J. Math. Soc. Japan
Vol. 56, No. 4, 2004

Regular projectively Anosov flows on the Seifert fibered 3-manifolds

Dedicated to Professor Yukio Matsumoto on his sixtieth birthday

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(Received Apr. 22, 2003)
(Revised Aug. 26, 2003)

Abstract. This paper concerns projectively Anosov flows φ_t with smooth stable and unstable foliations $\mathscr{F}^s$ and $\mathscr{F}^u$ on a Seifert fibered 3-manifold M over a hyperbolic orbifold. We show that if the foliations $\mathscr{F}^s$ and $\mathscr{F}^u$ do not have compact leaves, then after changing the parameter, φ_t is differentiably isotopic to a quasi-Fuchsian flow lifted to a finite cover.

1. Introduction and the statement of the result.

A non singular flow φ_t on a closed 3-dimensional manifold M is a projectively Anosov flow if there exist a continuous Riemannian metric on M, a continuous splitting $\hat{E}^u \oplus \hat{E}^s$ of $TM/T\varphi$ invariant under the action of $\hat{T}\varphi_t$ on $TM/T\varphi$, and a positive real number C such that the following inequality holds for $t \geq 0$, $v^u \in \hat{E}^u \backslash \{0\}$ and $v^s \in \hat{E}^s \backslash \{0\}$:

$$\frac{\|(\hat{T}\varphi_t)v^u\|}{\|(\hat{T}\varphi_t)v^s\|} \geq e^{Ct} \frac{\|v^u\|}{\|v^s\|}.$$

This definition was given in [**7**], where Eliashberg and Thurston called it a conformally Anosov flow. The same flow was investigated by Mitsumatsu [**19**] (see also [**20**]) and was called a projectively Anosov flow.

The invariant line bundles $\hat{E}^u$ and $\hat{E}^s$ give rise to the invariant plane fields E^u and E^s over M. As is remarked in [**7**], the plane fields E^u and E^s are continuous and integrable, but frequently they are not uniquely integrable. It is, however, interesting to investigate the case where the plane fields E^u and E^s are smooth, and then E^u and E^s determine codimension 1 smooth foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ of M. In this case, we call the projectively Anosov flow regular.

There are a large variety of the Anosov flows with stable and unstable foliations of class C^1 but not C^2. On the other hand, the Anosov flows with smooth (at least C^3) stable and unstable foliations (regular Anosov flows) are classified by Ghys ([**12**], [**13**]). Up to finite cover and parameter change, they are either isotopic to the suspension flow of the Anosov diffeomorphisms of the torus or to the quasi-Fuchsian flows on the Seifert

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2000 *Mathematics Subject Classification*. Primary 57R30, 58F18; Secondary 58F15, 53C30, 53C15, 53C12.

Key Words and Phrases. projectively Anosov flows, smooth stable foliations, without compact leaves, transversely projective structure, Seifert fibered 3-manifolds.

The author is partially supported by Grant-in-Aid for Scientific Research (No. 12304003 and No. 16204004), Japan Society for Promotion of Science, Japan, and by the 21st Century COE Program at Graduate School of Mathematical Sciences, the University of Tokyo.

fibered 3-manifolds over hyperbolic orbifolds. These are the most important examples of regular projectively Anosov flows.

For a regular projectively Anosov flow, the smooth foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ may have compact leaves which are tori ([**7**]). In [**20**], Noda investigated the regular projectively Anosov flows with compact leaves, and he also gave the classification of those flows with compact leaves on torus bundles over the circle.

In this paper, we look at regular projectively Anosov flows without compact leaves on the Seifert fibered 3-manifolds over hyperbolic orbifolds, and show that they are in fact the Anosov flows (quasi-Fuchsian flows). More precisely, we show the following theorem.

Theorem 1.1. *Let φ_t be a regular projectively Anosov flow on a Seifert fibered* 3-*manifold M over a hyperbolic orbifold. Assume that the unstable foliation $\mathscr{F}^u$ and the stable foliation $\mathscr{F}^s$ do not have compact leaves. Then after changing the parameter of the flow, φ_t is differentiably isotopic to a quasi-Fuchsian flow lifted to a finite cover.*

Noda showed in [**21**] that a Seifert fibered 3-manifold over a hyperbolic orbifold does not admit regular projectively Anosov flows with compact leaves in the unstable foliation $\mathscr{F}^u$ or the stable foliation $\mathscr{F}^s$. Hence the above theorem gives the classification of regular projectively Anosov flows on a Seifert fibered 3-manifold. Note that Barbot ([**1**]) classified the Anosov flows (which are not necessarily regular) on a large family of graph manifolds including Seifert fibered 3-manifold M over a hyperbolic orbifold.

Note also that Theorem 1.1 for the regular projectively Anosov flows without compact leaves on the unit tangent bundle of a closed hyperbolic surface was shown in [**22**].

By the assumption of our theorem that the foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ do not have compact leaves, the theorems of Thurston ([**24**]), Levitt ([**16**]), Eisenbud-Hirsch-Neumann ([**6**]), Matsumoto ([**17**]) and Brittenham ([**2**]) assert that each foliation $\mathscr{F}^u$ and $\mathscr{F}^s$ can be individually isotoped to be transverse to the fibers. Hence the lifted foliations $\tilde{\mathscr{F}}^u$ and $\tilde{\mathscr{F}}^s$ of the universal covering space $\tilde{M}$ are the product foliations.

To show our results, we first look at the leaf spaces Q^u and Q^s of the lifted foliations $\tilde{\mathscr{F}}^u$ and $\tilde{\mathscr{F}}^s$ and the orbit space of the lifted flow $\tilde{\varphi}_t$ on the universal covering space $\tilde{M}$ together with the action of the fundamental group $\pi_1(M)$. This procedure is the same as in [**22**], and the orbit foliation $\tilde{\varphi}$ of $\tilde{M}$ is again shown to be Hausdorff. The Hausdorffness follows from two facts, namely, that the flow is projectively Anosov and that the lifted foliations $\tilde{\mathscr{F}}^u$ and $\tilde{\mathscr{F}}^s$ are product foliations. We review it in Section 2. In a recent paper [**18**], it is also shown that without the assumption of being projectively Anosov, the transverse intersection of $\mathscr{F}^u$ and $\mathscr{F}^s$ of the unit tangent bundle of the hyperbolic surface is not unique and the orbit foliation $\tilde{\varphi}$ of $\tilde{M}$ is not Hausdorff.

To proceed further, we need to know the topology of the leaves of $\mathscr{F}^u$ and $\mathscr{F}^s$. We show in Section 3 that each leaf of $\mathscr{F}^u$ or $\mathscr{F}^s$ is homeomorphic either to a plane or to a cylinder. By a simple argument of Poincaré-Hopf type, this follows from the fact that the lifted flow on each lifted leaf is Hausdorff.

The information on the topology of the leaves of smooth foliations has a remarkable consequence by the unpublished famous work by Duminy [**5**] (announced in

[**15**], a proof is given by Cantwell-Conlon [**3**]). Duminy showed that the end set of a semiproper leaf of an exceptional minimal set of a codimension 1, C^2 foliation of a closed manifold is homeomorphic to the Cantor set. In our situation, Duminy's result implies that all leaves of $\mathscr{F}^u$ or $\mathscr{F}^s$ are dense.

Then in Sections 4, 5 and 6, we look at the action of $\pi_1(M)$ on the orbit space $\tilde{M}/\tilde{\varphi}$ and we study the shape of the image $p(\tilde{M})$. If there is a closed orbit of φ, then we have a fixed point in $\tilde{M}/\tilde{\varphi}$ for the action of the element of $\pi_1(M)$ represented by the closed orbit. Even if we assume that there are no closed orbits, we show that there is an element of $\pi_1(M)$ whose action on Q^u and on Q^s have fixed points. Then, in both cases, the image $p(\tilde{M})$ should not be very large and should look like a band from $(-\infty, -\infty)$ to (∞, ∞) in $Q^u \times Q^s$. We see that the boundary components of the band are graphs of homeomorphisms $Q^u \to Q^s$, for otherwise the action of $\pi_1(M)$ on Q^u or on Q^s has exceptional minimal set and this contradicts Duminy's theorem.

In Section 7, we construct an action on the circle of the fundamental group of the base 2-orbifold of the Seifert fibered 3-manifold. The fact that the flow is projectively Anosov implies that the action on the circle is as is described by Barbot [**1**], that is, this induces a convergence group action on the circle. Using the results of Tukia [**27**], Casson-Jungreis [**4**] or Gabai [**10**] as in [**1**], the action is shown to be topologically conjugate to a Fuchsian action in Section 8.

In Section 9, we use these results to show our main theorem. Since the holonomy of the foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ are topologically conjugate to the lifts of those of the Anosov foliations for the geodesic flow on the unit tangent bundle of the base orbifold, M is a finite covering space of the unit tangent bundle of the base orbifold and the foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ are topologically conjugate to the Anosov foliations lifted to the finite cover M. Since our foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ are smooth foliations topologically conjugate to the lifted Anosov foliations, the result of Ghys in [**13**] says that there are hyperbolic metrics g_u and g_s such that $\mathscr{F}^u$ and $\mathscr{F}^s$ are differentiably conjugate to the the Anosov foliations with respect to g_u and g_s lifted to the finite cover M. This enable us to follow the argument of Ghys [**12**] to show our theorem.

The author is grateful to the Erwin Schrödinger Institute for its warm hospitality where he could almost finish this work. He is also grateful to Franz Kamber for organizing an excellent workshop at ESI in 2002. The author thanks the referee for the detailed reading and the valuable comments.

2. Lifted flow and foliations in the universal covering.

Let φ_t be a regular projectively Anosov flow on a 3-manifold M. Let $\mathscr{F}^u$ and $\mathscr{F}^s$ be the unstable foliation and the stable foliation for φ_t, respectively. Let $\tilde{\varphi}_t, \tilde{\mathscr{F}}^u$ and $\tilde{\mathscr{F}}^s$ denote the induced flow and foliations on the universal covering space $\tilde{M}$ of M.

We look at the leaf spaces $Q^u = \tilde{M}/\tilde{\mathscr{F}}^u$ and $Q^s = \tilde{M}/\tilde{\mathscr{F}}^s$ ([**8**], [**1**]). For the purpose of this paper, we restrict our attention to the case where Q^u and Q^s are Hausdorff, i.e., $\tilde{\mathscr{F}}^u$ and $\tilde{\mathscr{F}}^s$ are diffeomorphic to the product foliation of $\boldsymbol{R}^3$ with leaves $\boldsymbol{R}^2 \times \{*\}$. Then Q^u and Q^s are diffeomorphic to the real line $\boldsymbol{R}$. The projections $p^u : \tilde{M} \to Q^u$ and $p^s : \tilde{M} \to Q^s$ are both $\pi_1(M)$ equivariant and determine the foliations $\tilde{\mathscr{F}}^u$ and $\tilde{\mathscr{F}}^s$, respectively.

We consider the juxtaposition map of projections;

$$p = (p^u, p^s) : \tilde{M} \to Q^u \times Q^s.$$

The map p to the plane is a $\pi_1(M)$ equivariant submersion and it determines the structure of the orbit foliation φ of the flow φ_t.

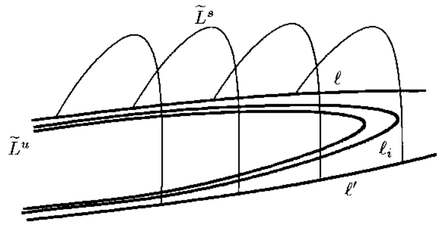

Figure 1.

The problem we need to treat first is to know whether the orbit foliation $\tilde{\varphi}$ of the lifted flow on $\tilde{M}$ is Hausdorff. We have the following proposition ([**22**]) which we also include the proof.

Proposition 2.1 ([**22**]). *Let φ_t be a regular projectively Anosov flow on a 3-manifold M. Assume that the lifted stable foliation $\tilde{\mathscr{F}}^s$ of the universal covering space $\tilde{M}$ is diffeomorphic to the product foliation of $\boldsymbol{R}^3$. Then the lifted orbit foliation $\tilde{\varphi}$ restricted to each leaf $\tilde{L}^u$ of the lifted unstable foliation $\tilde{\mathscr{F}}^u$ is Hausdorff.*

Proof. Assume that there are two distinct orbits ℓ and ℓ' of $\tilde{\varphi}$ on $\tilde{L}^u$ such that a sequence $\{\ell_i\}$ of orbits of $\tilde{\varphi}$ on $\tilde{L}^u$ converges to them simultaneously. Let $\tilde{L}_i^s$ denote the unstable leaf passing ℓ_i. Then by the assumption that $\tilde{\mathscr{F}}^s$ is the product foliation, the leaf $\tilde{L}_i^s$ converges to a leaf $\tilde{L}^s$. Thus ℓ and ℓ' are components of the intersection of $\tilde{L}^u$ and $\tilde{L}^s$. See Figure 1.

We take points x and x' on ℓ and ℓ', respectively. Then we take bi-foliated rectangles T and T' at x and x' transverse to $\tilde{\varphi}$, respectively. There are curves γ^u and γ^s on $\tilde{L}^u$ and $\tilde{L}^s$ joining x and x'. Then we obtain holonomies $h^u_{\gamma^u}$ and $h^s_{\gamma^s}$ for the foliations $\tilde{\mathscr{F}}^u$ and $\tilde{\mathscr{F}}^s$ along γ^u and γ^s, respectively. Since $\tilde{L}^u$ and $\tilde{L}^s$ are contractible, the holonomies do not depend on the paths on the leaves.

Take a Riemannian metric on M adapted for the projectively Anosov flow φ. We lift it to the universal covering space $\tilde{M}$. We look at the intersections x_i, x_i' of the orbit ℓ_i and the bi-foliated rectangles T, T'. We may assume that $\tilde{\varphi}_{t_i}(x_i) = x_i'$ for positive t_i. Then we see that $t_i \to \infty$ as $i \to \infty$. This implies that

$$\frac{\|(\hat{T}\tilde{\varphi}_{t_i})v^u\|}{\|(\hat{T}\tilde{\varphi}_{t_i})v^s\|} \geq e^{Ct_i}\frac{\|v^u\|}{\|v^s\|}$$

for $v^u \in (T\tilde{\mathscr{F}}^u/T\tilde{\varphi})\backslash\{0\}$ and $v^s \in (T\tilde{\mathscr{F}}^s/T\tilde{\varphi})\backslash\{0\}$ at x_i. Thus this ratio tends to the infinity as $i \to \infty$.

This ratio can also be calculated as the ratio of the derivatives of the holonomies $h^u_{\gamma^u}$ and $h^s_{\gamma^s}$ at that points, and hence the ratio is bounded. This is a contradiction. $\square$

In our case, both the lifted unstable foliation $\tilde{\mathscr{F}}^u$ and the lifted stable foliation $\tilde{\mathscr{F}}^s$ are diffeomorphic to the product foliation of $\boldsymbol{R}^3$, and Proposition 2.1 implies the following lemma.

LEMMA 2.2. *If both the lifted unstable foliation $\tilde{\mathscr{F}}^u$ and the lifted stable foliation $\tilde{\mathscr{F}}^s$ are diffeomorphic to the product foliation of $\boldsymbol{R}^3$, then the orbit foliation $\tilde{\varphi}$ of $\tilde{M}$ is Hausdorff and $p : \tilde{M} \to Q^u \times Q^s$ is a fibration to the image with fiber being the orbit of $\tilde{\varphi}_t$. The image $p(\tilde{M})$ has the following properties.*

(i) *The image $p(\tilde{M})$ is a simply connected domain in $Q^u \times Q^s$.*

(ii) *The intersection $p(\tilde{M}) \cap (Q^u \times \{y^s\})$ for $y^s \in Q^s$ or $p(\tilde{M}) \cap (\{x^u\} \times Q^s)$ for $x^u \in Q^u$ is either empty or homeomorphic to the real line.*

3. Poincaré-Hopf type invariant.

First we review an invariant for the immersed curves on a foliated surface, which should have been well known.

Let φ be a nonsingular flow on an oriented 2-manifold L. Let X be the generating vector field for the flow φ. For a smooth immersion $\gamma : S^1 \to L$, one can count the degree of $t \mapsto \gamma'(t)/\|\gamma'(t)\|$ with respect to the trivialization of $T_{\gamma(t)}L$ given by $X(\gamma(t))/\|X(\gamma(t))\|$ and its normal vector in the positive orientation. This integer $NR_\varphi(\gamma)$ (the number of rotations of γ with respect to φ) depends on the regular homotopy class of the curve γ on L.

We have the following well known lemma.

LEMMA 3.1 (Poincaré-Hopf Theorem). *Let φ be a nonsingular flow on an oriented 2-manifold L. Let S be a smoothly embedded compact surface with boundary ∂S in L. Then*

$$\sum_{\gamma \subset \partial S} NR_\varphi(\gamma) = \chi(S),$$

where $\gamma \subset \partial S$ is given the induced orientation and $\chi(S)$ denotes the Euler characteristic of S.

PROPOSITION 3.2. *Let φ_t be a nonsingular flow on an oriented 2-manifold L. Assume that the induced orbit foliation $\tilde{\varphi}$ on the universal covering space $\tilde{L}$ of L is the product foliation $(\boldsymbol{R}^2, \boldsymbol{R} \times \{*\})$. Then L is homeomorphic either to the plane, to the cylinder or to the torus.*

PROOF. If L is homeomorphic neither to the plane, to the cylinder nor to the torus, one can find a pair P of pants (a 2-disk with two 2-disks deleted) embedded as an essential submanifold, i.e., the embedding induces the injection in the fundamental groups. Lemma 3.1 says that the sum of the invariant $NR_\varphi(\gamma)$ over the three boundaries of P is -1, hence one of the boundary components, say γ_0, has non-zero invariant.

On the other hand, $(\tilde{L}, \tilde{\varphi})$ is the product foliation and the action of the element α of $\pi_1(L)$ represented by the closed curve γ_0 preserves the product foliation with orientation. Then γ_0 is regularly homotopic in L to a closed curve γ_1 which is an orbit or a curve transverse to φ. (γ_1 may not be a simple closed curve in the argument.) The reason is as follows. By the assumption, $\pi_1(L)$ acts on the leaf space $\tilde{L}/\tilde{\varphi}$ homeomorphic to $\boldsymbol{R}$. If the action of α on $\tilde{L}/\tilde{\varphi}$ has a fixed point, then this fixed orbit is a lift of a closed orbit

γ_1 in L. If the action of α has no fixed point on $\tilde{L}/\tilde{\varphi}$, we can take a curve $\tilde{\gamma}_1$ in $\tilde{L}$ transverse to $\tilde{\varphi}$ which is invariant under the action of α. This $\tilde{\gamma}_1$ defines a closed curve γ_1 on L. Let $\hat{\gamma}_0$ and $\hat{\gamma}_1$ be the lifts of γ_0 and γ_1 in $\tilde{L}/\alpha$ which are simple closed curves. Since the simple closed curves $\hat{\gamma}_0$ and $\hat{\gamma}_1$ are in the same homotopy class α in $\tilde{L}/\alpha$, $\hat{\gamma}_0$ and $\hat{\gamma}_1$ are in the same regular homotopy class in $\tilde{L}/\alpha$. Hence γ_0 and γ_1 are in the same regular homotopy class in L. Thus $NR_\varphi(\gamma_1) = NR_\varphi(\gamma_0)$.

Since γ_1 is tangent or transverse to φ, $NR_\varphi(\gamma_1) = 0$. This contradicts that $NR_\varphi(\gamma_0) \neq 0$. □

When we look at a flow on a 3-manifold tangent to an oriented foliation, this invariant should play an essential role. In fact the above proposition has the following corollary.

Corollary 3.3. *Let φ be a regular projectively Anosov flow on a closed* 3-*dimensional manifold M. Assume that the lifted foliation $\tilde{\mathscr{F}}^s$ of the universal covering space $\tilde{M}$ of M is the product foliation. Then each leaf of $\mathscr{F}^u$ is homeomorphic either to the plane, to the cylinder or to the torus.*

Now we cite the theorem of Duminy [**5**] (announced in [**15**], a proof is given by Cantwell-Conlon [**3**]).

Theorem 3.4 (Duminy). *The end set of a semiproper leaf of an exceptional minimal set of a codimension* 1, C^2 *foliation of a closed manifold is homeomorphic to the Cantor set.*

Corollary 3.5. *Let φ be a regular projectively Anosov flow on a closed* 3-*dimensional manifold M with the associated foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ being without compact leaves. Assume that the lifted foliations $\tilde{\mathscr{F}}^u$ and $\tilde{\mathscr{F}}^s$ of the universal covering space $\tilde{M}$ of M are product foliations. Then each leaf of $\mathscr{F}^u$ or $\mathscr{F}^s$ is dense and homeomorphic either to the plane or to the cylinder.*

4. Properties of the orbit space.

Our situation is as follows. We have foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ of the Seifert fibered 3-manifold M without compact leaves. By the theorems of Thurston ([**24**]), Levitt ([**16**]), Eisenbud-Hirsch-Neumann ([**6**]), Matsumoto ([**17**]) and Brittenham ([**2**]), the foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ can be isotoped to be transverse to the fibers. Hence the lifted foliations $\tilde{\mathscr{F}}^u$ and $\tilde{\mathscr{F}}^s$ of the universal covering space $\tilde{M}$ are the product foliations. Let $Q^u = \tilde{M}/\tilde{\mathscr{F}}^u$ and $Q^s = \tilde{M}/\tilde{\mathscr{F}}^s$ denote the leaf spaces. Then we have the projection $p : \tilde{M} \to Q^u \times Q^s$. Since the intersection $\mathscr{F}^u \cap \mathscr{F}^s$ is projectively Anosov, p is a fibration to the image by Lemma 2.2.

Now the transverse structure of the foliations and the flow is given by the diagonal $\pi_1(M)$ action on the image $p(\tilde{M})$ in the product $Q^u \times Q^s$. The fundamental group $\pi_1(M)$ has the center Z whose generator is represented by the general fiber of the Seifert fibration. Since the foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ can be isotoped to be transverse to the general fiber, the generator of the center acts by a non trivial translation on the leaf space Q^u as well as on the leaf space Q^s. We fix a generator c of the center Z and fix the transverse orientations of $\mathscr{F}^u$ and $\mathscr{F}^s$ so that the action of the generator c of Z is the translation by 1 on Q^u and on Q^s.

Then using Lemma 2.2 and Corollary 3.5, we see the following lemmas.

Lemma 4.1. *The image $p(\tilde{M}) \subset Q^u \times Q^s$ has the following property.*

(i) *The image $p(\tilde{M})$ is a simply connected domain in $Q^u \times Q^s$.*

(ii) *The intersection $p(\tilde{M}) \cap Q^u \times \{y^s\}$ $(y_s \in Q^s)$ or $p(\tilde{M}) \cap \{x^u\} \times Q^s$ $(x_u \in Q^u)$ is non empty and homeomorphic to the real line.*

(iii) *For $x_1^u, x_2^u \in Q^u$ and $x_1^s, x_2^s \in Q^s$ such that $x_1^u < x_2^u < x_1^u + 1$ and $x_1^s < x_2^s < x_1^s + 1$, if (x_1^u, x_2^s) and (x_2^u, x_1^s) belong to $p(\tilde{M})$, then (x_1^u, x_1^s) and (x_2^u, x_2^s) belong to $p(\tilde{M})$.*

(iv) *If A is a component of the boundary of $p(\tilde{M})$, then A is the graph of a homeomorphism $Q^u \to Q^s$.*

Proof. The statements (i) and (ii) follow from the first part of Lemma 2.2 and the fact that the center acts as non trivial translations on Q^u and on Q^s.

The statement (iii), which is a kind of convexity property, is shown as follows. By the invariance under the action of the center, $(x_1^u + 1, x_2^s + 1)$ belongs to $p(\tilde{M})$ as well. Note that any path from (x_1^u, x_2^s) to $(x_1^u + 1, x_2^s + 1)$ in $p(\tilde{M})$ passes across the union of half lines:

$$\{x_2^u\} \times [x_2^s, \infty) \cup [x_2^u, \infty) \times \{x_2^s\}.$$

Hence either $(x_2^u, x_3^s) \in p(\tilde{M})$ for $x_3^s \geq x_2^s$ or $(x_3^u, x_2^s) \in p(\tilde{M})$ for $x_3^u \geq x_2^u$. In either case, by statement (ii), (x_2^u, x_2^s) belongs to $p(\tilde{M})$.

In a similar way, (x_1^u, x_1^s) belongs to $p(\tilde{M})$.

For the statement (iv), the statements (i) and (ii) imply that a component A of the boundary of $p(\tilde{M})$ is the completed graph of a non decreasing function $Q^u \to Q^s$. Here the completed graph means that gaps of the graph are filled by vertical segments.

Now A is invariant under the action of $\pi_1(M)$. If A is not the graph of a homeomorphism, there is either a vertical segment or a horizontal segment in A.

Assume that there is a vertical segment $J \subset A$. We take a maximal vertical segment containing J and contained in A, and by changing the name let J denote the maximal one. Since the action of $\pi_1(M)$ on $Q^u \times Q^s$ is the diagonal action, an element of $\pi_1(M)$ sends a vertical segment to a vertical segment in A. Then the orbit $\pi_1(M)(\mathrm{Int}\, J)$ is an invariant set under the action of $\pi_1(M)$. Again since the action of $\pi_1(M)$ on $Q^u \times Q^s$ is the diagonal action, $p^s(\pi_1(M)(\mathrm{Int}\, J))$ is invariant under the action of $\pi_1(M)$ on Q^s. Since $p^s(\pi_1(M)(\mathrm{Int}\, J))$ is a disjoint union of open intervals, the closure of an orbit of a point of the complement of $p^s(\pi_1(M)(\mathrm{Int}\, J))$ is not equal to the whole Q^s. This implies that a leaf of $\mathscr{F}^s$ is not dense. This contradicts Corollary 3.5.

If there is a horizontal segment $J \subset A$, then we see in the same way that a leaf of $\mathscr{F}^u$ is not dense, and this contradicts Corollary 3.5.

Thus the statement (iv) is proved. □

Lemma 4.2. *Neither $\mathscr{F}^u$ nor $\mathscr{F}^s$ is a foliation without holonomy.*

Proof. If $\mathscr{F}^u$ is without holonomy, then the global holonomy of the foliated bundle over the orbifold is contained in a subgroup conjugate to the group of rotations. This implies that each leaf is a covering associated to a subgroup containing the commutator subgroup of the base space, and hence has nontrivial genus. This contradicts Corollary 3.5. □

5. Closed orbits.

We look at the closed orbits of our regular projectively Anosov flows. The existence of a closed orbit gives a lot of restrictions on the shape of the orbit space.

Lemma 5.1. *If there is a closed orbit γ of φ, the actions of the element $[\gamma] \in \pi_1(M)$ on Q^u and Q^s have fixed points and there is a fixed point $(x_0^u, x_0^s) \in p(\tilde{M})$ of the action of $[\gamma]$ in the image $p(\tilde{M})$ such that for a positive real number ε, the interval $(x_0^u, x_0^u + \varepsilon) \subset Q^u$ or $(x_0^s, x_0^s + \varepsilon) \subset Q^s$ does not contain fixed points for $[\gamma]$.*

Proof. If there is a closed orbit γ of φ, then by taking a suitable lift $\tilde{\gamma} \subset \tilde{M}$, $p(\tilde{\gamma}) = (p^u(\tilde{\gamma}), p^s(\tilde{\gamma}))$ is a fixed point for the action of $[\gamma] \in \pi_1(M)$.

If the germ of the action of $[\gamma]$ at $p^u(\tilde{\gamma}) \in Q^u$ is the identity, γ has a neighborhood in L^s saturated by closed orbits, where L^s is the leaf of the stable foliation $\mathscr{F}^s$ containing γ. Since all leaves are dense and homeomorphic either to the plane or to the cylinder by Corollary 3.5, all leaves of the unstable foliation $\mathscr{F}^u$ are cylinders. Then it is easy to see that the unstable foliation $\mathscr{F}^u$ is without holonomy and topologically conjugate to a linear foliation by cylinders on T^3, contradicting the assumption on the manifold M (or Lemma 4.2).

Thus the germ of the action of $[\gamma]$ on the positive side of $p^u(\tilde{\gamma}) \in Q^u$ is not that of the identity.

If there are no fixed points on $(p^u(\tilde{\gamma}), p^u(\tilde{\gamma}) + \varepsilon)$, we take $x_0^u = p^u(\tilde{\gamma})$. Otherwise $p^u(\tilde{\gamma})$ is an accumulation point of the fixed point set of the action of $[\gamma]$ on Q^u and we can find a desired x_0^u $(> p^u(\tilde{\gamma}))$ such that $(x_0^u, p^u(\tilde{\gamma})) \in p(\tilde{M})$.

In a similar way, we can take $x_0^s \in Q^s$ and (x_0^u, x_0^s) is the desired point.

Note that $p^{-1}(x_0^u, p^u(\tilde{\gamma}))$ corresponds to a closed orbit γ' which is in the same stable leaf as γ and parallel to γ. In the same way, $p^{-1}(x_0^u, x_0^s)$ corresponds to a closed orbit γ'' which is in the same unstable leaf as γ' parallel to γ'. $\square$

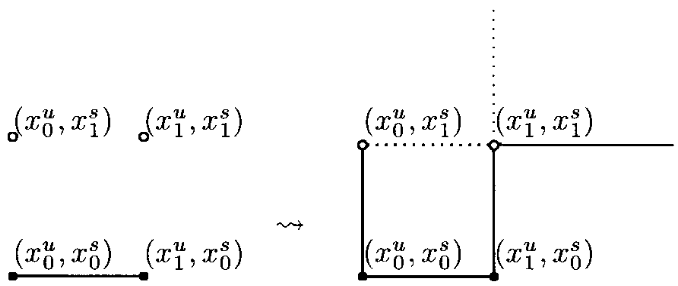

Figure 2. Proof of Lemma 5.2 (i).

We will see that there is a unique closed orbit on each cylindrical leaf of the unstable foliation $\mathscr{F}^u$ or of the stable foliation $\mathscr{F}^s$. But it is necessary to see the consequences of the unrealizable case where there are two closed orbit on a leaf. We have the following technical lemma.

Lemma 5.2. *For $x_0^u < x_1^u \leq x_0^u + 1$ and $x_0^s < x_1^s \leq x_0^s + 1$, let $[x_0^u, x_1^u] \subset Q^u$ and $[x_0^s, x_1^s] \subset Q^s$ be invariant intervals for the action of $[\gamma] \in \pi_1(M)$ such that the action of $[\gamma]$ has no fixed points in the open intervals (x_0^u, x_1^u) and (x_0^s, x_1^s).*

(i) *If the interval* $[x_0^u, x_1^u] \times \{x_0^s\}$ *is contained in* $p(\tilde{M})$, *then the interval* $\{x_1^u\} \times [x_0^s, x_1^s]$ *is contained in* $p(\tilde{M})$.
If $\{x_0^u\} \times [x_0^s, x_1^s]$ *is contained in* $p(\tilde{M})$, *then the interval* $[x_0^u, x_1^u] \times \{x_1^s\}$ *is contained in* $p(\tilde{M})$.

(ii) *If the point* (x_0^u, x_0^s) *belongs to* $p(\tilde{M})$, *then the point* (x_1^u, x_1^s) *belongs to* $p(\tilde{M})$.
If the point (x_1^u, x_1^s) *belongs to* $p(\tilde{M})$, *then the point* (x_0^u, x_0^s) *belongs to* $p(\tilde{M})$.

Proof. (i). Assume that $[x_0^u, x_1^u] \times \{x_0^s\}$ is contained in $p(\tilde{M})$. By the invariance of $p(\tilde{M})$ under the action of $[\gamma]$, the half intervals $\{x_0^u\} \times [x_0^s, x_1^s)$ and $\{x_1^u\} \times [x_0^s, x_1^s)$ are contained in $p(\tilde{M})$. Hence by Lemma 4.1 (ii), $[x_0^u, x_1^u] \times [x_0^s, x_1^s)$ is contained in $p(\tilde{M})$.

If the point (x_1^u, x_1^s) does not belong to $p(\tilde{M})$, by Lemma 4.1 (iii), the point (x_0^u, x_1^s) does not belong to $p(\tilde{M})$. See Figure 2. By the invariance under the action of the center, $p(\tilde{M})$ intersects $\{x_1^u\} \times [x_1^s, \infty) \cup [x_1^u, \infty) \times \{x_1^s\}$, but by Lemma 4.1 (ii), $p(\tilde{M})$ does not intersect $\{x_1^u\} \times [x_1^s, \infty)$, and $p(\tilde{M})$ intersects $[x_1^u, \infty) \times \{x_1^s\}$. Hence again by Lemma 4.1 (ii), $[x_0^u, x_1^u] \times \{x_1^s\}$ is a horizontal segment on the boundary of $p(\tilde{M})$. This contradicts Lemma 4.1 (iv) and the assertion (i) is shown.

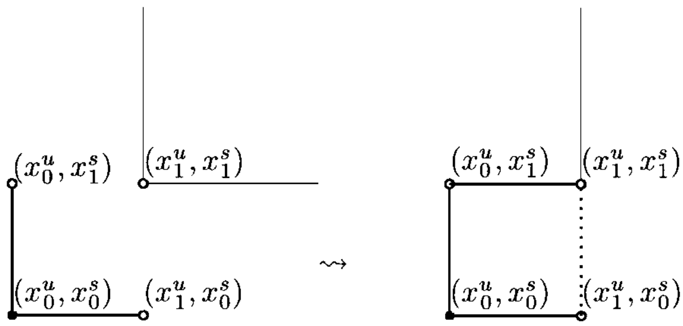

Figure 3. Proof of Lemma 5.2 (ii).

The case where $\{x_0^u\} \times [x_0^s, x_1^s]$ is contained in $p(\tilde{M})$ is treated in a similar way.

(ii). By the invariance of $p(\tilde{M})$ under the action of $[\gamma]$, $[x_0^u, x_1^u) \times \{x_0^s\} \cup \{x_0^u\} \times [x_0^s, x_1^s)$ is contained in $p(\tilde{M})$. If the point (x_1^u, x_1^s) does not belong to $p(\tilde{M})$, then by (i) just shown, the points (x_1^u, x_0^s) and (x_0^u, x_1^s) do not belong to $p(\tilde{M})$. By the invariance under the action of the center, $p(\tilde{M})$ intersects $\{x_1^u\} \times (x_1^s, \infty) \cup (x_1^u, \infty) \times \{x_1^s\}$.

If $p(\tilde{M})$ intersects $\{x_1^u\} \times (x_1^s, \infty)$, then by Lemma 4.1 (iii), $p(\tilde{M})$ does not intersect $\{x_1^u\} \times (-\infty, x_1^s)$ and does intersect $(x_0^u, x_1^u) \times \{x_1^s\}$. See Figure 3. By the invariance under the action of $[\gamma]$, $(x_0^u, x_1^u) \times \{x_1^s\}$ is contained in $p(\tilde{M})$, and by Lemma 4.1 (iii), $[x_0^u, x_1^u) \times [x_0^s, x_1^s)$ is contained in $p(\tilde{M})$. Then $\{x_1^u\} \times [x_0^s, x_1^s]$ is a vertical segment on the boundary of $p(\tilde{M})$. This contradicts Lemma 4.1 (iv).

If $p(\tilde{M})$ intersects $(x_1^u, \infty) \times \{x_1^s\}$, we find in a similar way that $[x_0^u, x_1^u] \times \{x_1^s\}$ is a horizontal segment on the boundary of $p(\tilde{M})$ and this contradicts Lemma 4.1 (iv). Thus the assertion (ii) is shown. □

Lemma 5.3. *Let* γ *be a closed orbit, and* $(x_0^u, x_0^s) = p(\tilde{\gamma}) \in Q^u \times Q^s$ *is the point invariant under the action of* $[\gamma] \in \pi_1(M)$. *Then the point* (x_0^u, x_0^s) *is neither an attractor nor a repeller for each of the* 4 *quadrants.*

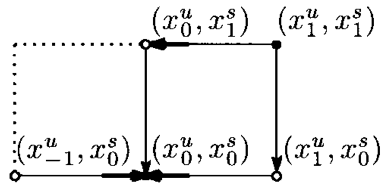

Figure 4. Proof of Lemma 5.3.

Proof. Assume that the point (x_0^u, x_0^s) is an attractor in one of the 4 quadrants for the action of $[\gamma]$. Since the flow in M is projectively Anosov, the action of $[\gamma]$ on the tangent space $T_{(x_0^u, x_0^s)}Q^u \times Q^s$ at the point (x_0^u, x_0^s) sends the direction of the vectors nearer to the direction of Q^s. Thus x_0^u is a linearly nontrivial attractor for the action of $[\gamma]$ on Q^u.

Then we have the nearby fixed points x_{-1}^u and x_1^u for the action of $[\gamma]$ on Q^u.

These points (x_{-1}^u, x_0^s) and (x_1^u, x_0^s) do not belong to $p(\tilde{M})$. The reason is as follows. If the point (x_1^u, x_0^s) is in the image $p(\tilde{M})$, $p^{-1}([x_0^u, x_1^u] \times \{x_0^s\})$ projects to an embedded annulus on a leaf L^s with the boundary components being closed orbits. Then the action at the closed orbit corresponding to $p^{-1}(x_1^u, x_0^s)$ is repelling in the direction of Q^u (that is, in the direction of L^s) and attracting in the direction of Q^s (that is in the direction of L^u). This contradicts the definition of $\mathscr{F}^u$ and $\mathscr{F}^s$. Thus the point (x_1^u, x_0^s) is not in the image $p(\tilde{M})$.

By the same reason, the point (x_{-1}^u, x_0^s) is not in the image $p(\tilde{M})$.

Now we assume that the action of $[\gamma]$ is attracting in a neighborhood of (x_0^u, x_0^s) in the upper half plane $Q^u \times [x_0^s, \infty)$. Let x_1^s $(x_0^s < x_1^s)$ be the adjacent fixed point for the action of $[\gamma]$. Then the point (x_0^u, x_1^s) belongs to $p(\tilde{M})$ by the following reason. By the invariance of $p(\tilde{M})$ under the action, $(x_{-1}^u, x_1^u) \times [x_0^s, x_1^s)$ is contained in $p(\tilde{M})$. See Figure 4. Then by Lemma 5.2 (ii) just shown, the point (x_1^u, x_1^s) belongs to $p(\tilde{M})$. If the point (x_0^u, x_1^s) does not belong to $p(\tilde{M})$, $[x_{-1}^u, x_0^u) \times \{x_1^s\}$ is a horizontal segment on the boundary of $p(\tilde{M})$ contradicting Lemma 4.1 (iv). Hence the point (x_0^u, x_1^s) belongs to $p(\tilde{M})$.

Now again by Lemma 5.2 (ii), if the point (x_0^u, x_1^s) belongs to $p(\tilde{M})$, the point (x_{-1}^u, x_0^s) belongs to $p(\tilde{M})$. This contradicts the above.

If we assume that the action of $[\gamma]$ is attracting in a neighborhood in the lower half plane $Q^u \times (-\infty, x_0^s]$, then we see that the point (x_0^u, x_{-1}^s) belongs to $p(\tilde{M})$, Lemma 5.2 (ii) implies that the point (x_1^u, x_0^s) belongs to $p(\tilde{M})$ and we have contradiction.

If the point (x_0^u, x_0^s) is a repeller in one of the 4 quadrants, we argue in a similar way. That is, then x_0^s is a linearly nontrivial repeller for the action of $[\gamma]$ on Q^s, and for the nearby fixed points x_{-1}^s and x_1^s, the points (x_0^u, x_{-1}^s) and (x_0^u, x_1^s) do not belong to $p(\tilde{M})$, and however, (x_1^u, x_0^s) or (x_{-1}^u, x_0^s) belongs to $p(\tilde{M})$. This contradicts Lemma 5.2 (ii). $\square$

Thus we obtain the following lemma.

Lemma 5.4. *Let γ be a closed orbit. Let $(x_0^u, x_0^s) = p(\tilde{\gamma})$ be the fixed point for the action of $[\gamma]$. Then for $[\gamma]$ or $[\gamma]^{-1}$, x_0^u is a repeller for the action on Q^u and x_0^s is an attractor for the action on Q^s. For the adjacent fixed points $x_{-1}^u < x_0^u < x_1^u$ and $x_{-1}^s < x_0^s < x_1^s$, $\{(x_{-1}^u, x_{-1}^s), (x_1^u, x_1^s)\} \subset p(\tilde{M})$ and*

$$\{(x_{-1}^u, x_0^s), (x_0^u, x_{-1}^s), (x_0^u, x_1^s), (x_1^u, x_0^s)\} \cap p(\tilde{M}) = \varnothing.$$

The cylindrical leaves L^u and L^s containing γ do not contain other closed orbits.

Proof. We proceed as in the proof of Lemma 5.1. Then for the closed orbit γ'', the action of $[\gamma] = [\gamma'] = [\gamma'']$ is hyperbolic in a quadrant. If $\gamma' \neq \gamma''$, we find an attracting or repelling orbit in a quadrant between γ' and γ'', or γ' is attracting or repelling in a quadrant. By Lemma 5.3, $\gamma' = \gamma''$. In a similar way, $\gamma = \gamma'$.

The same argument shows the statement for the adjacent fixed points. □

If there are closed orbits, then we see that the holonomy of the closed orbit is topologically hyperbolic by Lemma 5.4 and we also determined the shape of the orbit space. See Figure 5.

There are periodic homeomorphisms $f_+ : Q^u \to Q^s$ and $f_- : Q^s \to Q^u$ such that

$$p(\tilde{M}) = \{(x^u, x^s); f_-^{-1}(x^u) < x^s < f_+(x^u)\},$$

where periodic means $f_\pm(x+1) = f_\pm(x) + 1$.

Since $p(\tilde{M}) \subset Q^u \times Q^s$ is invariant under the action of $\pi_1(M)$, as the image of the rectangle $(x^u, f_-(x^s)) \times (x^s, f_+(x^u))$ by the action of $\alpha \in \pi_1(M)$, we have the following equality:

$$(\alpha x^u, \alpha f_-(x^s)) \times (\alpha x^s, \alpha f_+(x^u)) = (\alpha x^u, f_-(\alpha x^s)) \times (\alpha x^s, f_+(\alpha x^u)).$$

Hence we have

$$\alpha f_-(x^s) = f_-(\alpha x^s) \quad \text{and} \quad \alpha f_+(x^u) = f_+(\alpha x^u).$$

Then we see that the action of α on Q^u commutes with $f_- f_+$ and the action of α on Q^s commutes with $f_+ f_-$.

To summarize what we have shown, we have the following proposition.

Proposition 5.5. *If a cylindrical leaf L^u of $\mathscr{F}^u$ contains a closed orbit γ, γ is in the intersection of L^u and a cylindrical leaf L^s of $\mathscr{F}^s$ and γ is the unique closed orbit on L^u and on L^s.*

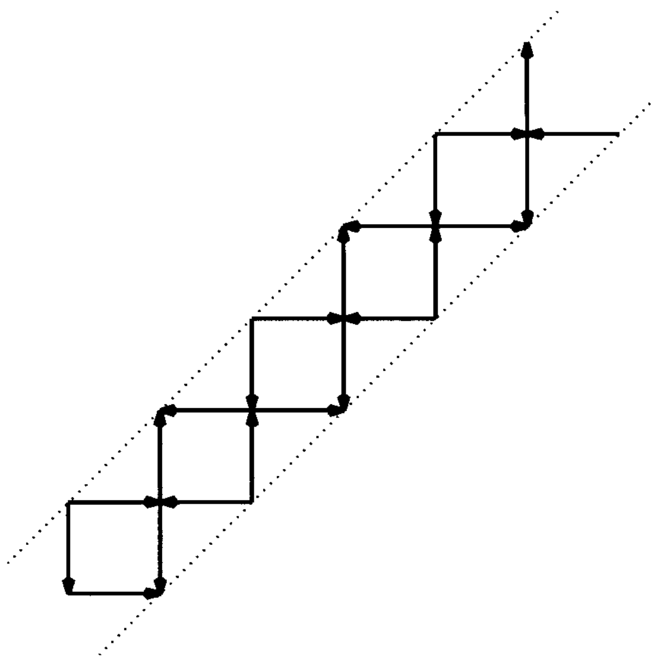

Figure 5. Topologically hyperbolic action.

There are periodic homeomorphisms $f_+ : Q^u \to Q^s$ *and* $f_- : Q^s \to Q^u$ *such that*

$$p(\tilde{M}) = \{(x^u, x^s); f_-^{-1}(x^u) < x^s < f_+(x^u)\},$$

where periodic means $f_\pm(x+1) = f_\pm(x) + 1$.

Let $\tilde{\gamma}$ *be a lift of* γ, *and* $\tilde{L}^u$ *and* $\tilde{L}^s$ *be the lifts containing* $\tilde{\gamma}$. *Then*

$$p(\tilde{L}^u) = (\{p^u(\tilde{\gamma})\} \times Q^s) \cap p(\tilde{M}) = \{p^u(\tilde{\gamma})\} \times (f_-^{-1}(p^u(\tilde{\gamma})), f_+(p^u(\tilde{\gamma}))),$$

$$p(\tilde{L}^s) = (Q^u \times \{p^s(\tilde{\gamma})\}) \cap p(\tilde{M}) = (f_+^{-1}(p^s(\tilde{\gamma})), f_-(p^s(\tilde{\gamma}))) \times \{p^s(\tilde{\gamma})\}.$$

For integers m, $(f_-f_+)^m(p^u(\tilde{\gamma}))$ *are attracting fixed points and* $(f_-f_+)^m f_-(p^s(\tilde{\gamma}))$ *are repelling fixed points for the action of* $[\gamma]$ *on* Q^u, *and* $(f_+f_-)^m f_+(p^u(\tilde{\gamma}))$ *are attracting fixed points and* $(f_+f_-)^m(p^s(\tilde{\gamma}))$ *are repelling fixed points for the action of* $[\gamma]$ *on* Q^s. *Since the actions commute with the action of the center* c, *there is an integer* k *such that* $(f_-f_+)^k(x^u) = x^u + 1$ *and* $(f_+f_-)^k(x^s) = x^s + 1$.

6. Cylindrical leaves without closed orbits.

Assume that a cylindrical leaf L^u of the unstable foliation $\mathscr{F}^u$ does not contain closed orbits of φ. Then by Proposition 2.1, the orbits of φ traverse from one end of L^u to the other. Hence we can take a simple closed transverse curve σ for the orbits on L^u.

Lemma 6.1. *Assume that a cylindrical leaf* L^u *of the unstable foliation* $\mathscr{F}^u$ *does not contain closed orbits of* φ. *Let* σ *be a simple closed transverse curve for the orbits on* L^u.

(i) *By a suitable choice of the lift* $\tilde{L}^u$, *the action of* $[\sigma]$ *on* Q^u *has a fixed point* $x^u = p^u(\tilde{L}^u)$.

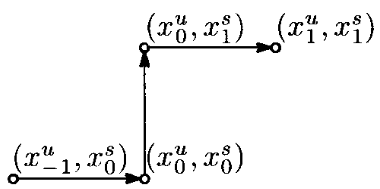

Figure 6. Cylindrical leaves without closed orbits.

(ii) *The action of* $[\sigma]$ *on* $p^s(\tilde{L}^u)$ *has no fixed points.*

(iii) *The action of* $[\sigma]$ *on* Q^s *has fixed points.*

Proof. The statements (i), (ii) follows from the choice of σ and $\tilde{L}^u$.

To show the statement (iii), assume that the action of $[\gamma]$ on Q^s has no fixed points. Then $p(\tilde{L}^u) = \{p^u(\tilde{L}^u)\} \times Q^s$ and since $p(\tilde{M})$ is invariant under the action of the center, Lemma 4.1 (ii) implies $p(\tilde{M}) = Q^u \times Q^s$.

Then we take a point (x^u, x^s) on $p(\tilde{L}^u)$ and draw a curve δ from (x^u, x^s) to (x^u+1, x^s+1). We have a parallelogram P bounded by the curves $\delta, [\sigma]\delta$, and segments $\{x^u\} \times [x^s, [\sigma]x^s]$, $\{x^u+1\} \times [x^s+1, [\sigma]x^s+1]$. If the curve $[\sigma]\delta$ intersects the curve δ, we replace $[\sigma]\delta$ by $[\sigma]^m\delta$ for a large m, and the argument goes without change. The boundary of P can be lifted to $\tilde{M}$ so that they connects $\widetilde{(x^u, x^s)}, c\widetilde{(x^u, x^s)}$, $c[\gamma]\widetilde{(x^u, x^s)}, [\gamma]\widetilde{(x^u, x^s)}$, where c denotes the generator of the center Z of $\pi_1(M)$. Then P

can be lifted to $\tilde{M}$, and in M it defines an immersed torus transverse to the flow. By modifying the immersed transverse torus as in Fried [**9**], we obtain an oriented embedded surface transverse to the flow. Since the embedded surface is transverse to $\mathscr{F}^u$ and $\mathscr{F}^s$, its connected component is again a torus. Since the orbit under $[\sigma]$ and c of P covers $Q^u \times Q^s$, the obtained embedded surface intersects all orbits of φ_t. Thus M is a torus bundle over the circle and is not a Seifert fibered 3-manifold over a hyperbolic orbifold. □

LEMMA 6.2. *Assume that all cylindrical leaves of the unstable foliation $\mathscr{F}^u$ and the stable foliation $\mathscr{F}^s$ do not contain closed orbits of φ. Let σ be a simple closed transverse curve for the orbits on a cylindrical leaf L^u of the unstable foliation $\mathscr{F}^u$. Then the following holds.*

(i) *$p^s(\tilde{L}^u)$ is a bounded interval (x_0^s, x_1^s), where x_0^s and x_1^s are fixed points for the action of $[\sigma]$ on Q^s, and $[\sigma]$ has no fixed points on (x_0^s, x_1^s).*

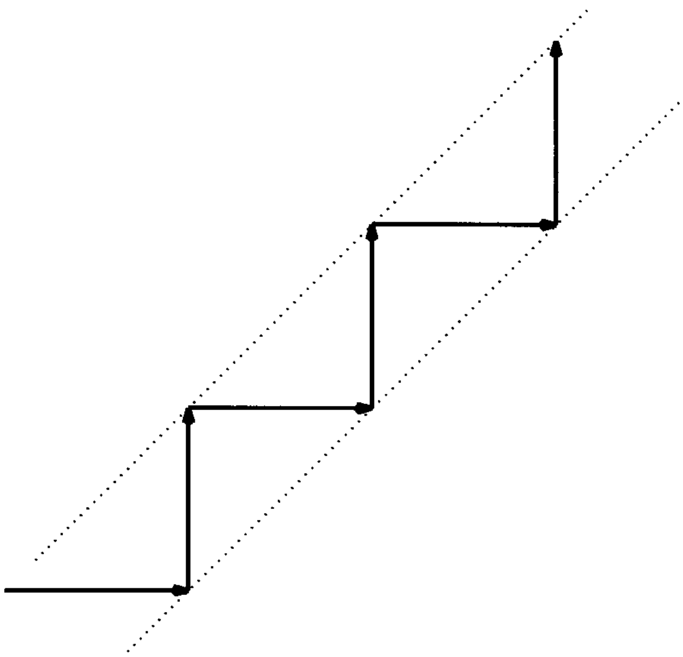

Figure 7. Topologically parabolic action.

(ii) *There are fixed points $x_{-1}^u < x_0^u < x_1^u$ for the action of $[\sigma]$ on Q^u such that*

$$Q^u \times \{x_0^s\} \cap p(\tilde{M}) = (x_{-1}^u, x_0^u) \times \{x_0^s\} \quad \textit{and}$$

$$Q^u \times \{x_1^s\} \cap p(\tilde{M}) = (x_0^u, x_1^u) \times \{x_1^s\}.$$

PROOF. (i) is shown in the previous Lemma 6.1. If the action of $[\sigma]$ on Q^s has fixed points, then the action of $[\sigma]$ commutes with the action of the center and $p^s(\tilde{L}^u)$ is a bounded interval (x_0^s, x_1^s), where x_0^s and x_1^s are fixed points for the action of $[\sigma]$ on Q^s. Then $p(\tilde{L}^u) = \{x_0^u\} \times (x_0^s, x_1^s)$, where $x_0^u = p^u(\tilde{L}^u)$ is a fixed point for the action of $[\sigma]$ on Q^u, and $[\sigma]$ has no fixed points on (x_0^s, x_1^s). We look at the intersection of $Q^u \times \{x_0^s\}$ and $p(\tilde{M})$, or $Q^u \times \{x_1^s\}$ and $p(\tilde{M})$. See Figure 6. Then we see that $Q^u \times \{x_1^s\}$

intersects $p(\tilde{M})$ on $(x_0^u, x_1^u) \times \{x_1^s\}$ for $x_0^u < x_1^u$ and $Q^u \times \{x_0^s\}$ intersects $p(\tilde{M})$ on $(x_{-1}^u, x_0^u) \times \{x_0^s\}$ for $x_{-1}^u < x_0^u$. Then x_{-1}^u and x_1^u are the fixed points of the action of $[\sigma]$ on Q^u. Since we are assuming that there are no closed orbits, there are no fixed points for this action on (x_{-1}^u, x_0^u) or (x_0^u, x_1^u).

Considering Lemma 4.1 (iv), if the action of $[\sigma]$ on $(x_0^s, x_1^s) \subset Q^s$ is increasing, then the action of $[\sigma]$ on (x_{-1}^u, x_0^u) or (x_0^u, x_1^u) is increasing. □

To summarize what we know for the (unrealized) case where there are no closed orbits, we have the following proposition.

Proposition 6.3. *Suppose that all cylindrical leaves of the unstable foliation $\mathscr{F}^u$ and the stable foliation $\mathscr{F}^s$ do not contain closed orbits of φ. Then there are periodic homeomorphisms $f_+ : Q^u \to Q^s$ and $f_- : Q^s \to Q^u$ such that*

$$p(\tilde{M}) = \{(x^u, x^s); f_-^{-1}(x^u) < x^s < f_+(x^u)\}.$$

Let σ be a simple closed transverse curve for the orbits on L^u. Then

$$p(\tilde{L}^u) = (\{p^u(\tilde{L}^u)\} \times Q^s) \cap p(\tilde{M}) = \{p^u(\tilde{L}^u)\} \times (f_-^{-1}(p^u(\tilde{L}^u)), f_+(p^u(\tilde{L}^u))).$$

For integers m, $(f_- f_+)^m(p^u(\tilde{L}^u))$ are fixed points for the action of $[\sigma]$ on Q^u, and $(f_+ f_-)^m f_+(p^u(\tilde{L}^u))$ are fixed points for the action of $[\sigma]$ on Q^s. Both of the actions of $[\sigma]$ on Q^u and on Q^s are simultaneously non decreasing or non increasing. Since the actions commute with the action of the center c, there is an integer k such that $(f_- f_+)^k(x^u) = x^u + 1$ and $(f_+ f_-)^k(x^s) = x^s + 1$.

7. Conjugate actions on the circle.

By Propositions 5.5 and 6.3, we know of the shape of $p(\tilde{M}) \subset Q^u \times Q^s$. That is, we always have periodic homeomorphisms $f_+ : Q^u \to Q^s$ and $f_- : Q^s \to Q^u$ such that

$$p(\tilde{M}) = \{(x^u, x^s); f_-^{-1}(x^u) < x^s < f_+(x^u)\}.$$

Moreover there is an integer k such that $(f_- f_+)^k(x^u) = x^u + 1$ and $(f_+ f_-)^k(x^s) = x^s + 1$.

Now we look at the circles obtained from Q^u and Q^s by identifying by the actions of $f_- f_+$ and $f_+ f_-$, respectively. Let $S_u^1 = Q^u/(f_- f_+)$ and $S_s^1 = Q^s/(f_+ f_-)$ denote them. The graphs of f_+ and f_- are identified to give rise to a graph of a homeomorphism $f : Q^u/(f_- f_+) \to Q^s/(f_+ f_-)$. The diagonal action of $\pi_1(M)$ on $Q^u \times Q^s$ induces an action of $\pi_1(M)$ on $S_u^1 \times S_s^1 = Q^u/(f_- f_+) \times Q^s/(f_+ f_-)$ for which the graph of f is an invariant subset. Note that this action factors through the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\Sigma)$ of the base space Σ of the Seifert fibered 3-manifold M.

The actions of $\pi_1(M)$ on S_u^1 and S_s^1 are conjugate by the homeomorphism $f : S_u^1 \to S_s^1$.

We arrived at the delicate point that $f_- f_+$ and $f_+ f_-$ may not be differentiable. They are topological translations whose k-th powers are the translation by 1. The argument by Ghys [**13**] on the projective structure on the general fiber of the Siefert fibered 3-manifold asserts that in fact they are differentiable. See Section 9.

8. Convergence group action.

A convergence group is a subgroup G of the group of orientation preserving homeomorphisms of the circle with the following property: For any infinite sequence $\{g_i\}$ of G, there are a pair of points x and y (possibly $x = y$) and a subsequence $\{g_{i_j}\}$ such that $g_{i_j} \to x$ uniformly on the compact sets in $S^1 - \{y\}$ and $g_{i_j}^{-1} \to y$ uniformly on the compact sets in $S^1 - \{x\}$. By the result of Tukia [**27**], Casson-Jungreis [**4**] or Gabai [**10**], the convergence groups are topologically conjugate to the Fuchsian group.

For our case, we have a little easier criterion to be a convergence group which is found in [**1**].

Theorem 8.1 ([**1**]). *A finitely generated group G of* $\mathrm{Homeo}_+(S^1)$ *satisfying the following condition is topologically conjugate to a subgroup of $PSL(2;\boldsymbol{R})$.*

(i) *All orbits are dense.*
(ii) *Any element g of G has at most two fixed points and if g has two fixed points, one is attracting and the other is repelling.*
(iii) *The isotropy subgroup of a point is either trivial or cyclic.*
(iv) *For a fixed point $(x_0, y_0) \in S^1 \times S^1 \backslash \Delta$ for the diagonal action of $\alpha \in G$ on $S^1 \times S^1$, its G orbit is discrete on $S^1 \times S^1 \backslash \Delta$.*
(v) *G is not a free group.*

Moreover if G is not conjugate to a group generated by an irrational rotation, G is a convergence group.

We are going to show that the image of $\pi_1(M)$ in $\mathrm{Homeo}(S^1_u)$ is a convergence group.

If we identify $S^1_u \times S^1_s$ with $S^1_u \times S^1_u$, by the homeomorphism

$$(\mathrm{id}, f^{-1}) : S^1_u \times S^1_s \to S^1_u \times S^1_u,$$

the action of an element α of $\pi_1(M)$ on $S^1_u \times S^1_u$ is given by

$$(\alpha(x), f^{-1}(\alpha(f(y)))),$$

and the action is the diagonal action. That is, the action of $\pi_1(M)$ on $S^1_u \times S^1_s$ is conjugate to the diagonal action.

Now we use the action of $\pi_1(M)$ on $S^1_u \times S^1_s$ to show that the image of $\pi_1(M)$ in $\mathrm{Homeo}(S^1_u)$ satisfies the conditions of Theorem 8.1.

Lemma 8.2. *If the action of an element α of $\pi_1(M)$ has a fixed point $[x^u] \in Q^u/(f_- f_+) = S^1_u$, then $f([x^u]) \in Q^s/(f_+ f_-) = S^1_s$ is a fixed point. If the action is not trivial, either the action of α has no other fixed points on $Q^u/(f_- f_+) = S^1_u$, or the action of α has only one other fixed point $[x^u_1] \in Q^u/(f_- f_+) = S^1_u$ and one of the two fixed points is attracting and the other is repelling.*

Proof. Let k be the integer such that $(f_- f_+)^k(x^u) = x^u + 1$. If there is a fixed point $[x^u] \in Q^u/(f_- f_+) = S^1_u$, there is an integer ℓ such that the action of $\alpha^k c^\ell$ on Q^u has a fixed point. Since $p(\tilde{M})$ is invariant under the diagonal action of $\alpha^k c^\ell$, the action of $\alpha^k c^\ell$ on Q^s has fixed points. If there are two other fixed points on $Q^u/(f_- f_+) = S^1_u$, the configuration of the fixed points gives two closed orbits on a leaf of $\mathscr{F}^s$ or $\mathscr{F}^u$. This contradicts Lemma 5.4. $\square$

Lemma 8.3. *If the action of an element α of $\pi_1(M)$ on S^1_u has a periodic point of period greater than* 1, *the action of α is finite order.*

Proof. Let k be the integer such that $(f_-f_+)^k(x^u) = x^u + 1$. Assume that the action of α is not finite order and $[x^u] \in S^1_u$ is a non trivial periodic point. Then there is an integer $m > 1$ such that $\alpha^m[x^u] = [x^u]$ and $[x^u], \alpha[x^u], \dots, \alpha^{m-1}[x^u]$ are distinct points in S^1_u. Since these $[x^u], \alpha[x^u], \dots, \alpha^{m-1}[x^u]$ are fixed points for α^m and they are conjugate by the action of α, we have a contradiction to Lemma 8.2. □

Lemma 8.4. *Assume that there is an element α of $\pi_1(M)$ such that the action of α on S^1_u has no periodic points. Then the followings hold.*

(i) *The action of α on S^1_s has no periodic points.*
(ii) *The action of α on S^1_u or S^1_s is topologically conjugate to an irrational rotation.*
(iii) *There is a closed orbit γ for the projectively Anosov flow φ.*

Proof. By Lemma 8.2, the assertion (i) follows.

The assertion (ii) follows from the assumption that foliations are smooth and the Denjoy theorem for the action of α on $Q^u/\boldsymbol{Z}$ or $Q^s/\boldsymbol{Z}$ which is a k-fold covering of S^1_u or S^1_s. Since the action on $Q^u/\boldsymbol{Z}$ or $Q^s/\boldsymbol{Z}$ is topologically conjugate to an irrational rotation, so is that on S^1_u or S^1_s.

To show the assertion (iii), first note that the action of α and c generates a group topologically conjugate to a dense subgroup of translations of Q^u and Q^s. Hence there are integers m, n such that $\alpha^m c^n$ is a positive translation C^0 close to the identity.

If we do not find a closed orbit, we may assume that the situation is as in Proposition 6.3. Then for an element $\beta \in \pi_1(M)$ which has a fixed point x^u on Q^u, the action of β is parabolic like and we assume it is not decreasing on $\{x^u\} \times Q^s$.

Then if $r = \alpha^m c^n$ is taken close to the identity, there are points x^u_1 a little larger than x^u and x^u_2 a little smaller than $f_-f_+(x^u)$ such that $r(x^u_1) = \beta(x^u_1)$ and $r(x^u_2) = \beta(x^u_2)$, respectively. There are also points x^s_1 a little larger than $f_-^{-1}(x^u)$ and x^s_2 a little smaller than $f_+(x^u)$ such that $r(x^s_1) = \beta(x^s_1)$ and $r(x^s_2) = \beta(x^s_2)$, respectively. Then $(x^u_1, x^s_2) \in p(\tilde{M})$ is a fixed point for the action of $r^{-1}\beta$ on $p(\tilde{M})$. Thus this corresponds to a periodic orbit of φ. □

Lemma 8.5. *Either the action of an element α of $\pi_1(M)$ is conjugate to a rotation of finite order, or the action of α has* 1 *or* 2 *fixed points on $Q^u/(f_-f_+)$. If the action has* 2 *fixed points, one is attracting and the other is repelling.*

Proof. We show that there are no elements α of $\pi_1(M)$ such that the action of α on S^1_u has no periodic points. Then the statement follows from Lemmas 8.2 and 8.3.

If such α exists, by Lemma 8.4 (iii), there is a closed orbit γ. It is easy to show that the image of the lifts of γ which is the $\pi_1(M)$ orbit of the image $p(\tilde{\gamma})$ of a lift is discrete in $p(\tilde{M})$. (For, a finite family of compact disks in $p(\tilde{M})$ has a lift in $\tilde{M}$ and project to M which have only finitely many intersections with γ.) On the other hand, the action of α and c makes an accumulation of the $\pi_1(M)$ orbit of $p(\tilde{\gamma})$ to $p(\tilde{\gamma})$. This contradiction shows the lemma. □

Proposition 8.6. *The image G of $\pi_1(M)$ or $\pi_1^{\mathrm{orb}}(\Sigma)$ in* $\mathrm{Homeo}(S^1_u)$ *is a convergence group.*

Proof. We verify the conditions of Theorem 8.1. The condition (i) is Corollary 3.5. The condition (ii) is Lemma 8.5. The condition (iii) also follows from Corollary 3.5, because the isotropy subgroup is the holonomy of the leaves of $\mathscr{F}^u$ or $\mathscr{F}^s$, which are homeomorphic either to the plane or to the cylinder. The condition (iv) is satisfied because the fixed point $(x_0, y_0) \in S^1_u \times S^1_u \backslash \Delta$ corresponds to a closed orbit γ. The $\pi_1(M)$ orbit of $\widetilde{(x_0, y_0)}$ in the universal cover $\tilde{X}$ of $X = S^1_u \times S^1_u \backslash \Delta$ is the image of the lifts of γ and is discrete in $p(\tilde{M})$ identified with $\tilde{X}$. Since the image is invariant under the action of the center Z and $\tilde{X}/Z$ is a k fold covering of $X = S^1_u \times S^1_u \backslash \Delta$, the G orbit of (x_0, y_0) is discrete in $S^1_u \times S^1_u \backslash \Delta$. The condition (v) is satisfied because G is image of the fundamental group of the base orbifold.

Since we showed in Lemma 8.5 that G is not conjugate to a group generated by an irrational rotation, G is a convergence group. □

Thus the image G of $\pi_1(M)$ in $\mathrm{Homeo}(S^1_u)$ is a convergence group. By the result of Tukia [**27**], Casson-Jungreis [**4**] or Gabai [**10**], this group is topologically conjugate to a Fuchsian group.

A Fuchsian group corresponds to a hyperbolic orbifold Σ_0 and it is isomorphic to the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\Sigma_0)$.

Lemma 8.7. *The base orbifold Σ of Seifert fibered* 3*-manifold M is diffeomorphic to Σ_0.*

Proof. We have the surjective homomorphism $\pi_1^{\mathrm{orb}}(\Sigma) \to \pi_1^{\mathrm{orb}}(\Sigma_0)$. If the kernel is non trivial, we take a closed curve γ representing a nontrivial element of the kernel. Then the global holonomy of the foliation $\mathscr{F}^u$ (or $\mathscr{F}^s$) along γ is the identity. As in the proof of Lemma 5.1, using Corollary 3.5, we see that all leaves of the unstable foliation $\mathscr{F}^u$ are cylinders, that the unstable foliation $\mathscr{F}^u$ is without holonomy and topologically conjugate to a linear foliation by cylinders on T^3, contradicting the assumption on the manifold M (or Lemma 4.2). □

Since the Fuchsian group action is determined by the base orbifold Σ of the Seifert fibered 3-manifold, it cannot have the parabolic element whose action was drawn in Figure 7.

9. Proof of the main theorem.

Let Σ be a hyperbolic orbifold with the hyperbolic structure g_0. We have the Anosov geodesic flow on the unit tangent bundle $T_1\Sigma$ of the orbifold Σ. The stable foliation and the unstable foliation for the Anosov geodesic flow are both defined by the holonomy homomorphism which is a Fuchsian group representation:

$$\pi_1^{\mathrm{orb}}(\Sigma) \to PSL(2; \boldsymbol{R}) \subset \mathrm{Homeo}(\boldsymbol{R}P^1).$$

Note that the holonomy homomorphisms for the stable foliation and for the unstable foliation are conjugate by an elliptic element of $PSL(2; \boldsymbol{R})$ and the holonomy homomorphisms for different hyperbolic structures are topologically conjugate.

The homomorphism $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{Homeo}(S^1_u)$ derived from $\mathscr{F}^u$ determines a foliation of a Seifert fibered space over Σ. By the argument of Section 8, $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{Homeo}(S^1_u)$ is topologically conjugate to a Fuchsian group representation $\pi_1^{\mathrm{orb}}(\Sigma) \to PSL(2; \boldsymbol{R})$.

Hence $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{Homeo}(S_u^1)$ determines a topological foliation of $T_1\Sigma$ transverse to the fibers.

By the argument of Section 7, the action of $\pi_1^{\mathrm{orb}}(\Sigma)$ on $S_u^1 = Q^u/(f_-f_+)$ lifts to the action on the k-fold cyclic covering $Q^u/\boldsymbol{Z}$ of $S_u^1 = Q^u/(f_-f_+)$. That is, we have a homomorphism $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{Homeo}(Q^u/\boldsymbol{Z})$. Hence the Fuchsian group action of $\pi_1^{\mathrm{orb}}(\Sigma)$ on $\boldsymbol{RP}^1$ also lifts to the action on the k-fold cyclic covering $(\boldsymbol{RP}^1)_k$ of $\boldsymbol{RP}^1$. This determines the transversely projective foliations $F_{g_0}^u$ or $F_{g_0}^s$ of $M_k = (T_1\Sigma)_k$ which is the k-fold cyclic covering of $T_1\Sigma$ in the direction of the fibers. Since the lifted homomorphism $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{Homeo}(Q^u/\boldsymbol{Z})$ determines the smooth foliation $\mathscr{F}^u$, this is a homomorphism to the diffeomorphism group: $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{Diffeo}(Q^u/\boldsymbol{Z})$.

Now Ghys showed in [**13**] the following:

"*A smooth foliation $\mathscr{F}$ of M_k topologically isotopic to $F_{g_0}^u$ is transversely projective and hence differentiably isotopic to F_g^u for a hyperbolic structure g of Σ.*"

By this result, $\mathscr{F}^u$ is differentiably isotopic to $F_{g_u}^u$.

In the same way, there is a hyperbolic structure g_s such that $\mathscr{F}^s$ is differentiably isotopic to $F_{g_s}^s$.

For a pair g_u and g_s of hyperbolic structures on Σ, we have the quasi-Fuchsian flow on $T_1\Sigma$ which is the Anosov flow with the unstable foliation and the stable foliation differentiably isotopic to those of the geodesic flows determined by the hyperbolic structures g_u and g_s, respectively [**12**]. We can lift the quasi-Fuchsian flow to M_k and obtain an Anosov flow ϕ on M_k such that the unstable foliation and the stable foliation are differentiably isotopic to $F_{g_u}^u$ and $F_{g_s}^s$, respectively.

We are going to show that our projectively Anosov flow φ is differentiably isotopic to the quasi-Fuchsian flow ϕ.

There is a homeomorphism $H \in \mathrm{Homeo}(\boldsymbol{RP}^1)$ which conjugates the holonomy homomorphism h^u of the unstable foliation of the geodesic flow determined by the hyperbolic structures g_u to that h^s of the stable foliation of the geodesic flow determined by the hyperbolic structures g_s:

$$H(h_\alpha^u(H^{-1}(x^s))) = h_\alpha^s(x^s) \quad \text{for } \alpha \in \pi_1^{\mathrm{orb}}(\Sigma).$$

This homeomorphism H lifts to homeomorphisms of the k-fold cyclic covering $(\boldsymbol{RP}^1)_k$ as well as those of the universal covering $\widetilde{\boldsymbol{RP}}^1$.

We identify $Q^u/\boldsymbol{Z}$ with $(\boldsymbol{RP}^1)_k$ by the diffeomorphism conjugating the holonomy of $\mathscr{F}^u$ to that of $F_{g_u}^u$. Since the action of f_-f_+ on $Q^u/\boldsymbol{Z}$ is order k and commutes with all the holonomy, the action of f_-f_+ is conjugated to the $1/k$ rotation on $(\boldsymbol{RP}^1)_k$. For, if the action of $\alpha \in \pi_1^{\mathrm{orb}}(\Sigma)$ has hyperbolic fixed points, it has k attracting fixed points in $Q^u/\boldsymbol{Z}$ which are in the orbit of the $1/k$ rotation in $(\boldsymbol{RP}^1)_k$. Such fixed points are dense in the circle. This shows that f_-f_+ is smooth on $Q^u/\boldsymbol{Z}$.

In the same way, we identify $Q^s/\boldsymbol{Z}$ with $(\boldsymbol{RP}^1)_k$ by the diffeomorphism conjugating the holonomy of $\mathscr{F}^s$ to that of $F_{g_s}^s$ and the action of f_+f_- is also conjugated to the $1/k$ rotation on $(\boldsymbol{RP}^1)_k$.

Hence $S_u^1 = Q^u/(f_-f_+)$ and $S_s^1 = Q^s/(f_+f_-)$ have the differentiable structure and in fact the homomorphisms $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{Homeo}(S_u^1)$ and $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{Homeo}(S_s^1)$ are homomorphisms to the diffeomorphism groups: $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{Diffeo}(S_u^1)$ and $\pi_1^{\mathrm{orb}}(\Sigma) \to$

$\mathrm{Diffeo}(S^1_s)$, respectively. As is shown in Section 7, these two homomorphisms are conjugate by the homeomorphism $f : S^1_u \to S^1_s$.

The result of Ghys implies that $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{Diffeo}(S^1_u)$ and $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{Diffeo}(S^1_s)$ are conjugate to the holonomy homomorphisms of Anosov foliations for the geodesic flows with respect to g_u and g_s, respectively. We identify S^1_u, S^1_s with $\boldsymbol{RP}^1$ by the conjugating diffeomorphisms, and we compare the homeomorphisms H and f. Since $H^{-1}f$ commute with h^u_α for any $\alpha \in \pi_1^{\mathrm{orb}}(\Sigma)$, f coincides with H:

$$\begin{array}{ccc} S^1_u & \xrightarrow{f} & S^1_s \\ \| & & \| \\ \boldsymbol{RP}^1 & \xrightarrow[H]{} & \boldsymbol{RP}^1. \end{array}$$

For the foliations $F^u_{g_u}$ and $F^s_{g_s}$ of M_k, we take the lifts $\tilde{F}^u_{g_u}$ and $\tilde{F}^s_{g_s}$ of them to the universal covering space $\tilde{M}_k$. The lifted foliations $\tilde{F}^u_{g_u}$ and $\tilde{F}^s_{g_s}$ are product foliations and we look at the map to the product of their leaf spaces:

$$p_0 : \tilde{M}_k \to \tilde{M}_k/\tilde{F}^u_{g_u} \times \tilde{M}_k/\tilde{F}^s_{g_s}.$$

Since the product action on $\boldsymbol{RP}^1 \times \boldsymbol{RP}^1$ of the Fuchsian group actions on $\boldsymbol{RP}^1$ with respect to g_u and g_s leaves the graph of H invariant, the image $p_0(\tilde{M}_k)$ is the region between the two adjacent graphs of lifts of H. (See [**12**]).

The action of $\pi_1(M)$ on $p(\tilde{M})$ is effective, because the action of $\pi_1^{\mathrm{orb}}(\Sigma)$ on S^1_u or S^1_s is effective and the action of the class of general fiber is a nontrivial translation. Since the image of $\pi_1(M)$ in $\mathrm{Homeo}(p(\tilde{M}))$ coincides with the image of $\pi_1(M_k)$ in $\mathrm{Homeo}(p(\tilde{M}))$, we obtain the isomorphism; $\pi_1(M) \cong \pi_1(M_k)$.

Now we can prove that our projecctively Anosov flow φ is differentiablly isotopic to the quasi-Fuchsian flow ϕ by looking at the transverse structures of the flows φ and ϕ.

Proof of Theorem 1.1. We follow the argument by Ghys [**12**].

We compare the map $p : \tilde{M} \to Q^u \times Q^s = \tilde{M}/\tilde{\mathscr{F}}^u \times \tilde{M}/\tilde{\mathscr{F}}^s$ defined by $\tilde{\mathscr{F}}^u$ and $\tilde{\mathscr{F}}^s$ with $p_0 : \tilde{M}_k \to \tilde{M}_k/\tilde{F}^u_{g_u} \times \tilde{M}_k/\tilde{F}^s_{g_s}$ defined by $\tilde{F}^u_{g_u}$ and $\tilde{F}^s_{g_s}$. As we discussed, the images of p and p_0 coincide. The actions of $\pi_1(M) \cong \pi_1(M_k)$ on the images also coincide. They define the transverse structure of the orbit foliations φ and ϕ. This implies that the holonomy groupoids for φ and ϕ are equivalent.

For a closed orbit c of φ on M, we look at a lift $\tilde{c}$ of c in $\tilde{M}$ and its image in $Q^u \times Q^s$. The curve c represents an element $\alpha \in \pi_1(M)$ and the image of $\tilde{c}$ in $Q^u \times Q^s$ is a fixed point of the action of α and the action is topologically hyperbolic. Hence the holonomy covering of c is contractible.

For a closed orbit c of ϕ on M_k, we have a lift $\tilde{c}$ of c in $\tilde{M}_k$ and its image in $\tilde{M}_k/\tilde{F}^u_{g_u} \times \tilde{M}_k/\tilde{F}^s_{g_s}$. In a similar way, the holonomy covering of c is contractible.

Thus both (M, φ) and (M_k, ϕ) are the classifying space for the groupoid ([**14**]). Hence we have a homotopy equivalence $M \to M_k$ which sends the orbit of φ to the orbit of ϕ and is transversely a diffeomorphism. As in [**11**] (see also [**1**], [**18**]), one can deform this homotopy equivalence to a diffeomorphism which sends an orbit of φ to an orbit of ϕ.

In fact the Seifert fibered 3-manifold M is determined by its fundamental group. Since the resultant diffeomorphism can be thought inducing the identity on the fundamental group, it is isotopic to the identity. $\square$

REMARK 9.1. Since the fibrations $\tilde{M} \to p(\tilde{M})$ and $\tilde{M}_k \to p_0(\tilde{M}_k)$ are locally trivial fibration with fiber being $\boldsymbol{R}$, we can construct an equivariant lift $\tilde{M} \to \tilde{M}_k$ as described in [**18**]:

$$\begin{tikzcd} \tilde{M} \arrow[r] \arrow[d, "p"'] & \tilde{M}_k \arrow[d, "p_0"] \\ p(\tilde{M}) \arrow[r, equal] & p_0(\tilde{M}_k). \end{tikzcd}$$

This also shows that we have a homotopy equivalence $M \to M_k$ which sends the orbit of φ to the orbit of ϕ which is transversely a diffeomorphism.

References

[1] T. Barbot, Flot d'Anosov sur les variétés graphées au sens de Waldhausen, Ann. Inst. Fourier (Grenoble), **46** (1996), 1451–1517.

[2] M. Brittenham, Essential laminations in Seifert fibered spaces, Topology, **32** (1993), 61–85.

[3] J. Cantwell and L. Conlon, Endsets of exceptional leaves; A theorem of Duminy, Proceedings of Foliations: Geometry and Dynamics, Warsaw, 2000, World Scientific, Singapore, 2002, 225–261.

[4] A. Casson and D. Jungreis, Convergence groups and Seifert fibered 3-manifolds, Invent. Math., **118** (1994), 441–456.

[5] G. Duminy, Thesis.

[6] D. Eisenbud, U. Hirsch and W. Neumann, Transverse foliations of Seifert bundles and self-homeomorphism of the circle, Comment. Math. Helv., **56** (1981), 638–660.

[7] Y. Eliashberg and W. Thurston, Confoliations, Univ. Lecture Ser., **13**, Amer. Math. Soc., 1978.

[8] S. Fenley, Anosov flows in 3-manifolds, Ann. of Math. (2), **139** (1994), 79–115.

[9] D. Fried, Transitive Anosov flows and pseudo-Anosov maps, Topology, **22** (1983), 299–303.

[10] D. Gabai, Convergence groups are Fuchsian groups, Ann. of Math. (2), **136** (1992), 447–510.

[11] É. Ghys, Flots d'Anosov sur les 3-variétés fibrées en cercles, Ergodic Theory Dynam. Systems, **4** (1984), 67–80.

[12] É. Ghys, Déformations de flots d'Anosov et de groupes fuchsiens, Ann. Inst. Fourier (Grenoble), **42** (1992), 209–247.

[13] É. Ghys, Rigidité différentiable des groupes fuchsiens, Inst. Hautes Études Sci. Publ. Math., **78** (1993), 163–185.

[14] A. Haefliger, Groupoïde d'holonomie et classifiants, Structure Transverse des Feuilletages, Astérisque, **116** (1984), 70–97.

[15] G. Hector, Architecture des feuilletages de classe C^2, Astérisque, **107–108** (1983), 243–258.

[16] G. Levitt, Feuilletages des variétés de dimension 3 qui sont des fibres en cercles, Comment. Math. Helv., **53** (1978), 572–594.

[17] S. Matsumoto, Foliations of Seifert fibered spaces over S^2, Foliations Tokyo, 1983, (ed. I. Tamura) Adv. Stud. Pure Math., **5**, Kinokuniya, Tokyo; North-Holland, Amsterdam-New York-Oxford, 1985, pp. 325–339.

[18] S. Matsumoto and T. Tsuboi, Transverse intersections of foliations in three-manifolds, Monographie de L'Enseignement Math., **38** (2001), 503–525.

[19] Y. Mitsumatsu, Anosov flows and non-Stein symplectic manifolds, Ann. Inst. Fourier (Grenoble), **45** (1995), 1407–1421.

[20] T. Noda, Projectively Anosov flows with differentiable (un)stable foliations, Ann. Inst. Fourier (Grenoble), **50** (2000), 1617–1647.

[21] T. Noda, Regular projectively Anosov flows with compact leaves, Ann. Inst. Fourier (Grenoble), **54** (2004), 353–363.

[22] T. Noda and T. Tsuboi, Regular projectively Anosov flows without compact leaves, Proceedings of Foliations: Geometry and Dynamics, Warsaw, 2000, World Scientific, Singapore, 2002, 403–419.

[23] I. Tamura and A. Sato, On transverse foliations, Inst. Hautes Études Sci. Publ. Math., **54** (1981), 205–235.

[24] W. Thurston, Foliations of 3-manifolds that are circle bundles, University of California at Berkeley, Thesis, 1972.

[25] W. Thurston, Three-dimensional geometry and topology, Vol. 1, Princeton Math. Ser., **35**, Princeton Univ. Press, Princeton, NJ, 1997.

[26] W. Thurston, Three-manifolds, Foliations and Circles, I, e-Print archive, math/9712268.

[27] P. Tukia, Homeomorphic conjugates of Fuchsian groups, J. Reine Angew. Math., **391** (1988), 1–54.

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