# Global smoothing of singular weak Fano 3-folds

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**Abstract.** In this paper, we will study smoothability of a weak Fano 3-fold with only canonical singularities which is obtained as an image of a crepant primitive birational contraction from a smooth weak Fano 3-fold. Main part is on a contraction of type III.

### 0. Introduction.

We will work over C in this paper.

DEFINITION 0.1. Let X be a 3-dimensional normal Gorenstein projective variety which has only canonical singularities.

- (i) We call X a weak Fano 3-fold when  $-K_X$  is nef and big.
- (ii) We call X a Fano 3-fold when  $-K_X$  is ample.

DEFINITION 0.2. Let X be a 3-dimensional normal Gorenstein projective variety which has only canonical singularities,  $(\Delta, 0)$  a germ of the 1-dimensional disk, and  $f: \mathscr{X} \to (\Delta, 0)$  be a small deformation of X over  $(\Delta, 0)$ . We call f a smoothing of X when the fiber  $\mathscr{X}_s = f^{-1}(s)$  is smooth for any  $s \in (\Delta, 0) \setminus \{0\}$ .

Let X be a smooth weak Fano 3-fold and  $\phi: X \to \overline{X}$  a birational projective contraction to a weak Fano 3-fold with only canonical singularities. If  $\overline{X}$  has a smoothing, then we have another smooth weak Fano 3-fold  $\overline{\mathscr{X}}_s$ . We want to connect weak Fano 3-folds by deformations and birational contractions as above. We call this problem Reid's fantasy for weak Fano 3-folds.

REMARK. "Original" Reid's fantasy is for Calabi-Yau 3-folds.

Thus we consider the following problem.

**PROBLEM.** Let X be a weak Fano 3-folds with only canonical singularities. When does X have a smoothing?

Known results on Problem are as follows.

1. (Namikawa, Mukai Cf. [Na 3] and [Mu])

Let X be a Fano 3-fold with only terminal singularities. Then X has a smoothing.

2. (Namikawa, Takagi Cf. [Na 3], [Ta] and [Mi]) Let X be a weak Fano 3-fold with only terminal singularities. Assume that there exists a birational proper morphism  $\phi : X \to \overline{X}$  to a Fano 3-fold with only

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canonical singularities such that dim  $\phi^{-1}(x) \le 1$  for any  $x \in \overline{X}$ . Then X has a smoothing.

3. ([**Mi**])

Let X be a weak Fano 3-fold with only terminal singularities. Then there exists a small deformation of X over  $(\Delta, 0)$   $\mathfrak{f} : \mathscr{X} \to (\Delta, 0)$  such that the fiber  $\mathscr{X}_s = \mathfrak{f}^{-1}(s)$  has only ordinary double points for any  $s \in (\Delta, 0) \setminus \{0\}$ .

4. ([**Mi**])

Let X be a weak Fano 3-fold with only terminal singularities. If X is Q-factorial, then X has a smoothing.

If the condition "Q-factorial" is dropped, then there exists an example which does not have a smoothing.

Extending the method in Section 3 of [Mi], we will show the following theorems in Section 1 and 2 of this paper.

THEOREM 0.3. Let X be a weak Fano 3-fold with only terminal singularities,  $\{p_1, p_2, \ldots, p_l\} \subset \operatorname{Sing}(X)$  the ordinary double points on X, and  $f: Z \to X$  a small partial resolution of X such that  $C_i =: f^{-1}(P_i) \cong \mathbf{P}^1$  and that f is an isomorphism over  $X \setminus \{p_1, p_2, \ldots, p_l\}$ . If there is a relation in  $H_2(Z, \mathbf{C}) : \sum_{i=1}^l \alpha_i [C_i] = 0$  with  $\alpha_i \neq 0$  for all i, then X has a smoothing.

THEOREM 0.4. Let X be a weak Fano 3-fold with only isolated canonical singularities. Assume that

- (i) X is **Q**-factorial,
- (ii) for any  $p \in \text{Sing}(X)$ , the Kuranishi space of (X, p) is smooth, and

(iii) for any  $p \in \text{Sing}(X)$ , (X, p) has a smoothing.

Then X has a smoothing.

REMARK (Cf. [Na 1], [Na 2], [Na 4] and [Na-St]).

Namikawa proved the same statements of Theorem 0.3 and Theorem 0.4 for Calabi-Yau 3-fold. But the condition in Theorem 0.3 is a necessary and sufficient condition of smoothability in the case of Calabi-Yau 3-fold.

In order to consider Reid's fantasy for weak Fano 3-folds, we study "Smoothing problem" of a weak Fano 3-fold with only canonical singularities obtained as an image of a crepant primitive birational contraction from a smooth weak Fano 3-fold.

DEFINITION 0.5. Let X be a smooth weak Fano 3-fold, and  $\phi: X \to \overline{X}$  a crepant birational projective morphism. We call  $\phi$  primitive when its relative Picard number  $\rho(X/\overline{X}) = 1$ . Moreover, letting E be the exceptional locus of  $\phi$ , we will define as follows.

- (i)  $\phi$  is a crepant primitive birational contraction of type I when dim(E) = 1.
- (ii)  $\phi$  is a crepant primitive birational contraction of type II when dim(E) = 2 and dim  $\phi(E) = 0$ .
- (iii)  $\phi$  is a crepant primitive birational contraction of type III when dim(E) = 2and dim  $\phi(E) = 1$ .

We treat a crepant primitive birational contraction of type III from a smooth weak Fano 3-fold in Section 3, which is the main part of this paper. On a crepant primitive

birational contraction of type III from a smooth weak Fano 3-fold, we have the following theorem.

THEOREM 0.6. Let X be a smooth weak Fano 3-fold and  $\phi: X \to \overline{X}$  a crepant primitive birational contraction of type III contracting a divisor E to a curve C. Then

- (i) C is smooth.
- (ii)  $\phi|_E : E \to C$  is a conic bundle, and each fiber is a non-singular conic, a union of two lines meeting at a point, or a double line.
- (iii) If the general fiber of  $\phi|_E$  is a non-singular curve, then E is normal and E has only rational double points.
- (iv) If the general fiber of  $\phi|_E$  is two lines meeting at a point, then singularities of E on the double line are pinch point singularities (of the form  $x^2 + tz^2 = 0$  in  $(C^3, 0)$ ).

PROOF. We can show this by the same method in [Wi 1], [Wi 2] and Section 3 of [Wi 3].  $\Box$ 

DEFINITION 0.7. Let X be a smooth weak Fano 3-fold and  $\phi: X \to \overline{X}$  a crepant primitive birational contraction of type III contracting a divisor E to a curve C. We call  $p \in C$  is a dissident point if the fiber of  $\phi|_E: E \to C$  over p is not isomorphic to general fiber. We call the fiber over a dissident point the dissident fiber.

Theorem 0.6 enables us to define the following.

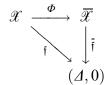
DEFINITION 0.8. Let X be a smooth weak Fano 3-fold and  $\phi: X \to \overline{X}$  a crepant primitive birational contraction of type III contracting a divisor E to a curve C.

- (i) The case *E* is normal: We call  $\phi$  a contraction of (g, d)-type when g = g(C)and  $d = -K_{\overline{X}} \cdot C$ . Moreover we call  $\phi$  without dissident fibers when  $\phi|_E$ :  $E \to C$  is a  $P^1$ -bundle and  $\phi$  with dissident fibers when  $\phi|_E$  is not a  $P^1$ -bundle.
- (ii) The case E is non-normal: Let  $\tilde{E}$  be the normalization of E, and  $\tilde{E} \rightarrow \tilde{C} \rightarrow C$  the Stein factorization. We call  $\phi$  a contraction of  $(g, \tilde{g}, d)$ -type when  $g = g(C), \ \tilde{g} = g(\tilde{C}), \ \text{and} \ d = -K_{\bar{X}} \cdot C.$

We will prove the following theorem on deformations of  $\overline{X}$  and  $\phi$ .

THEOREM 0.9. Let X be a smooth weak Fano 3-fold and  $\phi: X \to \overline{X}$  a crepant primitive birational contraction of type III contracting a divisor E to a curve C.

- (i) X has a smoothing unless  $\phi$  is of (0,0), (0,1),  $(0,\tilde{g},0)$ , or  $(0,\tilde{g},1)$ -type, or (0,3)-type without dissident fibers, or (1,1)-type without dissident fibers.
- (ii) Let  $\mathscr{X} \to Def(X)$  be the Kuranishi family of X. Assume that  $\phi$  is a contraction of (0,0), (0,1),  $(0,\tilde{g},0)$  or  $(0,\tilde{g},1)$ -type. Then E will deform in the family.
- (iii) Assume that  $\phi$  is a contraction of (0,3)-type without dissident fibers, or (1,1)type without dissident fibers. Then there exists a small deformation of  $\phi$  over  $(\Delta, 0)$



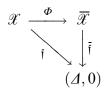
such that, for any  $t \in (\Delta, 0) \setminus \{0\}$ ,

 $\Phi_t: \mathscr{X}_t \to \overline{\mathscr{X}}_t$ 

is a crepant primitive birational contraction of type I which is a contraction of a single  $P^1$ .

Key ideas of the proof of (i) of this theorem are

(1) We will find a small deformation of  $\phi$  over  $(\Delta, 0)$ 



such that, for any  $t \in (\varDelta, 0) \setminus \{0\}$ ,

$$\Phi_t: \mathscr{X}_t \to \overline{\mathscr{X}}_t$$

is a crepant primitive birational contraction of type I, and

(2) We will count the number of curves which are contracted by such  $\Phi_t$ . It depends on not only g or  $\tilde{g}$  but also d. We remark that there are similar results for Calabi-Yau 3-folds (Cf. [Wi 1], [Gr 1], [Gr 2]). But there are differences in the case of type III with  $d \neq 0$ .

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NOTATIONS.

- (1) In this paper,  $(\Delta, 0)$  means a germ of the 1-dimensional disk.
- (2)  $Def(\bullet)$  means the Kuranishi space of  $\bullet$  and  $T_{\bullet}^{1}$  means its tangent space at 0. We also use this notation in the case that  $\bullet$  is a morphism.
- (3) Let  $G \cong \mathbb{Z}/2\mathbb{Z}$  acting on a *C*-vector space *F* (resp. a coherent sheaf  $\mathscr{F}$  on a scheme *X* over *C*. In this case *G* acts on *X* trivially.). Let  $\sigma$  be the generator of *G*. We set  $F^G = \{s \in F \mid \sigma s = s\}$  (resp. we define  $\mathscr{F}^G$  by  $\mathscr{F}^G(U) = \{s \in \mathscr{F}(U) \mid \sigma s = s\}$  for an open set *U* of *X*).
- (4) Let  $G \cong \mathbb{Z}/2\mathbb{Z}$  acting on a  $\mathbb{C}$ -vector space F (resp. a coherent sheaf  $\mathscr{F}$  on a scheme X over  $\mathbb{C}$ . In this case G acts on X trivially.). Let  $\sigma$  be the generator of G. We set  $F^{[-1]} = \{s \in F \mid \sigma s = -s\}$  (resp. we define  $\mathscr{F}^{[-1]}$  by  $\mathscr{F}^{[-1]}(U) = \{s \in \mathscr{F}(U) \mid \sigma s = -s\}$  for an open set U of X).

## 1. On a contraction of Type I.

We will prove Theorem 0.3 in this section, and we have a theorem on a contraction of type I as a corollary of Theorem 0.3. We prove the following theorem first.

THEOREM 1.1. Let X be a weak Fano 3-fold with only ordinary double points,  $\{p_1, p_2, \ldots, p_l\} = \operatorname{Sing}(X)$ , and  $v : Z \to X$  be a small resolution of X such that v is an isomorphism over  $X \setminus \operatorname{Sing}(X)$ . Let  $C_i = v^{-1}(P_i)$ . Assume that there exists a relation in  $H_2(Z, \mathbb{C}) : \sum_{i=1}^l \alpha_i [C_i] = 0$  with  $\alpha_i \neq 0$  for all i, then X has a smoothing.

**PROOF.** Let  $U = X \setminus \text{Sing}(X)$ ,  $X_i$  a sufficiently small neighborhood of  $p_i$ , and  $U_i = X_i \setminus \{p_i\}$ . Under these setting, we consider the following commutative diagram:

We remark that  $\gamma$  and  $\gamma_i$ 's are defined by a section of  $H^0(Z, \omega_Z^{-1})$ , and the upper horizontal sequence is exact. By the assumption, there exist elements  $(\eta'_i | i = 1, 2, ..., l) \in \bigoplus_{i=1}^l H^2_{C_i}(Z, \Omega_Z^2)$  such that  $\eta'_i \neq 0$  and  $\beta'((\eta'_i | i = 1, 2, ..., l)) = 0$ . Thus there exists  $\eta \in H^1(U, \Theta_U)$  such that  $\alpha(\eta)_i = \gamma_i(\eta'_i)$  for i = 1, 2, ..., l.

Case 1. (The case that there exists a smooth member  $S \in |-K_X|$ .)

We may assume that  $\gamma$  and  $\gamma_i$  are defined by  $v^*S$ . In this case,  $\gamma_i$  is an isomorphism for any *i*. Thus  $\gamma_i(\eta'_i) \neq 0$ . Since Def(X) is smooth as in [Mi],  $\eta$  can be realized as a smoothing of X.

Case 2. (The case that  $|-K_X|$  does not have a smooth member.)

Let  $\phi_{ac}: X \to X_{ac}$  be a multi-anti-canonical morphism. In this case, as in Section 3 of [**Mi**], we may assume that  $Bs|-K_X| = \{p_1\}$ ,  $\phi_{ac}$  is an isomorphism near  $p_1$ ,  $X_{ac}$ is isomorphic to  $X_{2,6} \subset P(1,1,1,1,2,3)$  which is a weighted complete intersection of multi-degree  $\{2,6\}$ , and its defining homogeneous equation of degree 2 of  $X_{2,6}$  in P(1,1,1,1,2,3) is given by  $X_0^2 + X_1^2 + X_2^2 + X_3^2 = 0$ . By the structure of  $X_{ac}$ , there exists an element  $\zeta' \in T_{X_{ac}}^1$  such that  $\zeta$  is locally a non-trivial deformation at  $\phi_{ac}(p_1)$  and is locally the trivial deformation at any other singularities.

Let  $U' = X_{ac} \setminus \text{Sing}(X_{ac})$ ,  $X'_j$  a sufficiently small neighborhood of each connected component of Sing(X),  $U'_j = X'_j \setminus (\text{Sing}(X_{ac}) \cap X'_j)$ , and  $E = \phi_{ac}^{-1}(\text{Sing}(X_{ac})) \cap U$ . We may assume that  $U_1 \cong U'_1$ . Under these setting, we consider the following diagram:

In this diagram, the upper horizontal sequence is exact. By the choice of  $\zeta'$ , we have that  $r(\zeta')_j = 0$  for  $j \neq 1$ . Since  $\phi_{ac}$  is an isomorphism near  $p_1$ , we have that  $\tau_i(r(\zeta')_1) = 0$ . Since  $\tau(\zeta'|_{U'}) = 0$ , there exists an element  $\zeta$  such that  $\zeta|_{U'} = \zeta'|_{U'}$ . Thus we have that  $\alpha(\zeta)_1 \neq 0$ .

Suppose that  $\gamma$  and  $\gamma_i$  are defined by  $S \in |-K_X|$  such that  $S \cap \text{Sing}(X) = \{p_1\}$ , We know that  $\gamma_i$  is an isomorphism for  $i \neq 1$ . Thus  $\alpha(\eta)_i \neq 0$  for  $i \neq 1$ . Thus there exists a complex number  $\varepsilon$  such that  $\alpha(\eta + \varepsilon\zeta)_i \neq 0$  for all *i*. Since Def(X) is smooth as in [Mi], there exists a realization of  $\eta + \varepsilon\zeta$  which is a smoothing of X.

PROOF OF THEOREM 0.3. By Theorem 1.1 and its proof, it is enough to show that all singularities of X which are not ordinary double points are smoothed by a suitable deformation of X. There is a deformation of X to a 3-fold with only ordinary double points by (2) of Main Theorem of [Mi]. Considering Section 3 of [Mi] (refined in Section 2 of this paper), it follows from the method in the first part of the proof of Theorem 2.5 (3)  $\Rightarrow$  (2) of [Na 2].

We can show the following theorem as in Theorem 5.1 of [Gr 1].

THEOREM 1.2. Let X be a smooth weak Fano 3-fold, and  $\phi: X \to \overline{X}$  a crepant primitive birational contraction of type I. Then  $\overline{X}$  has a smoothing unless  $\phi$  is a contraction of a single  $\mathbf{P}^1$  to an ordinary double point.

### 2. On a contraction of Type II.

In this section, we prove Theorem 0.4. To prove this theorem we need the following which we can prove by a little refinement of the method in Section 3 of [Mi].

THEOREM 2.1. Let X be a weak Fano 3-fold with only isolated canonical singularities. Assume that, for any singularity p, the Kuranishi space of (X, p) is smooth. Then the Kuranishi space Def(X) of X is smooth.

**PROPOSITION 2.2.** Let X be a weak Fano 3-fold with only isolated canonical singularities. Assume that X is **Q**-factorial. Then there exists a smooth member  $S \in |-K_X|$ .

We prove now Theorem 0.4.

PROOF OF THEOREM 0.4. Let  $\{p_1, p_2, \ldots, p_n\} = \operatorname{Sing}(X)$ . Let  $v : \tilde{X} \to X$  be a resolution of X such that v is an isomorphism over  $U := X \setminus \operatorname{Sing}(X)$  and its exceptional divisors  $E_i := v^{-1}(p_i)$  have simple normal crossings. Let  $X_i$  be a sufficiently neighborhood of  $p_i$ ,  $U_i = X_i \setminus \{p_i\}$ . We know the following proposition.

**PROPOSITION 2.3** (Cf. The proof of Proposition 4 of [Na 4]). If  $(X, p_i)$  is not a rigid singularity, then the homomorphism

$$l_i: H^2_{E_i}(\tilde{X}, \Omega^2_{\tilde{X}}) \to H^2(\tilde{X}, \Omega^2_{\tilde{X}})$$

is not injective.

By Proposition 2.2, there exists a smooth member  $S \in |-K_X|$ , then we have the following commutative diagram defined by  $v^*S$ :

By this diagram and Proposition 2.3,  $t'_i$  is not injective for any *i*. We consider the following commutative diagram:

We remark that the upper horizontal sequence is exact, and the homomorphism  $\delta_i$  is factorized as follows:

$$H^1(U_i, \mathcal{O}_{U_i}) \to H^2_{E_i}(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^2_{E_i}(\tilde{X}, \Omega^2_{\tilde{X}} \otimes \nu^* \omega_X^{-1}).$$

By the exactness, there exists an element  $\eta \in H^1(U, \Theta_U)$  such that  $\alpha'(\eta)_i \neq 0$  for any *i*. By Theorem 2.1, we can prove Theorem 0.4 by the Namikawa's stratification method (cf. the proof of Theorem 5 of [Na 4]).

### 3. On a contraction of type III.

We use the following theorem of Takagi in this section,

THEOREM 3.1 (Takagi) (Cf. [Ta]). Let X be a weak Fano 3-fold with only canonical singularities. The complete linear system  $|-2K_X|$  is base-point free.

PROPOSITION 3.2. Let X be a smooth weak Fano 3-fold, and  $\phi: X \to \overline{X}$  a crepant primitive birational contraction of type III contracting a divisor E to a curve C. Let  $\tilde{E}$  be the normalization of E (when E is normal  $\tilde{E} = E$ ),  $\tilde{E} \to \tilde{C} \to C$  the Stein factorization, and  $f: \tilde{E} \to X$  the induced map. Then the image of the natural map  $Def(f) \to Def(X)$ has codimension

- (i)  $\geq \max\{g, g+d-1\}$  when  $\phi$  is a contraction of (g, d)-type.
- (ii)  $\geq \max{\{\tilde{g}, \tilde{g} + 2d 1\}}$  when  $\phi$  is a contraction of  $(g, \tilde{g}, d)$ -type.

**PROOF.** We first show codimension  $\geq g$  when  $\phi$  is (g, d)-type (resp.  $\geq \tilde{g}$  when  $\phi$  is  $(g, \tilde{g}, d)$ -type). (This was proved in Proposition 4.2 of [**Pa**] in the case that E is  $P^1$ -bundle or any fiber of  $\phi|_E$  is union of two lines meeting at a point, and this proof is a modification of it.) To show this, we need the following lemma:

LEMMA 3.3. Let  $\tilde{\Omega}_{\overline{X}}^2$  be the double dual of  $\Omega_{\overline{X}}^2$ . We have that  $H^0(\overline{X}, \tilde{\Omega}_{\overline{X}}^2) = 0$ .

PROOF OF LEMMA. Let  $U = \overline{X} \setminus C$ , which is a smooth locus of  $\overline{X}$ . By Theorem 3.1,  $|-2K_{\overline{X}}|$  is base-point free. Since  $\overline{X}$  has generically  $cA_1$  or  $cA_2$  singularities, there exists a member  $D \in |-2K_{\overline{X}}|$  such that D is smooth except  $D \cap C$  and D has an  $A_1$  or  $A_2$  singularity at each point of  $D \cap C$ . Let  $\pi : Y = \operatorname{Spec}(\mathcal{O}_{\overline{X}} \oplus \mathcal{O}_{\overline{X}}(K_{\overline{X}})) \to \overline{X}$  be the double cover of  $\overline{X}$  ramified along D, then Y is a Calabi-Yau 3-fold with only canonical singularities. Let  $V = \pi^{-1}(U)$  and  $G = \mathbb{Z}/2\mathbb{Z}$ . We have that  $(\pi_*\Omega_V^2)^G = \Omega_U^2$ . So we have  $H^0(V, \pi_*\Omega_V^2) = H^0(V, \Omega_V^2) = H^0(V, \mathcal{O}_V) = H^0(Y, \mathcal{O}_Y) = 0$  by the result of Kawamata [Ka]. Thus  $H^0(\overline{X}, \tilde{\Omega}_{\overline{X}}^2) = H^0(U, \Omega_U^2) = H^0(V, \pi_*\Omega_V^2)^G = 0$ .

Step 1. When  $\phi$  is (g,d)-type, the codimension  $\geq g$ .

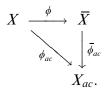
**PROOF.** This is a modification of the argument from the proof of Proposition 6.5 of [Na 1]. Let  $T_f^1$  be the tangent space of Def(f). We have an exact sequence by [Ra]

$$T_f^1 \to T_X^1 \oplus T_E^1 \to H^1(X, f^* \Theta_X).$$

This induces an exact sequence

$$T_f^1 \xrightarrow{\alpha} T_X^1 \xrightarrow{\beta} H^1(E, N_{E/X}).$$

We know that Def(X) is smooth by  $[\mathbf{Pa}]$  and  $[\mathbf{Mi}]$ , thus it is enough to show that  $\operatorname{rank}(\beta) \ge g$ . Using the identification  $\Theta_X \cong \Omega_X^2 \otimes \omega_X^{-1}$  and  $N_{E/X} \cong \omega_E \otimes \omega_X^{-1}$ , we may view  $\beta$  as a map  $\beta : H^1(X, \Omega_X^2 \otimes \omega_X^{-1}) \to H^1(E, \omega_E \otimes \omega_X^{-1})$ . While we can choose  $S \in |-K_X|$  such that  $F = S \cap E = \sum_{i=1}^d F_i$ , where  $F_i \cong \mathbf{P}^1$  is a fiber of E over C for each i, and  $F_i \cap F_j = \emptyset$  if  $i \ne j$ . In fact there exists a member  $S \in |-K_X|$  which has only rational double points by  $[\mathbf{Re}]$ , so C is not a base locus of  $|-K_X|$ . If there is a base point p on C, we consider the anti-canonical model  $X_{ac}$  of X:



As in Section 3 in [Mi],  $Bs|-K_{X_{ac}}| = \{\overline{\phi}_{ac}(p)\}\)$ , and general member  $S_{ac} \in |-K_{X_{ac}}|\)$  has a single ordinary double point only at  $\overline{\phi}_{ac}(p)$ . By Proposition 2.1 of [Pa], for a general member  $S \in |-K_X|$ ,  $S \to S_{ac}$  is the minimal resolution of the ordinary double point. Thus we can choose a member of  $|-K_X|$  as desired. We consider the following commutative diagram whose vertical arrows are defined by an  $S \in |-K_X|$ :

$$\begin{array}{cccc} H^1(X, \Omega_X^2) & \stackrel{\beta}{\longrightarrow} & H^1(E, \omega_E) \\ & & & & \downarrow^a \\ & & & \downarrow^a \\ H^1(X, \Omega_X^2 \otimes \omega_X^{-1}) & \stackrel{\beta}{\longrightarrow} & H^1(E, \omega_E \otimes \omega_X^{-1}) \end{array}$$

By the exact sequence

$$0 \to \omega_E \to \omega_E \otimes \omega_X^{-1} \to \omega_E \otimes \omega_X^{-1}|_F \to 0,$$

and  $H^0(\omega_E \otimes \omega_X^{-1}|_F) = H^0(\omega_F) = 0$ , we have that *a* is injective. Thus we have that

$$\operatorname{rank}(\tilde{\beta}) \le \operatorname{rank}(a \circ \tilde{\beta}) = \operatorname{rank}(\beta \circ b) \le \operatorname{rank}(\beta).$$

So it is enough to show that  $g(C) \leq \operatorname{rank}(\tilde{\beta})$ .

Let  $v: X' \to X$  be a embedded resolution of the pair (X, E). Let  $\mu = \phi \circ v$ , E' the proper transform of E by v. There is a commutative diagram

$$\begin{array}{cccc} H^{1}(E, \mathcal{O}_{E}) & \stackrel{\delta}{\longrightarrow} & H^{2}(X, \Omega^{1}_{X}) \\ & & & \downarrow \\ & & & \downarrow \\ H^{1}(C, \mathcal{O}_{C}) & \stackrel{\simeq}{\longrightarrow} & H^{1}(E', \mathcal{O}_{E'}) & \stackrel{\delta'}{\longrightarrow} & H^{2}(X', \Omega^{1}_{X'}). \end{array}$$

The vertical arrow of left-hand side is an isomorphism because E has only rational double points by Theorem 0.6, and the first horizontal arrow at the bottom is an isomorphism because a general fiber of  $\phi|_E : E \to C$  is isomorphic to  $P^1$ . We remark that  $\delta$  is the dual map of  $\tilde{\beta}$ . If we can show that  $\delta'$  is injective, we have that

$$\operatorname{rank}(\boldsymbol{\beta}) = \operatorname{rank}(\boldsymbol{\delta}) \ge \operatorname{rank}(\boldsymbol{\delta}') \ge g(C)$$

Thus it is enough to show that  $H^1(X', \Omega^2_{X'}) \to H^1(E', \omega_{E'})$  is surjective. By the Hodge symmetry it is enough to show that  $H^2(X', \Omega^1_{X'}) \to H^2(E', \Omega^1_{E'})$  is surjective. By the following 2 exact sequences

$$\begin{split} 0 &\to \mathscr{O}_{E'}(-E') \to \Omega^1_{X'}|_{E'} \to \Omega^1_{E'} \to 0 \\ 0 &\to \Omega^1_{X'}(-E') \to \Omega^1_{X'} \to \Omega^1_{X'}|_{E'} \to 0, \end{split}$$

it is enough to show that  $H^3(X', \Omega^1_{X'}(-E')) = 0$ . By the Serre duality

$$H^{3}(X', \Omega^{1}_{X'}(-E')) \cong H^{0}(X', \Theta_{X'}(K_{X'}+E')) \cong H^{0}(X', \Omega^{2}_{X'}(E')).$$

There is an injection

 $\mu_* \varOmega^2_{X'}(E') \hookrightarrow \tilde{\varOmega}^2_{\overline{X}}$ 

because both sheaves are isomorphic to each other outside a subset of codimension  $\geq 2$ and  $\tilde{\Omega}_{\bar{X}}^2$  is a reflexible sheaf. Then we have  $H^0(X', \Omega_{X'}^2(E')) = 0$  because  $H^0(\bar{X}, \tilde{\Omega}_{\bar{X}}^2) = 0$ by the Lemma.

Step 2. When the  $\phi$  is  $(g, \tilde{g}, d)$ , the codimension  $\geq \tilde{g}$ .

PROOF. This is a modification of the argument from the proof of Proposition 1.2 of [Gr 2]. By Theorem 0.6,  $\tilde{\phi}_{\tilde{E}} : \tilde{E} \to \tilde{C}$  is a  $P^1$ -bundle over  $\tilde{C}$ . Define  $N_f$  by the exact sequence

$$0 \to \mathcal{O}_{\tilde{E}} \to f^* \mathcal{O}_{\tilde{X}} \to N_f \to 0.$$

We remark that  $N_f$  is torsion free, locally free away from the inverse image of pinch points of E. Thus we have that the double dual  $\tilde{N}_f$  of  $N_f$  is isomorphic to  $\omega_{\tilde{E}} \otimes f^* \omega_X^{-1}$ . We have an exact sequence as in [**Ra**],

$$T_f^1 \to T_X^1 \oplus T_{\tilde{E}}^1 \to H^1(X, f^* \Theta_X).$$

This induces an exact sequence

$$T_f^1 \xrightarrow{\alpha} T_X^1 \xrightarrow{\beta'} H^1(\tilde{E}, N_f).$$

Let  $\beta$  be a composition homomorphism  $T_X^1 \to H^1(\tilde{E}, N_f) \to H^1(\tilde{E}, \tilde{N}_f)$ . We know that Def(X) is smooth by [**Pa**] and [**Mi**], thus it is enough to show that  $rank(\beta) \ge \tilde{g}$ . Using the identifications  $\Theta_X \cong \Omega_X^2 \otimes \omega_X^{-1}$  and  $\tilde{N}_f \cong \omega_{\tilde{E}} \otimes f^* \omega_X^{-1}$ , we may view  $\beta$  as a map  $\beta : H^1(X, \Omega_X^2 \otimes \omega_X^{-1}) \to H^1(E, \omega_{\tilde{E}} \otimes f^* \omega_X^{-1})$ . We consider the following commutative diagram:

We remark that injectivity of the first vertical map of the right-hand side is because of the same reason in Step 1. Thus it is enough to show that  $\operatorname{rank}(\tilde{\beta}) \geq \tilde{g}$ . Since  $h^1(\tilde{E}, \omega_{\tilde{E}}) = \tilde{g}$  (because  $\tilde{\phi}_{\tilde{E}} : \tilde{E} \to \tilde{C}$  is a  $P^1$ -bundle over  $\tilde{C}$ ), it is enough to show that  $\tilde{\beta}$  is surjective. By the Hodge symmetry, it is enough to show that  $H^2(X, \Omega_X^1) \to$  $H^2(\tilde{E}, \Omega_{\tilde{E}}^1)$  is surjective.

We consider the following 2 exact sequences,

$$0 \to \mathscr{F}_1 \to \Omega^1_X \to f_* f^* \Omega^1_X \to \mathscr{F}_2 \to 0$$

where  $\mathscr{F}_2$  has support on the singularities of E, and

$$0 \to \mathscr{F}_3 \to f^*\Omega^1_X \to \Omega^1_{\tilde{E}} \to \mathscr{F}_4 \to 0$$

where  $\mathscr{F}_4$  has support on the pinch point of  $\check{E}$ . By the second exact sequence, we have that the map  $H^2(\check{E}, f^*\Omega^1_X) \to H^2(\check{E}, \Omega^1_{\check{E}})$  is surjective. Thus it is enough to show that  $H^3(X, \mathscr{F}_1) = 0$ . By the Serre duality,  $H^3(X, \mathscr{F}_1) \cong H^0(X, \mathscr{F}_1^{\vee} \otimes \omega_X)^{\vee}$ . There is an injection

$$\phi_*\mathscr{F}_1^{\vee} \otimes \omega_X \hookrightarrow \tilde{\Omega}_{\bar{X}}^2$$

because both sheaves are isomorphic to each other outside a subset of codimension  $\geq 2$ and  $\tilde{\Omega}_{\bar{X}}^2$  is a reflexible sheaf. Thus we have that  $H^0(X, \mathscr{F}_1^{\vee} \otimes \omega_X) \subseteq H^0(\bar{X}, \tilde{\Omega}_{\bar{X}}^2) = 0$  by Lemma 3.3.

We next show that the codimension  $\geq g + d - 1$  when  $\phi$  is (g, d)-type (resp.  $\geq \tilde{g} + 2d - 1$ when  $\phi$  is  $(g, \tilde{g}, d)$ -type). If d = 0, then  $g \geq g + d - 1$  (resp.  $\tilde{g} \geq \tilde{g} + 2d - 1$ ). So we may assume  $d \neq 0$ . Define  $N_f$  by the exact sequence

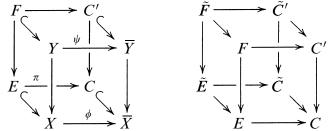
$$0 \to \mathcal{O}_{\tilde{E}} \to f^* \mathcal{O}_{\tilde{X}} \to N_f \to 0$$

as in Step 2. We remark that  $N_f = N_{E/X}$  when E is normal.

LEMMA 3.4. The homomorphism  $H^1(X, \Theta_X) \to H^1(\tilde{E}, \omega_{\tilde{E}} \otimes f^* \omega_X^{-1})$  is surjective, where the homomorphism is induced by the composition of homomorphisms

$$\Theta_X \to f^* \Theta_X \to N_f \to N_f \cong \omega_{\tilde{E}} \otimes f^* \omega_X^{-1}.$$

**PROOF** OF LEMMA. By Theorem 3.1, there exists a member  $\overline{D} \in |-2K_{\overline{X}}|$  such that  $\overline{D} \cap C \cap \{\text{dissident points}\} = \emptyset$  and  $D := \phi^* D \in |-2K_X|$  is smooth. Taking double cover of X and  $\overline{X}$  branched along D and  $\overline{D}$ , we have the following commutative diagrams.



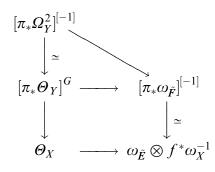
By the results of Wilson, Namikawa, and Gross of Calabi-Yau 3-folds (Cf. [Wi 1], [Wi 2], [Na 1] and [Gr 2]), we have that the homomorphism

$$H^1(Y, \Omega_Y^2) \to H^1(\tilde{F}, \omega_{\tilde{F}})$$

is surjective. We have that

$$\gamma: \omega_{\widetilde{F}} \xrightarrow{\simeq} \pi^*(\omega_{\widetilde{E}} \otimes f^* \omega_X^{-1}).$$

Let  $\sigma$  be the involution on Y. We remark that  $\sigma \circ \gamma = -\gamma$ . We have that  $(\pi_* \omega_{\tilde{F}})^{[-1]} \cong [\pi_* \pi^* (\omega_{\tilde{E}} \otimes f^* \omega_X^{-1})]^G \cong [(\omega_{\tilde{E}} \otimes f^* \omega_X^{-1}) \otimes (\mathcal{O}_{\tilde{E}} \oplus f^* \omega_X)]^G \cong \omega_{\tilde{E}} \otimes f^* \omega_X^{-1}$ . We consider the following commutative diagram.



This induces a commutative diagram

Since the upper horizontal homomorphism is surjective, so is the horizontal homomorphism at the bottom.  $\hfill \Box$ 

As in Step 1 and 2, we have a commutative diagram with an exact row

and we know that Def(X) is smooth. By Lemma 3.4, we know that  $\operatorname{Im} \beta = h^1(\tilde{E}, \omega_{\tilde{E}} \otimes f^* \omega_X^{-1})$ . Thus it is enough to show that  $h^1(\tilde{E}, \omega_{\tilde{E}} \otimes f^* \omega_X^{-1}) \ge g + d - 1$  (resp.  $\tilde{g} + 2d - 1$ ).

As in Step 1, we can show that there exists a member  $S \in |-K_X|$  such that  $F = f^*S = \sum_{i=1}^d F_i$  (resp.  $\sum_{i=1}^{2d} F_i$ ), where  $F_i \cong \mathbf{P}^1$  is a fiber of  $\tilde{E}$  over  $\tilde{C}$  for each *i*, and  $F_i \cap F_j = \emptyset$  if  $i \neq j$ . Since  $\omega_{\tilde{E}} \otimes f^* \omega_X^{-1}|_{F_i} \cong \omega_{F_i}$ , we have an exact sequence induced by the choice of S

$$0 \to \omega_{\tilde{E}} \to \omega_{\tilde{E}} \otimes f^* \omega_X^{-1} \to \bigoplus_{i=1}^d \omega_{F_i} \to 0.$$

$$\left(\text{resp. } 0 \to \omega_{\tilde{E}} \to \omega_{\tilde{E}} \otimes f^* \omega_X^{-1} \to \bigoplus_{i=1}^{2d} \omega_{F_i} \to 0.\right)$$

This induces the following exact sequence:

$$\begin{split} \bigoplus_{i=1}^{d} H^{0}(F_{i},\omega_{F_{i}}) &\to H^{1}(\tilde{E},\omega_{\tilde{E}}) \to H^{1}(\tilde{E},\omega_{\tilde{E}} \otimes f^{*}\omega_{X}^{-1}) \\ &\to \bigoplus_{i=1}^{d} H^{1}(F_{i},\omega_{F_{i}}) \to H^{2}(\tilde{E},\omega_{\tilde{E}}). \end{split}$$

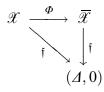
$$\begin{pmatrix} \operatorname{resp.} \ \bigoplus_{i=1}^{2d} H^{0}(F_{i},\omega_{F_{i}}) \to H^{1}(\tilde{E},\omega_{\tilde{E}}) \to H^{1}(\tilde{E},\omega_{\tilde{E}} \otimes f^{*}\omega_{X}^{-1}) \\ &\to \bigoplus_{i=1}^{2d} H^{1}(F_{i},\omega_{F_{i}}) \to H^{2}(\tilde{E},\omega_{\tilde{E}}). \end{pmatrix}$$

$$\to \bigoplus_{i=1}^{2d} H^{1}(F_{i},\omega_{F_{i}}) \to H^{2}(\tilde{E},\omega_{\tilde{E}}). \end{split}$$

Since  $h^0(F_i, \omega_{F_i}) = 0$ ,  $h^1(F_i, \omega_{F_i}) = 1$ ,  $h^1(\tilde{E}, \omega_{\tilde{E}}) = g$  (resp.  $=\tilde{g}$ ), and  $h^2(\tilde{E}, \omega_{\tilde{E}}) = 1$ , Proposition 3.2 follows from this exact sequence.

By Proposition 3.2, *E* will not deform under a generic deformation of *X*, unless  $\phi$  is a contraction of (0,0), (0,1),  $(0,\tilde{g},0)$ , or  $(0,\tilde{g},1)$ -type. Thus there exists a deformation of  $\phi$  which is a crepant primitive birational contraction of type I. We want to count the number of curves contracted by the contraction of type I in the following proposition.

**PROPOSITION 3.5.** Let X be a smooth weak Fano 3-fold and  $\phi : X \to \overline{X}$  a crepant primitive birational contraction of type III. Assume that  $\phi$  is neither (0,0), (0,1),  $(0,\tilde{g},0)$ , nor  $(0,\tilde{g},1)$ -type. Then there exists a small deformation of  $\phi$  over  $(\varDelta,0)$ 



such that, for any  $t \in (\Delta, 0) \setminus \{0\}$ ,

$$\Phi_t: \mathscr{X}_t \to \overline{\mathscr{X}}_t$$

is a crepant primitive contraction of type I which contracts

- (i) just  $2g 2 + d\mathbf{P}^{1}$ 's when  $\phi$  is a contraction of (g, d)-type without dissident fibers with  $d \ge 2$ .
- (ii) at least  $2\tilde{g} 2 + 2d\mathbf{P}^{1}$ 's when  $\phi$  is a contraction of  $(g, \tilde{g}, d)$ -type.
- (iii)  $l\mathbf{P}^{1}$ 's where  $2g 2 \le l \le 2g 1$  when  $\phi$  is a contraction of (g, 1)-type without dissident fibers.
- (iv) a single  $\mathbf{P}^1$  when  $\phi$  is a contraction of (1,1)-type without dissident fibers.

PROOF. We will divide the proof into 3 cases.

Case 1 ( $\phi$  is of (g, d)-type without dissident fibers).

Let  $Z \cong \mathbf{P}^1$  be any fiber of  $\phi|_E$  over  $p \in C$  and  $i: Z \hookrightarrow X$  the natural closed embedding. We have that  $N_{Z/X} \cong N_{Z/E} \oplus (N_{E/X}|_Z)$  where  $N_{Z/E} \cong \mathcal{O}_Z$ . We consider the following commutative diagram:

We remark that the upper horizontal sequence is exact as in [**Ra**]. Thus for  $\eta \in T_X^1$ , Z extends sideways to first order in the first order deformation corresponding to  $\eta$  if and only if  $\tau \circ \beta(\eta) = 0$ . Using the identification  $N_{E/X} \cong \omega_E \otimes \omega_X^{-1}$ , we may view  $\tau$  as a map

$$\tau: H^1(E, \omega_E \otimes \omega_X^{-1}) \to H^1(\omega_E \otimes \omega_X^{-1}|_Z).$$

By the relative duality, we know that

$$R^{1}\phi_{*}\omega_{E} \cong R^{1}\phi_{*}(\omega_{E/C} \otimes \phi^{*}\omega_{C}) \cong (R^{1}\phi_{*}\omega_{E/C}) \otimes \omega_{C} \cong \omega_{C}$$

Thus we have the isomorphisms

$$R^{1}\phi_{*}(\omega_{E}\otimes\omega_{X}^{-1})\cong R^{1}\phi_{*}(\omega_{E}\otimes\phi^{*}(\omega_{\overline{X}}^{-1}|_{C}))\cong (R^{1}\phi_{*}\omega_{E})\otimes\omega_{\overline{X}}^{-1}\cong\omega_{C}\otimes\omega_{\overline{X}}^{-1}.$$

Considering the Leray spectral sequence, we have an identification

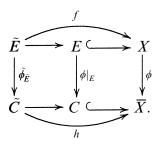
$$\begin{array}{cccc} H^{1}(E, \omega_{E} \otimes \omega_{X}^{-1}) & \stackrel{\tau}{\longrightarrow} & H^{1}(Z, \omega_{E} \otimes \omega_{X}^{-1}|_{Z}) \\ & & & \downarrow \simeq & \\ & & & \downarrow \simeq & \\ H^{0}(C, \omega_{C} \otimes \omega_{\overline{X}}^{-1}) & \stackrel{\overline{\tau}}{\longrightarrow} & H^{0}(p, \omega_{C} \otimes \omega_{\overline{X}}^{-1}|_{p}). \end{array}$$

By the assumption, there exists an element  $\xi \in H^0(C, \omega_C \otimes \omega_{\overline{X}}^{-1})$  such that  $\eta \neq 0$ . Then there exists an element  $\eta \in T_X^1$  such that  $\beta(\eta) = \xi$  by Lemma 3.4. From the above argument, the fiber over a point where  $\overline{\tau}(\xi)$  vanishes extends sideways to first order in the first order deformation corresponding to  $\eta$ . Because  $\overline{\tau}(\xi)$  vanishes at 2g - 2 + dpoints,  $2g - 2 + d\mathbf{P}^1$ 's extends sideways in a general first order deformation.

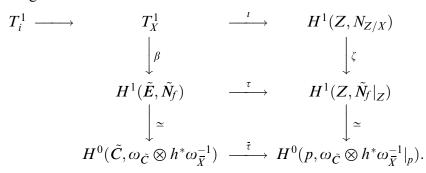
Next, we consider the obstruction of the morphism  $i: Z \hookrightarrow X$ . We consider the following commutative diagram:

We remark that the upper horizontal sequence is exact. If  $d \neq 1$ , the linear system  $|K_C - (K_{\bar{X}}|_C)|$  is base-point free. Thus  $\bar{\tau}$  is surjective. We know that  $\beta$  is surjective by Lemma 3.4, thus  $\iota$  is surjective. If d = 1, in this case  $g \neq 0$  by the assumption, we have that  $|K_C - (K_{\bar{X}}|_C)| = |K_C| + q$  where  $q = -K_{\bar{X}} \cdot C$ . Thus  $\bar{\tau}$  is surjective if  $p \neq q$ . This completes the proof in this case except (iv).

Case 2 ( $\phi$  is of  $(g, \tilde{g}, d)$ -type). Let  $\tilde{E}$  be the normalization of E, and  $\tilde{E} \to \tilde{C} \to C$  the Stein factorization. We consider the following commutative diagram:



Let  $Z \cong \mathbf{P}^1$  be any fiber of  $\tilde{\phi}_{\tilde{E}}$  over  $p \in \tilde{C}$  and  $i: Z \hookrightarrow X$  be the natural closed embedding. There is a natural morphism  $N_{Z/X} \to \tilde{N}_f|_Z$  for any fiber Z. We remark that  $N_{Z/X} \cong N_{Z/\tilde{E}} \oplus (\tilde{N}_f|_Z)$  where  $N_{Z/\tilde{E}} \cong \mathcal{O}_Z$  for a general fiber Z. We consider the following commutative diagram:



We remark that  $\zeta$  is an isomorphism for a general fiber Z and we can show that the lower vertical arrows are isomorphisms by the similar reason in Step 1. The above commutative diagram tells us that Z will not extend in a first order deformation corresponding to  $\eta$  if  $\overline{\tau} \circ \beta(\eta) \neq 0$  for any fiber Z, and that Z extends sideways to first order in the first order deformation corresponding to  $\eta$  if  $\overline{\tau} \circ \beta(\eta) = 0$  for a general fiber Z. Since the degree of  $K_{\tilde{C}} + h^*(-K_{\bar{X}})$  is  $2\tilde{g} - 2 + 2d$  and  $d \neq 0$ ,  $|K_{\tilde{C}} + f^*(-K_{\bar{X}})|$  is basepoint free. Thus we can show this proposition by the same method in Step 1 in this case. We remark that we only considered a fiber of  $\tilde{\phi}_{\tilde{E}}$  in this proof, thus we need "at least" in this statement.

Case 3 ((iv) of this proposition). If any fiber will not deform, then the Kähler cone of X is not locally constant at  $0 \in Def(X)$ . But it contradicts Page 63 of [Pa] and Theorem 3.1.

**PROPOSITION 3.6.** Let X be a smooth weak Fano 3-fold, and  $\phi: X \to \overline{X}$  a crepant primitive birational contraction of type III. Assume that  $\phi$  is a contraction of (g, d)-type with dissident fibers and is neither (0,0) nor (0,1)-type. Then there exists a small deformation of  $\phi$  over  $(\Delta, 0)$ 



such that, for any  $t \in (\Delta, 0) \setminus \{0\}$ ,

 $\Phi_t: \mathscr{X}_t \to \overline{\mathscr{X}}_t$ 

is a crepant primitive birational contraction of type I which is not a contraction of a single  $P^1$  to an ordinary double point.

PROOF. We can show this proposition by the same method in the proof of Theorem 1.3 of [Gr 2].

**PROPOSITION 3.7.** Let X be a smooth weak Fano 3-fold,  $\phi: X \to \overline{X}$  a crepant primitive birational contraction of type III, and  $\mathscr{X} \to Def(X)$  the Kuranishi family of X. Assume that  $\phi$  is a contraction of (0,0), (0,1),  $(0,\tilde{g},0)$ , or  $(0,\tilde{g},1)$ -type. Then E will deforms in the family.

**PROOF.** The case d = 0 was proved by Paoletti (cf. the proof of Lemma 3.6 of **[Pa]**), thus we may assume d = 1.

We first treat the case  $\phi$  is of (0,1)-type. When  $\phi$  is without dissident fibers,  $h^1(E, \mathcal{O}_E(E)) = h^1(E, \omega_E \otimes \omega_X^{-1}) = h^0(C, \omega_C \otimes \omega_{\overline{X}}^{-1}) = 0$ . Consider the exact sequence in Step 1 of Proposition 3.6,

$$0 \to \omega_E \to \omega_E \otimes \omega_X^{-1} \to \omega_F \to 0$$

where  $F \cong \mathbf{P}^1$ . This induces a long exact sequence

$$0 = H^{1}(E, \omega_{E} \otimes \omega_{X}^{-1}) \to H^{1}(F, \omega_{F}) \to H^{2}(E, \omega_{E})$$
$$\to H^{2}(E, \omega_{E} \otimes \omega_{X}^{-1}) \to 0.$$

Thus we can show that  $h^2(E, \mathcal{O}_E(E)) = h^2(E, \omega_E \otimes \omega_X^{-1}) = 0$ . Thus *E* will deform in the Kuranishi family of *X*.

When *E* is of (0, 1)-type with dissident fibers. As in the case *E* is (0, 1)-type without dissident fibers, we can prove that  $h^2(E, \mathcal{O}_E(E)) = 0$  if  $h^1(E, \mathcal{O}_E(E)) = 0$ . So it is enough to show that  $h^1(E, \mathcal{O}_E(E)) = 0$ . We can consider the following commutative diagram:

where  $\mu: \hat{E} \to E$  is the minimal resolution of E, and  $\phi': E' \to C$  is a  $P^1$ -bundle. Since  $\omega_{\hat{E}} \simeq \mu^* \omega_E$ , we have that

$$H^1(\hat{E}, \omega_{\hat{E}} \otimes \mu^* \omega_X^{-1}) = H^1(\hat{E}, \mu^*(\omega_E \otimes \omega_X^{-1})).$$

Since  $\mu_*\mu^*(\omega_E \otimes \omega_X^{-1}) \simeq \omega_E \otimes \omega_X^{-1}$ , we have that

$$H^{1}(E, \mathcal{O}_{E}(E)) \cong H^{1}(E, \omega_{E} \otimes \omega_{X}^{-1}) \hookrightarrow H^{1}(\hat{E}, \omega_{\hat{E}} \otimes \mu^{*} \omega_{X}^{-1})$$

by the Leray spectral sequence.

Thus it is enough to show that

$$H^1(\hat{E}, \omega_{\hat{E}} \otimes \mu^* \omega_X^{-1}) \cong H^1(\hat{E}, \mu^* \omega_X) = 0.$$

Since  $\phi^* \omega_{\overline{X}} \simeq \omega_X$ , we have that  $v^* (\phi')^* \omega_{\overline{X}} \simeq \mu^* \phi^* \omega_{\overline{X}} \simeq \mu^* \omega_X$ , thus

$$H^1(\hat{E}, \mu^* \omega_X) = H^1(\hat{E}, \nu^* (\phi')^* \omega_{\overline{X}}).$$

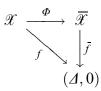
Since  $v_*v^*((\phi')^*\omega_{\overline{\chi}}) \simeq (\phi')^*\omega_{\overline{\chi}}$ , we have that

$$H^1(E',(\phi')^*\omega_{\overline{X}}) \cong H^1(\hat{E},\nu^*(\phi')^*\omega_{\overline{X}})$$

by the Leray spectral sequence. As is the case  $\phi$  is a (0, 1)-type without dissident fibers, we can show that

$$H^1(E',(\phi')^*\omega_{\overline{X}}) \cong H^1(E',\omega_{E'}\otimes(\phi')^*\omega_{\overline{X}}^{-1}) = 0.$$

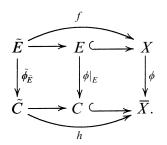
Thus the rest is the case  $\phi$  is of  $(0, \tilde{g}, 1)$ -type. If there exists a small deformation of  $\phi$  over  $(\Delta, 0)$ 



such that, for any  $t \in (\Delta, 0) \setminus \{0\}$ ,

 $\Phi_t:\mathscr{X}_t\to\overline{\mathscr{X}}_t$ 

is a crepant primitive birational contraction of type I. We consider the following commutative diagram as in Case 2 of the proof of Proposition 3.6:



Then  $\Phi_t$  contracts at least  $2\tilde{g}P^1$ 's for  $t \in (\varDelta, 0) \setminus \{0\}$  which are deformations of fibers of  $\tilde{\phi}_{\tilde{E}}$ , and these fibers are chosen generically in fibers of  $\tilde{\phi}_{\tilde{E}}$  as in Case 2 of the proof of Proposition 3.6. We remark that the morphism  $\tilde{C} \to C$  is a finite morphism branched over at least 2 points on C by Hurwitz formula, and these points on C are dissident points. By [**Pa**], we know that the fiber of  $\tilde{\phi}_{\tilde{E}}$  whose image by  $\phi \circ f$  is a dissident point deforms in the Kuranishi family of X. It is a contradiction.

Combining these propositions and Theorem 1.2, we can prove Theorem 0.9.

## References

- [Gr 1] M. Gross, Deforming Calabi-Yau threefolds, Math. Ann., 308 (1997), 187-220.
- [Gr 2] M. Gross, Primitive Calabi-Yau threefolds, J. Differential Geom., 45 (1997), 288-318.
- [Ka] Y. Kawamata, Minimal models and the Kodaira dimension of algebraic fiber spaces, J. Reine Angew. Math., 363 (1985), 1–46.
- [Mi] T. Minagawa, Deformations of weak Fano 3-folds with only terminal singularities (1999), To appear in Osaka J. Math.
- [Mu] S. Mukai, Gorenstein Fano threefolds, Proceedings of Alg. Geom. Symposium, Saitama, 1994, 87–90.
- [Na 1] Y. Namikawa, On deformations of Calabi-Yau 3-folds with terminal singularities, Topology, 33 (1994), 429-446.
- [Na 2] Y. Namikawa, Stratified local moduli of Calabi-Yau threefolds, Preprint, 1995.
- [Na 3] Y. Namikawa, Smoothing Fano 3-folds, J. Algebraic Geom., 6 (1997), 307-329.
- [Na 4] Y. Namikawa, Deformation theory of Calabi-Yau threefolds and certain invariants of singularities, J. Algebraic Geom., 6 (1997), 753–776.
- [Na-St] Y. Namikawa and J. Steenbrink, Global smoothing of Calabi-Yau threefolds, Invent. Math., 122 (1995), 403–419.
- [Pa] R. Paoletti, The Kähler cone in families of quasi-Fano threefolds, Math. Z., 227 (1998), 45-68.
- [Ra] Z. Ran, Deformations of maps, Alg. curves and proj. geom., C. Ballico and C. Ciliberto Eds. LMN1389, Springer-Verlag, 1989, 246–253.

- [Re] M. Reid, Projective morphism according to Kawamata, Preprint.
- [Ta] H. Takagi, On classifications of *Q*-Fano 3-folds with Gorenstein index 2 and Fano index 1/2, Preprint, 1999.
- [Wi 1] P. M. H. Wilson, The Kähler cone on Calabi-Yau threefolds, Invent. Math., 107 (1992), 561-583.
- [Wi 2] P. M. H. Wilson, Erratum to "The Kähler cone on Calabi-Yau threefolds", Invent. Math., 114 (1993), 231–233.
- [Wi 3] P. M. H. Wilson, Symplectic deformations of Calabi-Yau threefolds, J. Differential Geom., 45 (1997), 611–637.

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