J. Math. Soc. Japan
Vol. 55, No. 3, 2003

On global smooth solutions to the initial-boundary value problem for quasilinear wave equations in exterior domains

Dedicated to Professor Atsushi Yoshikawa on his sixtieth birthday

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(Received Nov. 19, 2001)
(Revised Apr. 8, 2002)

Abstract. We consider the initial-boundary value problem for the standard quasilinear wave equation:

$$u_{tt} - \operatorname{div}\{\sigma(|\nabla u|^2)\nabla u\} + a(x)u_t = 0 \quad \text{in } \Omega \times [0, \infty)$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{and} \quad u|_{\partial\Omega} = 0$$

where Ω is an exterior domain in R^N, $\sigma(v)$ is a function like $\sigma(v) = 1/\sqrt{1+v}$ and $a(x)$ is a nonnegative function. Under two types of hypotheses on $a(x)$ we prove existence theorems of global small amplitude solutions. We note that $a(x)u_t$ is required to be effective only in localized area and no geometrical condition is imposed on the boundary $\partial\Omega$.

1. Introduction.

In this paper we consider the initial-boundary value problem for the quasilinear wave equation:

$$u_{tt} - \operatorname{div}\{\sigma(|\nabla u|^2)\nabla u\} + a(x)u_t = 0 \quad \text{in } \Omega \times [0, \infty) \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{and} \quad u|_{\partial\Omega} = 0 \tag{1.2}$$

where Ω is an exterior domain in the N dimensional Euclidean space R^N with a smooth boundary $\partial\Omega$ and $\sigma(v)$ is a function like $\sigma(v) = 1/\sqrt{1+v}$. Concerning the dissipation $a(x)u_t$ we make two types of assumptions specified later, which are intended to make the effect of this term as weaker as possible.

When $a(x) \equiv 1$ Matsumura [**11**] proved the global existence of smooth solutions for the Cauchy problem in the whole space R^N and this result was generalized by Shibata [**23**] to the exterior problems with $N \geq 3$. Here we first establish a global existence result under a weaker assumption on $a(x)$ which admits $a(x)$ to vanish in a large area. We make no restriction on the shape of obstacle V.

When $a(x) \equiv 0$ and $N = 1, 2$ we can not generally expect the global existence of smooth solutions of (1.1)–(1.2) even if the initial-data are small and smooth. Indeed, when $\Omega = R^N$ nonexistence was proved by Lax [**8**] and John [**5**] for the case $N = 1$ and Hoshiga [**4**] for the case $N = 2$. For the case $N \geq 3$ Kleinermann and Ponce [**7**], Shatah [**22**] proved global existence of small amplitude solutions when $\Omega = R^N$ and

2000 *Mathematics Subject Classification.* 35B35, 35L70.
Key Words and Phrases. Decay, Global solution, Localized dissipation, Quasilinear wave equation, Exterior domain.

Shibata and Tsutsumi [24] proved similar results for exterior problems under the assumption that the obstacle $V \equiv R^N/\Omega$ is convex. However, if Ω is a general domain no result on global existence is known. The reason is that when V is trapping the local energy never decay uniformly (Ralston [20]) and hence it is difficult to expect global solutions for such an exterior domain. In this paper, by introducing a dissipative term $a(x)u_t$, we want to treat general exterior domains in odd dimensions and prove global existence theorem for the problem (1.1)–(1.2). Our result admits the case $a(x) \equiv 0$ when V is star-shaped.

To specify our assumption on $a(x)$ we define $\Gamma(x_0)$, a part of the boundary $\partial\Omega$, as follows:

$$\Gamma(x_0) = \{x \in \partial\Omega \mid \nu(x) \cdot (x - x_0) > 0\}, \quad x_0 \in R^N,$$

where $\nu(x)$ is the outward normal at $x \in \partial\Omega$.

This set was introduced by D. Russell [21] motivated by Morawetz [13] and often used in control or stabilization theory for the wave equation in bounded domains (cf. Chen [1], Lions [10], Zuazua [25], Lasiecka and Triggiani [10], Nakao [15] etc.). In this paper we use this set for exterior domains.

Now, we make the following assumption on $a(\cdot)$.

Hyp.A. There exist $x_0 \in R^N$ and an open set ω in $\overline{\Omega}$ such that

$$\text{closure of } \Gamma(x_0) \subset \omega \quad \text{and} \quad a(x) \geq \varepsilon_0 > 0 \quad \text{for } x \in \omega$$

with some constant $\varepsilon_0 > 0$.

We first consider the problem with the following additional assumption on $a(x)$:

Hyp.B. There exist $L \gg 1$ and $\varepsilon_0 > 0$ such that

$$a(x) \geq \varepsilon_0 > 0 \quad \text{for } |x| \geq L.$$

We note that if V is star-shaped with respect to x_0, then the set $\Gamma(x_0)$ is empty and hence Hyp.A imposes no restriction on $a(x)$ and the case $a(x) \equiv 0$ is allowed. Hyp.B means that the dissipation $a(x)u_t$ is effective near infinity.

The first object of this paper is to prove the global existence of small amplitude solutions under the hypotheses A and B. It should be noted again that in our case $a(x)$ may vanish in a large area in Ω.

Quite recently, in [18], we have proved the total energy decay and L^2 boundedness

$$E(t) \equiv \frac{1}{2}\int_\Omega (|u_t|^2 + |\nabla u|^2)\,dx \leq CI_0^2(1+t)^{-1} \quad \text{and} \quad \|u(t)\|_2 \leq CI_0 < \infty$$

$(I_0^2 = E(0) + \|u_0\|^2)$ for the linear wave equation with $\sigma = 1$ under the Hypotheses A and B, and applied these estimates to semilinear wave equations with nonlinear term $f(u)$. Here, we again apply the same idea to our problem (1.1)–(1.2). But, here, we must establish such estimates for the linear equations with variable coefficients and also we must treat the nonlinear term more carefully. If we consider the problem in a bounded domain, the situation is simpler and hence we can treat a more delicate case where $a(x)$ is degenerate also in ω (cf. [17]). As a related work we mention also Mochizuki [12], where the Cauchy problem in R^N for the Kirchhoff type quasilinear wave equation with a localized dissipation near infinity has been considered.

Secondly, we consider the problem (1.1)–(1.2) under the assumption that Hyp.A holds and the support of $a(\cdot)$ is compact. In this case, the dissipation does not work for large x and we can no longer expect any uniform decay of total energy. While, recently in [**16**], we have proved a local energy decay for the linear wave equation with such a localized dissipation and extended the result by Morawetz [**13**] to general domains. On the basis of this result, we have further derived in [**19**] L^p estimates of solutions for the linear exterior problem and applied them to semilinear wave equations in odd dimensional exterior domains. In the present paper, we apply again such an estimate to the quasilinear wave equations. Shibata and Tsutsumi [**24**] already derived L^p estimates of ∇u for linear wave equations and treated more general nonlinear problems. But, they assumed that the obstacle V is non-trapping in the sense of Vainberg, particulerly, convex. See also Hayashi [**3**] where radially symmetric solutions of fully nonlinear wave equations outside a ball are considered. We would emphasize again that we make no geometrical conditions on V due to the dissipation $a(x)u_t$ and further this term can be dropped when V is star-shaped. We also note that although our equation is restricted to a typical case, the smoothness condition imposed on the initial data is weaker than those in [**24**], which comes from a carefull treatment of the nonlinear terms.

2. Preliminaries and statement of the results.

Let V be a compact set (obstacle) in R^N which may consist of several closed domains and set $\Omega = R^N/V$. We use only familiar functional spaces and omit the definitions, but we note that $\|\cdot\|_p$ denotes L^p norm and $W_0^{m,p}(\Omega)(W^{m,p}(\Omega))$ is a completion of $C_0^\infty(\Omega)(C_0^\infty(\bar{\Omega}))$ with respect to the norm $\sum_{k=0}^m \|D_x^k u\|_p$, where D_x^k denotes partial differentiations in x of the order k. We set $H^m = W^{m,2}$ and $H_0 = L^2$.

Concerning $\sigma(\cdot)$ we make the following assumptions.

Hyp.C. $\sigma(\cdot)$ is a differentiable function on $R^+ = [0, \infty]$ and satisfies the conditions:

$$\sigma(v) \geq k_0 > 0 \quad \text{and} \quad \sigma(v) - 2|\sigma'(v)|v \geq k_0 > 0 \quad \textit{if} \;\; 0 \leq v \leq R,\; R > 0,$$

where $k_0 = k_0(R)$ is a positive constant. (We may assume $\sigma(0) = 1$.)

The following result concerning local in time solutions is standard (cf. Kato [**6**]).

Proposition 1. *Let $m > M \equiv [N/2] + 1$ be an integer and assume that $\sigma(\cdot) \in C^{m+1}([0,\infty))$, $a(\cdot) \in C^{m+1}(\bar{\Omega})$ and $\partial\Omega$ is of C^{m+1} class. Let $(u_0, u_1) \in H^{m+1}(\Omega) \times H^m(\Omega)$ satisfy the compatibility condition of m-th order associated with the problem* (1.1)–(1.2). *Then, there exists $T \equiv T(\|u_0\|_{H^{m+1}} + \|u_1\|_{H^m}) > 0$ such that the problem admits a unique solution $u(t)$ on $[0, T)$ belonging to*

$$X_m^T \equiv \bigcap_{k=0}^m C^k([0,T); H^{m+1-k}(\Omega) \cap H_0^1(\Omega)) \bigcap C^{m+1}([0,T); L^2(\Omega)).$$

We set $X_m = X_m^\infty$. From Proposition 1 it suffices for the existence of global solutions in X_m to derive a priori estimate

$$\sup_{0 \leq t < T} \sum_{k=0}^{m+1} \|D_t^k u(t)\|_{H^{m+1-k}(\Omega)} < \infty$$

with all $T > 0$ where $u(t)$ is an assumed local solution on $[0, T)$. In what follows we assume that $\partial\Omega$ is sufficiently smooth, i.e., $\partial\Omega$ is of C^{m+1} class.

Our first main result reads as follows.

THEOREM 1. *Let N be any integer ≥ 1 and assume that $\sigma(\cdot) \in C^{m+1}(R^+)$ and $a(\cdot) \in C^{m+1}(\bar{\Omega})$ with an integer $m > [N/2] + 1$. Then, under hypotheses A, B and C, there exists $\delta > 0$ such that if $(u_0, u_1) \in H^{m+1} \times H^m$ satisfies the compatibility condition of the m-th order and smallness condition $I_m \equiv \|u_0\|_{H^{m+1}} + \|u_1\|_{H^m} < \delta$, the problem* (1.1)–(1.2) *admits a unique solution $u(t)$ in the class X_m. Further, the following estimates hold*:

$$\|D_t^{k+1}u(t)\|_{H^{m-k}}^2 + \|D_t^k \nabla u(t)\|_{H^{m-k}}^2 \leq CI_m^2(1+t)^{-k-1} \quad \text{for } 0 \leq k \leq m$$

and

$$\|\nabla u(t)\|_{H^{m-k}}^2 \leq CI_m^2(1+t)^{-1} \quad \text{for } 0 \leq k \leq m.$$

The results by the second approach are stated separately in the cases $N \geq 4$ and $N = 3$.

THEOREM 2. *Let $N \geq 4$. When N is even we assume that V is convex. Assume that σ and $a(\cdot)$ are of C^{3M} class. We assume that (u_0, u_1) belongs to $H^{3M+1} \cap W^{2M+1,1} \times H^{3M} \cap W^{2M,1}$ and satisfies the compatibility conditions of the $3M$-th order associated with the quasilinear problem* (1.1)–(1.2) *and also the linear problem with $\sigma \equiv 1$. Further, we assume that $a(\cdot)$ satisfies Hyp.A and $\operatorname{supp} a(\cdot)$ is compact. Then, under Hyp.C, there exists $\delta > 0$ such that if*

$$I_{3M} \equiv \|u_0\|_{H^{3M+1}} + \|u_0\|_{W^{2M+1,1}} + \|u_1\|_{H^{3M}} + \|u_1\|_{W^{2M,1}} \leq \delta,$$

there exists a unique solution $u(t)$ in the class

$$Y_{3M} \equiv \bigcap_{k=0}^{3M} C^k([0,\infty); H^{3M+1-k} \cap H_0^1) \bigcap C^{3M+1}([0,\infty); L^2)$$
$$\bigcap W^{k,\infty}([0,\infty); W^{M+1-k,\infty}(\Omega)),$$

satisfying

$$\sum_{k=0}^{3M} \|D_t^k \nabla u(t)\|_{H^{3M-k}} \leq CI_{3M} < \infty$$

and

$$\sum_{k=0}^{M} \|D_t^k \nabla u(t)\|_{W^{M-k,\infty}} \leq CI_{3M}(1+t)^{-d}$$

with $d = (N-1)/2$.

More interesting is the case $N = 3$, where the situation is also more delicate.

THEOREM 3. *Let $N = 3$. Assume that σ and $a(\cdot)$ are of C^{4M+2} class. We assume that (u_0, u_1) belongs to $H^{4M+3} \cap W^{4M+2,q} \times H^{4M+2} \cap W^{4M+1,q}$ and satisfies the compatibility conditions of the $4M+2$-th order associated with the quasilinear problem* (1.1)–(1.2) *and also the linear problem with $\sigma \equiv 1$. We assume that $a(\cdot)$ satisfies Hyp.A and $\operatorname{supp} a(\cdot)$ is compact. Then, under Hyp.C, there exists δ such that if*

$$\tilde{I}_{4M+2} \equiv \|u_0\|_{H^{4M+3}} + \|u_0\|_{W^{4M+2,q}} + \|u_1\|_{H^{4M+2}} + \|u_1\|_{W^{4M+1,q}} \leq \delta,$$

there exists a unique solution $u(t)$ *in the class*

$$Y_{4M+3} \equiv \bigcap_{k=0}^{4M+2} C^k([0,\infty); H^{4M+3-k} \cap H_0^1) \bigcap C^{4M+2}([0,\infty); L^2),$$

satisfying

$$\sum_{k=0}^{4M+3} \|D_t^k \nabla u(t)\|_{H^{4M+3-k}} \le C\tilde{I}_{4M+2} < \infty$$

and

$$\sum_{k=0}^{M+1} \|D_t^k \nabla u(t)\|_{W^{M+1-k,p}} \le C\tilde{I}_{4M+2}(1+t)^{-d(p)}$$

with $d(p) = (p-2)(1-\varepsilon)/p$, $0 < \varepsilon \ll 1$, *where we should take* $6 \le p < \infty$ *and* $q = p/(p-1)$.

Remark. Concerning the regularity of the initial data, in [**21**] (u_0, u_1) is required to belong to $H^{20} \times H^{19}$ if $N = 3$, while here we impose $(u_0, u_1) \in H^{11} \times H^{10}$.

3. Total energy decay for the linear wave equations with variable coefficients.

For preparation of the proof of Theorem 1 we consider in this section the linear wave equations with variable coefficients:

$$u_{tt} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{i,j}(x,t)\frac{\partial u}{\partial x_j}\right) + a(x)u_t = f(x,t) \quad \text{in } \Omega \times (0,\infty) \tag{3.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{and} \quad u|_{\partial\Omega} = 0. \tag{3.2}$$

We assume

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \ge k_0|\xi|^2, \quad \xi \in R^N,$$

with some $k_0 > 0$. We also assume that a_{ij}, $\partial\Omega$ are of C^1 class and $f \in W_{loc}^{1,2}([0,\infty); L^2(\Omega))$.

It is a standard result that for each $(u_0, u_1) \in H^2 \cap H_0^1 \times H_0^1$ there exists a unique solution $u(t)$ in X_1, H^2-solution, for the problem (3.1)–(3.2). We establish some decay estimates for such H^2-solutions.

Multiplying the equation (3.1) by u_t and integrating by parts, we have the identity:

$$\frac{d}{dt}E(t) + \int_\Omega a|u_t|^2\,dx = -\frac{1}{2}\int_\Omega \sum_{i,j} \dot{a}_{ij}u_{x_i}u_{x_j}\,dx + \int_\Omega fu_t\,dx, \tag{3.3}$$

where $\dot{a}_{ij} = \partial a_{ij}/\partial t$ and

$$E(t) \equiv \frac{1}{2}\int_\Omega \left(|u_t|^2 + \sum_{i,j} a_{ij}u_{x_i}u_{x_j}\right)dx.$$

Next, multiplying the equation by $\boldsymbol{h}\cdot\nabla u$, $\boldsymbol{h}=(h_1,\dots,h_n)$, and integrating we have (cf. [**16**])

$$\frac{d}{dt}(u_t,\boldsymbol{h}\cdot\nabla u)+\frac{1}{2}\int_\Omega \nabla\cdot\boldsymbol{h}|u_t|^2\,dx-\frac{1}{2}\sum_{i,j}\int_\Omega((\boldsymbol{h}\cdot\nabla)a_{ij}+(\nabla\cdot\boldsymbol{h}a_{ij})u_{x_i}u_{x_j})\,dx$$

$$-\frac{1}{2}\sum_{i,j}\int_{\partial\Omega}a_{ij}v_iv_j\left|\frac{\partial u}{\partial v}\right|^2(v\cdot\boldsymbol{h})\,dS+\sum_{i,j}\int_\Omega a_{i,j}u_{x_j}\nabla u\cdot\partial\boldsymbol{h}/\partial x_i\,dx+\int_\Omega au_t\boldsymbol{h}\cdot\nabla u\,dx$$

$$=\int_\Omega f\boldsymbol{h}\cdot\nabla u\,dx. \tag{3.4}$$

Further, multiplying the equation by ηu, $\eta\in W^{1,\infty}(\Omega)$, we get

$$\sum_{i,j}\int_\Omega \eta a_{ij}u_{x_i}u_{x_jx_j}\,dx+\sum_{i,j}\int_\Omega a_{ij}u_{x_j}\eta_{x_i}u\,dx+\frac{d}{dt}(u_t,\eta u)+\int_\Omega au_t\eta u\,dx$$

$$=\int_\Omega(\eta|u_t|^2+f\eta u)\,dx. \tag{3.5}$$

In particular, taking $\eta\equiv 1$, (3.5) is reduced to

$$\sum_{i,j}\int_\Omega a_{ij}u_{x_i}u_{x_j}\,dx+\frac{d}{dt}(u_t,u)+\int_\Omega au_t\eta u\,dx=\int_\Omega(|u_t|^2+fu)\,dx. \tag{3.5$'$}$$

Now, we take a function $\phi(r)$, $r=|x-x_0|$, such that

$$\phi(r)=\begin{cases}\varepsilon_0 & \text{if } r\le L+|x_0|\\ \dfrac{(L+|x_0|)\varepsilon_0}{r} & \text{if } r\ge L+|x_0|.\end{cases}$$

Then, setting $\boldsymbol{h}=\phi(r)(x-x_0)$ in (3.4) we have

$$\frac{d}{dt}(u_t,\phi(r)(x-x_0)\cdot\nabla u)+\frac{1}{2}\int_\Omega(N\phi+\phi' r)|u_t|^2\,dx$$

$$-\frac{1}{2}\sum_{i,j}\int_\Omega(a_{ij}(N\phi+\phi' r)+\phi(r)(x-x_0)\cdot\nabla a_{ij})u_{x_i}u_{x_j}\,dx$$

$$-\frac{1}{2}\sum_{i,j}\int_{\partial\Omega}a_{ij}v_iv_j\left|\frac{\partial u}{\partial v}\right|^2\phi(r)v\cdot(x-x_0)\,dS$$

$$+\sum_{i,j}\int_\Omega a_{i,j}u_{x_i}u_{x_j}\left(\phi'\frac{(x_i-x_i^0)(x_j-x_j^0)}{r}+\phi\right)dx$$

$$+\int_\Omega au_t\phi(x-x_0)\cdot\nabla u\,dx=\int_\Omega f\phi(x-x_0)\cdot\nabla u\,dx. \tag{3.4$'$}$$

Combining (3.3), (3.4)$'$ and (3.5)$'$ we obtain for $k>0$, $\alpha>0$,

$$\frac{d}{dt}\left\{(u_t, \phi(x-x_0)\cdot\nabla u) + \alpha(u_t, u) + \frac{1}{2}\int_\Omega a|u|^2\,dx + kE(t)\right\}$$
$$-\sum_{i,j}\int_\Omega\left\{\left(\frac{(N\phi+\phi' r)}{2} + \phi'\frac{(x_i-x_i^0)(x_j-x_j^0)}{r} + \phi + \alpha\right)a_{ij}\right.$$
$$\left. + \frac{1}{2}\phi(x-x_0)\cdot\nabla a_{ij}\right\}u_{x_i}u_{x_j}\,dx$$
$$+\int_\Omega\left(\frac{N\phi+\phi' r}{2} - \alpha + ka(x)\right)|u_t|^2\,dx$$
$$\le \frac{k}{2}\sum_{i,j}\int_\Omega \dot{a}_{ij}u_{x_ix_j}\,dx + \frac{1}{2}\sum_{i,j}\int_{\Gamma(x_0)} a_{ij}\nu_i\nu_j\left|\frac{\partial u}{\partial \nu}\right|^2\phi\nu\cdot(x-x_0)\,dS$$
$$+\int_\Omega(\phi(x-x_0)\cdot\nabla u + u + ku_t)f\,dx + \int_\Omega|au_t\phi(x-x_0)\cdot\nabla u|\,dx. \tag{3.6}$$

Here, we see

$$\sum_{i,j}\left\{\left(-\frac{(N\phi+\phi' r)}{2} + \phi'\frac{(x_i-x_i^0)(x_j-x_j^0)}{r} + \phi + \alpha\right)a_{ij} - \frac{1}{2}\phi(x-x_0)\cdot\nabla a_{ij}\right\}u_{x_i}u_{x_j}$$
$$\ge \left\{-\frac{(N\phi+\phi' r)}{2} + \phi' r + \phi + \alpha - \frac{(L+|x_0|)\varepsilon_0 l_0}{2}\right\}\sum_{i,j}a_{ij}u_{x_i}u_{x_j} \tag{3.7}$$

where

$$l_0 = k_0^{-1}\left(\sum_{i,j}\|\nabla a_{ij}\|_\infty^2\right)^{1/2}.$$

Also, assuming $(L+|x_0|)l_0 < 1/4$, we can choose $\alpha > 0$ and $k \gg 1$ such that

$$-\frac{N\phi+\phi' r}{2} - \frac{(L+|x_0|)\varepsilon_0 l_0}{2} + \phi' r + \phi + \alpha \ge \frac{\varepsilon_0}{8}$$

and

$$\frac{N\phi+\phi' r}{2} - \alpha + \frac{k}{2}a(x) \ge \frac{\varepsilon_0}{8}.$$

Indeed, for example, we can take $\alpha = (N/2 - 1/4)\varepsilon_0$ and $k \ge N\varepsilon_0$.

Finally, noting

$$\left|\sum_{i,j}\frac{1}{2}\int_\Omega \dot{a}_{ij}u_{x_i}u_{x_j}\,dx\right| \le \frac{1}{2}l_1\int_\Omega\sum_{i,j}a_{i,j}u_{x_i}u_{x_j}\,dx$$

with $l_1 = k_0^{-1}(\sum_{i,j}\|\dot{a}_{ij}\|_\infty^2)^{1/2}$, we assume l_1 is sufficiently small so that

$$kl_1 \le \varepsilon_0/4.$$

Then, we obtain from (3.6) and (3.7) that

$$\frac{d}{dt}\left\{(u_t, \phi(x-x_0)\cdot\nabla u) + \alpha(u_t, u) + \frac{1}{2}\int_\Omega a|u|^2\,dx + kE(t)\right\} + \frac{\varepsilon_0}{8}E(t) + \frac{k}{2}\int_\Omega a|u_t|^2\,dx$$

$$\le C\int_{\Gamma(x_0)}\left|\frac{\partial u}{\partial \nu}\right|^2 dS + C\int_\Omega (|\nabla u| + |u| + k|u_t|)|f|\,dx \tag{3.8}$$

with a constant $C > 0$.

To control the first term of the right-hand side of (3.8) we again use (3.5). This time, we take a vector field $\boldsymbol{h}$ such that

$$\boldsymbol{h}\cdot\nu \ge 0, \quad \boldsymbol{h} = \nu \quad \text{on } \Gamma(x_0) \quad \text{and} \quad \operatorname{supp}\boldsymbol{h} \subset \tilde{\omega}$$

where $\tilde{\omega}$ is a bouned open set in R^N with $\overline{\Gamma(x_0)} \subset \tilde{\omega}\cap\Omega \subset \omega$. Then we can derive

$$\int_{\Gamma(x_0)}\left|\frac{\partial u}{\partial \nu}\right|^2 dS \le C\int_{\tilde{\omega}\cap\Omega}(|\nabla u|^2 + |u_t|^2)\,dx + C\int_\omega |f|\,|\nabla u|\,dx - c_0\frac{d}{dt}(u_t, \boldsymbol{h}\cdot\nabla u) \tag{3.9}$$

with a constant $C > 0$ and a certain $c_0 > 0$. Further, taking a function $\eta \in W^{1,\infty}(\bar{\Omega})$ such that

$$\eta = 0 \quad \text{on } \omega^c, \quad \eta = 1 \quad \text{on } \Omega\cap\tilde{\omega} \quad \text{and} \quad |\nabla\eta|/\eta \in L^\infty(\Omega)$$

we have from (3.6) that

$$\int_{\tilde{\omega}\cap\Omega}|\nabla u|^2\,dx + c_1\frac{d}{dt}(u_t, \eta u) + c_1\int_\Omega a\eta|u|^2\,dx \le C\int_\omega(|u_t|^2 + |u|^2)\,dx \tag{3.10}$$

for some $c_1 > 0$, $C > 0$.

It follows from (3.8)–(3.10) that

$$\frac{dX(t)}{dt} + \frac{\varepsilon_0}{8}E(t) + \frac{k}{2}\int_\Omega a|u_t|^2\,dx \le C\left(\int_\omega |u|^2\,dx + \int_\Omega(|\nabla u| + |u| + k|u_t|)|f|\,dx\right) \tag{3.11}$$

with some $C > 0$ and a large $k > 0$, where we set

$$X(t) = (u_t, (\phi(x-x_0) + c_0\boldsymbol{h})\cdot\nabla u) + ((\alpha + 2c_1\eta)u_t, u) + \frac{1}{2}\int_\Omega a|u|^2\,dx + kE(t).$$

We summarize the above argument in the following:

Proposition 2. *For any large k, say $k > N\varepsilon_0$, if*

$$\sup_{0\le t<\infty}\left\{\left(\sum_{i,j}\|\nabla a_{ij}\|_\infty^2\right)^{1/2} + \left(\sum_{i,j}\|\dot{a}_{ij}\|_\infty^2\right)^{1/2}\right\} \le \delta_0 \equiv \min\{(L + |x_0|)/4, k_0\varepsilon_0/8k\} \tag{3.12}$$

then, the inequality (3.11) *holds for the energy finite solutions $u(t)$ of the problem* (3.1)–(3.2).

As a corollary of Proposition 2 we can prove some convenient unique continuation properties for the wave equation with variable coefficients. (See [**17**].) In particular, the following will be used later.

Proposition 3. *Let $u(t)$ be an H^2-solution of the wave equation*

$$u_{tt} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{i,j}(x,t)\frac{\partial u}{\partial x_j}\right) = 0 \quad in\ \Omega \times [0,T], \quad u|_{\partial\Omega} = 0$$

satisfying the condition

$$u_t(x,t) = 0 \quad for\ 0 \le t \le T, \quad x \in \omega \cup \Omega_R^c.$$

Then, there exist $\delta_1 > 0$ and $T_0 > 0$ such that if $T > T_0$ and

$$\sup_{0\le t\le T}\sup_{i,j}(\|\nabla\dot{a}_{ij}(t)\|_\infty + \|\dot{a}_{ij}(t)\|_\infty + \|\nabla a_{ij}(t)\|_\infty) \le \delta_1, \quad \left(\dot{} \equiv \frac{\partial}{\partial t}\right) \tag{3.13}$$

we have $u(x,t) = 0$ on $[0,T] \times \Omega$.

Outline of the proof. For convenience of the readers we give an outline of the proof of Proposition 3. Setting $u_t = U$ we have

$$U_{tt} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{i,j}(x,t)\frac{\partial U}{\partial x_j}\right) = F \quad \text{in}\ \Omega \times [0,T]$$

with

$$F(x,t) = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\dot{a}_{i,j}(x,t)\frac{\partial U}{\partial x_j}\right).$$

Applying (3.11) to this equation we have

$$\frac{dX}{dt} + kE(t) \le C\int_{\Omega_R}(|\nabla U| + |U| + k|U_t|)|F|\,dx$$

where X and E are defined with u replaced by U and we have used the assumption $U = 0$ on $\omega \cup \Omega_R^c \times [0,T]$. Since $U = 0$ on Ω_R^c and $U|_{\partial\Omega} = 0$ we see that $X(t)$ is equivalent to

$$E(t) = \frac{1}{2}\int_{\Omega_R}(|U_t(t)|^2 + |\nabla U(t)|^2)\,dx.$$

Thus, integrating the above inequality we have

$$\int_0^T E(s)\,ds \le C\left(E(0) + \int_0^T\int_{\Omega_R}(|U_t| + |\nabla U| + |U|)|F|\,dxds\right)$$

and hence,

$$\int_0^T E(s)\,ds \le C\left(E(t^*) + \int_0^T\int_{\Omega_R}|F|^2\,dxds\right)$$

where $E(t^*) = \inf_{0\le s\le T} E(s)$. Here, by use of the equation and elliptic theory,

$$\int_{\Omega_R} |F(t)|^2\,dx \le C\delta_1 \left(\int_{\Omega_R} (|D_x^2 u|^2 + |\nabla u|^2 \right) dx$$

$$\le C\delta_1 \int_{\Omega_R} (|u_{tt}| + |\nabla u|^2 + |u|^2)\,dx$$

$$\le C\delta_1 \int_{\Omega_R} (|U_t|^2 + |\nabla u|^2)\,dx.$$

Further, by the equation we see

$$\|\nabla u\|^2 \le C \int_{\Omega_R} |u_{tt}|\,|u|\,dx \le C\|U_t\|\,\|\nabla u\|$$

and hence,

$$\int_{\Omega_R} |F(t)|^2\,dx \le C\delta_1 E(t).$$

Taking δ_1 small we arrived at the inequality

$$\int_0^T E(s)\,ds \le CE(t^*).$$

This implies for large T, $E(t) \equiv 0$, $0 \le t \le T$, which implies $u(x,t) = u(x)$, independent of t. Then, by the equation we see

$$\|\nabla u\|^2 = 0$$

which combined with $u|_{\partial\Omega} = 0$ implies $u(x) \equiv 0$.

Now, let us return to the inequality (3.11). By Proposition 3 we can prove the following delicate inequality. Note that we may assume $T_0 = T_1$.

Proposition 4. *Under the assumptions* (3.12) *and* (3.13) *in Propositions* 2 *and* 3, *for any* $\varepsilon > 0$ *there exists a constant* $C_\varepsilon > 0$ *such that*

$$\int_t^{t+T} \int_\omega |u|^2\,dxds \le C_\varepsilon \int_t^{t+T} \int_\Omega (a|u_t|^2 + |f|^2)\,dxds + \varepsilon \int_t^{t+T} E(t)\,dt \tag{3.14}$$

for any $t > 0$.

Proof. We use a contradiction method (cf. Zuazua [**25**], Nakao [**17**]). If (3.20) was false, there would exist a sequence $\{t_n\} \subset R^+$ and a sequence of solutions $\{u_j\}$ such that

$$\int_{t_n}^{t_n+T} \int_\omega |u_n|^2\,dxds > n \int_{t_n}^{t_n+T} \int_\Omega (a|u_{nt}|^2 + |f|^2)\,dxds + \frac{\varepsilon}{16} \int_{t_n}^{t_n+T} E_n(t)\,dt, \tag{3.15}$$

where $E_n(t)$ is defined by $E(t)$ with $u(t)$ replaced by $u_n(t)$. Setting

$$\lambda_n^2 = \int_{t_n}^{t_n+T} \int_\omega |u_n|^2\,dxds \quad \text{and} \quad v_n(t) = u_n(\cdot + t_n)/\lambda_n$$

we have

$$v_{ntt} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t+t_n)\frac{\partial v_n}{\partial x_j}\right) + a(x)v_{nt} = f(x, t+t_n)/\lambda_n \quad \text{in } \Omega \times [0, T]$$

and by (3.15)

$$\lim_{n\to 0}\int_0^T \int_\Omega a(x)|v_{nt}|^2\, dxds = \lim_{n\to 0}\int_0^T \int_\Omega |f(t+t_n)|^2\, dxds/\lambda_n^2 = 0 \quad \text{and} \quad \varepsilon\int_0^T \tilde{E}_n(t)\, dt \le 1$$

where $\tilde{E}_n(t)$ is defined by $E(t)$ with $u(t)$ replaced by $v_n(t)$. There exists a subsequence $\{n'\}$ which we denote again $\{n\}$ such that

$$a_{ij}(x, \cdot + t_n) \to \tilde{a}_{ij}(x, t) \text{ weakly* in } W^{1,\infty}([0, T] \times \Omega),$$

$v_n \to v$ weakly* in $L^2([0, T]; H^1_{0,loc}(\Omega))$ and strongly in $L^2([0, T] \times \Omega_R)$, $R > L$, and

$$Dv_n \to Dv \text{ weakly in } L^2([0, T]; L^2(\Omega)), \quad D = (D_x, D_t).$$

Therefore, v is a solution of the equation

$$v_{tt} - \sum_{i,j} \frac{\partial}{\partial x_i}\left(\tilde{a}_{ij}\frac{\partial v}{\partial x_j}\right) = 0 \quad \text{in } [0, T] \times \Omega$$

with

$$v_t = 0 \quad \text{on } [0, T] \times \omega \cup \Omega_R^c$$

and

$$\int_0^T \int_\omega |v|^2\, dxds = 1. \tag{3.16}$$

But, by the assumption on a_{ij} we see

$$\sup_{0\le t\le T}\sup_{i,j}(\|\nabla\dot{\tilde{a}}_{ij}(t)\|_\infty + \|\dot{\tilde{a}}_{ij}(t)\|_\infty + \|\nabla\tilde{a}_{ij}(t)\|_\infty) \le \delta_1. \qquad \square$$

Applying Proposition 3 (see the Remark below) to v we have $v(x, t) \equiv 0$ in $[0, T] \times \Omega$ if $T > T_0$, which contradicts to (3.16).

Remark. By use of the mollifier $\rho(t)$ with respect to t we may assume that $v \in C^2([0, T]; L^2_{loc}(\Omega))$, $|\nabla v| \in C([0, T]; L^2(\Omega))$, $|D^2 u| \in C([0, T]; L^2(\Omega))$. Further, we know that v belongs to $L^2([0, T], \dot{H}^1_0(\Omega))$, where $\dot{H}^1_0$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla u\|$, $u \in C_0^\infty(\Omega)$. From these facts, the proof of Proposition 3 can be applied to v.

Without loss of generality we may assume $\delta_0 \le \delta_1$. Then, by combining Proposition 2 and Proposition 4 we arrive at the following inequality, which is the basis of the estimations for the quasilinear wave equations.

Proposition 5. *Under the assumptions on a_{ij} in Propositions 2, 3 the solutions $u(t) \in X_2$ of the problem (3.1)–(3.2) satisfy*

$$X(t+T) - X(t) + \frac{\varepsilon_0}{16}\int_t^{t+T} E(s)\,ds + \frac{k}{2}\int_t^{t+T}\int_\Omega a|u_t|^2\,dxds$$

$$\leq C\int_t^{t+T}\int_\Omega (|\nabla u| + |u| + k|u_t|)|f|\,dxds, \tag{3.17}$$

where we set

$$X(t) = (u_t, (\phi(x-x_0)\cdot + c_0\boldsymbol{h})\cdot\nabla u) + ((\alpha + 2c_1\eta)u_t, u) + \frac{1}{2}\int_\Omega a|u|^2\,dx + kE(t). \tag{3.18}$$

Here, we note that taking a large $k > 0$, $X(t)$ is equivalent to $E(t) + \|u(t)\|^2$ since

$$\int_{\Omega_L} |u|^2\,dx \leq C\left(\int_{\Omega_{2L}/\Omega_L} |u|^2\,dx + \int_{\Omega_{2L}} |\nabla u|^2\,dx\right)$$

with some $C > 0$.

We also note that by a standard density argument, (3.17) is valid for finite energy solutions $u(t)$.

4. Energy decay for the quasilinear wave equation.

Let $u(t)$ be a local solution on $[0, \tilde{T})$, $0 < \tilde{T} \leq \infty$ of the problem (1.1)–(1.2) in Proposition 1. In this section we first derive the L^2-boundedness and decay estimate for $E(t)$. Next, we also derive the decay of the energy for $U = u_t$.

Proposition 6. *There exists $\delta_2 > 0$ such that if*

$$\sup_{0\leq t<\tilde{T}} (\|D_x^2 D_t u(t)\|_\infty + \|D^2 u(t)\|_\infty + \|Du(t)\|_\infty) \leq \delta_2, \quad D = (D_x, D_t), \tag{4.1}$$

then

$$\int_0^{\tilde{T}} E(s)\,ds + \sup_{0\leq t<\tilde{T}} \|u(t)\|^2 \leq CI_0^2 \tag{4.2}$$

and

$$E(t) \leq CI_0^2(1+t)^{-1}, \quad 0 \leq t < \tilde{T}, \tag{4.3}$$

where

$$E(t) = \frac{1}{2}\int_\Omega \left(|u_t(t)|^2 + \int_0^{|\nabla u(t)|^2} \sigma(\xi)\,d\xi\right) dx.$$

Proof. We may assume $\tilde{T} > T_0$. Otherwise, we get the results by (3.3) with $f \equiv 0$. We use the notation ε_0 and k for $\varepsilon_0/16$ and $k/2$, respectively. Then, by Proposition 5, we have

$$X(t) + \varepsilon_0\int_0^{\tilde{T}} E(s)\,ds + k\int_0^{\tilde{T}} a|u_t|^2\,dxds \leq X(0) \leq CI_0^2$$

which implies, in particular,

$$\|u(t)\|^2 \leq CI_0^2 < \infty \quad \text{and} \quad \int_0^{\tilde{T}} E(s)\,ds \leq CI_0^2, \tag{4.4}$$

provided that

$$\|D_x D_t \sigma(|\nabla u|^2)\|_\infty + \|D\sigma(|\nabla u|^2)\|_\infty \le \delta_1$$

which holds under (4.1).

Next, we use

$$\frac{d}{dt}E(t) + \int_\Omega a|u_t|^2\,dx = 0$$

to see

$$\frac{d}{dt}\{(1+t)E(t)\} = (1+t)\frac{d}{dt}E(t) + E(t) \le E(t),$$

and hence, by the latter inequality of (4.4),

$$(1+t)E(t) \le \int_0^{\tilde T} E(s)\,ds + E(0) \le CI_0^2.$$

We proceed to the estimation of the second order derivatives. For this, we assume for a moment,

$$\begin{aligned} &\sup_{0\le t<\tilde T}(1+t)^{k+1}\|D_t^{k+1}u(t)\|_{H^{m-k}}^2 + \|D_t^k\nabla u(t)\|_{H^{m-k}}^2 \\ &\quad + \int_0^{\tilde T}(1+t)^k(\|D_t^{k+1}u(t)\|_{H^{m-k}}^2 + \|D_t^k\nabla u(t)\|_{H^{m-k}}^2)\,dt \le K^2 \\ &\text{for } 0\le k\le m, 0<t<\tilde T \end{aligned} \tag{4.5}$$

and

$$\sup_{0\le t<\tilde T}(1+t)\|\nabla u(t)\|_{H^{m-k}}^2 + \int_0^{\tilde T}\|\nabla u(t)\|_{H^{m-k}}^2\,dt \le K^2 \quad \text{for } 0\le k\le m, 0\le t<\tilde T \tag{4.6}$$

with some $K>0$.

First, we note that if $u(t)$ is a local in time solution in X_m^T, $m>[N/2]+1$, then

$$\|D_t^{k+1}u(0)\| + \|D_t^k\nabla u(0)\| \le C(I_m)(\|u_0\|_{H^{k+1}} + \|u_1\|_{H^k}), \quad 0\le k\le m,$$

which is a standard fact for quasilinear evolution equations (cf. Kato [**6**]). □

PROPOSITION 7. *We assume that a local solution $u(t)\in X_m(\tilde T)$ satisfies* (4.5) *and* (4.6). *Then, under the assumption* (4.1), $u(t)$ *satisfies the estimates*

$$\int_0^{\tilde T}(1+t)E_1(t)\,dt \le CI_1^2, \quad E_1(t)\le C(1+K^2)I_1^2(1+t)^{-2} \tag{4.7}$$

and

$$\int_0^{\tilde T}\|\Delta u(t)\|^2\,dt \le CI_1^2\|\Delta u(t)\|_2^2 \le C(1+K^2)I_1^2(1+t)^{-1} \tag{4.8}$$

where

$$E_1(t) = \frac{1}{2}\int_\Omega(|u_{tt}(t)|^2 + |\nabla u_t(t)|^2)\,dx.$$

Proof. Setting $U = u_t$ we have

$$U_{tt} - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial U}{\partial x_j}\right) + a(x)U_t = 0 \tag{4.9}$$

where

$$a_{ij} = \sigma(|\nabla u|^2)\delta_{ij} + 2\sigma' u_{x_i} u_{x_j}.$$

Hence, applying Proposition 5 to (4.9) we obtain, under (4.1),

$$X_1(t+T) - X_1(t) + \varepsilon_0 \int_t^{t+T} E_1(s)\,ds + k\int_t^{t+T}\int_\Omega a|u_t|^2\,dxds \le 0 \tag{4.10}$$

if $t + T < \tilde{T}$, where X_1 is defined by X with u replaced by U and we use the notation

$$E_1(t) = \frac{1}{2}\int_\Omega \left(|u_{tt}(t)|^2 + \sum a_{i,j} u_{tx_i} u_{tx_j}(t)\right) dx.$$

By (4.3) we know already that

$$\|U(t)\|^2 = \|u_t(t)\|^2 \le CI_0^2(1+t)^{-1}.$$

From (4.10) we see easily

$$\int_0^{\tilde{T}} E_1(t)\,ds \le CI_1^2 < \infty. \tag{4.11}$$

Let us show the further inequality

$$\int_0^{\tilde{T}} (1+t)E_1(t)\,dt \le CI_1^2 < \infty. \tag{4.12}$$

Indeed, by (4.10),

$$(1+t+T)X_1(t+T) - (1+t)X_1(t) + \varepsilon_0\int_t^{t+T}(1+s)E_1(s)\,ds \le TX_1(t)$$

and hence, taking n such that $nT < \tilde{T}$,

$$\begin{aligned}\varepsilon_0\int_0^{nT}(1+s)E_1(s)\,ds &\le T\sum_{j=1}^{n-1} X_1(jT) + (1+T)X_1(0)\\ &\le CT\sum_{j=0}^{n-1}(E_1(jT) + \|U(jT)\|^2).\end{aligned} \tag{4.13}$$

Here, noting the inequality

$$\begin{aligned}&\frac{d}{dt}E_1(t) + \int_\Omega a|U_t|^2\,dx\\ &\quad = \frac{1}{2}\int_\Omega \sum_{i,j}\dot{a}_{ij}U_{x_i}U_{x_j}\,dx \le C\int_\Omega |\nabla u|\,|\nabla u_t|\,|\nabla U(t)|^2\,dx \le C\delta_1 E_1(t),\end{aligned} \tag{4.14}$$

we have

$$E_1 jT \le E_1(s^* + (j-1)T) + C\delta_1\int_{(j-1)T}^{jT} E_1(s)\,ds$$

for any $0 \le s^* \le T$, and hence,

$$TE_1(jT) \le \int_{(j-1)T}^{jT} E_1(s)\,ds + C\delta_1 T\int_{(j-1)T}^{jT} E_1(s)\,ds = (1 + C\delta_1 T)\int_{(j-1)T}^{jT} E_1(s)\,ds,$$

where we have taken s^* such that $\min_{(j-1)T \le s \le jT} E_1(s) = E_1(s^* + (j-1)T)$.

Thus, we have from (4.13) and (4.11) that

$$\varepsilon_0 \int_0^{nT} (1+s)E_1(s)\,ds \le C(1+T)(E_1(0) + E(0)) + C(1+\delta_1 T)\int_0^{\tilde{T}} E_1(s)\,ds \le CI_1^2$$

which implies the former estimate of (4.7).

Further, by use of (4.14),

$$\begin{aligned}\frac{d}{dt}\{(1+t)^2E_1(t)\} &\le 2(1+t)E_1(t) + (1+t)^2\frac{d}{dt}E_1(t)\\ &\le 2(1+t)E_1(t) + C(1+t)^2\|\nabla u(t)\|_\infty\|\nabla u_t(t)\|_\infty E_1(t). \qquad (4.15)\end{aligned}$$

Here, by Gagliardo Nirenberg inequality (cf. [**2**]) and the assumption (4.6) we see

$$\|\nabla u(t)\|_\infty \le C\|\nabla u(t)\|_2^{1-\theta}\|\nabla u(t)\|_{H^m}^{\theta} \le CK(1+t)^{-1/2}$$

with a certain $0 < \theta < 1$. Similarly,

$$\|\nabla u_t(t)\|_\infty \le CK(1+t)^{-1}.$$

Therefore, we have from (4.15)

$$\frac{d}{dt}\{(1+t)^2E_1(t)\} \le 2(1+t)E_1(t) + CK^2(1+t)E_1(t)$$

and integrating,

$$(1+t)^2E_1(t) \le CE_1(0) + C(1+K^2)\int_0^{\tilde{T}}(1+s)E_1(s)\,ds \le C(1+K^2)I_1^2$$

which implies the latter estimate of (4.7). To show (4.8) we have only to return to the original equation and use a regularity theory of elliptic equations. □

5. Estimation of higher order derivatives of solutions.

On the basis of Propositions 6 and 7 we derive in this section the estimates of the higher order derivatives of the (local) solutions $u(t) \in X_m^{\tilde{T}}$. Throughout of this section we assume (4.1), (4.5) and (4.6).

Proposition 8. *For $2 \le k \le m$ we have*

$$\int_0^{\tilde{T}}(1+t)^kE_k(t)\,dt + \sup_{0\le t<\tilde{T}}(1+t)^{k+1}E_k(t) \le Cq(I,K) \qquad (5.1)$$

where $E_k(t)$ is defined by

$$\frac{1}{2}\int_\Omega\left(|D_t^{k+1}u(t)|^2 + \sum a_{i,j}D_t^ku_{x_i}D_t^ku_{x_j}(t)\right)dx$$

and $q(I,K)$ denotes a polynomial of $I = (I_0, I_1, \dots, I_m)$ and K such that $q(0,K) = 0$. We note that $E_k(t)$ is equivalent to

$$\|D_t^{k+1}u(t)\|^2 + \|\nabla D_t^ku(t)\|^2.$$

Proof. We know already that (5.1) is valid if $k=1$. To show this for k, $2 \le k \le m$, we use induction and assume that (5.1) is valid for all $1 \le j \le k-1$. We use the notation I for I_m.

Differentiating the equation k times with respect to t and setting $U = D_t^k u(t)$, we have

$$U_{tt} - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial U}{\partial x_j}\right) + a(x)U_t = F(x,t) \tag{5.2}$$

with

$$a_{ij} = \sigma\delta_{ij} + 2\sigma'(Du)^2$$

and

$$F = \nabla \sum_{j=1}^{k-1} C_j D_t^j \sigma D_t^{k-j} \nabla u + 2\nabla \sum_{j=1}^{k-1} C_j D_t^j (\sigma'(Du)^2) D_t^{k-j} Du$$

where D_t^l denotes any partial differentiations with respect to t of order l and sum of them. We often use the notation D for $D_x = \nabla$ and (D_x, D_t).

We apply Proposition 5 to (5.2) to obtain

$$X_k(t+T) - X_k(t) + \varepsilon_0 \int_t^{t+T} E_k(s)\,ds \le C \int_t^{t+T} \int_\Omega (|\nabla U| + |U_t| + |U|)|F|\,dxds$$

and

$$X_k(t+T) - X_k(t) + \frac{\varepsilon_0}{2} \int_t^{t+T} E_k(s)\,ds \le C \int_t^{t+T} (\|U\|\,\|F\| + \|F\|^2)\,ds \tag{5.3}$$

where $t+T < \tilde{T}$.

To estimate $\|F(t)\|$ we rewrite F as follows.

$$\begin{aligned} F = {} & \sum_{j=1}^{k-1} C_j \Gamma Du D_t^j D^2 u D_t^{k-j} Du + \sum_{j=1}^{k-1} C_j (\Gamma'(Du)^2 + \Gamma) Du_t D_t^{j-1} D^2 u D_t^{k-j} Du \\ & + \sum_{j=2}^{k-1} \sum_{l=2}^{j} C_{jl} D_t^l (\Gamma Du) D_t^{j-l} D^2 u D_t^{k-j} Du + \sum_{j=1}^{k-1} C_j \Gamma Du D_t^j Du D_t^{k-j} D^2 u \\ & + \sum_{j=1}^{k-1} C_{jl} (\Gamma'(Du)^2 + \Gamma) Du_t D_t^{j-l} Du D_t^{k-j} D^2 u \\ & + \sum_{j=2}^{k-1} \sum_{l=2}^{j} C_{jl} (\Gamma'(Du)^2 + \Gamma) Du_t D_t^{j-l} Du D_t^{k-j} D^2 u \\ \equiv {} & J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \end{aligned} \tag{5.4}$$

where

$$\Gamma \equiv \Gamma(Du) = 6\sigma' + 4\sigma'' \cdot (Du)^2.$$

Estimation of J_1 and J_4.

Taking appropriate $p_1 \geq 1$ and $p_2 \geq 2$ with $1/p_1 + 1/p_2 = 1/2$ we have

$$\begin{aligned}\|J_1\| &\leq C\sum_{j=1}^{k-1}\|Du\|_\infty \|D_t^j D^2 u\|_{p_1}\|D_t^{k-j}Du\|_{p_2}\\ &\leq C\sum_{j=1}^{k-1}\|Du\|^{1-\theta}\|Du\|_{H^m}^{\theta}\|D_t^j D^2 u\|_{H^{m-j}}\|D_t^{k-j}Du\|_{H^{m+1-k+j}}\\ &\leq CI_0^{1-\theta}K^{2+\theta}(1+t)^{-\nu}\end{aligned}$$

with a certain $0 < \theta < 1$ and

$$\nu = 1/2 + (1+j)/2 + (1+k-j)/2 = (k+3)/2.$$

Similarly, we see

$$\int_0^{\tilde{T}}(1+t)^{k+2}\|J_1(t)\|^2\,dt \leq CK^2\int_0^{\tilde{T}}(1+t)\|Du(t)\|^{2(1-\theta)}\|Du(t)\|_{H^m}^{2\theta}\,dt \leq CI_0^{2(1-\theta)}K^{2(2+\theta)}.$$

Similar estimates hold for J_4.

Estimation of J_2 and J_5.

We see

$$\begin{aligned}\|J_2\| &\leq C\sum_{j=1}^{k-1}\|Du_t D_t^{j-1}D^2 u D_t^{k-j}Du\|\\ &\leq \|Du_t\|_\infty\|D_t^{j-1}D^2u\|_{p_1}\|D_t^{k-j}Du\|_{p_2}\\ &\leq C\sum_{j-1}^{k-1}\|Du_t\|_2^{1-\theta}\|Du_t\|_{H^{m-1}}^{\theta}\|D_t^{j-1}D^2u\|_{H^{m-j}}\|D_t^{k-j}Du\|_{H^{m-k+j}}\\ &\leq C(1+K^2)I_1^{1-\theta}(1+t)^{-1}K(1+t)^{-j/2}K(1+t)^{(-k+j-1)/2}\\ &\leq C(1+K^2)K^2I_1^{1-\theta}(1+t)^{-\nu}\end{aligned}$$

with a certain $0 < \theta < 1$ and $\nu = (k+3)/2$. Similarly, we have

$$\int_0^{\tilde{T}}(1+t)^{k+2}\|J_2(t)\|^2\,dt \leq C(1+K^2)K^4I_1^{2(1-\theta)}.$$

The same estimates as for J_2 hold for J_5.

Estimation of J_3 and J_6.

Setting $\tilde{\Gamma} = \Gamma Du$, we have

$$\|J_3\| \leq C\sum_{j=2}^{k-1}\sum_{l=2}^{j}\sum_{r=1}^{l}\left\|\tilde{\Gamma}^{(r)}\sum_{S_r}|(D_t^{\alpha_1}Du)^{\gamma_1}\cdots(D_t^{\alpha_s}Du)^{\gamma_s}|\,|D_t^{j-l}D^2uD_t^{k-j}Du|\right\|$$

where

$$S_r = \left\{ (\alpha_1, \dots, \alpha_s, \gamma_1, \dots, r_s) \mid 1 \le \alpha_1 < \alpha_2 \cdots < \alpha_s, \sum_{i=1}^{s} \alpha_i \gamma_i = l \text{ and } \sum_{i=1}^{s} \gamma_s = r \right\}.$$

Hence, by choosing appropriate $\{p_i\}$, $p_i \ge 2$, (cf. [**14**]), we have

$$\|J_3\| \le C \sum_{j=2}^{k-1} \sum_{l=2}^{j} \sum_{r=1}^{l} \sum_{S_r} \|D_t^{\alpha_1} Du\|_{p_1}^{\gamma_1} \cdots \|D_t^{\alpha_s} Du\|_{p_s}^{\gamma_s} \|D_t^{j-l} D^2 u\|_{p_{s+1}} \| \, \|D_t^{k-j} Du\|_{p_{s+2}}$$

$$\le C \sum_{j=2}^{k-1} \sum_{l=2}^{j} \sum_{r=1}^{l} \sum_{S_r} \|D_t^{\alpha_1} Du\|_{H^{m-\alpha_1}}^{\gamma_1 \theta_1} \|D_t^{\alpha_1} Du\|^{\gamma_1(1-\theta_1)} \| \cdots$$

$$\|D_t^{\alpha_s} Du\|_{H^{m-\alpha_s}}^{\gamma_s \theta_s} \|D_t^{\alpha_s} Du\|^{\gamma_s(1-\theta_s)} \|D_t^{j-l} D^2 u\|_{H^{m-j+l-1}} \| \, \|D_t^{k-j} Du\|_{H^{m-k+j}}.$$

Therefore, by the assumption of induction,

$$\|J_3\| \le C q_k(I_m, K)(1+t)^{-\sum_i \gamma_i(\alpha_i+1)/2 - (k-l+2)/2}$$

$$\le C q(I, K)(1+t)^{-(r+k+2)/2} \le C q(I, K)(1+t)^{-\nu}$$

with $\nu = (k+3)/2$ and a certain quantity $q(I, K)$ satisfying $q(0, K) = 0$.

We also have

$$\int_0^{\tilde{T}} (1+t)^{k+2} \|J_3(t)\|^2 \, dt \le C q(I, K) < \infty.$$

J_6 is also treated quite similarly and satisfies the same final two estimates. Thus, we obtain

$$\sup_{0 \le t < \tilde{T}} (1+t)^{k+3} \|F(t)\|^2 + \int_0^{\tilde{T}} (1+t)^{2+k} \|F(t)\|^2 \, dt \le q(I, K) < \infty. \tag{5.5}$$

Therefore, under the smallness assumption on δ_2 we obtain from (5.3) that

$$X_k(t+T) - X_k(t) + \frac{\varepsilon_0}{2} \int_t^{t+T} E_k(t) \le C q(I, K)(1+t)^{-k-3/2}, \tag{5.6}$$

where we have again used the assumption of induction

$$\|U(t)\|^2 \le 2E_{k-1}(t) \le q(I, K)(1+t)^{-k}.$$

On the basis of the integral inequality (5.6) we shall prove the desired energy estimates (5.1). For this we first show that

$$\int_0^{\tilde{T}} (1+t)^k E_k(t) \, dt \le C q(I, K) < \infty. \tag{5.7}$$

Indeed, by the same argument deriving the estimate of $E_1(t)$ (see (4.12)), we can show from (5.6) that

$$\int_0^{\tilde{T}} (1+t) E_k(t) \, dt \le C q(I, K) < \infty.$$

So, for induction, let us assume

$$\int_0^{\tilde{T}} (1+t)^j E_k(t)\,dt \le Cq(I,K) < \infty$$

for some j, $0 \le j < k$. Then, we see from (5.5) that

$$(1+t+T)^{j+1} X_k(t+T) - (1+t)^{j+1} X_k(t) + \frac{\varepsilon_0}{2}\int_t^{t+T} (1+s)^{j+1} E_k(t)$$
$$\le Cq(I,K)(1+t)^{-3/2} + C(1+t+T)^j X_k(t).$$

Since $X_k(t)$ is equivalent to $E_k(t) + \|D_t^k u(t)\|^2$ we conclude from the assumption of induction that

$$\int_0^{\tilde{T}} (1+t)^{j+1} E_k(t)\,dt \le Cq(I,K) < \infty.$$

Thus, we conclude (5.7).

Finally, we return to the equation (5.2) to get the energy inequality

$$\frac{d}{dt} E_k(t) \le C(\|F(t)\|\sqrt{E_k(t)} + \tilde{\delta}_0(t) E_k(t)). \tag{5.8}$$

Here

$$\tilde{\delta}_0(t) \equiv \sup_{i,j} \|a_{ij}(t)\|_\infty \le C\|\nabla u(t)\|_\infty \|\nabla u_t(t)\|_\infty \le Cq(I,K)(1+t)^{-3/2}$$

and

$$\|F(t)\|\sqrt{E_k(t)} \le \frac{1}{2}((1+t)\|F\|^2 + (1+t)^{-1} E_k(t)).$$

Thus, we obtain from (5.5) and (5.7) that

$$\frac{d}{dt}\{(1+k)^{1+k} E_k(t)\} = (1+k)(1+t)^k E_k(t) + (1+t)^{1+k}\frac{d}{dt} E_k(t)$$
$$\le C(1+t)^k E_k(t) + C(1+t)^{2+k}\|F(t)\|^2 + Cq(I,K)(1+t)^{k-1/2} E_k(t)$$

which together with (5.7) and (5.5) implies

$$(1+t)^{1+k} E_k(t) \le (1+k) E_k(0) + C\int_0^{\tilde{T}} (1+t)^{2+k}\|F(t)\|^2\,dt$$
$$+ C(1+q(I,K))\int_0^{\tilde{T}} (1+s)^k E_k(s)\,ds \le Cq(I,K) < \infty.$$

Thus, (5.1) is now proved. □

Once the energy decay for the higher derivatives are derived, the following assertion is standard.

Proposition 9. *Under the assumptions* (4.1), (4.5) *and* (4.6) *we have further*

$$\sup_{0\le t<\tilde T}(1+t)^{k+1}(\|D_t^{k+1}u(t)\|^2_{H^{m-k}}+\|D_t^k\nabla u(t)\|^2_{H^{m-k}})$$
$$+\int_0^{\tilde T}(1+t)^k(\|D_t^{k+1}u(t)\|^2_{H^{m-k}}+\|D_t^k\nabla u(t)\|^2_{H^{m-k}})\,dt$$
$$\le q(I,K)\quad \text{for } 0\le k\le m \tag{5.9}$$

and

$$\sup_{0\le t<\tilde T}(1+t)\|\nabla u(t)\|^2_{H^{m-k}}+\int_0^{\tilde T}\|\nabla u(t)\|^2_{H^{m-k}}\,dt\le Cq(I,K)\quad \text{for } 0\le k\le m. \tag{5.10}$$

Proof. The preceding Proposition means that (5.10) is valid for $k=m$ and further

$$\sup_{0\le t<\tilde T}(1+t)^{j+1}(\|D_t^{j+1}u(t)\|^2+\|D_t^k\nabla u(t)\|^2)$$
$$+\int_0^{\tilde T}(1+t)^j(\|D_t^{j+1}u(t)\|^2+\|D_t^j\nabla u(t)\|^2)\,dt\le q(I,K)\quad \text{for } 0\le j\le m. \tag{5.11}$$

We show (5.9) by induction and for this we assume that

$$\sup_{0\le t<\tilde T}\{(1+t)^{j+1}\|D_t^{j+1}u(t)\|^2_{H^{m-j}}+\|D_t^j\nabla u(t)\|^2_{H^{m-j}}\}$$
$$+\int_0^{\tilde T}(1+t)^j(\|D_t^{j+1}u(t)\|^2_{H^{m-j}}+\|D_t^j\nabla u(t)\|^2_{H^{m-j}})\,dt\le q(I,K),\quad k+1\le j\le m. \tag{5.12}$$

To prove (5.9) it suffices to show for $0\le k\le m-1$,

$$\sup_{0\le t<\tilde T}(1+t)^{k+1}\|D_t^k\nabla u(t)\|^2_{H^{m-k}}+\int_0^{\tilde T}(1+t)^k(\|D_t^k\nabla u(t)\|^2_{H^{m-k}}\,dt\le q(I,K) \tag{5.9$'$}$$

under the conditions that (5.9) with $k=m$, (5.11) and (5.12) hold.

Differentiating the equation k times with respect to t we have

$$\begin{aligned}\Delta(D_t^k u)&=-D\{D_t^k((\sigma-1)\nabla u)\}+D_t^{k+2}u(t)+a(x)D_t^{k+1}u(t)\\&=\Gamma(|\nabla u|^2)D_t^kD^2u(t)+\left\{\sum_{j=1}^k C_{j,k}D_t^j\Gamma D_t^{k-j}D^2u\right\}+D_t^{k+2}u(t)+a(x)D_t^{k+1}u(t)\\&\equiv F(t)\equiv F_0(t)+F_1(t)+F_2(t)+F_3(t)\end{aligned} \tag{5.13}$$

where $\Gamma=\sigma-1$. Since we may assume $\sigma(0)=1$ we have $\Gamma(0)=0$. Then, by the elliptic regularity theory we know

$$\|D_t^k\nabla u(t)\|_{H^{m-k}}\le C(\|D_t^k\nabla u(t)\|_{H^{m-1-k}}+\|F(t)\|_{H^{m-1-k}}). \tag{5.14}$$

First, we note that by Gagliardo-Nirenberg inequality,

$$\|D^l D_t^j \nabla u(t)\| \le C\|D_t^j \nabla u(t)\|^{1-\theta}\|D_t^j \nabla u(t)\|_{H^{m-j}}^{\theta}$$
$$\le Cq(I,K)(1+t)^{-(1+j)(1-\theta)/2}(1+t)K^{\theta}(1+t)^{-(1+j)\theta/2}$$
$$\le Cq(I,K)(1+t)^{-(1+j)/2}$$

for $0 \le l \le m-1-j$ and hence

$$\|D_t^j \nabla u(t)\|_{H^{m-1-j}}^2 \le Cq(I,K)(1+t)^{-1-j} \tag{5.15}$$

for all j, $0 \le j \le m-1$.

Similarly we have

$$\int_0^{\tilde{T}} (1+t)^j \|D_t^j \nabla u(t)\|_{H^{m-1-j}}^2 \le Cq(I,K), \quad 0 \le j \le m-1.$$

Further, we easily see by (5.12) that

$$\|F_2(t)+F_3(t)\|_{H^{m-1-k}} \le C(\|D_t^{k+2}u(t)\|_{H^{m-1-k}} + \|D_t^{k+1}u(t)\|_{H^{m-1-k}})$$
$$\le Cq(I,K)(1+t)^{-(1+k)/2}$$

and

$$\int_0^{\tilde{T}} (1+t)^k \|F_2(t)+F_3(t)\|_{H^{m-1-k}}^2 \, dt \le C\int_0^{\tilde{T}} (1+t)^k (\|D_t^{k+2}u(t)\|_{H^{m-1-k}}^2 + \|D_t^{k+1}u(t)\|_{H^{m-1-k}}^2)\, dt$$
$$\le Cq(I,K).$$

By the use of (4.5), (4.6) and (5.14) we can carry out a similar argument obtaining (5.5) to get

$$\sup_{0 \le t < \tilde{T}} (1+t)^{k+1}\|F_0(t)\|_{H^{m-1-k}}^2 + \int_0^{\tilde{T}} (1+t)^k \|F_0(t)\|_{H^{m-1-k}}^2 \, dt \le Cq(I,K) < \infty$$

where we have used $\Gamma(0) = 0$.

Treatment of the term $F_1(t)$ is also delicate. But, noting that

$$\|F_1(t)\|_{H^{m-1-k}} \le C\sum_{j=1}^{k} \|D_t^j \Gamma D_t^{k-j} D^2 u\| + C\sum_{j=1}^{k} \|D^{m-1-k}(D_t^j \Gamma D_t^{k-j} D^2 u)\|$$

and repeating again a similar argument estimating J_1 through J_6 as in (5.5) we can prove that

$$\sup_{0 \le t < \tilde{T}} (1+t)^{k+1}\|F_1(t)\|^2 + \int_0^{\tilde{T}} (1+t)^k \|F_1(t)\|^2 \, dt \le Cq(I,K).$$

Thus, we conclude $(5.9)'$. □

6. Completion of the proof of Theorem 1.

The apriori estimates in preceding sections are sufficient for the proof of Theorem 1. Under the assumptions (4.1), (4.5) and (4.6) we have shown that any local solution $u(t) \in X_m^{\tilde{T}}$ satisfies

$$\sup_{0 \le t < \tilde{T}} (1+t)^{k+1} (\|D_t^{k+1} u(t)\|_{H^{m-k}}^2 + \|D_t^k \nabla u(t)\|_{H^{m-k}}^2)$$

$$+ \int_0^{\tilde{T}} (1+t)^k (\|D_t^{k+1} u(t)\|_{H^{m-k}}^2 + \|D_t^k \nabla u(t)\|_{H^{m-k}}^2)\, dt \le q(I, K) < \infty \tag{6.1}$$

for $0 \le k \le m$, $0 \le t < \tilde{T}$, where $q(I,K)$ is some quantity depending on I, K in such a way that $q(0,K) = 0$.

Thus, fixing $K > 0$ arbitrarily and making the additional assumption

$$q(I,K) < K^2 \tag{6.2}$$

we can conclude that under (4.1), the local solutions exist in fact on $[0,\infty)$ and the estimate (6.1) holds on $[0,\infty)$.

Finally, we note that since $m > [N/2] + 2$, under (4.5) and (4.6),

$$\begin{aligned}
&\|D^2 D_t u(t)\|_\infty + \|D^2 u(t)\|_\infty + \|Du(t)\| \\
&\quad \le C(\|D_t u(t)\|^{1-\theta_1} \|D_t u(t)\|_{H^{m-1}}^{\theta_1} + \|D^2 u(t)\|^{1-\theta_2} \|D^2 u(t)\|_{H^{m-1}}^{\theta_2} \\
&\qquad + \|Du(t)\|^{1-\theta_3} \|Du(t)\|_{H^m}^{\theta_3}) \\
&\quad \le C \sum_{i=1}^3 (I_0 + I_1)^{1-\theta_i} K^{\theta_i}
\end{aligned}$$

with a certain $0 < \theta_i < 1$, $i = 1,2,3$. Thus, (4.1) is satisfied under the further additional assumption

$$C \sum_{i=1}^3 (I_0 + I_1)^{1-\theta_i} K^{\theta_i} < \delta_2 \tag{6.3}$$

for the fixed $K > 0$.

Since both of (6.2) and (6.3) are valid for small I, the proof of Theorem 1 is now complete.

Remark. By a careful observation we see that $q(I,K)$ is replaced by $q(I_0,K) + CI_m^2$. Hence, for any $K > CI_m$, there exists $\delta(K)$ such that $q(I_0,K) + CI_m^2 < K^2$ if $I_0 \le \delta(K)$. (6.4) also holds under these conditions. Thus, the set of initial data assuring the global existence in Theorem 1 is in fact unbounded in $H^{m+1} \times H^m$.

7. Proof of Theorems 2 and 3.

In this section we assume only Hyp.A. That is, we consider the case where $a(x)u_t$ is effective only at a neighbourhood of $\Gamma(x_0)$ and $\operatorname{supp} a(\cdot)$ is compact, say,

$$\operatorname{supp} a(\cdot) \subset B_L, \quad L > 0. \tag{7.1}$$

First, let us consider the linear wave equation with a localized dissipation:

$$u_{tt} - \Delta u + a(x) u_t = 0 \quad \text{in } \Omega \times [0,\infty), \tag{7.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{and} \quad u|_{\partial\Omega} = 0. \tag{7.3}$$

Under our assumption on $a(x)$ we know a local energy decay for the solutions of the linear problem (7.2)–(7.3) ([**14**]) and using this we can prove L^p estimates for the linear equation ([**17**]). In particular, for the odd dimensional cases we have the following. We denote $W^{m,p}(\Omega)$ norm by $\|\cdot\|_{m,p}$.

Proposition 10. *Let $N \geq 3$ be an odd integer and m be a nonnegative integer. Let $a \in C^{2M+m}(\overline{\Omega})$ and Hyp.A with (7.1) be valid. We set $M = [N/2] + 1$. Assume that $u_0 \in H_0^1(\Omega) \cap H^{2M+m}(\Omega) \cap W^{2M+m,1}(\Omega)$, $u_1 \in H^{2M+m-1}(\Omega) \cap W^{2M+m-1,1}$ and these data satisfy the compatibility condition of the $M+m-1$-th order. Then, there exists a unique solution $u(t)$ of the problem (7.2)–(7.3) in $\bigcap_{k=0}^{2M+m-1} C^k([0,\infty);\, H^{2M+m-k}(\Omega) \cap H_0^1(\Omega)) \cap C^{2M+m}([0,\infty);\, L^2(\Omega))$ and it satisfies*

$$\sum_{k=0}^{m} \|D_t^k u(t)\|_{m-k,p} \leq C(I_{m+M,1} + I_{m+M,2})(1+t)^{-b} \tag{7.4}$$

if $2 < p \leq p_0 \equiv 2(N+1)/(N-1)$

and

$$\sum_{k=0}^{m} \|D_t^k u(t)\|_{m-k,\infty} \leq C(I_{m+2M-1,1} + I_{m+2M-1,2})(1+t)^{-d}, \tag{7.5}$$

where we set

$$I_{l,1} = (\|u_0\|_{l+1,1} + \|u_1\|_{l,1}),$$

$$I_{l,2} = (\|u_0\|_{l+1,2} + \|u_1\|_{l,2}),$$

$$b = \begin{cases} (N-1)(1/2 - 1/p) & \text{if } N \text{ is odd and } N \geq 5 \\ 1 - 2/p - \delta & \text{if } N = 3 \end{cases}$$

and

$$d = \begin{cases} (N-1)/2 & \text{if } N \text{ is odd and } N \geq 5 \\ 1 - \delta & \text{if } N = 3. \end{cases}$$

We note that $\delta > 0$ in the above can be chosen arbitrarily small.

Remark. In Proposition 10, if V is convex and $N \geq 4$ we can take $b = (N-1)(1/2 - 1/p)$ and $d = (N-1)/2$.

When $N \geq 4$ we set

$$Y_{3M}^T \equiv \bigcap_{k=0}^{3M} C^k([0,T); H^{3M+1-k} \cap H_0^1) \bigcap C^{3M+1}([0,T); L^2)$$

and

$$V_{3M}(K,T) = \Bigg\{ u \in Y_{3M}^T \mid \sum_{k=0}^{3M+1} \|D_t^k \nabla u(t)\|_{H^{3M+1-k}} \leq K$$

$$\text{and } \sum_{k=0}^{M} \|D_t^k \nabla u(t)\|_{M+1-k,\infty} \leq K(1+t)^{-d} \Bigg\}$$

for $K > 0$.

When $N = 3$ we set

$$Y_{4M+2}^T \equiv \bigcap_{k=0}^{4M+2} C^k([0,T); H^{4M+3-k} \cap H_0^1) \bigcap C^{4M+3}([0,T); L^2)$$

and

$$V_{4M+2}(K,T) = \Bigg\{ u \in Y_{4M+2}^T \mid \sum_{k=0}^{4M+2} \|D_t^k \nabla u(t)\|_{H^{4M+2-k}} \le K$$

$$\text{and } \sum_{k=0}^{M+1} \|D_t^k \nabla u(t)\|_{M+1-k,p} \le K(1+t)^{-d(p)} \Bigg\},$$

for $K > 0$ and $6 \le p < \infty$, where $d(p) = d(1-2/p) = (1-\varepsilon)(1-2/p)$, $0 < \varepsilon \ll 1$.

The local existence of the solution for each (u_0, u_1) satisfying the compatibility condition is standard (cf. Kato [**6**]). So, for the proof of Theorems 2, 3 it suffices to derive the desired estimates. We treat mainly the case $N = 3$ because the other cases are proved in a similar and simpler manner.

We set $m = 3M$ if $N \ge 4$ and $m = 4M+2$ if $N = 3$. We may assume $u(\cdot) \in V_m(K,T)$ for some $K > 0$, $T > 0$, which will be shown later.

For a moment we assume that

$$\|\nabla u(t)\|_\infty + \|\nabla D_t u(t)\|_\infty \le \delta_1 \tag{7.6}$$

with some small $\delta_1 > 0$, which will be satisfied if $I \equiv \|u_0\|_{H^{m+1}} + \|u_1\|_{H^m}$ is small.

We begin with the following observation which will make a proof a little simpler for the case $N = 3$.

Proposition 11. *Let $N = 3$. If $u(t)$ is a (local) solution $\in V_m(T)$ it satisfies*

$$\sum_{k=0}^{M+1} \|D_t^k D u(t)\|_{M+1-k,\infty} \le K(1+t)^{-d_1} \tag{7.7}$$

with

$$d_1 = \frac{11(p-2)(1-\varepsilon)}{11p+6}$$

and

$$\|D_t^{M+2} D u(t)\|_{2M-k,\infty} \le Cq(K)K(1+t)^{-d_2} \tag{7.8}$$

with $d_2 = 7d_1/9$, where $q(K)$ is a quantity (in fact polynomial) depending on K and we recall that $D = (\nabla, D_t)$ and $2 < p \le p_0 = 2(N+1)/(N-1)$.

Proof. We first note that $M = 2$ if $N = 3$. By Gagliardo-Nirenberg inequality we see

$$\|D_t^k D u(t)\|_{M+1-k,\infty} \le C\|D_t^k D u(t)\|_{M+1-k,p}^{1-\theta} \|D_t^k D u(t)\|_{4M+2-k,2}^{\theta} \le CK(1+t)^{d(1-\theta)}$$

with

$$\theta = \frac{1}{p}\left(\frac{3M+1}{N} + \frac{1}{p} - \frac{1}{2}\right)^{-1} = \frac{6}{11p+6}$$

which implies (7.7).

To prove (7.8) we return to the equation. Note that $2M = M + 2 = 4$. Then, we see by a standard argument based on Leibniz's formula,

$$\|D_t^{M+2}Du(t)\|_{M-2,\infty} = \|D_t^M D\{\nabla(\sigma\nabla u) + au_t\}\|_\infty$$
$$\leq Cq(K)\left(\sum_{i=1}^{2}(\|D_t^i Du(t)\|_{M,\infty} + \|D_t^i Du(t)\|_{M-1,\infty}\right).$$

Here, by (7.7),

$$\|D_t Du(t)\|_{M,\infty} + \sum_{i=1}^{2}\|D_t^i Du(t)\|_{M-1,\infty} \leq K(1+t)^{-d_1}$$

and further, by Gagliardo-Nirenberg inequality,

$$\|D_t^2 Du(t)\|_{M,\infty} \leq C\|D_t^2 Du(t)\|_{M-1,\infty}^{1-\theta}\|D_t^2 Du(t)\|_{4M,2}^{\theta} \leq CK(1+t)^{-(1-\theta)d_1}, \quad \theta = \frac{2}{9}.$$

Thus we have (7.8). □

We proceed to the proof of Theorem 3. Let $N = 3$. Setting $D_t^m u(t) = U(t)$, $m = 4M + 2$, we have

$$U_{tt} - \operatorname{div}\{D_t^m(\sigma\nabla u)\} + a(x)U_t = 0. \tag{7.9}$$

Then, as in (5.3), we can derive

$$\frac{d}{dt}E_m(t) \leq \int_\Omega |D_t\sigma|\,|D_t^m\nabla u|^2\,dx + \int_\Omega \sum_{k=1}^{m-1}|D_t^k(\Gamma)|\,|D_t^{m-k}\nabla u|\,|\nabla D_t^{m+1}u|\,dx \equiv J_1 + J_2 \tag{7.10}$$

where $\Gamma \equiv \Gamma(\nabla u) = \sigma(|\nabla u|^2) + 2\sigma'(\nabla u)^2$.

Here, taking account of (7.7), we see

$$J_1 \leq C\|\nabla u(t)\|_\infty\|D_t\nabla u(t)\|_\infty\|D_t^m\nabla u(t)\|^2 \leq CK^3(1+t)^{-2d_1}\sqrt{E_m(t)}.$$

The treatment of J_2 is delicate. We first observe

$$J_2 \leq C\Bigg\{\|\nabla u(t)\|_\infty\|\nabla u_t(t)\|_\infty\|D_t^{m-1}\nabla u(t)\|\,\|D_t^m\nabla u(t)\|$$
$$+ \int_\Omega\sum_{k=2}^{m-1}\sum_{i=1}^{k}\sum_{S_i}|\Gamma^{(i)}|\,|D_t(\nabla u(t))^2|^{\nu_1}\cdots|D_t^k(\nabla u(t))^2|^{\nu_k}|D_t^{m-k}\nabla u(t)|\,|D_t^{m+1}\nabla u(t)|\,dx\Bigg\}$$
$$\equiv J_2^{(1)} + J_2^{(2)}.$$

It is easy to see

$$J_2^{(1)} \leq CK^3(1+t)^{-2d_1}\sqrt{E_m(t)}.$$

Further, we see

$$J_2^{(2)} \leq C\sum_{k=2}^{m-1}\sum_{i=1}^{k}\sum_{S_i}\left(\int_\Omega |D_t(\nabla u(t))^2|^{2\nu_1}\cdots|D_t^k(\nabla u(t))^2|^{2\nu_k}|D_t^{m-k}\nabla u(t)|^2\,dx\right)^{1/2}\sqrt{E_m(t)}.$$

Here, since

$$D_t^k(\nabla u(t))^2 = \sum_{j=0}^{k} C_{kj} D_t^j \nabla u D_t^{k-j} \nabla u$$

and $m = 4M + 2$, we observe that each product of the above integrand contains the term

$$|D_t^j \nabla u D_t^l \nabla u|^2 \quad \text{with } j \leq M+1 \quad \text{and} \quad l \leq M+2 = 2M.$$

Indeed, one of the most delicate terms appears in the case $k = 2M + 2$, that is

$$|D_t^{2M+2}(\nabla u)^2|^2 |D_t^{2M} \nabla u|^2.$$

For this term, however, we see

$$D_t^{2M+2}(\nabla u)^2 = \sum_{0 \leq j \leq M+1} C_j D_t^j \nabla u D_t^{2M+2-j} \nabla u.$$

Hence, by (7.8), we can prove that

$$J_2^{(2)} \leq Cq(K)K^2(1+t)^{-d_1-d_2}\sqrt{E_m(t)}.$$

We note that when $N \geq 4$, we can use the estimate

$$\|D_t^j \nabla u(t)\|_\infty \leq K(1+t)^{-d}$$

to prove

$$J_2^{(2)} \leq Cq(K)K^2(1+t)^{-d}\sqrt{E_m(t)}, \quad 0 \leq j \leq M+1,$$

with $d = (N-1)/2 > 1$.

Thus, we obtain

$$\frac{d}{dt} E_m(t) \leq Cq(K)K^2(1+t)^{-d_1-d_2}\sqrt{E_m(t)}.$$

Since $d_1 + d_2 > 1$, this implies

$$E_m(t) \leq C(q(K)K^4 + E_m(0)). \tag{7.11}$$

The same estimate is valid for the case $N \geq 4$ where we take $m = 3M + 1$. Note that a standard argument shows

$$\sqrt{E_m(0)} \leq q(K) I_{m,2}$$

where

$$I_{m,2} = \|u_0\|_{H^{m+1}} + \|u_1\|_{H^m}.$$

Now, returning to the equation and combining elliptic regularity theory with (7.11) we can prove as in Proposition 8 that

$$\sum_{k=0}^{4M+2} \|D_t^k D u(t)\|_{H^{4M+2-k}} \leq q(K)(K^2 + I_{m,2}). \tag{7.12}$$

We summarize the estimates (7.11) and (7.12) in the following Proposition:

Proposition 12. *Let $u(\cdot) \in V_m(K, T)$ be a solution of the problem* (1.1)–(1.2) *as in Proposition* 1. *Then, we have*

$$\sum_{k=0}^{m} \|D_t^k D u(t)\|_{H^{m-k}} \le q(K)(K^2 + I_{m,2}), \quad D = (\nabla, D_t), \tag{7.13}$$

where $m = 3M + 1$ if $N \ge 4$ and $m = 4M + 2$ if $N = 3$.

Next, we proceed to the estimation of $\|D_t \nabla u(t)\|_{M+1-k,p}$, $0 \le k \le M+1$, $6 \le p < \infty$. For this we begin with L^p estimates for the linear equation. By the known result (7.5) and Sobolev's embedding theorem $W^{l+1,1} \subset W^{2M+l,2}$, we see for the solutions $u(t)$ of the linear equation that

$$\|D_t^k \nabla u(t)\|_{l-k,\infty} \le C(I_{2,l+M} + I_{1,l+M})(1+t)^{-d} \le (\|u_0\|_{3M+l+1,1} + \|u_1\|_{3M+l,1})(1+t)^{-d}$$

with any nonnegative integer l.

On the other hand, by use of the energy identity, we see easily that

$$\|D_t^k \nabla u(t)\|_{l-k,2} \le C(\|u_0\|_{l+1,2} + \|u_1\|_{l,2}) \le C(\|u_0\|_{3M+l+1,2} + \|u_1\|_{3M+l,2}).$$

Thus, by interpolation, we obtain for all $2 \le p \le \infty$,

$$\|D_t^k \nabla u(t)\|_{l-k,p} \le C(\|u_0\|_{3M+l+1,q} + \|u_1\|_{3M+l,q})(1+t)^{-d(p)}, \quad 0 \le k \le l, \tag{7.14}$$

where $1/q + 1/p = 1$.

We use this estimate with $l = M+1$ and $N = 3$.

Let $u(\cdot) \in V_m(K, T)$ be a solution of the quasilinear equation with initial data (u_0, u_1) and let us denote the solution of the linear equation with the same initial data by $U(t; u_0, u_1)$. Then, by constant variation formula,

$$u(t) = U(t; u_0, u_1) + \int_0^t U(t-s; 0, \tilde{F})\, ds, \tag{7.15}$$

where

$$\tilde{F} = \nabla \cdot \{\sigma(|\nabla u|^2) - 1)\nabla u(t)\} \equiv \Gamma(|\nabla u|^2) \cdot (Du)^2 D^2 u.$$

Thus, by (7.14) and (7.15) we have

$$\|D_t^k \nabla u(t)\|_{M+1-k,p} \le C I_{4M+2,q}(1+t)^{-d(p)} + \int_0^t (1+t-s)^{-d(p)} \|\tilde{F}(s)\|_{4M+1,q}\, ds. \tag{7.16}$$

Here,

$$\begin{aligned} D^{4M+1}\tilde{F} = & \sum_{j=0}^{4M+1} \sum_{\alpha \le M+1, \beta \le M+1} \tilde{\Gamma}_{\alpha,\beta} D^\alpha \nabla u D^\beta \nabla u D^{4M+2-j} \nabla u \\ & + \sum_{j=0}^{3M} \sum_{\alpha \le M+1, \beta \ge M+2} \tilde{\Gamma}_{\alpha,\beta} D^\alpha \nabla u D^\beta \nabla u D^{4M+2-j} \nabla u \\ & + \sum_{j=3M+1}^{4M+1} \sum_{\alpha \ge M+2, \beta \ge M+2} \tilde{\Gamma}_{\alpha,\beta} D^\alpha \nabla u D^\beta \nabla u D^{4M+2-j} \nabla u \\ & \equiv J_1 + J_2 + J_3. \end{aligned} \tag{7.17}$$

Noting that $\tilde{\Gamma} = \tilde{\Gamma}(D^\gamma u)$, $\gamma \le 2M$, and $\|\tilde{\Gamma}\|_\infty \le C(K) < \infty$, we estimate J_i, $1 \le i \le 3$, as follows. First, we see

$$\begin{aligned}\|J_1\|_q &\le C \sum_{j=0}^{4M+1} \left(\int_\Omega |D^\alpha \nabla u|^q |D^\beta \nabla u|^q |D^{4M+3-j} u|^q \, dx \right)^{1/q} \\ &\le C(K) \sum_{j=0}^{4M+1} \|D^{4M+3-j} u\|^2 \left(\int_\Omega |D^\alpha u|^{2q/(2-q)} \, dx \right)^{(2-q)/2q} \left(\int_\Omega |D^\beta u|^{2q/(2-q)} \, dx \right)^{(2-q)/2q} \\ &\le CK \|D^\alpha \nabla u\|_2^{(1-\theta)} \|D^\alpha \nabla u\|_p^\theta \|D^\beta \nabla u\|_2^{(1-\theta)} \|D^\beta \nabla u\|_p^\theta \\ &\le C(K) K^3 I_0^{2(1-\theta)} (1+t)^{-2d(p)\theta} \end{aligned} \tag{7.18}$$

with $\theta = (p+2)/2(p-2)$. Here, we note that

$$2d(p)\theta = (p+2)(1-\varepsilon)/p > 1.$$

For J_2 we see

$$\begin{aligned}\|J_2\|_q &\le C(K) \sum_{j=0}^{3M} \sum_{\alpha \le M+1, \beta \ge M+2} \left(\int_\Omega |D^\alpha \nabla u|^q |D^\beta \nabla u|^q |D^{4M+2-j} \nabla u|^q \, dx \right)^{1/q} \\ &\le C(K) \sum_{j \le 3M} \sum_{\alpha \le M+1, \beta \ge M+2} \|D^\alpha \nabla u\|_p \|D^\beta \nabla u\|_{2pq/(p-q)} \|D^{4M+2-j} \nabla u\|_{2pq/(p-q)}. \end{aligned} \tag{7.19}$$

Here, we see by Gagliardo-Nirenberg inequality,

$$\|D^\beta \nabla u\|_{2pq/(p-q)} \le C \|D^{M+1} \nabla u\|_{2pq/(p-q)}^{1-\theta_1} \|\nabla u\|_{H^{4M+2}}^{\theta_1}$$

with

$$\theta_1 = \left(\frac{\beta - M - 1}{N} + \frac{p-q}{2pq} - \frac{2pq}{p-q} \right) \left(\frac{4M+2-\beta}{N} + \frac{p-q}{2pq} - \frac{1}{2} \right) = \frac{p(\beta - M - 1)}{p(4M+2-\beta) - N},$$

and

$$\|D^{M+1} \nabla u\|_{2pq/(p-q)} \le CK \|\nabla u\|^{1-\tilde{\theta}_1} \|D^{M+1} \nabla u\|_p^{\tilde{\theta}_1}$$

with $\tilde{\theta}_1 = 2/(p-2)$. Thus, we have

$$\|D^\beta \nabla u\|_{2pq/(p-q)} \le C(K)(1+t)^{-d(p)\theta_2} \tag{7.20}$$

with $\theta_2 = (1-\theta_1)\tilde{\theta}_1$.

Quite similarly, since $4M + 2 - j \ge M + 2$ we can show that

$$\|D^{4M+2-j} \nabla u\|_{2pq/(p-q)} \le C(K) K (1+t)^{-d(p)\bar{\theta}_2} \tag{7.21}$$

with $\bar{\theta}_2 = (1 - \bar{\theta}_1)\tilde{\theta}_1$, where $\bar{\theta}_1$ is defined by θ_1 with β replaced by $4M + 2 - j$.

Thus, we obtain from (7.19), (7.20) and (7.21),

$$\|J_2\|_q \le C(K) K^3 (1+t)^{-(1+\theta_2+\bar{\theta}_2)d(p)}. \tag{7.22}$$

Here, we observe that

$$\theta_1, \tilde{\theta}_1 \le 3p/(3pM - N) = p/(2p-1)$$

and

$$(1+\theta_2+\bar{\theta}_2)d(p) \geq (1+2(p-3)/3p(2p-1))(1-\varepsilon) > 1.$$

Finally, we can show as in the treatment of J_2,

$$\begin{aligned}\|J_3\|_q &\leq C(K)\sum_{j=3M+1}^{4M+1}\sum_{\alpha\geq M+2,\beta\geq M+2}\|D^{4M+2-j}\nabla u\|_p\|D^{\alpha}\nabla u\|_{2pq/(p-q)}\|D^{\beta}\nabla u\|_{2pq/(p-q)}\\ &\leq C(K)K^3(1+t)^{-(1+\theta_2+\tilde{\theta}_2)d(p)} \qquad (7.23)\end{aligned}$$

where $\tilde{\theta}_2$ is defined by θ_2 with β replaced by α, and we know

$$(1+\theta_2+\tilde{\theta}_2)d(p) \geq (1+2(p-3)/3p(2p-1))(1-\varepsilon) > 1.$$

Also we easily see

$$\|\tilde{F}\|_q \leq C\|Du\|_p^2\|Du\|_{4q/(2-q)} \leq CK^3(1+t)^{-2d(p)}. \qquad (7.24)$$

Since $\|\tilde{F}\|_{4M+1,q} \leq C(\|D^{4M+1}\tilde{F}\|_q+\|\tilde{F}\|_q)$, returning to the integral inequality (7.14) we obtain

$$\begin{aligned}\|D_t^k\nabla u(t)\|_{M+1-k,p} &\leq CI_{4M+2,q}(1+t)^{-d(p)}+c(K)K^3\int_0^t(1+t-s)^{-d(p)}(1+s)^{-\tilde{d}}\,ds, \tilde{d}>1,\\ &\leq (I_{4M+2,q}+C(K)K^3)(1+t)^{-d(p)}. \qquad (7.25)\end{aligned}$$

We summarize the results concerning the estimate of $\|D_t^k\nabla u(t)\|_{M+1-k,p}$ as follows.

Proposition 13. *Let $N=3$ and let $u(\cdot)\in V_{4M+2}(K,T)$ be a solution of the problem* (1.1)–(1.2) *as in Proposition* 1. *Then, we have*

$$\sum_{k=0}^{M+1}\|D_t^kDu(t)\|_{M+1-k,p} \leq (q(K)K+CI_{4M+2,q})(1+t)^{-d(p)} \qquad (7.26)$$

where $q(K)$ is a quantity depending on K continuously in such a way that $q(0)=0$.

When $N\geq 4$, for the solution $u(t)\in V_{3M}(K,T)$ we have

$$\begin{aligned}\|D_t^k\nabla u(t)\|_{M+1-k,\infty} &\leq C(I_{3M,2}+I_{3M,1})(1+t)^{-d}\\ &\quad+\int_0^t(1+t-s)^{-d}(\|\tilde{F}(s)\|_{3M,2}+\|\tilde{F}(s)\|_{3M,1})\,ds. \qquad (7.27)\end{aligned}$$

By use of the inequality

$$\sum_{k=0}^{M}\|D_t^k\nabla u(t)\|_{M+1-k,\infty} \leq K(1+t)^{-d}$$

we can easily prove under the assumption (7.6) that

$$\|\tilde{F}(s)\|_{3M,2}+\|\tilde{F}(s)\|_{3M,1} \leq CK^3(1+t)^{-d} \qquad (7.28)$$

and hence we obtain

$$\sum_{k=0}^{M+1} \|D_t^k Du(t)\|_{M+1-k,\infty} \le (q(K)K + CI_{3M,1})(1+t)^{-d}. \tag{7.29}$$

Completion of the proof of Theorems 2, 3.

Let us consider the case $N = 3$. By the proofs of Propositions 13, 14 we easily see that if $K > 2(I_{4M+2,2} + I_{4M+1,q})$ the local solution $u(t)$ belongs $V_{3M}(K,T)$ for some $T > 0$ and by Propositions 13 and 14 we know that this is valid for all $T > 0$ provided that

$$q(K)K + C(I_{4M+2,2} + I_{4M+1,q}) < K. \tag{7.30}$$

Since $q(K)$ continuously depends on K and $q(0) = 0$ the above condition (7.30) is satisfied if we take $K = C_1(I_{4M+2,2} + I_{4M+1,q})$ with $C_1 \gg C$ and if $I_{4M+2,2} + I_{4M+1,q} \le \delta$ for a small constant $\delta > 0$. Thus we arrived at the desired estimates for all $T > 0$.

The case $N \ge 4$ (Theorem 3) is also proved quite similarly by use of Proposition 13 and the estimate (7.29). □

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