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A surgery formula for the discrete Godbillon-Vey invariant

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Abstract. Transversely piecewise linear foliations of codimension one on closed 3-manifolds are deformed by surgery. We will show the surgery formula for the discrete Godbillon-Vey invariant of these foliations.

1. Introduction.

The Godbillon-Vey invariant was found for C^2 foliations of codimension one as a closed 3-form in 1971 ([**7**]). This is the first invariant defined for the foliations. Thurston succeeded in describing it as a 2-cocycle of the groups of C^2 diffeomorphisms of the circle using the area surrounded by closed curves ([**1**]). Because the second derivatives of diffeomorphisms are necessary to define it, we had no invariant for foliations which have only lower differentiabilities. In 1987, Ghys and Sergiescu [**6**] defined a 2-cocycle of the groups of piecewise linear homeomorphisms of the circle using the "discrete area." This invariant is called the *discrete Godbillon-Vey invariant*. This is extended to an invariant for transversely piecewise linear foliations of codimension one on closed 3-manifolds by Tsuboi [**11**], which is denoted by $\overline{\mathrm{GV}}$. These foliations can be deformed by surgery which is defined in Section 3. The surgery is topologically $(1,1)$ Dehn surgery. In this paper, we will describe the relation between $\overline{\mathrm{GV}}$ and the surgery.

Theorem. *Let M be an oriented closed* 3*-manifold and $\mathscr{F}_0$, a transversely oriented, transversely piecewise linear foliation of codimension one on M. Suppose that there are a leaf L and a simple closed curve $C \subset L$ with a holonomy h_λ where $h_\lambda(z) = e^{-\lambda}z$ $(\lambda > 0)$. If $\mathscr{F}$ is obtained from $\mathscr{F}_0$ by operating the surgery along C, then*

$$\overline{\mathrm{GV}}(\mathscr{F}) = \overline{\mathrm{GV}}(\mathscr{F}_0) - \lambda^2.$$

In the last of this paper, using this theorem, we calculate the discrete Godbillon-Vey invariants of the unstable foliations of the geodesic flows of some hyperbolic orbifolds. From this result, we calculate the discrete Godbillon-Vey invariants in case of the hyperbolic closed surfaces.

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2. Transversely piecewise linear foliations.

We begin in more general situation. Let M be a closed oriented 3-dimensional manifold and $\mathscr{F}$, a transversely oriented, transversely piecewise C^2 foliation of codimension one on M.

Lemma 1. *There exist finite leaves $L_1, L_2, L_3, \dots, L_l$ and a compact subset $K \subset L_1 \cup L_2 \cup \cdots \cup L_l$ such that $\mathscr{F}$ is of class C^2 outside K.*

Proof. For a foliated atlas, the changes of transverse coordinates are piecewise C^2 homeomorphisms of intervals. Let K be the union of the closures of intersections of plaques which correspond to the bending points of the changes of transverse coordinates. Then $\mathscr{F}$ is of class C^2 outside K. If K intersects infinitely many leaves, there exists a change of transverse coordinates whose bending points are accumulated since M is closed. Hence K is contained in only finite leaves $L_1, L_2, L_3, \dots, L_l$. By the same argument, for all $i = 1, 2, \dots, l$, $K \cap L_i$ is compact in L_i. □

Therefore, $\mathscr{F}$ is defined by a 1-form ω on $M - K$ and there exists a 1-form η defined on $M - K$ such that $d\omega = \omega \wedge \eta$.

Let $L = L_1 \cup L_2 \cup \cdots \cup L_l$.

Lemma 2. *ω and η are smoothly extended to the boundary of the closure of $M - L$.*

Proof. Let B be the bending points of a piecewise C^2 homeomorphism f defined on an interval $I \subset \boldsymbol{R}$. Because B has no accumulating points in $\boldsymbol{R}$, each restriction of f to a component of $I - B$ smoothly extends to the boundary. Hence $\mathscr{F}|_{M-L}$ induces a smooth foliation on the closure $\overline{M - L}$. Then ω and η are smoothly extended to $\partial(\overline{M - L})$. □

The extension of η is denoted by $\bar{\eta}$. We set $\partial(\overline{M - L}) = L_1^+ \cup L_1^- \cup L_2^+ \cup \cdots \cup L_l^+ \cup L_l^-$ where $L_i^{\pm} \cong L_i$ and the transverse orientation of $\mathscr{F}$ is inward on L_i^+ and outward on L_i^-. Then $\bar{\eta}|_{L_i^{\pm}}$ is regarded as the 1-form on L_i which is denoted by $\eta_{L_i}^{\pm}$. For each leaf $L \in \mathscr{F} - \{L_1, L_2, \dots, L_l\}$, we set $\eta_L^+ = \eta_L^- = \eta|_L$. Let E_L be the 2-form $\eta_L^+ \wedge \eta_L^-$ for $L \in \mathscr{F}$.

Lemma 3. (1) $\eta_{L_i}^+|_{L_i - K} = \eta_{L_i}^-|_{L_i - K}$.

(2) *For $L \in \mathscr{F}$, E_L is closed and compactly supported on L.*

Proof. Since η is originally a smooth 1-form defined on $M - K$, $\eta_{L_i}^+ = \eta = \eta_{L_i}^-$ on $L_i - K$. □

Lemma 4. *For $L \in \mathscr{F}$, the compactly supported de Rham cohomology class represented by E_L depends only on $\mathscr{F}$.*

Proof. $\eta_L^{\pm}$ is uniquely determined by ω as a 1-form on L. If we take other $\tilde{\omega}$ to define $\mathscr{F}$, $\tilde{\omega} = v\omega$ outside a compact set which is contained in finite leaves where v is a C^2 function and never vanishes.

$$d\tilde{\omega} = dv \wedge \omega + v\, d\omega = v\omega \wedge \left(\eta - \frac{dv}{v}\right) = \tilde{\omega} \wedge (\eta - d\log|v|).$$

So, we set $\tilde{\eta} = \eta - d\log|v|$.

We can define $v_L^{\pm}$ in the same way as $\eta_L^{\pm}$. Then

$$\tilde{\eta}_L^{\pm} = \eta_L^{\pm} - d\log|v_L^{\pm}|.$$

$$\begin{aligned}\tilde{\eta}_L^{+} \wedge \tilde{\eta}_L^{-} &= (\eta_L^{+} - d\log|v_L^{+}|) \wedge (\eta_L^{-} - d\log|v_L^{-}|)\\ &= \eta_L^{+} \wedge \eta_L^{-} - d(\log|v_L^{+}|\eta_L^{-} - \log|v_L^{-}|\eta_L^{+}) + d\log|v_L^{+}| \wedge d\log|v_L^{-}|.\end{aligned}$$

Because $v_L^{+} = v_L^{-} = v$ outside a compact set of L, $\log|v_L^{+}|\eta_L^{-} - \log|v_L^{-}|\eta_L^{+}$ has a compact support. Hence the second term is an exact 1-form of the de Rham complex with compact supports. The third term is equal to

$$d\left\{\frac{1}{2}(\log|v_L^{+}|d\log|v_L^{-}| - \log|v_L^{-}|d\log|v_L^{+}|)\right\}.$$

So, this term is also exact. □

The closed 1-forms $\eta_{L_i}^{+}$ and $\eta_{L_i}^{-}$ represent the infinitesimal holonomy class of L_i. $\eta_{L_i}^{+} - \eta_{L_i}^{-}$ is also a compactly supported 1-form on L_i. Let $H_1^{lf}(L_i;\boldsymbol{Z})$ be the locally finite first homology group of L_i. If $D \in H_1^{lf}(L_i;\boldsymbol{Z})$ satisfies that $\mu = \int_D(\eta_{L_i}^{+} - \eta_{L_i}^{-}) \neq 0$, then there are bending points along D. For every indivisible element $D \in H_1^{lf}(L_i;\boldsymbol{Z})$, there is a simple closed curve C which represents the Poincaré dual of D in $H_1(L_i;\boldsymbol{Z})$. In particular, the intersection number $C \cdot D$ is equal to 1.

From now on, we suppose that $\mathscr{F}$ is transversely piecewise linear. For $\lambda > 0$, let h_λ be a linear map, $z \mapsto e^{-\lambda}z$, on a neighborhood of 0.

Theorem 1. *If the holonomy of C is h_λ, then*

$$\int_{K_0} E_{L_i} = -\lambda\mu$$

where K_0 is the component of $K \cap L_i$ which contains C.

Proof. It is sufficient to prove the statement in the case of a cylindrical leaf since K_0 is contained in a tubular neighborhood of C. Then we may assume that D is a simple curve which connects two ends of the cylindrical leaf and that C is a simple closed curve which intersects D with a point. In this case, we can realize a neighborhood of this leaf as follows.

Put

$$N = \{(x,y,z) \in \boldsymbol{R}^3 \mid x > 0\} = \{[x,Y,\phi] \in \boldsymbol{R}^3 \mid x > 0, Y \geq 0, -\pi \leq \phi \leq \pi\}$$

where $[x,Y,\phi]$ is the cylindrical coordinate of $\boldsymbol{R}^3$, i.e.,

$$[x,Y,\phi] = (x, Y\cos\phi, Y\sin\phi).$$

We also define

$$N_+ = \{[x,Y,\phi] \in N \mid 0 \leq \phi \leq \pi\},$$

$$N_- = \{[x,Y,\phi] \in N \mid -\pi \leq \phi \leq 0\}$$

and

$$N_-^0 = N_- - \{(x,0,0) \mid x > 0\}.$$

N, N_+ and N_- are given the trivial foliation whose leaves are half planes $(0, \infty) \times \boldsymbol{R} \times \{z\}$.

We define two diffeomorphisms $S_\lambda : N \to N$ and $F_\mu : N_-^0 \to N_-^0$ by $S_\lambda([x, Y, \phi]) = [e^\lambda x, e^\lambda Y, \phi]$ and $F_\mu([x, Y, \phi]) = [e^{(1+\phi/\pi)\mu} x, Y, \phi]$.

We use these maps to make a transversely piecewise linear foliation in a solid torus. We glue N_+ and N_-^0 by identifying $[x, Y, \pi]$ (resp. $[x, Y, 0]$) and $[x, Y, -\pi]$ (resp. $F_\mu([x, Y, 0])$) in order to obtain N^0. Since $S_\lambda | N_-^0$ and F_μ commute and they preserve the trivial foliation, S_λ induces a foliation preserving diffeomorphisms $S_\lambda^0 : N^0 \to N^0$ and $N_0 = N^0/[x, Y, \phi] \sim S_\lambda^0([x, Y, \phi])$ is a foliated solid torus. Let Π be the projection $N \to N_0$.

Now we notice a neighborhood of the cylindrical leaf L_0 induced from the x-y plane. We will see that L_0 satisfies the condition of the theorem.

$\omega = dz/r$ defines the trivial foliation of N and is invariant by S_λ where $r = \sqrt{x^2 + y^2 + z^2}$.

$$d\omega = \omega \wedge d \log r.$$

Hence

$$\eta^+ = d \log r.$$

Let N_L be $\{(x, y, z) \in N \mid y < 0\}$ and N_R, $\{(x, y, z) \in N \mid y > 0\}$. Then

$$\eta^- |_{\Pi(N_- \cap N_L)} = d \log r,$$

and

$$\eta^- |_{\Pi(N_- \cap N_R)} = d \log(r \circ F_\mu).$$

Let D_0 be the arc $\Pi(\{(\cos\theta, \sin\theta, 0) \in N_+ \mid -\pi/2 < \theta < \pi/2\})$ in L_0. D_0 is given the same orientation as the unit circle in the x-y plane.

$$\begin{aligned}\int_{D_0} (\eta^+ - \eta^-) &= \int_{-\pi/2}^{0} (d \log r - d \log r) + \int_0^{\pi/2} (d \log r - d \log(r \circ F_\mu)) \\ &= \int_0^{\pi/2} (d \log \sqrt{\cos^2\theta + \sin^2\theta} - d \log \sqrt{e^{2\mu}\cos^2\theta + \sin^2\theta})\, d\theta \\ &= \int_0^{\pi/2} \frac{(e^{2\mu} - 1)\cos\theta \sin\theta}{e^{2\mu}\cos^2\theta + \sin^2\theta} d\theta \\ &= \mu.\end{aligned}$$

Let C_0 be the simple closed curve in L_0 which is induced from the x-axis. C_0 is also given the same orientation as the x-axis. Then the intersection number $C_0 \cdot D_0$ is equal to 1. It is easy to see that the holonomy of C_0 is the linear map h_λ.

Hence, L_0, C_0 and D_0 satisfy the condition of the theorem.

On the other hand, we can easily find a neighborhood of L in M which is equivalent to a neighborhood of L_0.

To finish the proof, we calculate $\int_{L_0} E$.

$$\begin{aligned}\int_{L_0} E &= \int_{L_0} \eta^+ \wedge \eta^- \\ &= \int_{L_0 \cap \Pi(\overline{N_L})} \eta^+ \wedge \eta^- + \int_{L_0 \cap \Pi(\overline{N_R})} \eta^+ \wedge \eta^- \\ &= \int_{L_0 \cap \Pi(\overline{N_R})} d \log r \wedge d \log(r \circ F_\mu) \\ &= \int_{L_0 \cap \Pi(\overline{N_R})} d \log\sqrt{x^2+y^2} \wedge d \log\sqrt{e^{2\mu}x^2+y^2} \\ &= \int_{L_0 \cap \Pi(\overline{N_R})} \frac{xy}{(x^2+y^2)(e^{2\mu}x^2+y^2)}(1-e^{2\mu})\,dxdy \\ &= (1-e^{2\mu})\int_1^{e^\lambda}\frac{dl}{l}\int_0^{\pi/2}\frac{\cos\theta\sin\theta}{e^{2\mu}\cos^2\theta+\sin^2\theta}d\theta \\ &= -\lambda\mu. \end{aligned}$$

□

3. Discrete Godbillon-Vey number and surgery formula.

DEFINITION 1. We define the discrete Godbillon-Vey number for $\mathscr{F}$ by

$$\overline{\mathrm{GV}}(\mathscr{F}) = \sum_{i=1}^{l}\int_{L_i} E_i.$$

REMARK 1. In [**6**], [**5**], the discrete Godbillon-Vey cocycle is defined as a two cocycle of $\mathrm{PL}_+(S^1)$ which is the group of orientation preserving piecewise linear homeomorphisms of S^1. For $g_1, g_2 \in \mathrm{PL}_+(S^1)$,

$$\mathrm{GSGV}(g_1, g_2) = \frac{1}{2}\sum_{x\in S^1}\begin{vmatrix} \log g_2'(x+0) & \log(g_1\circ g_2)'(x+0) \\ \log g_2'(x+0) - \log g_2'(x-0) & \log(g_1\circ g_2)'(x+0) - \log(g_1\circ g_2)'(x-0)\end{vmatrix}.$$

Let $\mathscr{F}$ be a foliated S^1 bundle over an oriented closed surface Σ and $\varphi : \pi_1(\Sigma) \to \mathrm{PL}_+(S^1)$, the global holonomy of $\mathscr{F}$. Then

$$\mathrm{GSGV}(\varphi_*([\Sigma])) = \overline{\mathrm{GV}}(\mathscr{F})$$

where $[\Sigma] \in H_2(\Sigma; \boldsymbol{Z})$ is the fundamental class of Σ. This is proved by using foliated S^1 products as we have done in the case of the Godbillon-Vey invariant.

It is easy to define the surgery now.

DEFINITION 2. For $\lambda > 0$, the surgery along a simple closed curve in a leaf with the holonomy h_λ, is defined by the operation used in order to obtain N_0 from $N/[x, Y, \phi] \sim S_\lambda[x, Y, \phi]$ with $\mu = \lambda$ in the proof of Theorem 1.

REMARK 2. This is topologically $(1,1)$ Dehn surgery. Goodman and Fried define the surgery for Anosov flows [**8**] [**4**] (see also [**3**]).

From Theorem 1, we show

THEOREM 2. *Let $\mathscr{F}_0$ be a transversely oriented, transversely piecewise linear foliation of codimension one on an oriented closed* 3*-manifold. If $\mathscr{F}$ is obtained from $\mathscr{F}_0$ by the surgery, then*

$$\overline{\mathrm{GV}}(\mathscr{F}) = \overline{\mathrm{GV}}(\mathscr{F}_0) - \lambda^2.$$

We apply the surgery formula to some examples now. There is other good application in [**2**].

EXAMPLES.

Let p, q, r be positive integers satisfying that $(1/p) + (1/q) + (1/r) < 1$ and $S(p, q, r)$, the 2-sphere with three singular points whose cone angles are $2\pi/p, 2\pi/q$ and $2\pi/r$. We consider $S(p, q, r)$ as a quotient space of the Poincaré disk by a triangle group. Then its geodesic flow is defined in its unit tangent circle bundle $M(p, q, r)$, which is a Seifert fibered space, and of Anosov. Let $\mathscr{F}^u_{p,q,r}$ be the unstable foliation of the geodesic flow. This flow is obtained from an element of $SL(2; \boldsymbol{Z})$ (see [**10**]). For example, if $p = q = 2g + 1$ and $r = g + 1$ $(g = 2, 3, 4, \dots)$, then the geodesic flow is obtained by two surgeries along closed orbits of the suspension flow of the diffeomorphism of 2-torus induced by

$$A_{2g+1,\,2g+1,\,g+1} = \begin{pmatrix} 2g^2 - 1 & 4g \\ g(g^2 - 1) & 2g^2 - 1 \end{pmatrix} \in SL(2, \boldsymbol{Z}).$$

We can calculate the discrete Godbillon-Vey invariant from the surgery formula. Since the unstable foliation of the suspension flow has no bending points and the holonomy of closed orbits are $h_{\log \lambda_g}$,

$$\overline{\mathrm{GV}}(\mathscr{F}^u_{2g+1,\,2g+1,\,g+1}) = -2(\log \lambda_g)^2$$

where λ_g is the larger eigenvalue of $A_{2g+1,\,2g+1,\,g+1}$.

Let Σ_g be a closed surface of genus g. Σ_g is given a hyperbolic metric. The geodesic flow of Σ_g is of Anosov in the unite tangent bundle $T_1\Sigma_g$. The unstable foliation of the geodesic flow is denoted by $\mathscr{F}_g{}^u$. There is a $(2g + 2)$-fold covering $T_1\Sigma_g \to M(2g + 1, 2g + 1, g + 1)$ which preserves geodesic flows and unstable foliations. Therefore,

$$\overline{\mathrm{GV}}(\mathscr{F}_g{}^u) = -2(2g + 2)(\log \lambda_g)^2.$$

In [**9**], we showed this from the definition of the discrete Godbillon-Vey cocycle by another monotonous way.

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