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Cobordism group of Morse functions on surfaces

Dedicated to Professor Tatsuo Suwa on his sixtieth birthday

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Abstract. We define the cobordism group of Morse functions on manifolds and show that it is an infinite cyclic group for dimension two in the oriented case. We also give an explicit Morse function which gives a generator of the group.

1. Introduction.

The purpose of this paper is to determine the structure of the 2-dimensional oriented cobordism group of Morse functions.

Let us recall a brief history of the cobordism theory of smooth maps. Thom [**18**] described the cobordism groups of embeddings in terms of homotopy groups of certain spaces, using the so-called Pontrjagin-Thom construction. Wells [**19**] defined the cobordism groups of immersions and studied them again by using a Pontrjagin-Thom type construction. Eliashberg [**3**] generalized Wells' result to cobordism groups of smooth maps satisfying certain differential relations of order one. Cobordism groups of smooth maps with a given set of local and global singularities were introduced and studied by Rimányi and Szűcs [**14**], who showed that these groups are isomorphic to the homotopy groups of certain "universal spaces", where the isomorphisms are obtained again by a Pontrjagin-Thom type construction. Recall that they considered only the non-negative codimension case, i.e., the case where the dimension of the target is greater than or equal to that of the source.

In this paper, as a simple but important example for the strictly negative codimension case, we study the cobordism group $\mathscr{M}(2)$ of Morse functions on oriented surfaces and show that it is an infinite cyclic group, using a totally different method. Note that the group $\mathscr{M}(2)$ corresponds to the 2-dimensional oriented cobordism group for fold singularities of codimension -1 in a sense similar to that of [**14**].

Recently Ando [**1**], [**2**] studied the n-dimensional oriented cobordism group for fold singularities of codimension zero. Our method is totally different from Ando's and uses the notion of the Stein factorization. The second author [**15**], [**16**] studied the cobordism group of Morse functions with only minima and maxima as their critical points using the Stein factorizations, which are manifolds for such functions. Our situation admitting critical points of any index is more complicated, since the Stein factorizations are no longer manifolds.

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The paper is organized as follows. In §2, we give a precise definition of the (fold) cobordism group of Morse functions on n-dimensional manifolds and state our main theorem. We also define the Stein factorization of a Morse function and introduce the notion of a Reeb function, which is a function on a certain graph. In §3, we define and study the cobordism group of Reeb functions from a graph theoretical viewpoint. Our approach is to reduce the cobordism relation to certain moves for functions on graphs. In §4, we complete the proof of our main theorem by using a method of Mata-Lorenzo [**12**] for realizing a certain 2-dimensional polyhedron as the Stein factorization of a generic map of a 3-manifold into the plane.

Throughout the paper, all manifolds and maps are of class C^∞. The symbol "$\cong$" denotes an appropriate isomorphism between algebraic objects. For a space X, id_X denotes the identity map of X.

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2. Preliminaries.

A smooth real-valued function on a smooth manifold is called a *Morse function* if its critical points are all non-degenerate. Note that its restriction to the set of critical points may not necessarily be injective in general. For a positive integer n, we denote by $M(n)$ the set of all Morse functions on closed (possibly disconnected) oriented n-dimensional manifolds. We adopt the convention that the function on the empty set $\varnothing$ is an element of $M(n)$ for all n.

Before defining the cobordism group of Morse functions, let us recall the notion of fold singularities. Let $f : M \to N$ be a smooth map between smooth manifolds with $n = \dim M \geq \dim N = p$. A *singular point* of f is a point $q \in M$ such that the differential $df_q : T_qM \to T_{f(q)}N$ has rank strictly smaller than p. We denote by $S(f)$ the set of all singular points of f and call it the *singular set* of f. A singular point $q \in S(f)$ is a *fold point* if there exist local coordinates $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_p)$ around q and $f(q)$ respectively such that f has the form

$$y_i \circ f = \begin{cases} x_i, & 1 \leq i \leq p-1, \\ \pm x_p^2 \pm x_{p+1}^2 \pm \cdots \pm x_n^2, & i = p. \end{cases}$$

If the signs appearing in $y_p \circ f$ all coincide, then we say that q is a *definite fold point*, otherwise an *indefinite fold point*. We denote by $S_0(f)$ (or $S_1(f)$) the set of all definite (resp. indefinite) fold points of f.

Definition 2.1. Two Morse functions $f_0 : M_0 \to \boldsymbol{R}$ and $f_1 : M_1 \to \boldsymbol{R}$ in $M(n)$ are said to be *cobordant* (or *fold cobordant*) if there exist a compact oriented $(n+1)$-dimensional manifold X and a smooth map $F : X \to \boldsymbol{R} \times [0,1]$ which has only fold points as its singularities such that

(1) the oriented boundary ∂X of X is the disjoint union $M_0 \amalg (-M_1)$, where $-M_1$ denotes the manifold M_1 with the orientation reversed, and

(2) we have

$$F|_{M_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)} : M_0 \times [0,\varepsilon) \to \boldsymbol{R} \times [0,\varepsilon), \quad \text{and}$$

$$F|_{M_1 \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]} : M_1 \times (1-\varepsilon,1] \to \boldsymbol{R} \times (1-\varepsilon,1]$$

for some sufficiently small $\varepsilon > 0$, where we identify the collar neighborhoods of M_0 and M_1 in X with $M_0 \times [0,\varepsilon)$ and $M_1 \times (1-\varepsilon,1]$ respectively.

In this case, we call F a *cobordism* between f_0 and f_1.

If a Morse function in $M(n)$ is cobordant to the function on the empty set, we say that it is *null-cobordant*.

It is easy to show that the above relation defines an equivalence relation on the set $M(n)$ for each n. Furthermore, it is easy to see that the set of all equivalence classes forms an additive group under the disjoint union: the neutral element is the class corresponding to null-cobordant Morse functions, and the inverse of a class represented by a Morse function $f : M \to \boldsymbol{R}$ is given by the class of $-f : -M \to \boldsymbol{R}$. We denote by $\mathscr{M}(n)$ the set of all (fold) cobordism classes of elements of $M(n)$ and call it the *cobordism group of Morse functions* (or *fold cobordism group of Morse functions*) *on oriented manifolds of dimension n*, or the *n-dimensional oriented cobordism group of Morse functions*.

Remark 2.2. Let M be a closed oriented n-dimensional manifold. It is easy to see that if two Morse functions f and g on M are connected by a one-parameter family of Morse functions, then they are cobordant. In particular, every Morse function is cobordant to a Morse function whose critical values are all distinct. Such a Morse function is said to be *stable*. For details, see [**5**, Chapter III, §2].

Our main result of this paper is the following.

Theorem 2.3. *The* 2*-dimensional oriented cobordism group of Morse functions* $\mathscr{M}(2)$ *is an infinite cyclic group.*

We will also give an explicit example of a Morse function which gives a generator of $\mathscr{M}(2)$ (see Figure 10 in §4).

In the subsequent sections, we will study the group structure of $\mathscr{M}(2)$ using the following notion of Stein factorizations (for more details, see [**11**], for example).

Definition 2.4. Suppose that a smooth map $f : M \to N$ with $n = \dim M \geq \dim N = p$ is given. Two points in M are *equivalent* if they lie on the same component of an f-fiber. Let W_f denote the quotient space of M with respect to this equivalence relation and $q_f : M \to W_f$ the quotient map. Then it is easy to see that there exists a continuous map $\bar{f} : W_f \to N$ such that $f = \bar{f} \circ q_f$. The space W_f or the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ & {\scriptstyle q_f}\searrow \quad \nearrow{\scriptstyle \bar{f}} & \\ & W_f & \end{array}$$

is called the *Stein factorization* of f.

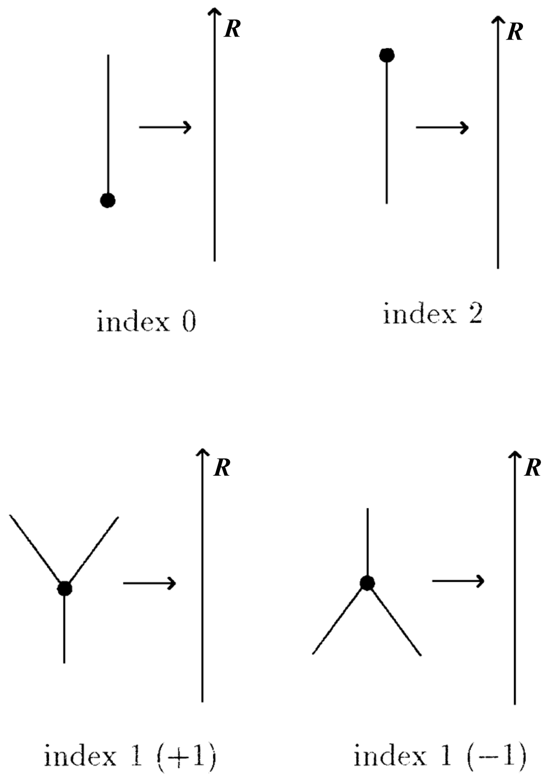

Figure 1. Behavior of $\bar{f}$ around each vertex of the Reeb graph W_f.

If $f : M \to \boldsymbol{R}$ is a Morse function on a closed manifold M, then W_f has the natural structure of a 1-dimensional CW complex. In this case, we call W_f the *Reeb graph* of f (for example, see [**4**]). Furthermore, we call the continuous map $\bar{f} : W_f \to \boldsymbol{R}$ a *Reeb function*.

3. Cobordism group of Reeb functions.

In this section, we study the Reeb graphs and Reeb functions associated with Morse functions on surfaces. Let us begin by defining the following class of Morse functions.

Definition 3.1. Let $f : M \to \boldsymbol{R}$ be a Morse function on a closed oriented surface M. We say that f is *simple* if every component of an arbitrary f-fiber contains at most one critical point. Note that a stable Morse function is always simple.

The following lemma can easily be proved (for example, see [**17**, §3]).

Lemma 3.2. *Let $f : M \to \boldsymbol{R}$ be a simple Morse function on a closed orientable surface M. Then its Reeb graph W_f is a finite graph whose vertices are the q_f-images of the critical points of f such that*

(1) *the vertices corresponding to critical points of index* 0 *or* 2 *have degree* 1, *and those of index* 1 *have degree* 3,

and the Reeb function $\bar{f} : W_f \to \boldsymbol{R}$ satisfies the following:

(2) *around each vertex of W_f, $\bar{f}$ is equivalent to one of the functions as depicted in Figure* 1, *and*

(3) *$\bar{f}$ is an embedding on each edge.*

To each vertex of degree three of the Reeb graph we associate the sign $+1$ or -1 as in Figure 1.

Let us consider an abstract generalization of the Reeb function $\bar{f} : W_f \to \boldsymbol{R}$ appearing in the Stein factorization of a simple Morse function on a closed orientable surface.

Definition 3.3. Let G be a finite graph which may possibly be disconnected, and $r : G \to \boldsymbol{R}$ a continuous function such that

(1) the degree of each vertex of G is equal either to 1 or to 3,

(2) around each vertex of G, r is equivalent to one of the functions as depicted in Figure 1, and

(3) r is an embedding on each edge.

To each vertex of degree three we associate the sign $+1$ or -1 as in Figure 1. We call G an *abstract Reeb graph* and $r : G \to \boldsymbol{R}$ an *abstract Reeb function*. We adopt the convention that the empty graph and the function on it are also an abstract Reeb graph and an abstract Reeb function, respectively.

By Lemma 3.2, the Reeb graph W_f and the Reeb function $\bar{f} : W_f \to \boldsymbol{R}$ associated with a simple Morse function f on a closed orientable surface are an abstract Reeb graph and an abstract Reeb function, respectively.

Let $f_0 : M_0 \to \boldsymbol{R}$ and $f_1 : M_1 \to \boldsymbol{R}$ be cobordant simple Morse functions on closed oriented surfaces and $F : X \to \boldsymbol{R} \times [0,1]$ a cobordism between them. By slightly changing F in the interior of X, we may assume that $F^{-1}(\boldsymbol{R} \times ([0,\varepsilon) \cup (1-\varepsilon, 1]))$ coincides with the collar neighborhood of ∂X as in Definition 2.1 (2). Note that X is a compact oriented 3-manifold with boundary and that F has only fold points as its singularities. Hence, the singular set $S(F)$ is a compact 1-dimensional manifold properly embedded in X and $F|_{S(F)}$ is an immersion. By perturbing F slightly on the complement of the collar neighborhood of ∂X, we may assume that $F|_{S(F)}$ is an immersion with normal crossings outside of the closure of the collar neighborhood (for details, see [**5**]). In particular, each F-fiber contains at most two singular points. Therefore, for a point $x \in q_F(S(F))$, $q_F^{-1}(x) (\subset F^{-1}(\bar{F}(x)))$ contains one or two singular points, where q_F and $\bar{F}$ are the maps appearing in the Stein factorization of F:

$$\begin{array}{ccc} X & \xrightarrow{F} & \boldsymbol{R} \times [0,1] \\ & {\scriptstyle q_F}\searrow \quad \nearrow{\scriptstyle \bar{F}} & \\ & W_F. & \end{array}$$

In the former case, we call x a *simple point*, otherwise a *non-simple point*.

Combining Lemma 3.2 and results of Kushner, Levine and Porto [**10**], [**11**], we have the following.

Lemma 3.4. *The Stein factorization W_F is a compact 2-dimensional polyhedron. Furthermore, around each point $x \in W_F$, the map $\bar{F} : W_F \to \boldsymbol{R} \times [0,1]$ is equivalent to one of the maps as depicted in Figure 2, where the maps in question are vertical projections to a horizontal plane and they correspond to the following cases.*

(1) $x \in q_F(\partial X) \setminus q_F(S(F))$. (2) $x \in q_F(\operatorname{Int} X) \setminus q_F(S(F))$.

(3) $x \in q_F(\partial X \cap S_0(F))$. (4) $x \in q_F(\operatorname{Int} X \cap S_0(F))$.

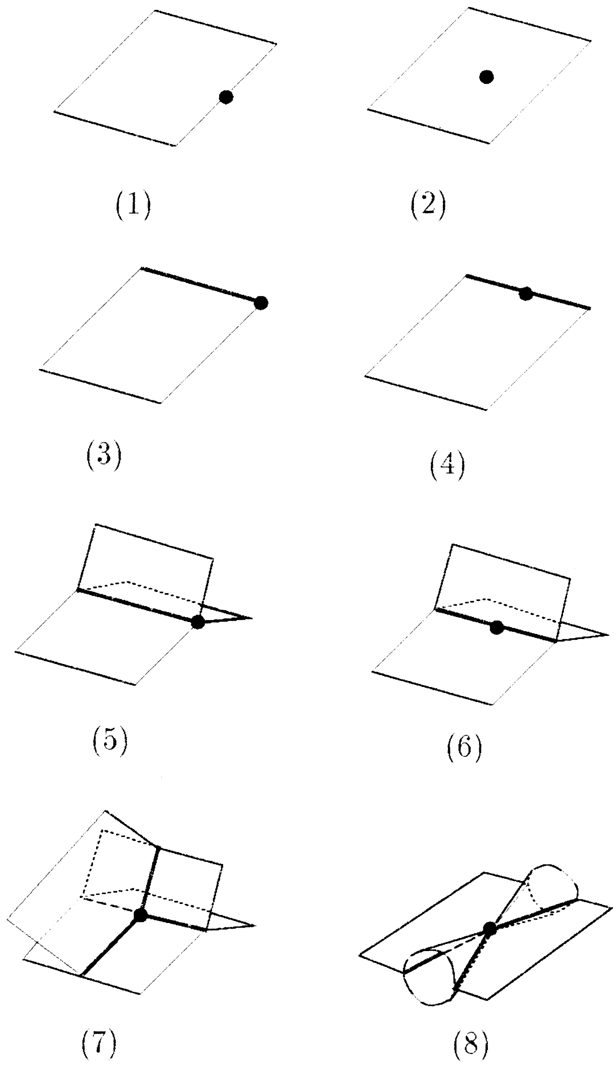

Figure 2. Behavior of $\bar{F} : W_F \to \boldsymbol{R} \times [0,1]$ around each point of W_F.

(5) $x \in q_F(\partial X \cap S_1(F))$. (6) $x \in q_F(\mathrm{Int}\, X \cap S_1(F))$ *and* x *is simple.*
(7), (8) $x \in q_F(\mathrm{Int}\, X \cap S_1(F))$ *and* x *is non-simple.*
Furthermore, the thick lines indicate $q_F(S(F))$.

By composing F with a diffeomorphism $\boldsymbol{R} \times [0,1] \to \boldsymbol{R} \times [0,1]$ which is the identity on $\boldsymbol{R} \times ([0,\varepsilon] \cup [1-\varepsilon, 1])$, we may arrange so that $\pi_2 \circ F|_{S(F)}$ is a Morse function with finitely many critical points whose F-images are not self-intersections of $F|_{S(F)}$, where $\pi_2 : \boldsymbol{R} \times [0,1] \to [0,1]$ is the projection to the second factor. For $t \in [0,1]$ which is not a critical value of the Morse function $\pi_2 \circ F|_{S(F)}$, $M_t = (\pi_2 \circ F)^{-1}(t)$ is a closed oriented surface and $\pi_1 \circ F|_{M_t} : M_t \to \boldsymbol{R}$ is a Morse function, where $\pi_1 : \boldsymbol{R} \times [0,1] \to \boldsymbol{R}$ is the projection to the first factor. By carefully examining the changes in the Reeb function of $\pi_1 \circ F|_{M_t}$ as t varies from 0 to 1, we easily obtain the following.

Corollary 3.5. *If two simple Morse functions* $f_0 : M_0 \to \boldsymbol{R}$ *and* $f_1 : M_1 \to \boldsymbol{R}$ *on closed oriented surfaces are cobordant, then the Reeb function* $\bar{f}_1 : W_{f_1} \to \boldsymbol{R}$ *is obtained from* $\bar{f}_0 : W_{f_0} \to \boldsymbol{R}$ *by a finite iteration of moves as depicted in Figure* 3 *up to homotopy*

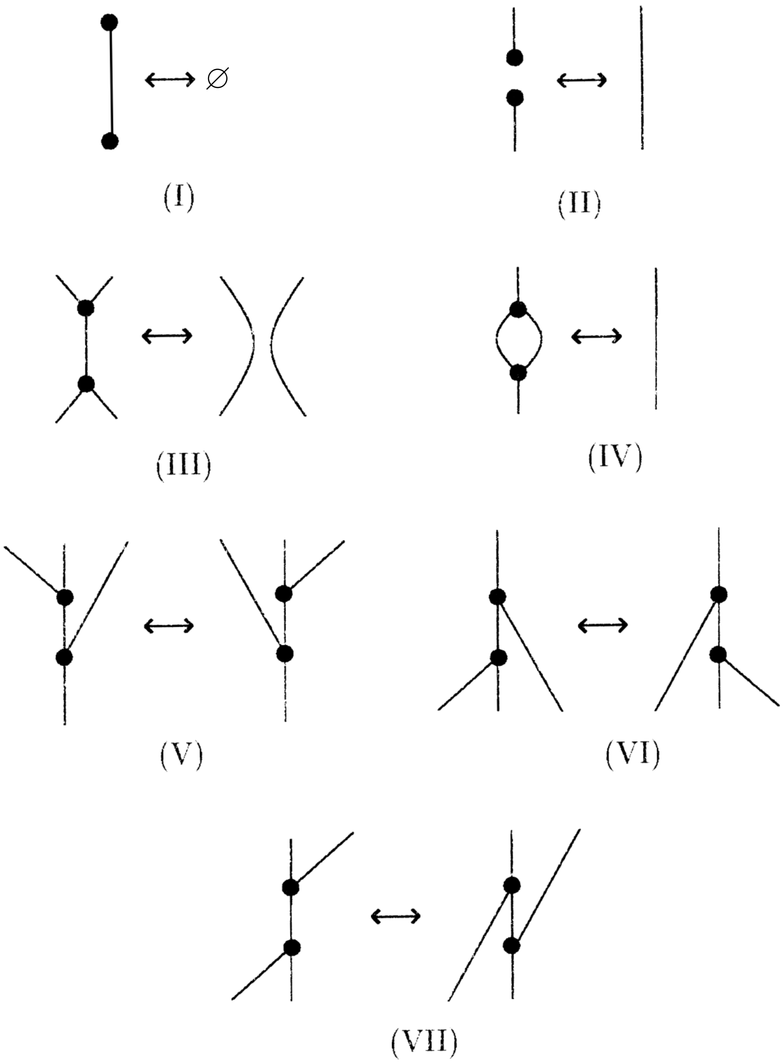

Figure 3. Seven moves for Reeb functions.

in the space of abstract Reeb functions, where the functions in question are the height functions.

Details of the proof of the above corollary are left to the reader.

Definition 3.6. Let $r_i : G_i \to \boldsymbol{R}$, $i = 0, 1$, be abstract Reeb functions. We say that r_0 and r_1 are *cobordant* if there exist a 2-dimensional compact polyhedron P and a continuous map $R : P \to \boldsymbol{R} \times [0, 1]$ such that

(1) $G_i = R^{-1}(\boldsymbol{R} \times \{i\})$, $i = 0, 1$, and they are subcomplexes of P with regular neighborhoods of the forms $G_0 \times [0, \varepsilon]$ and $G_1 \times [1 - \varepsilon, 1]$, respectively, for some sufficiently small $\varepsilon > 0$, where G_0 corresponds to $G_0 \times \{0\}$ and G_1 to $G_1 \times \{1\}$,

(2) we have

$$R|_{G_0 \times [0,\varepsilon)} = r_0 \times \mathrm{id}_{[0,\varepsilon)} : G_0 \times [0, \varepsilon) \to \boldsymbol{R} \times [0, \varepsilon), \quad \text{and}$$

$$R|_{G_1 \times (1-\varepsilon, 1]} = r_1 \times \mathrm{id}_{(1-\varepsilon, 1]} : G_1 \times (1 - \varepsilon, 1] \to \boldsymbol{R} \times (1 - \varepsilon, 1],$$

and

(3) around each point of P, the polyhedron P and the map R is equivalent to one of the maps as depicted in Figure 2.

In this case, we call R a *cobordism* between r_0 and r_1. Furthermore, the set $\Sigma(P)$ of the points in P which have regular neighborhoods as in Figure 2 (3)–(8) is called the *singular set* of P, and the points in P which have regular neighborhoods as in Figure 2 (7) and (8) are called *non-simple vertices* of P.

Note that by Lemma 3.4, Reeb functions associated with cobordant simple Morse functions are cobordant in the above sense.

If an abstract Reeb function is cobordant to the function on the empty graph, we say that it is *null-cobordant*.

It is easy to show that the above relation defines an equivalence relation on the set of abstract Reeb functions. Furthermore, it is easy to see that the set of all equivalence classes forms an additive group under the disjoint union: the neutral element is the class corresponding to null-cobordant abstract Reeb functions, and the inverse of a class represented by an abstract Reeb function $r : G \to \boldsymbol{R}$ is given by the class of $-r : G \to \boldsymbol{R}$. We denote by $\mathscr{R}$ the set of all cobordism classes of abstract Reeb functions and call it the *cobordism group of Reeb functions*.

By an argument similar to the proof of Corollary 3.5, we can show the following.

Lemma 3.7. *Two abstract Reeb functions $r_0 : G_0 \to \boldsymbol{R}$ and $r_1 : G_1 \to \boldsymbol{R}$ are cobordant if and only if r_1 is obtained from r_0 by a finite iteration of moves as depicted in Figure* 3 *up to homotopy in the space of abstract Reeb functions, where the functions in question are the height functions.*

Take an arbitrary element of the 2-dimensional oriented cobordism group of Morse functions $\mathscr{M}(2)$. By Remark 2.2, we can always find a stable and hence simple Morse function $f : M \to \boldsymbol{R}$ as a representative. By Lemma 3.4, the cobordism class of the Reeb function $\bar{f} : W_f \to \boldsymbol{R}$ does not depend on the choice of such a representative. Thus the map

$$\rho : \mathscr{M}(2) \to \mathscr{R}$$

sending the cobordism class represented by a simple Morse function to the cobordism class of its associated Reeb function is well-defined. Furthermore, it is clearly a homomorphism of abelian groups.

The main result of this section is the following.

Proposition 3.8. *The cobordism group $\mathscr{R}$ of Reeb functions is an infinite cyclic group generated by the cobordism class of the abstract Reeb function as depicted in Figure* 4.

Proof. For an abstract Reeb function $r : G \to \boldsymbol{R}$, let $\sigma(r)$ be the sum of the signs over all vertices of G of degree three (see Figure 1 for the definition of the sign of each degree three vertex). By Lemma 3.7, if two abstract Reeb functions r_0 and r_1 are cobordant, then we have $\sigma(r_0) = \sigma(r_1)$, since each of the seven moves as in Figure 3 and any homotopy in the space of abstract Reeb functions leave the sum of the signs invariant. Hence, $\bar{\sigma} : \mathscr{R} \to \boldsymbol{Z}$ defined by sending the cobordism class of r to $\sigma(r)$ is well-defined. Furthermore, it is clearly a homomorphism of abelian groups.

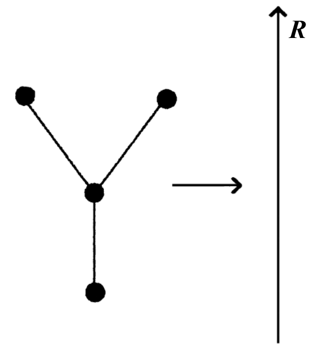

Figure 4. Abstract Reeb function representing a generator of $\mathscr{R}$.

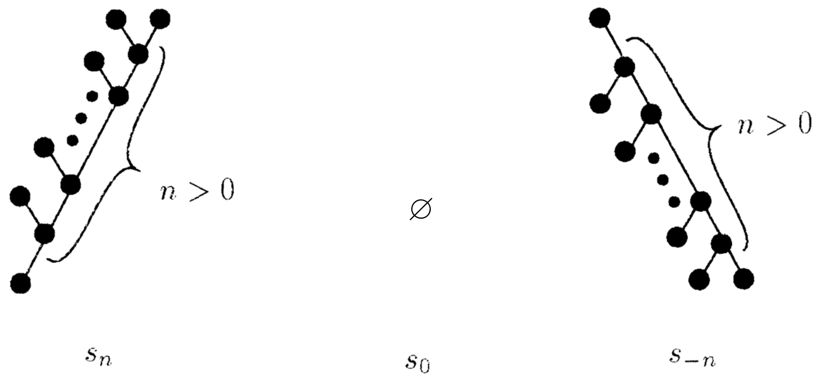

Figure 5. Standard abstract Reeb functions.

Let us consider the family

$$\mathscr{S} = \{s_n : n \in \boldsymbol{Z}\}$$

of standard abstract Reeb functions as depicted in Figure 5 and $i : \mathscr{S} \to \mathscr{R}$ the natural map induced by the inclusion. Then, we see easily that $\bar{\sigma} \circ i$ is bijective, and hence $\bar{\sigma}$ is surjective and i is injective.

Let us show that $\bar{\sigma}$ is injective by showing that i is surjective. For a given abstract Reeb function $r : G \to \boldsymbol{R}$, we will show that there exists an $s_n \in \mathscr{S}$ which is cobordant to r.

If G is the empty set, then r coincides with s_0. Suppose that G is non-empty. If some component of G is not a tree, then there exists a non-trivial loop ℓ in G which is not null-homotopic. We may assume that it is a simple loop and hence that every vertex on ℓ has degree three.

Lemma 3.9. *There exists an edge on ℓ which is of the form as depicted in Figure* 6 (1).

Proof. Suppose that all the edges of ℓ are of the forms (2), (3), or (4) of Figure 6. Let v be a vertex on ℓ which takes the maximum value of $r|_\ell$. Then an edge e_1 of ℓ incident to the vertex v must be of the form (3). Then the edge e_2 of ℓ adjacent to e_1 and not incident to v must also be of the form (3). Repeating the same argument, we have the conclusion that all the edges of ℓ are of the form (3). This is a

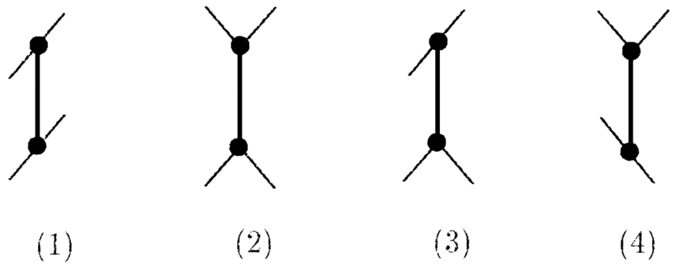

Figure 6. Four cases for an edge in ℓ.

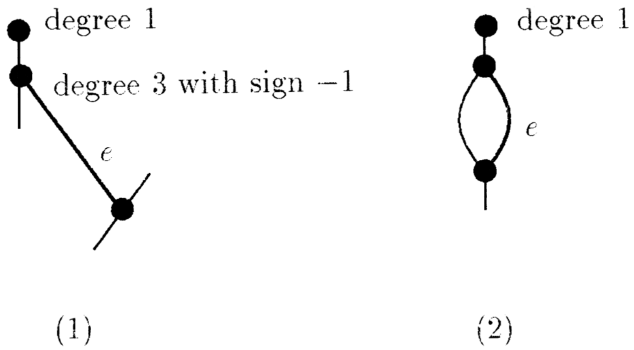

Figure 7. Sliding the edge e.

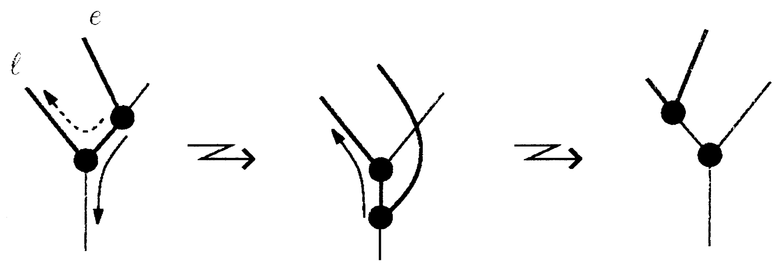

Figure 8. Sliding the lower vertex of e along ℓ.

contradiction, since an edge incident to the vertex corresponding to the minimum of $r|_\ell$ must be of the form (4). Hence an edge of ℓ must be of the form (1). □

Let us go back to the proof of Proposition 3.8. Let e be an edge of the loop ℓ as in Lemma 3.9. By using the moves (VI) and (VII) of Figure 3, which do not change the homotopy type of the graph, we can raise the upper vertex of e until e is as depicted in Figure 7 (1). Then by a homotopy in the space of abstract Reeb functions, we may further assume that $r|_\ell$ takes the maximum value at the upper vertex of e. Now, by using the moves (V) and (VII) of Figure 3, we can slide the lower vertex of e along ℓ (see Figure 8 for one of the procedures) until we get the situation as depicted in Figure 7 (2). Finally, by the move (IV), we can remove the edge e so that we obtain a graph whose first Betti number is smaller than the original one by one. Repeating this procedure, we may assume that every component of the graph G is a tree.

If G is not connected, then by using a homotopy in the space of abstract Reeb

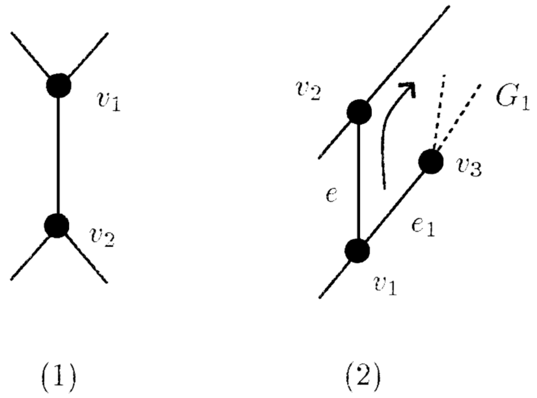

Figure 9. Adjacent vertices of degree three with opposite signs.

functions together with the move (II) in Figure 3, we may assume that G is connected and hence is a tree.

Suppose that G contains two adjacent vertices v_1 and v_2 of degree three whose signs are equal to $+1$ and -1 respectively. If $r(v_2) > r(v_1)$, then we have the situation as depicted in Figure 9 (2). Let $v_3 \neq v_2$ be the upper vertex of the upper edge $e_1 \neq e$ incident to v_1, where e is the edge which has v_1 and v_2 as its end points. Since G is a tree, the graph $G \backslash e_1$ has two components. Let G_1 be the component of $G \backslash e_1$ which contains v_3. We can raise G_1 by a homotopy in the space of abstract Reeb functions so that $r(v_3) > r(v_2)$. Then we can apply the move (VII) of Figure 3 as indicated in Figure 9 (2) so that we have $r(v_2) < r(v_1)$ as in Figure 9 (1).

Then, applying the move (III), we can eliminate the vertices v_1 and v_2. Note that the resulting graph does not contain any non-trivial loop, although it has two connected components. We raise one of the two components by a homotopy in the space of abstract Reeb functions and then use the move (II) to connect the two components. In this way, we get a tree whose number of vertices of degree three is fewer than the original one by two. Repeating this procedure finitely many times, we may assume that all the vertices of degree three have the same sign and that the graph G is a tree.

If G has no vertex of degree three, then G must be of the form as in the left hand side of Figure 3 (I), which is cobordant to s_0.

Suppose that all the vertices of degree three have the sign $+1$. Let T be a simple arc in G connecting a maximal vertex and a minimal vertex such that $r|_T$ is an embedding. If there exists an edge disjoint from T, then by using the move (V) of Figure 3, we can slide its lower vertex until it reaches T. After repeating this procedure finitely many times, we get an abstract Reeb function which is homotopic to s_n in the space of abstract Reeb functions, where n is the sum of the signs over all degree three vertices of G. Hence, we have proved that r is cobordant to s_n.

When all the vertices of degree three have the sign -1, a similar argument can be applied. Thus, we have proved that $i : \mathscr{S} \to \mathscr{R}$ is surjective and hence that $\bar{\sigma} : \mathscr{R} \to \boldsymbol{Z}$ is a bijection. Since it is a homomorphism, it is an isomorphism. By our construction of the homomorphism $\bar{\sigma}$, it is clear that $\mathscr{R}$ is generated by the cobordism class of the abstract Reeb function as depicted in Figure 4. This completes the proof of Proposition 3.8. □

4. Proof of the main theorem.

In this section, we complete the proof of our main theorem.

Proof of Theorem 2.3. By Proposition 3.8, we have only to show that the homomorphism $\rho : \mathscr{M}(2) \to \mathscr{R}$ defined in the previous section is an isomorphism.

For a given abstract Reeb function $r : G \to \boldsymbol{R}$, it is not difficult to construct a closed oriented surface M and a simple Morse function $f : M \to \boldsymbol{R}$ such that $\bar{f} : W_f \to \boldsymbol{R}$ can be identified with r in the sense that there exists a homeomorphism $\psi : W_f \to G$ which makes the following diagram commutative (for details, see [**6**], [**7**], [**8**], [**9**], for example).

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & \boldsymbol{R} \\ {\scriptstyle q_f}\searrow & \nearrow{\scriptstyle \bar{f}} & \nwarrow{\scriptstyle r} \\ W_f & \xrightarrow{\ \psi\ } & G \end{array}$$

Hence, ρ is surjective.

In order to show that ρ is injective, suppose that $f : M \to \boldsymbol{R}$ and $g : N \to \boldsymbol{R}$ are simple Morse functions whose associated Reeb functions $\bar{f} : W_f \to \boldsymbol{R}$ and $\bar{g} : W_g \to \boldsymbol{R}$ are cobordant. Then by Definition 3.6, there exists a continuous map $R : P \to \boldsymbol{R} \times [0,1]$ of a 2-dimensional compact polyhedron P as in Definition 3.6 (1), (2) and (3), where $r_0 = \bar{f}$, $G_0 = W_f$, $r_1 = \bar{g}$ and $G_1 = W_g$. Such a map R can be regarded as an immersed-W "with boundary" in the sense of Mata-Lorenzo (see [**12**], [**13**]). Then, by using Mata-Lorenzo's argument, we can construct a compact oriented 3-manifold X with boundary and a smooth map $F : X \to \boldsymbol{R} \times [0,1]$ which realizes $R : P \to \boldsymbol{R} \times [0,1]$ in the sense that there exists a homeomorphism $\varphi : W_F \to P$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ F\ } & \boldsymbol{R} \times [0,1] \\ {\scriptstyle q_F}\searrow & \nearrow{\scriptstyle \bar{F}} & \nwarrow{\scriptstyle R} \\ W_F & \xrightarrow{\ \varphi\ } & P \end{array}$$

is commutative. This is shown as follows.

First, we decompose the polyhedron P as

$$P = N(\partial P) \cup N(V) \cup N(\Sigma) \cup S,$$

where $N(\partial P)$ is the regular neighborhood of $\partial P = G_0 \cup G_1$ as in Definition 3.6 (1), $N(V)$ is the union of the regular neighborhoods of non-simple vertices of P,

$$N(\Sigma) = \overline{\tilde{N}(\Sigma) \setminus (N(\partial P) \cup N(V))},$$

$\tilde{N}(\Sigma)$ is the regular neighborhood of the singular set $\Sigma(P)$ of P, and S is the closure of the complement of $N(\partial P) \cup N(V) \cup N(\Sigma)$ in P.

Over $N(\partial P)$, we consider

$$F_{\partial P} : (M \times [0, \varepsilon]) \cup (N \times [1-\varepsilon, 1]) \to \boldsymbol{R} \times [0,1]$$

defined by $F_{\partial P} = f \times \mathrm{id}_{[0,\varepsilon]}$ on $M \times [0,\varepsilon]$ and by $F_{\partial P} = g \times \mathrm{id}_{[1-\varepsilon,1]}$ on $N \times [1-\varepsilon, 1]$. Over $N(V)$, we construct a compact oriented 3-manifold X_V and a smooth map $F_V : X_V \to \boldsymbol{R} \times [0,1]$ with only indefinite fold points as its singularities such that W_{F_V} and $\bar{F}_V$ are identified with $N(V)$ and $R|_{N(V)}$ respectively (for details, see [**10**], [**11**]).

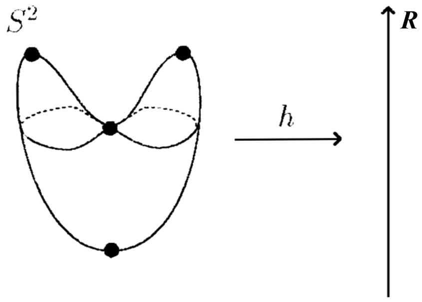

Figure 10. Morse function which represents a generator of $\mathscr{M}(2)$.

Then we can extend the smooth map $F_{\partial P} \cup F_V : (M \times [0, \varepsilon]) \cup (N \times [1 - \varepsilon, 1]) \cup X_V \to \boldsymbol{R} \times [0, 1]$ to a smooth map $F_\Sigma : X_\Sigma \to \boldsymbol{R} \times [0, 1]$ of a compact oriented 3-manifold X_Σ with only fold points as its singularities so that $\bar{F}_\Sigma : W_{F_\Sigma} \to \boldsymbol{R} \times [0, 1]$ is identified with $R|_{N(\partial P) \cup N(V) \cup N(\Sigma)}$. Finally, we can extend F_Σ to a smooth map $F : X \to \boldsymbol{R} \times [0, 1]$ of a compact oriented 3-manifold X with only fold points as its singularities so that $\bar{F} : W_F \to \boldsymbol{R} \times [0, 1]$ is identified with R, since the map q_{F_Σ} over $\partial S = (N(\partial P) \cup N(V) \cup N(\Sigma)) \cap S$ defines an orientable S^1-bundle and hence it is the projection of a trivial S^1-bundle.

By construction, we see that the smooth map $F : X \to \boldsymbol{R} \times [0, 1]$ gives a cobordism between f and g. Thus the map $\rho : \mathscr{M}(2) \to \mathscr{R}$ is injective.

This completes the proof of Theorem 2.3. □

Since for the Morse function $h : S^2 \to \boldsymbol{R}$ as depicted in Figure 10 the associated Reeb function represents a generator of $\mathscr{R}$, we see that h represents a generator of the 2-dimensional oriented cobordism group of Morse functions $\mathscr{M}(2) \cong \boldsymbol{Z}$.

Remark 4.1. For a given Morse function on a closed oriented surface, let us associate the sign $+1$ (or -1) for every degree one vertex of the Reeb graph corresponding to a critical point of index 0 (resp. 2). Then by considering the number of points in the inverse image of a point with respect to the Reeb function, we see easily that the sum of all the signs always vanishes. This shows that the sum of the signs over all vertices of degree three is equal to $c_2(f) - c_0(f)$, where $c_\lambda(f)$ denotes the number of critical points of f of index λ. Our proof of Theorem 2.3 shows that the map

$$\mathscr{M}(2) \to \boldsymbol{Z}$$

which associates $c_2(f) - c_0(f) \in \boldsymbol{Z}$ to the cobordism class of a Morse function f is well-defined and gives an isomorphism.

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