defined by (3.6). Thus applying Lemma 3.6 to the flow $\boldsymbol{X}^p$, we see from (7.22) that there exist an element $\gamma_+^p$ of $\mathscr{LMD}_+(\boldsymbol{X}^p)$ which satisfies (7.17) and (7.19). The same is true for the flow $\boldsymbol{Y}^p$. The converse is also proved in a similar way. □

We now prove Theorem 7.1.

Proof of Theorem 7.1. We assume that $[\boldsymbol{X}, \boldsymbol{Y}]$ has periodic stationarity property of period $p$. Then from Lemma 7.1, $[\boldsymbol{X}^p, \boldsymbol{Y}^p]$ has stationarity property. So, by Theorem 6.5, there exist an element $\gamma_+^p$ of $\mathscr{LMD}_+(\boldsymbol{X}^p)$ and an element $\gamma_-^p$ of $\mathscr{LMD}_-(\boldsymbol{Y}^p)$ such that for any integers $k, n$ $(1 \le k < n \le N)$

$$\gamma_+^p(n,k) = \gamma_+^p(n-1,k-1) + \delta_+^p(n)\gamma_-^p(n-1,n-k-1). \tag{7.23}$$

Substituting (7.17), (7.18) into the above equation, we have

$$\begin{aligned} \Gamma_+^{-1}(n,n)\Gamma_+(n,k) &= \Gamma_+^{-1}(n-1,n-1)\Gamma_+(n-1,k-1) \\ &\quad + \Gamma_+^{-1}(n,n)\Gamma_+(n,0)\Gamma_-^{-1}(n-1,n-1)\Gamma_-(n-1,n-k-1). \end{aligned} \tag{7.24}$$

Multiplying both-hand sides by $\Gamma_+(n,n)$ from the left-hand side in the above equation, we obtain

$$\begin{aligned} \Gamma_+(n,k) &= \Gamma_+(n,n)\Gamma_+^{-1}(n-1,n-1)\Gamma_+(n-1,k-1) \\ &\quad + \Gamma_+(n,0)\Gamma_-^{-1}(n-1,n-1)\Gamma_-(n-1,n-k-1). \end{aligned} \tag{7.25}$$

Applying (7.11) and (7.13) to the above equation, we have $(\mathrm{DDT_p})$(i). Relations $(\mathrm{DDT_p})$(ii) and $(\mathrm{FDT_p})$(i)–(iv) can be obtained in the same way. Conversely, we assume that there exist an element $\gamma_+$ of $\mathscr{LMD}_+(\boldsymbol{X})$ and an element $\gamma_-$ of $\mathscr{LMD}_-(\boldsymbol{Y})$ which satisfies ($\mathrm{DDT_p}$) and ($\mathrm{FDT_p}$). Then, it follows from (7.11)–(7.14) and (7.17)–(7.20) that there exist an element $\gamma_+^p$ of $\mathscr{LMD}_+(\boldsymbol{X}^p)$ and an element $\gamma_-^p$ of $\mathscr{LMD}_-(\boldsymbol{Y}^p)$ which satisfies (DDT) and (FDT). Thus $[\boldsymbol{X}^p, \boldsymbol{Y}^p]$ has stationarity property, so that $[\boldsymbol{X}, \boldsymbol{Y}]$ has periodic stationarity property of period $p$. The proof is now complete. $\square$

## 8. The test for models of covariance matrix functions.

In the present section, we shall propose a test for models of covariance matrix functions as an application of the theoretical results obtained in the previous sections. This is based on the same technique as in the test for stationarity in [**6**]. Let $\mathscr{Z} = (\mathscr{Z}(n); 0 \le n \le N)$ be a $d$-dimensional time series. We assume that this time series is a realization of an unknown $d$-dimensional stochastic process. We regard this stochastic process as a flow $\boldsymbol{X} = (X(n); 0 \le n \le N)$ in an real inner product space. Although we would like to identify this flow explicitly, here we consider an easier but important problem. Let $R = (R(m,n); 0 \le m, n \le N)$ be the covariance matrix function of the flow $\boldsymbol{X}$. Here $R$ is also unknown. And let $R^{\mathscr{Z}} = (R^{\mathscr{Z}}(m,n); 0 \le m, n \le N)$ be a model of the covariance matrix function presumed by the time series $\mathscr{Z}$. $R^{\mathscr{Z}}$ must be at least nonnegative definite.

For example, when we can obtain the same kinds of $d$-dimensional data $\mathscr{Z}^{(p)} = (\mathscr{Z}^{(p)}(n); 0 \le n \le N)$ $(1 \le p \le M)$ as the data $\mathscr{Z}$ by doing $M$ times observations, we can estimate a sample covariance matrix function $R^{\mathscr{Z}}$ by

$$R^{\mathscr{Z}}(m,n) \equiv \frac{1}{M}\sum_{p=1}^{M}(\mathscr{Z}^{(p)}(m) - \mu^{\mathscr{Z}}(m))^t(\mathscr{Z}^{(p)}(n) - \mu^{\mathscr{Z}}(n)),$$

where $\mu^{\mathscr{Z}}(n) \equiv (1/M)\sum_{p=1}^{M}\mathscr{Z}^{(p)}(n)$.

For simplicity, we assume that $R^{\mathscr{Z}}$ has strictly positive definite property. We will now propose a method to examine whether $R^{\mathscr{Z}}$ is appropriate or not as a model of the covariance matrix function $R$.

Algorithm 8.1.

[Step 1] *We calculate a forward* $\mathrm{KM_2O}$-*Langevin matrix*

$$\mathscr{LM}_+(R^{\mathscr{Z}}) = \{\gamma_+(R^{\mathscr{Z}})(n,k), \delta_+(R^{\mathscr{Z}})(n), V_+(R^{\mathscr{Z}})(m); 0 \le k < n \le N, 0 \le m \le N\} \tag{8.1}$$

*associated with the matrix function* $R^{\mathscr{Z}}$ *by Algorithm* 5.1.

[Step 2] *We derive from the time series* $\mathscr{Z}$ *a sample forward* $\mathrm{KM_2O}$-*Langevin fluctuation flow* $\nu_+(\mathscr{Z}) = (\nu_+(\mathscr{Z})(n); 0 \le n \le N)$ *as follows*:

$$\begin{cases} \nu_+(\mathscr{Z})(0) \equiv \mathscr{Z}(0), \\ \nu_+(\mathscr{Z})(n) \equiv \mathscr{Z}(n) + \sum_{k=0}^{n-1}\gamma_+(R^{\mathscr{Z}})(n,k)\mathscr{Z}(k) \quad (1 \le n \le N). \end{cases} \tag{8.2}$$

[Step 3] *Let* $W_+(n)$ $(0 \le n \le N)$ *be* $d \times d$ *matrices such that*

$$V_+(R^{\mathscr{Z}})(n) = W_+(n)^t W_+(n) \quad (0 \le n \le N). \tag{8.3}$$

*By standardizing the sample forward* $\mathrm{KM_2O}$*-Langevin fluctuation flow* $v_+(\mathscr{X})$, *we define a* $d$*-dimensional time series* $\xi_+ = (\xi_+(n); 0 \leq n \leq N)$, *i.e.*,

$$\xi_+(n) = {}^t(\xi_{+1}(n), \xi_{+2}(n), \ldots, \xi_{+d}(n)) \equiv W_+(n)^{-1} v_+(\mathscr{X})(n) \quad (0 \leq n \leq N). \tag{8.4}$$

*Furthermore we define a one-dimensional time series* $\xi = (\xi(n); 0 \leq n \leq (N+1)d - 1)$ *by*

$$\xi(n) \equiv \xi_{+j}(m) \quad n = md + j - 1 \quad (1 \leq j \leq d, 0 \leq m \leq N). \tag{8.5}$$

[Step 4] *It is easily verified that the following are equivalent*:

(i) *The time series* $\mathscr{X}$ *is a realization of some* $d$*-dimensional flow whose covariance matrix function is* $R^{\mathscr{X}}$.

(ii) *The time series* $\xi$ *is a realization of a white noise flow in a weak sense. Thus we verify the white noise property of the time series* $\xi$.

We will not go into details of the white noise test. An important conclusion here is that the problem of examining whether $R^{\mathscr{X}}$ is appropriate or not has now been reduced to the test of white noise property.

## A. Appendix.

We shall show the lemma which is used in the proof of Lemma 5.1 and Theorem 5.1.

Lemma A.1. *Let* $A$ *and* $C$ *be any* $d \times d$ *symmetric matrices, and* $B$ *be any* $d \times d$ *matrices. We define a* $2d \times 2d$ *matrix* $M$ *by*

$$M \equiv \begin{pmatrix} A & {}^tB \\ B & C \end{pmatrix}. \tag{A.1}$$

*If* $M$ *is nonnegative definite, then the following statements hold*:

(i) *There exists a* $d \times d$ *matrix* $X$ *such that* $XA + B = 0$;

(ii) *There exists a* $d \times d$ *matrix* $Y$ *such that* $YC + {}^tB = 0$.

Proof. Since $M$ is nonnegative definite, $M$ can be factorized as follows ($d$-dimensional block Cholesky factorization):

$$M = \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} I & -{}^tX \\ 0 & I \end{pmatrix}, \tag{A.2}$$

where $X$ and $F$ denote some $d \times d$ matrices. By direct calcultion, we see that the right-hand side above has the form

$$\begin{pmatrix} F & * \\ -XF & * \end{pmatrix}. \tag{A.3}$$

Therefore comparing (A.3) with (A.1), we have $XA + B = 0$, which implies (i). We shall prove (ii). Since

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & {}^tB \\ B & C \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} C & B \\ {}^tB & A \end{pmatrix}, \tag{A.4}$$

we see that

$$\begin{pmatrix} C & B \\ {}^tB & A \end{pmatrix} \geq 0. \tag{A.5}$$

Therefore (ii) follows from (i). □

## References


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