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Dirichlet problem for evolutionary surfaces of prescribed mean curvature in a non-convex domain

By Kazuya HAYASIDA and Yoshiaki IKEDA

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Abstract. We show the existence of solutions for Dirichlet problem of evolutionary surfaces of prescribed mean curvature. Usually the lateral boundary needs to satisfy a kind of convexity, more precisely H-convexity condition. But in this article we do not assume it on a portion S of the lateral boundary. Under some assumptions on the exterior forces and the shape of S we prove that the solution satisfies Dirichlet boundary condition on S in a weak sense.

1. Introduction.

Let $\Omega \subset R^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. We denote $D_i = \partial/\partial x_i$ and $D = (D_1, \dots, D_N)$. Let T be any fixed positive number and $Q_T = \Omega \times (0, T)$. We denote by $\partial_p Q_T$ the parabolic boundary of Q_T. We consider the Dirichlet problem

$$D \cdot \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) - \partial_t u = NH \quad \text{in } Q_T \tag{1.1}$$

with

$$u = \phi \quad \text{on } \partial_p Q_T. \tag{1.2}$$

Here we impose the compatibility condition on the boundary function ϕ:

$$D \cdot \left(\frac{D\phi}{\sqrt{1 + |D\phi|^2}} \right) - \partial_t \phi = NH \quad \text{on } \partial\Omega \times \{t = 0\}. \tag{1.3}$$

In the stationary case Serrin [**13**] solved first the above Dirichlet problem under some assumptions on H. In particular the following assumption is important

$$\frac{N}{N-1}|H| \leq \Lambda \quad \text{on } \partial\Omega, \tag{1.4}$$

where Λ is the boundary mean curvature. Afterward many authors solved the above Dirichlet problem in the stationary case under weaker assumptions, but (1.4) is essential

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(see e.g., [**4**], [**6**]) and necessary. Here the necessity is in the following sense: If (1.4) breaks at a point in $\partial\Omega$, there is at least a boundary function ϕ, for which (1.1)–(1.2) is not solvable (see Corollary 14.13 in [**5**]). The existence of solutions for the problem (1.1)–(1.2) was discussed by Lichnewsky and Temam [**11**]. In particular, when $H = 0$ and $\Lambda \geq 0$ on $\partial\Omega \times (0, T)$ and ϕ does not depend on t, they showed that the existence problem is affirmative. Their method is to construct the barrier. And if some nonlinear perturbation terms appear in (1.1), Nakao and Ohara [**12**] derived L^∞-gradient estimates and showed the behavior of solutions at $t = \infty$ under some assumptions. There are many papers treating various gradient estimates of quasilinear parabolic equations of mean curvature type: [**1**], [**2**], [**3**], [**10**] etc.

Recently one of the authors and Nakatani [**7**] tried to remove the assumption (1.4) for the stationary case without constructing the barrier. When (1.4) breaks on a portion Γ of $\partial\Omega$, some assumption was imposed for the approximating solutions and a few examples were given in [**7**] concerning the existence of weak solutions. But it is required that H is larger than some positive constant on Γ, which is determined from the shape of Γ.

Our aim is to extend the result in [**7**] to the non-stationary case. We shall prove the existence of weak solutions for the problem (1.1)–(1.2). The assumption (1.4) is not imposed on a portion $\Gamma \times (0, T)$ of the lateral boundary $\partial\Omega \times (0, T)$. But we impose some assumption for the approximating solution for the problem (1.1)–(1.2). Our theorem is stated in the next section. And an example satisfying the assumptions in our theorem is given in Section 3, where we treat the case when Ω is an annular domain in R^3. Then the inside portion Γ of $\partial\Omega$ does not satisfy (1.4). Our method is to derive a uniform energy estimate near $\Gamma \times (0, T)$ for each solution of the approximate problem of (1.1)–(1.2) (see Sections 5 and 6). Using the estimate, we show that the required solution satisfies the boundary condition on $\Gamma \times (0, T)$ in a weak sense, where (1.4) is not satisfied.

We proceed along the line in [**7**]. But, even in the stationary case, our method in this article is not the same. In both [**7**] and this article we suppose a property, called Property (A) (see Section 2), for each point on Γ. In this article the definition of Property (A) is different from that in [**7**] and weaker than it. For further detailes refer to the note behind the definition of Property (A), and the Remark 2 in Section 2.

When Ω is a ball with radius $R > N$ and $H = 1/N$ particularly, the condition (1.4) breaks, because $\Lambda = 1/R$. In this case Kawohl and Kutev [**8**] solved the problem (1.1)–(1.2) in the viscosity sense. Their method is to use the theory of viscosity solutions and different from ours.

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2. Theorem.

Let $x = (x_1, \dots, x_N)$ be the space variable in R^N and t be the time variable in R^1. From now on, let Ω be a bounded domain in R^N and $\partial\Omega$ be of class C^3. Let T be any fixed positive number. Let Q_T be the cylindrical domain in the previous section.

The following assertion is well known as Nikodym's theorem:

Suppose that u, $D_i u \in L^1(Q_T)$, $i = 1, \dots, N$. Then u has its trace on $\partial\Omega \times (0, T)$ such that it belongs to $L^1(\partial\Omega \times (0, T))$.

We consider the initial value problem:

$$\begin{cases} D \cdot \left(\dfrac{Du}{\sqrt{1 + |Du|^2}} \right) - \partial_t u = g(x, t) & \text{in } Q_T \\ u(x, 0) = f(x) & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{2.1}$$

As usually we define the weak solution of (2.1) as follows. First we put

$$C^1_{(0)}(\Omega \times [0, T)) = \{\psi(x, t) \in C^1(\overline{Q_T}) \mid \psi = 0 \text{ on } \{\partial\Omega \times (0, T)\} \cup \{\Omega \times \{t = T\}\}.$$

Definition. Let u and $D_i u \in L^1(Q_T)$, $i = 1, \dots, N$. Let $f(x) \in C(\overline{\Omega})$ and $g(x, t) \in C(\overline{Q_T})$. Then we say that u is a weak solution of (2.1), if it holds that for any $\psi \in C^1_{(0)}(\Omega \times [0, T))$

$$\int_{Q_T} u \partial_t \psi \, dxdt - \int_{Q_T} \frac{Du \cdot D\psi}{\sqrt{1 + |Du|^2}} dxdt + \int_{\Omega} f(x) \psi(x, 0) \, dx = \int_{Q_T} g\psi \, dxdt. \tag{2.2}$$

Naturally u is a weak solution of (2.1), if it holds in the classical sense. We denote by $B_\delta(P)$ the open ball in R^N with its center P and with its radius δ. We set the following

Definition. We say that $P \in \partial\Omega$ has Property (A), if the following holds: There exist a positive number δ and a one-to-one mapping Φ

$$\Phi : B_\delta(P) \ni (x_1, \dots, x_N) \mapsto (\xi_1, \dots, \xi_N) \in R^N$$

satisfying

(I) Φ and Φ^{-1} are both of class C^3 such that

$$\frac{D(\xi_1, \dots, \xi_N)}{D(x_1, \dots, x_N)} > 0 \quad \text{in } B_\delta(P).$$

(II) $\Phi(P) = O$, $\Phi(B_\delta(P) \cap \Omega) \subset \{\xi_N > 0\}$ and $\Phi(B_\delta(P) \cap \partial\Omega) \subset \{\xi_N = 0\}$.
(III) $D_x \xi_i \cdot D_x \xi_N = 0$ on $B_\delta(P) \cap \partial\Omega$, if $i \neq N$.

For example Φ is the case of the polar coordinates transformation. In this article we put $m_{ij} = D_x \xi_i \cdot D_x \xi_j$. In [**7**] the following stronger assumption was imposed in place of the above (III):

$$m_{ij} = 0 \text{ on } B_\delta(P) \cap \partial\Omega, \quad \text{if } i \neq j.$$

Let $P \in \partial\Omega$ have Property (A) and ξ_i, $i = 1, \dots, N$ be the functions in its definition. We put $h_{ij} = D_{x_j} \xi_i$ and

$$J = \frac{D(x_1, \dots, x_N)}{D(\xi_1, \dots, \xi_N)}.$$

Then $m_{ij} = h_{ip}h_{jp}$, $\det(h_{ij}) = J^{-1} > 0$ and $\{m_{ij}\}$ is symmetric. From our assumption $m_{Nj} = 0$ on $\{\xi_N = 0\}$ for $j \neq N$. For $(\eta_1, \dots, \eta_N) \in R^N$

$$\sum_{i,j} m_{ij}\eta_i\eta_j = \sum_p \left(\sum_i h_{ip}\eta_i\right)^2 \geq 0.$$

If the left-hand side equals 0, $\sum_i h_{ip}\eta_i = 0$ for any p. Hence $\eta_1 = \cdots = \eta_N = 0$. This implies that $\{m_{ij}\}$ is positive definite.

Throughout this article we set the following assumptions: Let Γ_1 and Γ_2 be two relatively open subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_1} \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Further we impose Property (A) for each point on Γ_1 and set

$$d_0 = \sup_{P \in \Gamma_1} d(P), \tag{2.3}$$

where $d(P)$ is the least positive number satisfying

$$Jd(P) \geq \frac{1}{2N}\left[\sqrt{m_{NN}}D_{\xi_N}J + \frac{1}{\sqrt{m_{NN}}}D_{\xi_N}(Jm_{NN})\right] \quad \text{on } B_\delta(P) \cap \partial\Omega. \tag{2.3$'$}$$

The positive constant d_0 is the same as in [**7**], which depends only on the shape of Γ_1.

Let H belong to $C^1(\overline{Q_T})$, and suppose that ϕ belongs to $C^5(\overline{Q_T})$. We take an approximating sequence $\{\phi_\nu\}$ such that

$$\phi_\nu \to \phi \text{ in } H^{2+\alpha, 1+\alpha/2}(\overline{Q_T}) \quad (\nu \to \infty), \tag{2.4}$$

where the space $H^{2+\alpha, 1+\alpha/2}(\overline{Q_T})$ is the usual Schauder space (see e.g., [**9**]). And we take a positive sequence $\{\varepsilon_\nu\}$ such that $\varepsilon_\nu \to 0$ $(\nu \to \infty)$. Each ϕ_ν needs to satisfy the compatibility condition

$$\varepsilon_\nu \triangle \phi_\nu + D \cdot \left(\frac{D\phi_\nu}{\sqrt{1 + |D\phi_\nu|^2}}\right) - \partial_t\phi_\nu = NH \quad \text{on } \partial\Omega \times \{t = 0\}. \tag{2.5}$$

This is fulfilled if we set, for example

$$\phi_\nu(x,t) = \phi(x,t) + \varepsilon_\nu t(\triangle\phi)(x,0). \tag{2.6}$$

So it is known that for each ν there is a solution $u_\nu \in H^{2+\alpha, 1+\alpha/2}(\overline{Q_T})$ $(0 < \alpha < 1)$ of

$$\begin{cases} \varepsilon_\nu \triangle u_\nu + D \cdot \left(\dfrac{Du_\nu}{\sqrt{1 + |Du_\nu|^2}}\right) - \partial_t u_\nu = NH \quad \text{in } Q_T \\ u_\nu = \phi_\nu \quad \text{on } \partial_p Q_T \end{cases} \tag{2.7}$$

(see e.g., [**9**]).

The space $H^{2,1}(Q_T)$ also is referred to [**9**]. It is known that for any compact set $K \subset Q_T$ there is a constant C depending on K but not on ν such that $\sup_K |Du_\nu| \leq C$ (see e.g., [**3**]). So by the usual argument there are a subsequence $\{u_\mu\}$ of $\{u_\nu\}$ and a function $u \in H^{2,1}(Q_T)$ such that

(2.8) $$\partial_t^k D^\gamma u_\mu \rightrightarrows \partial_t^k D^\gamma u \text{ in } K \ (\mu \to \infty), \quad 2k + |\gamma| \le 2.$$

In Section 4 we see that u is a weak solution of

(2.9) $$\begin{cases} D \cdot \left(\dfrac{Du}{\sqrt{1+|Du|^2}} \right) - \partial_t u = NH & \text{in } Q_T \\ u(x,0) = \phi(x,0) & \text{on } \Omega \times \{t=0\}. \end{cases}$$

Under the above assumptions our aim is to prove

THEOREM. *Suppose the following assumptions*:

a) *For each* v, $\partial u_v / \partial \boldsymbol{n} \ge 0$ *on* $\Gamma_1 \times (0,T)$, *where* $\boldsymbol{n}$ *is the inward normal vector at* $\partial\Omega$.

b) $H \ne 0$ *on* $\overline{\Gamma_2} \times (0,T)$.

c) *There is a positive number* δ *such that*

$$(1+\delta)|H| \le \frac{N-1}{N} \Lambda \quad \text{on } \Gamma_2 \times (0,T).$$

d) $|\partial_t \phi| < \delta'$ *on* $\Gamma_2 \times (0,T)$, *where* δ' *is a positive number determined later from* δ *and* $\inf_{\Gamma_2 \times (0,T)} |H|$.

e) *For the positive constant* d_0 *in* (2.3)

(2.10) $$H + N^{-1} \partial_t \phi \ge d_0 \quad \text{on } \Gamma_1 \times (0,T).$$

Then the problem (2.9) *has a weak solution* $u \in H^{2,1}(Q_T) \cap L^1(0,T; W^{1,1}(\Omega)) \cap C(Q_T \cup (\Gamma_2 \times (0,T))$ *such that* $u = \phi$ *on* $\Gamma_2 \times (0,T)$ *and the trace of* $u - \phi$ *on* $\Gamma_1 \times (0,T)$ *equals* 0.

In the statement of the above theorem we give additional explanations. From b) we can take $\kappa > 0$ such that $\kappa \le |H|$ on $\Gamma_2 \times (0,T)$. Then the constant δ' in d) is given with $\delta' = (2N/3)(\kappa/(2+\delta))$ (see the proof of Proposition 7.1). In the next section we shall give an example satisfying the assumptions a)–e). The solution u in our theorem is identical with the function in (2.8). By the well-known method of barriers, we see that u is continuous near $\Gamma_2 \times (0,T)$ (see e.g., [**11**]). In [**11**] this was treated when $H = 0$ and ϕ does not depend on t. But in order to make sure we repeat a similar argument in Section 7. By the usual argument the function u belongs to $H^{2,1}(Q_T) \cap L^1(0,T; W^{1,1}(\Omega))$. Thus our main goal is to show that the trace of u equals ϕ on $\Gamma_1 \times (0,T)$.

REMARK 1. We have constructed the approximating boundary function as in (2.6). There may be another method to construct it. But the argument below is not changed, because (2.4) and (2.5) only are essential.

REMARK 2. Let Ω be an annular domain in R^3 with its center O. Let Γ_1 be the inside boundary of Ω with its radius R. We calculate the constant d_0 in (2.3).

We take the polar coordinates transformation:

$$x = (\xi_3 + R)\sin\xi_1 \cos\xi_2, \quad y = (\xi_3 + R)\sin\xi_1 \sin\xi_2, \quad z = (\xi_3 + R)\cos\xi_1.$$

Then $m_{33} = 1$ and $J = (\xi_3 + R)^2 \sin\xi_1 > 0$ for $0 < \xi_1 < \pi$. Thus (2.3′) is written as

$$d(P)(\xi_3 + R)^2 \sin\xi_1 \geq \frac{1}{3} D_{\xi_3} J \quad \text{on } \{\xi_3 = 0\}.$$

This implies that we can take as $d_0 = 2/(3R)$.

When Ω is an annular domain in R^N with its center O and Γ_1 is the inside boundary of Ω with its radius R, we see that $m_{NN} = 1$ and $J = (\xi_N + R)^{N-1}(\sin\xi_1)^{N-2}(\sin\xi_2)^{N-3}\cdots \sin\xi_{N-1}$. Thus $d_0 = (N-1)/(NR)$.

Since $\Lambda = -1/R$ on Γ_1, we can write $d_0 = -((N-1)/N)\Lambda$ there. Thus for the general case we conjecture that the assumption (2.10) may be replaced with

$$-H - N^{-1}\partial_t\phi \leq \frac{N-1}{N}\Lambda \quad \text{on } \Gamma \times (0, T),$$

where Γ is a portion of $\partial\Omega$ and Λ is the boundary mean curvature of Γ with negative value.

3. Example.

In this section we give an example applicable to our theorem. In our example Γ_1 and Γ_2 are separate each other. But when they are not so, we can not yet give any example. The problem is still open.

The approximating boundary function ϕ_ν will be given in another method different from (2.4). We write as $r = |x|$. Let $N = 3$ and $0 < R_1 < R_2 < 1$. Here R_1 is arbitrarily fixed, but R_2 is taken as close to R_1, if necessary. We consider the annular domain $\Omega = \{R_1 < r < R_2\}$. Let $\Gamma_1 = \{r = R_1\}$ and $\Gamma_2 = \{r = R_2\}$, so $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Obviously the boundary mean curvature Λ on $\Gamma_2(\Gamma_1)$ equals $1/R_2(-1/R_1)$, respectively. From the previous section d_0 on Γ_1 equals $2/(3R_1)$.

We take a function $H(r) \in C^1[R_1, R_2]$ such that $H(R_1) \geq 2/(3R_1)$ and $0 < H(R_2) < 2/(3R_2)$. For $\varepsilon > 0$ we define the operator Q_ε such that

$$\begin{aligned} Q_\varepsilon(u) &= \varepsilon(1 + |Du|^2)^{3/2} \triangle u + (1 + |Du|^2)\triangle u - D_i u \cdot D_j u \cdot D_i D_j u \\ &\quad - (u_t + 3H)(1 + |Du|^2)^{3/2}, \end{aligned}$$

where $D_i = D_{x_i}$ and $D = (D_1, D_2, D_3)$. If $u = u(r, t)$ in particular, we see that

$$\triangle u = u'' + 2r^{-1}u' \quad \text{and} \quad D_i u \cdot D_j u \cdot D_{ij} u = u'' u'^2.$$

Thus we can write

$$\begin{aligned} Q_\varepsilon(u) &= \varepsilon(1 + u'^2)^{3/2}(u'' + 2r^{-1}u') + u'' + 2(1 + u'^2)r^{-1}u' \\ &\quad - (u_t + 3H)(1 + u'^2)^{3/2}. \end{aligned}$$

Next we take a function $p(r) \in C^2[R_1, R_2]$ such that $p > 0$ in (R_1, R_2) and $p(R_1) = p(R_2) = 0$. Let m be any fixed positive number. And we consider the initial value problem

$$\begin{cases} (1+\varepsilon(1+W^2)^{3/2})W' + 2r^{-1}W(1+W^2)(1+\varepsilon\sqrt{1+W^2}) \\ -(m+3H)(1+W^2)^{3/2} = p \quad \text{in } (R_1, R_2), \\ W(R_1) = 0. \end{cases} \tag{3.1}$$

If $R_2 - R_1$ is small, this problem has a solution $W(R) \in C^2[R_1, R_2]$ for each $\varepsilon > 0$. We set $g(r) = \int_{R_1}^r W(s)\,ds$. Then $g \in C^3[R_1, R_2]$ and $g(R_1) = 0$. From (3.1) we see that

$$\begin{cases} g'' + \varepsilon(1+g'^2)^{3/2}(g'' + 2r^{-1}g') + 2r^{-1}(1+g'^2)g' \\ -(m+3H)(1+g'^2)^{3/2} > 0 \quad \text{in } (R_1, R_2), \\ \text{the right-hand side} = 0 \quad \text{at } r = R_1, R_2. \end{cases} \tag{3.2}$$

Finally we define

$$\phi_\varepsilon(r,t) = v_\varepsilon(r,t) = g(r) + mt$$

and $\phi(r,t) = \phi_0(r,t)$ by setting $\varepsilon = 0$. Then $v_\varepsilon \in C^3(\overline{Q_T})$ and $\phi_\varepsilon(r,t) \to \phi(r,t)$ in $C^3(\overline{Q_T})$ as $\varepsilon \to 0$. From (3.2) we have

$$Q_\varepsilon(v_\varepsilon) > 0 \text{ in } Q_T \quad \text{and} \quad Q_\varepsilon(\phi_\varepsilon) = 0 \text{ on } \Gamma_i \times \{t = 0\} \ (i = 1, 2).$$

Since ϕ_ε satisfies the compatibility condition, there is a function $u_\varepsilon \in H^{2,1}(\overline{Q_T})$ such that

$$Q_\varepsilon(u_\varepsilon) = 0 \text{ in } Q_T, \quad u_\varepsilon = \phi_\varepsilon \text{ on } \partial_p Q_T.$$

By the comparison theorem we see that $u_\varepsilon \ge v_\varepsilon$ in Q_T. Therefore

$$\frac{\partial v_\varepsilon}{\partial \boldsymbol{n}} = \frac{\partial \phi_\varepsilon}{\partial r} = g' = W = 0 \quad \text{and} \quad \frac{\partial v_\varepsilon}{\partial \boldsymbol{n}} \le \frac{\partial u_\varepsilon}{\partial \boldsymbol{n}} \quad \text{on } \Gamma_1 \times (0,T).$$

This means that

$$\frac{\partial u_\varepsilon}{\partial \boldsymbol{n}} \ge 0 \quad \text{on } \Gamma_1 \times (0,T).$$

The conditions c)–e) in our theorem are satisfied, if m is sufficiently small. Naturally it is possible.

4. Preliminaries.

The arguments and the results in this section are usual. Let u_ν be the solutions of the approximating problem (2.7). In this section we denote by $(\ ,\ )$ the $L^2(\Omega)$-inner product.

Setting $v_\nu = u_\nu - \phi_\nu$, we multiply the equation in (2.7) with v_ν. Then

$$-\varepsilon_\nu \int_0^T (\triangle u_\nu, v_\nu)\,dt - \int_0^T \left(D \cdot \left(\frac{Du_\nu}{\sqrt{1+|Du_\nu|^2}}\right), v_\nu\right) dt$$

$$+ \int_0^T (\partial_t u_\nu, v_\nu)\,dt = -N \int_0^T (H, v_\nu)\,dt,$$

$$-(\triangle u_\nu, v_\nu) = (Du_\nu, Dv_\nu) = (1, |Du_\nu|^2) - (Du_\nu, D\phi_\nu)$$

and

$$-\left(D\cdot\left(\frac{Du_\nu}{\sqrt{1+|Du_\nu|^2}}\right), v_\nu\right) = \left(\frac{Du_\nu}{\sqrt{1+|Du_\nu|^2}}, Dv_\nu\right)$$

$$= \left(\frac{Du_\nu}{\sqrt{1+|Du_\nu|^2}}, Du_\nu\right) - \left(\frac{Du_\nu}{\sqrt{1+|Du_\nu|^2}}, D\phi_\nu\right).$$

Furthermore

$$\int_0^T (\partial_t u_\nu, v_\nu)\,dt = \int_0^T (\partial_t u_\nu, u_\nu)\,dt - \int_0^T (\partial_t u_\nu, \phi_\nu)\,dt$$

$$= \frac{1}{2}\int_0^T (1, \partial_t u_\nu^2)\,dt + \int_0^T (u_\nu, \partial_t\phi_\nu)\,dt - (u_\nu, \phi_\nu)|_{t=0}^T$$

$$= \frac{1}{2}(1, u_\nu(\cdot\,, T)^2) - \frac{1}{2}(1, \phi_\nu(\cdot\,, 0)^2) + \int_0^T (u_\nu, \partial_t\phi_\nu)\,dt$$

$$- (u_\nu(\cdot\,, T), \phi_\nu(\cdot\,, T)) + (1, \phi_\nu(\cdot\,, 0)^2)$$

(by Cauchy's inequality)

$$\geq -\frac{1}{2}(1, \phi_\nu(\cdot\,, T)^2) + \frac{1}{2}(1, \phi(\cdot\,, 0)^2) + \int_0^T (u_\nu, \partial_t\phi_\nu)\,dt.$$

Hence we have

$$\varepsilon_\nu \int_0^T (1, |Du_\nu|^2)\,dt + \int_0^T \left(1, \frac{|Du_\nu|^2}{\sqrt{1+|Du_\nu|^2}}\right) dt$$

$$\leq \varepsilon_\nu \int_0^T (Du_\nu, D\phi_\nu)\,dt + \int_0^T \left(\frac{|Du_\nu|}{\sqrt{1+|Du_\nu|^2}}, |D\phi_\nu|\right) dt$$

$$+ \frac{1}{2}(1, \phi_\nu(\cdot\,, T)^2) - \frac{1}{2}(1, \phi_\nu(\cdot\,, 0)^2) - \int_0^T (u_\nu, \partial_t\phi_\nu)\,dt - N\int_0^T (H, v_\nu)\,dt.$$

From now on we take two positive numbers C_0 and M such that

$$\sup_{Q_T} |H| \leq C_0, \qquad \sum_{|\alpha|+k\leq 3}\left(\sup_{\nu,\, Q_T} |D^\alpha \partial_t^k \phi_\nu|\right) \leq M.$$

From the inequality $|s| - 1 \leq s^2/\sqrt{1+s^2}$ $(s \in R)$, we see that

$$|Du_\nu| - 1 \leq \frac{|Du_\nu|^2}{\sqrt{1+|Du_\nu|^2}}.$$

Therefore it follows that

$$\varepsilon_\nu \int_0^T (1, |Du_\nu|^2)\,dt + \int_0^T (1, |Du_\nu|)\,dt \le \frac{\varepsilon_\nu}{2} \int_0^T (1, |Du_\nu|^2)\,dt$$
$$+ C\left(1 + M^2 + M \int_0^T (1, |u_\nu|)\,dt + C_0 \int_0^T (1, |u_\nu|)\,dt + C_0 M\right),$$

where C is a positive constant independent of ν. In virtue of [**3**] there is a positive constant C_1 independent of ν such that

$$\sup_{Q_T} |u_\nu| \le C_1. \tag{4.1}$$

Thus we obtain

$$\varepsilon_\nu \int_0^T (1, |Du_\nu|^2)\,dt + \int_0^T (1, |Du_\nu|)\,dt \le C_2, \tag{4.2}$$

where $C_2 = C(1 + C_0 M + C_1 M + M^2 + C_0 C_1)$. This means that

$$\int_0^T \int_\Omega (|Du_\nu|^{1/2})^2\,dxdt \le C_2. \tag{4.3}$$

Throughout this article we denote by the same $\{u_\mu\}$ any subsequence of $\{u_\nu\}$. Let u be the function in (2.8). Then from the above and (2.8) we obtain

$$|Du_\mu|^{1/2} \to |Du|^{1/2} \text{ weakly in } L^2(Q_T) \quad (\mu \to \infty).$$

So $Du \in (L^1(Q_T))^N$. On the other hand from (4.1) and (2.8), $u \in L^\infty(Q_T) \cap H^{2,1}(Q_T)$. Easily for $\psi \in C^1_{(0)}(\Omega \times [0, T))$

$$\int_{Q_T} \frac{Du_\mu \cdot D\psi}{\sqrt{1 + |Du_\mu|^2}}\,dxdt \to \int_{Q_T} \frac{Du \cdot D\psi}{\sqrt{1 + |Du|^2}}\,dxdt \quad (\mu \to \infty).$$

Thus we see that u is a weak solution of (2.9).

From the above we have

Proposition 4.1. *The function u in* (2.8) *is a weak solution of* (2.9). *And it holds that $u \in L^\infty(Q_T) \cap H^{2,1}(Q_T)$ and $D_i u \in L^1(Q_T)$, $i = 1, \dots, N$.*

5. Main estimate (I).

Hereafter we suppose the assumptions in our theorem. Let u_ν be the solution of (2.7). Let P be any fixed point on Γ_1. Let $\xi = (\xi_1, \dots, \xi_N)$ be the new coordinate and $B_\delta(P)$ be the ball in Property (A).

In Sections 5 and 6 we write $D_x = (D_{x_1}, \dots, D_{x_N})$, $D_i = D_{\xi_i}$, $i = 1, \dots, N$ and $D = (D_1, \dots, D_N)$. Let ψ be any test function in $C^1_0(B_\delta(P) \cap \Omega)$. Then from (2.7) we have

$$\varepsilon_\nu \int_\Omega D_x u_\nu \cdot D_x \psi\,dx + \int_\Omega \frac{D_x u_\nu \cdot D_x \psi}{\sqrt{1 + |D_x u_\nu|^2}}\,dx + \int_\Omega \partial_t u_\nu \cdot \psi\,dx = -N \int_\Omega H\psi\,dx.$$

Let $\{m_{ij}\}$ and J be the quantities in Section 2. If we write $Eu = D_x u$, then $|Eu|^2 = m_{ij} D_i u \cdot D_j u$. The above equality becomes

$$\varepsilon_\nu \int_{\{\xi_N \geq 0\}} m_{ij} D_i u_\nu \cdot D_j \psi \cdot J\, d\xi + \int_{\{\xi_N \geq 0\}} m_{ij} \frac{D_i u_\nu \cdot D_j \psi}{\sqrt{1+|Eu_\nu|^2}} J\, d\xi$$

$$+ \int_{\{\xi_N \geq 0\}} \partial_t u_\nu \cdot \psi J\, d\xi = -N \int_{\{\xi_N \geq 0\}} H \psi J\, d\xi.$$

Hence (2.7) becomes

$$\varepsilon_\nu D_j (J m_{ij} D_i u_\nu) + D_j \left(\frac{J m_{ij}}{\sqrt{1+|Eu_\nu|^2}} D_i u_\nu \right) \tag{5.1}$$

$$-J \partial_t u_\nu = NHJ \quad \text{in } B_\delta(P) \cap \Omega.$$

From now on we put $v_\nu = u_\nu - \phi_\nu$. Then $v_\nu = 0$ on $\partial_p Q_T$. Let ζ be a non-negative function in $C_0^\infty(B_\delta(P))$. We denote by $(\ ,\ )(\langle\ ,\ \rangle)$ the inner product of $L^2(\{\xi_N \geq 0\})$ $(L^2(\{\xi_N = 0\}))$, respectively. Let α be the least eigenvalue of $\{m_{ij}\}$. We denote by C all constants not depending on ν.

We first prove

Proposition 5.1. *It holds that*

$$\frac{\alpha}{4} \left\{ \varepsilon_\nu \int_0^T (J\zeta, |DD_N v_\nu|^2)\, dt + \int_0^T \left(\frac{J\zeta}{(1+|Eu_\nu|^2)^{3/2}}, |DD_N u_\nu|^2 \right) dt \right\}$$

$$- \int_0^T (J \partial_t u_\nu, D_N(\zeta D_N v_\nu))\, dt - N \int_0^T (HJ, D_N(\zeta D_N v_\nu))\, dt$$

$$\leq \frac{1}{2} \int_0^T \left\langle \frac{\zeta}{\sqrt{1+|Eu_\nu|^2}} (D_N J \cdot m_{NN} + D_N(J m_{NN})), (D_N u_\nu)^2 \right\rangle dt$$

$$+ C(1 + C_2)(1 + M^2),$$

where M and C_2 are the constants in the previous section.

In this section we shall give the proof of Proposition 5.1. For simplicity we denote u_ν, v_ν, ϕ_ν and ε_ν as u, v, ϕ and ε, respectively, for some time. Multiplying (5.1) with $D_N(\zeta D_N v)$,

$$\varepsilon \int_0^T (D_j (J m_{ij} D_i u), D_N(\zeta D_N v))\, dt \tag{5.2}$$

$$+ \int_0^T \left(D_j \left(\frac{J m_{ij}}{\sqrt{1+|Eu|^2}} D_i u \right), D_N(\zeta D_N v) \right) dt$$

$$- \int_0^T (J \partial_t u, D_N(\zeta D_N v))\, dt = N \int_0^T (HJ, D_N(\zeta D_N v))\, dt.$$

First we calculate the first term on the left-hand side of (5.2). By integration by parts we can write for any fixed (i,j)

$$(D_j(Jm_{ij}D_iu), D_N(\zeta D_N v)) = (D_N(Jm_{ij}D_iu), D_j(\zeta D_N v)) + P_{ij},$$

where

$$P_{ij} = \begin{cases} -\langle D_j(Jm_{ij}D_iu), \zeta D_N v\rangle & (j \neq N) \\ 0 & (j = N). \end{cases}$$

Hence

$$\begin{aligned}(5.3)\qquad (D_j(Jm_{ij}D_iu), D_N(\zeta D_N v)) &= (D_N(Jm_{ij}D_iv), D_j(\zeta D_N v)) \\ &\quad + (D_N(Jm_{ij}D_i\phi), D_j(\zeta D_N v)) + \sum_{i,j} P_{ij} \\ &\equiv I_1 + I_2 + \sum_{i,j} P_{ij}, \quad \text{say.}\end{aligned}$$

This calculation needs that u is in C^3. But it is avoided by the regularized approximation.

Easily we have

$$\begin{aligned}I_1 &= (J\zeta m_{ij}D_N D_i v, D_j D_N v) + (Jm_{ij}D_j\zeta \cdot D_N D_i v, D_N v) \\ &\quad + (\zeta D_N(Jm_{ij}) \cdot D_i v, D_j D_N v) + (D_j\zeta \cdot D_N(Jm_{ij}) \cdot D_i v, D_N v).\end{aligned}$$

Since $\{m_{ij}\}$ is symmetric and positive definite, we have

$$(J\zeta m_{ij}D_N D_i v, D_N D_j v) \geq \alpha(J\zeta, |DD_N v|^2).$$

From Cauchy's inequality for $\delta > 0$

$$|(Jm_{ij}D_j\zeta \cdot D_N D_i v, D_N v)| \leq \delta(J\zeta, |DD_N v|^2) + C(\delta)(\zeta^{-1}|D\zeta|^2, |Dv|^2)$$

and

$$|(\zeta D_N(Jm_{ij}) \cdot D_i v, D_N D_j v)| \leq \delta(J\zeta, |DD_N v|^2) + C(\delta)(\zeta, |Dv|^2).$$

Hence we have

$$I_1 \geq \frac{\alpha}{2}(J\zeta, |DD_N v|^2) - C(\zeta + |D\zeta| + \zeta^{-1}|D\zeta|^2, |Dv|^2).$$

We write

$$I_2 = (\zeta D_N(Jm_{ij}D_i\phi), D_j D_N v) + (D_j\zeta \cdot D_N(Jm_{ij}D_i\phi), D_N v).$$

Easily

$$|(\zeta D_N(Jm_{ij}D_i\phi), D_j D_N v)| \leq \delta(J\zeta, |DD_N v|^2) + C(\delta)M^2,$$

$$|(D_j\zeta \cdot D_N(Jm_{ij}D_i\phi), D_N v)| \leq (|D\zeta|, |Dv|^2) + CM^2.$$

Therefore we obtain

$$(5.4)\qquad I_1 + I_2 \geq \frac{\alpha}{3}(J\zeta, |DD_N v|^2) - C(\zeta + |D\zeta| + \zeta^{-1}|D\zeta|^2, |Dv|^2) - CM^2.$$

Next we estimate $|P_{ij}|$. From our assumption $m_{Nj} = 0$ on $\{\xi_N = 0\}$ for $j \neq N$. So we may assume that $i, j < N$. We see that

$$P_{i,j} = (1, D_N(D_j(Jm_{ij}D_i\phi)\cdot\zeta D_N v))$$
$$= (D_j(Jm_{ij}D_i\phi), D_N(\zeta D_N v)) + (D_N D_j(Jm_{ij}D_i\phi), \zeta D_N v).$$

Hence

$$|P_{ij}| \le \delta(J\zeta, (D_N^2 v)^2) + C(\delta)(M^2 + (\zeta + |D\zeta|, |Dv|^2)).$$

Combining (5.3) and (5.4) with this inequality, we have

$$(D_j(Jm_{ij}D_i u), D_N(\zeta D_N v)) \ge \frac{\alpha}{4}(J\zeta, |DD_N v|^2) - CM^2$$
$$- C(\zeta + |D\zeta| + \zeta^{-1}|D\zeta|^2, |Dv|^2).$$

Therefore we conclude from (4.2) that

$$\varepsilon\int_0^T (D_j(Jm_{ij}D_i u), D_N(\zeta D_N v))\,dt \tag{5.5}$$
$$\ge \frac{\varepsilon\alpha}{4}\int_0^T (J\zeta, |DD_N v|^2)\,dt - CM^2 - CC_2.$$

Now we estimate the second term on the left-hand side of (5.2). By integration by parts for any fixed (i, j)

$$\left(D_j\left(\frac{Jm_{ij}}{\sqrt{1+|Eu|^2}}D_i u\right), D_N(\zeta D_N v)\right)$$
$$= \left(D_N\left(\frac{Jm_{ij}}{\sqrt{1+|Eu|^2}}D_i u\right), D_j(\zeta D_N v)\right) + Q_{ij},$$

where

$$Q_{ij} = \begin{cases} -\left\langle D_j\left(\dfrac{Jm_{ij}}{\sqrt{1+|Eu|^2}}D_i u\right), \zeta D_N v\right\rangle & (j \ne N) \\ 0 & (j = N). \end{cases}$$

Easily

$$D_N|Eu|^2 = 2m_{ij}D_i u\cdot D_N D_j u + D_N m_{ij}\cdot D_i u\cdot D_j u.$$

Hence

$$D_N\left(\frac{Jm_{ij}}{\sqrt{1+|Eu|^2}}D_i u\right) \tag{5.6}$$
$$= Jm_{ij}\left(\frac{D_N D_i u}{\sqrt{1+|Eu|^2}} - m_{pq}\frac{D_p u\cdot D_i u}{(1+|Eu|^2)^{3/2}}D_N D_q u\right)$$
$$-\frac{1}{2}Jm_{ij}D_N m_{pq}\cdot\frac{D_p u\cdot D_q u}{(1+|Eu|^2)^{3/2}}D_i u + \frac{D_N(Jm_{ij})}{\sqrt{1+|Eu|^2}}D_i u.$$

We write

$$(5.7)\qquad \left(D_j\left(\frac{Jm_{ij}}{\sqrt{1+|Eu|^2}}D_iu\right), D_N(\zeta D_Nv)\right)$$

$$= \left(D_N\left(\frac{Jm_{ij}}{\sqrt{1+|Eu|^2}}D_iu\right), D_j(\zeta D_Nu)\right)$$

$$-\left(D_N\left(\frac{Jm_{ij}}{\sqrt{1+|Eu|^2}}D_iu\right), D_j(\zeta D_N\phi)\right) + \sum_{i,j} Q_{ij}$$

$$\equiv I_3 + I_4 + \sum_{i,j} Q_{ij}, \quad \text{say.}$$

We first estimate $|I_4|$. From (5.6)

$$-I_4 = \left(\frac{J\zeta m_{ij}}{\sqrt{1+|Eu|^2}}, D_ND_iu\cdot D_ND_j\phi - m_{pq}\frac{D_pu\cdot D_iu}{1+|Eu|^2}D_ND_qu\cdot D_ND_j\phi\right)$$

$$+\left(\frac{D_N(Jm_{ij})}{\sqrt{1+|Eu|^2}}, \zeta D_iu\cdot D_ND_j\phi\right)$$

$$-\frac{1}{2}\left(\frac{J\zeta m_{ij}}{(1+|Eu|^2)^{3/2}}, D_Nm_{pq}\cdot D_pu\cdot D_qu\cdot D_iu\cdot D_ND_j\phi\right)$$

$$+\left(\frac{Jm_{ij}D_j\zeta}{\sqrt{1+|Eu|^2}}, D_ND_iu\cdot D_N\phi - \frac{m_{pq}D_pu}{1+|Eu|^2}D_ND_qu\cdot D_iu\cdot D_N\phi\right)$$

$$+\left(\frac{D_N(Jm_{ij})}{\sqrt{1+|Eu|^2}}, D_j\zeta\cdot D_iu\cdot D_N\phi\right)$$

$$-\frac{1}{2}\left(\frac{Jm_{ij}D_Nm_{pq}}{(1+|Eu|^2)^{3/2}}, D_j\zeta\cdot D_pu\cdot D_qu\cdot D_iu\cdot D_N\phi\right).$$

Hence we have

$$(5.8)\qquad |I_4| \le CM\left[1+\left(\frac{\zeta+|D\zeta|}{\sqrt{1+|Eu|^2}}, |DD_Nu|\right)\right].$$

From (5.6) again

$$(5.9)\qquad I_3 = \left(\frac{J\zeta m_{ij}}{\sqrt{1+|Eu|^2}}, D_N D_i u \cdot D_N D_j u - m_{pq}\frac{D_p u \cdot D_i u}{1+|Eu|^2} D_N D_q u \cdot D_N D_j u\right)$$

$$+\left(\frac{\zeta D_N(Jm_{ij})}{\sqrt{1+|Eu|^2}} D_i u, D_N D_j u\right)$$

$$-\frac{1}{2}\left(\frac{J\zeta m_{ij}}{(1+|Eu|^2)^{3/2}} D_N m_{pq} \cdot D_p u \cdot D_q u, D_i u \cdot D_N D_j u\right)$$

$$+\left(\frac{Jm_{ij} D_j\zeta}{\sqrt{1+|Eu|^2}}, D_N D_i u \cdot D_N u - \frac{m_{pq} D_p u}{1+|Eu|^2} D_N D_q u \cdot D_i u \cdot D_N u\right)$$

$$+\left(\frac{D_N(Jm_{ij})}{\sqrt{1+|Eu|^2}}, D_j\zeta \cdot D_i u \cdot D_N u\right)$$

$$-\frac{1}{2}\left(\frac{Jm_{ij} D_j\zeta}{(1+|Eu|^2)^{3/2}}, D_N m_{pq} \cdot D_p u \cdot D_q u \cdot D_i u \cdot D_N u\right)$$

$$\equiv \sum_{i=1}^{6} I_{3i}, \quad \text{say.}$$

Easily

$$(5.10)\qquad |I_{35}|, |I_{36}| \le C(|D\zeta|, |Du|).$$

Now we set $((\eta, \tilde{\eta})) = m_{ij}\eta_i\tilde{\eta}_j$ for two real vectors $\eta = (\eta_1, \dots, \eta_N)$ and $\tilde{\eta} = (\tilde{\eta}_1, \dots, \tilde{\eta}_N)$. Then $((\eta, \tilde{\eta})) = ((\tilde{\eta}, \eta))$ and $((\eta, \eta)) \ge \alpha|\eta|^2$. We define $|||\eta||| = \sqrt{((\eta, \eta))}$. The Schwarz inequality $|((\eta, \tilde{\eta}))| \le |||\eta||| \cdot |||\tilde{\eta}|||$ holds.

For any fixed $\xi \in R^N$ we set

$$(((\eta, \tilde{\eta}))) = ((\eta, \tilde{\eta})) - \frac{((\eta, \xi))((\tilde{\eta}, \xi))}{1+|||\xi|||^2} \quad \text{and} \quad [[\eta]] = \sqrt{(((\eta, \eta)))}.$$

Then

$$[[\eta]]^2 = |||\eta|||^2 - \frac{((\eta, \xi))^2}{1+|||\xi|||^2}$$

$$= \frac{1}{1+|||\xi|||^2}(|||\eta|||^2 + |||\eta|||^2|||\xi|||^2 - ((\eta, \xi))^2) \ge \frac{|||\eta|||^2}{1+|||\xi|||^2}.$$

So, the inner product $(((\ ,\ )))$ is symmetric and positive definite. Hence Schwarz inequality holds for such an inner product. These inner products were introduced in [**10**].

We see that

$$I_{31} = \left(\frac{J\zeta}{\sqrt{1+|Eu|^2}}, |||DD_N u|||^2 - \frac{((Du, DD_N u))^2}{1+|Eu|^2} \right) \tag{5.11}$$

$$\geq \left(\frac{J\zeta m_{ij}}{(1+|Eu|^2)^{3/2}}, D_i D_N u \cdot D_j D_N u \right).$$

We estimate $|I_{34}|$. This idea is due to [**10**]. Setting $\xi = (Du)$, $\mathbf{a} = (D\zeta)$ and $\mathbf{b} = (DD_N u)$, we have

$$\left| m_{ij} D_j \zeta \cdot D_N D_i u - \frac{m_{ij} m_{pq}}{1+|Eu|^2} D_j \zeta \cdot D_p u \cdot D_i u \cdot D_N D_q u \right|$$

$$= \left| ((\mathbf{a}, \mathbf{b})) - \frac{((\mathbf{a}, \xi))((\xi, \mathbf{b}))}{1+|||\xi|||^2} \right|$$

$$= |(((\mathbf{a}, \mathbf{b})))| \leq [[\mathbf{a}]]\,[[\mathbf{b}]].$$

Hence

$$|I_{34}| \leq \left(\frac{J|||\mathbf{a}||| \cdot |D_N u|}{\sqrt{1+|Eu|^2}}, \left(|||\mathbf{b}|||^2 - \frac{((\mathbf{b}, \xi))^2}{1+|||\xi|||^2} \right)^{1/2} \right).$$

This means that

$$|I_{34}| \leq \delta I_{31} + C(\delta)(\zeta^{-1}|D\zeta|^2, |Du|), \quad \delta > 0.$$

Accordingly

$$I_{31} + I_{34} \geq \frac{1}{2} I_{31} - C(\zeta^{-1}|D\zeta|^2, |Du|). \tag{5.12}$$

From (5.9)–(5.12) it follows that

$$I_3 \geq \frac{1}{2} \left(\frac{J\zeta m_{ij}}{(1+|Eu|^2)^{3/2}}, D_i D_N u \cdot D_j D_N u \right) \tag{5.13}$$

$$+ I_{32} + I_{33} - C(|D\zeta| + \zeta^{-1}|D\zeta|^2, |Du|).$$

Next we estimate $I_{32} + I_{33}$. We write

$$I_{32} = \frac{1}{2} \left(\frac{\zeta D_N(J m_{ij})}{\sqrt{1+|Eu|^2}}, D_N(D_i u \cdot D_j u) \right).$$

By integration by parts

$$
\begin{aligned}
(5.14) \qquad I_{32} = &-\frac{1}{2}\left(\frac{D_N(\zeta D_N(Jm_{ij}))}{\sqrt{1+|Eu|^2}}, D_i u \cdot D_j u\right) \\
&+\frac{1}{2}\left(\frac{\zeta D_N(Jm_{ij})}{(1+|Eu|^2)^{3/2}}, m_{pq} D_p u \cdot D_N D_q u \cdot D_i u \cdot D_j u\right) \\
&+\frac{1}{4}\left(\frac{\zeta D_N(Jm_{ij})}{(1+|Eu|^2)^{3/2}}, D_N m_{pq} \cdot D_p u \cdot D_q u \cdot D_i u \cdot D_j u\right) \\
&-\frac{1}{2}\left\langle\frac{\zeta D_N(Jm_{ij})}{\sqrt{1+|Eu|^2}}, D_i u \cdot D_j u\right\rangle \\
\equiv &\sum_{i=1}^{3} J_i - \frac{1}{2}\left\langle\frac{\zeta D_N(Jm_{ij})}{\sqrt{1+|Eu|^2}}, D_i u \cdot D_j u\right\rangle, \quad \text{say.}
\end{aligned}
$$

Easily

$$
|J_1|, |J_3| \leq C(\zeta + |D\zeta|, |Du|).
$$

We can write

$$
\begin{aligned}
J_2 = &\frac{1}{2}\left(\frac{\zeta J D_N m_{ij}}{(1+|Eu|^2)^{3/2}}, m_{pq} D_p u \cdot D_N D_q u \cdot D_i u \cdot D_j u\right) \\
&+\frac{1}{2}\left(\frac{\zeta D_N J \cdot m_{ij}}{(1+|Eu|^2)^{3/2}}, m_{pq} D_p u \cdot D_N D_q u \cdot D_i u \cdot D_j u\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
J_2 + I_{33} &= \frac{1}{2}\left(\frac{\zeta D_N J \cdot m_{ij}}{(1+|Eu|^2)^{3/2}}, m_{pq} D_p u \cdot D_N D_q u \cdot D_i u \cdot D_j u\right) \\
&= \frac{1}{2}\left(\frac{\zeta D_N J}{(1+|Eu|^2)^{3/2}}, |Eu|^2 m_{pq} D_p u \cdot D_N D_q u\right).
\end{aligned}
$$

That is,

$$
\begin{aligned}
(5.15) \qquad J_2 + I_{33} = &\frac{1}{2}\left(\frac{\zeta D_N J}{\sqrt{1+|Eu|^2}}, m_{pq} D_p u \cdot D_N D_q u\right) \\
&-\frac{1}{2}\left(\frac{\zeta D_N J}{(1+|Eu|^2)^{3/2}}, m_{pq} D_p u \cdot D_N D_q u\right).
\end{aligned}
$$

We write by K_1 the first term on the right-hand side of (5.15).

From now on let A_i, $i = 1, \dots,$ be the terms satisfying

$$|A_i| \le C\left[\left(\zeta, \frac{|DD_N u|}{\sqrt{1+|Eu|^2}}\right) + (\zeta + |D\zeta| + \zeta^{-1}|D\zeta|^2, |Du|)\right].$$

From (5.15) we can write

$$J_2 + I_{33} - K_1 = A_1.$$

Easily

$$K_1 = \frac{1}{4}\left(\frac{\zeta D_N J}{\sqrt{1+|Eu|^2}}, m_{pq} D_N(D_p u \cdot D_q u)\right).$$

By integration by parts

$$K_1 = -\frac{1}{4}\left(D_N\left(\frac{\zeta D_N J \cdot m_{pq}}{\sqrt{1+|Eu|^2}}\right), D_p u \cdot D_q u\right)$$

$$-\frac{1}{4}\left\langle \frac{\zeta D_N J}{\sqrt{1+|Eu|^2}}, m_{pq} D_p u \cdot D_q u\right\rangle \equiv \tilde{K}_1 + L, \quad \text{say.}$$

Hence we have

$$J_2 + I_{33} = \tilde{K}_1 + L + A_1. \tag{5.16}$$

Further we write

$$\tilde{K}_1 = -\frac{1}{4}\left(\frac{D_N(\zeta D_N J \cdot m_{pq})}{\sqrt{1+|Eu|^2}}, D_p u \cdot D_q u\right)$$

$$+\frac{1}{4}\left(\frac{\zeta D_N J \cdot m_{pq}}{(1+|Eu|^2)^{3/2}}, m_{rs} D_r u \cdot D_N D_s u \cdot D_p u \cdot D_q u\right)$$

$$+\frac{1}{8}\left(\frac{\zeta D_N J \cdot m_{pq}}{(1+|Eu|^2)^{3/2}}, D_N m_{rs} \cdot D_r u \cdot D_s u \cdot D_p u \cdot D_q u\right).$$

We denote by K_2 the second term on the right-hand side. Then we have

$$\tilde{K}_1 = K_2 + A_2. \tag{5.17}$$

The term K_2 is written as follows:

$$K_2 = \frac{1}{4}\left(\frac{\zeta D_N J}{(1+|Eu|^2)^{3/2}}, |Eu|^2 m_{rs} D_r u \cdot D_N D_s u\right).$$

Hence we have

$$K_2 = \frac{1}{4}\left(\frac{\zeta D_N J}{\sqrt{1+|Eu|^2}}, m_{rs} D_r u \cdot D_N D_s u\right) - \frac{1}{4}\left(\frac{\zeta D_N J}{(1+|Eu|^2)^{3/2}}, m_{rs} D_r u \cdot D_N D_s u\right).$$

This implies

$$K_2 = \frac{1}{2}(\tilde{K}_1 + L) + A_3. \tag{5.18}$$

From (5.17) and (5.18)

$$\begin{aligned}\frac{1}{2}\tilde{K}_1 &= (\tilde{K}_1 - K_2) + \left(K_2 - \frac{1}{2}\tilde{K}_1\right) \\ &= \frac{L}{2} + A_2 + A_3.\end{aligned}$$

So

$$\tilde{K}_1 + L = 2(L + A_2 + A_3).$$

Hence from (5.16)

$$J_2 + I_{33} = 2L + (A_1 + 2A_2 + 2A_3).$$

Combining this with (5.14), we have

$$I_{32} + I_{33} = 2L - \frac{1}{2}\left\langle \frac{\zeta D_N(Jm_{ij})}{\sqrt{1+|Eu|^2}}, D_i u \cdot D_j u \right\rangle + A_4.$$

Therefore from (5.13) we obtain

$$\begin{aligned} I_3 \geq \frac{1}{2}&\left(\frac{J\zeta m_{ij}}{(1+|Eu|^2)^{3/2}}, D_N D_i u \cdot D_N D_j u\right) \\ &- \frac{1}{2}\left\langle \frac{\zeta D_N J}{\sqrt{1+|Eu|^2}}, m_{ij} D_i u \cdot D_j u \right\rangle \\ &- \frac{1}{2}\left\langle \frac{\zeta D_N(Jm_{ij})}{\sqrt{1+|Eu|^2}}, D_i u \cdot D_j u \right\rangle + A_5. \end{aligned}$$

Hence from (5.8) we conclude that

$$\begin{aligned} I_3 + I_4 \geq \frac{1}{2}&\left(\frac{J\zeta m_{ij}}{(1+|Eu|^2)^{3/2}}, D_N D_i u \cdot D_N D_j u\right) \\ &- \frac{1}{2}\left\langle \frac{\zeta}{\sqrt{1+|Eu|^2}}(D_N J \cdot m_{ij} + D_N(Jm_{ij})), D_i u \cdot D_j u \right\rangle \\ &- C(1+M)\left[1 + (\zeta + |D\zeta| + \zeta^{-1}|D\zeta|^2, |Du|) + \left(\frac{\zeta + |D\zeta|}{\sqrt{1+|Eu|^2}}, |DD_N u|\right)\right]. \end{aligned} \tag{5.19}$$

Finally we estimate $|Q_{ij}|$. From the assumption on $\{m_{ij}\}$ it is sufficient to assume that $i, j \neq N$. Let us fix i and j. By integration by parts

$$Q_{ij} = \left\langle J m_{ij} D_i u, \frac{D_j(\zeta D_N v)}{\sqrt{1+|Eu|^2}} \right\rangle. \tag{5.20}$$

Since $D_i u = D_i \phi$ on $\{\xi_N = 0\}$, we see that

$$Q_{ij} = \left\langle J\zeta m_{ij} D_i \phi, \frac{D_j D_N u}{\sqrt{1+|Eu|^2}} \right\rangle + \left\langle J D_j \zeta \cdot m_{ij} D_i \phi, \frac{D_N u}{\sqrt{1+|Eu|^2}} \right\rangle$$

$$- \left\langle J m_{ij} D_i u, \frac{D_j(\zeta D_N \phi)}{\sqrt{1+|Eu|^2}} \right\rangle \equiv B_1 + B_2 + B_3, \quad \text{say.}$$

Easily, $|B_2|, |B_3| \leq C(1 + M^2)$.

Let $f(s), g(s)$ and $h(s)$ be any C^1 function. If $g, h > 0$, we have

$$\frac{f'}{\sqrt{g+hf^2}} = \frac{1}{\sqrt{h}} (\log(\sqrt{h} f + \sqrt{g+hf^2}))' - \frac{h'}{2h} \frac{f}{\sqrt{h} f + \sqrt{g+hf^2}} \tag{5.21}$$

$$- \frac{1}{2\sqrt{h}} \frac{h' f^2 + g'}{\sqrt{g+hf^2}(\sqrt{h} f + \sqrt{g+hf^2})}.$$

Here we set $s = \xi_j$, $f = D_N u$, $g = 1 + \sum_{p,q<N} m_{pq} D_p u \cdot D_q u$ and $h = m_{NN}$. Then

$$\frac{D_j D_N u}{\sqrt{1+|Eu|^2}} = \frac{1}{\sqrt{m_{NN}}} D_j(\log(\sqrt{m_{NN}} D_N u + \sqrt{1+|Eu|^2}))$$

$$- \frac{1}{2m_{NN}} \frac{D_j m_{NN} \cdot D_N u}{\sqrt{m_{NN}} D_N u + \sqrt{1+|Eu|^2}}$$

$$- \frac{1}{2\sqrt{m_{NN}}} \frac{D_j m_{NN} \cdot (D_N u)^2 + D_j g}{\sqrt{1+|Eu|^2}(\sqrt{m_{NN}} D_N u + \sqrt{1+|Eu|^2})}.$$

Hence

$$B_1 = \left\langle J\zeta D_i \phi \cdot \frac{m_{ij}}{\sqrt{m_{NN}}}, D_j(\log(\sqrt{m_{NN}} D_N u + \sqrt{1+|Eu|^2})) \right\rangle \tag{5.22}$$

$$- \frac{1}{2} \left\langle J\zeta D_i \phi \cdot \frac{m_{ij}}{m_{NN}}, \frac{D_j m_{NN} \cdot D_N u}{\sqrt{m_{NN}} D_N u + \sqrt{1+|Eu|^2}} \right\rangle$$

$$- \frac{1}{2} \left\langle J\zeta D_i \phi \cdot \frac{m_{ij}}{\sqrt{m_{NN}}}, \frac{D_j m_{NN} \cdot (D_N u)^2 + D_j g}{\sqrt{1+|Eu|^2}(\sqrt{m_{NN}} D_N u + \sqrt{1+|Eu|^2})} \right\rangle$$

$$\equiv B_{11} + B_{12} + B_{13}, \quad \text{say.}$$

By integration by parts

$$B_{11} = -\left\langle D_j\left(J\zeta D_i\phi \cdot \frac{m_{ij}}{\sqrt{m_{NN}}}\right), \log(\sqrt{m_{NN}}D_N u + \sqrt{1+|Eu|^2})\right\rangle.$$

Since $D_N u \geq 0$ on $\{\xi_N = 0\}$ from our assumption,

$$0 \leq \log(\sqrt{m_{NN}}D_N u + \sqrt{1+|Eu|^2}) \leq \log 2 + \frac{1}{2}\log(1+|Eu|^2).$$

So we have

$$|B_{11}| \leq CM[1 + \langle \zeta + |D\zeta|, \log(1+|Eu|^2)\rangle].$$

Using the inequalities

$$\langle \zeta + |D\zeta|, \log(1+|Eu|^2)\rangle = -(1, D_N((\zeta + |D\zeta|)\log(1+|Eu|^2))),$$

$$|D_N \log(1+|Eu|^2)| \leq C\left(1 + \frac{|DD_N u|}{\sqrt{1+|Eu|^2}}\right),$$

we get

$$|B_{11}| \leq CM\left[1 + \left(\zeta + |D\zeta|, \frac{|DD_N u|}{\sqrt{1+|Eu|^2}}\right) + (|D\zeta| + |D|D\zeta||, \sqrt{1+|Eu|^2})\right].$$

Clearly, $|B_{12}| \leq CM$ and

$$|D_j g| \leq C\sum_{p,q<N}(|D_j D_p\phi|\,|D_q u| + |D_p\phi|\,|D_q u|).$$

So, $|B_{13}| \leq CM(1+M)$. From the above and (5.22) we obtain

$$|B_1| \leq CM\left[1 + M + \left(\zeta + |D\zeta|, \frac{|DD_N u|}{\sqrt{1+|Eu|^2}}\right)\right] + CM(|D\zeta| + |D|D\zeta||, \sqrt{1+|Eu|^2}). \tag{5.23}$$

Hence it follows that

$$|Q_{ij}| \leq \text{the right-hand side of (5.23)}. \tag{5.24}$$

Combining (5.2), (5.4), (5.7), (5.19), (5.24) with (5.5), we conclude that

(5.25)
$$\frac{\varepsilon\alpha}{4}\int_0^T (J\zeta, |DD_N v|^2)\,dt + \frac{\alpha}{2}\int_0^T \left(\frac{J\zeta}{(1+|Eu|^2)^{3/2}}, |DD_N u|^2\right) dt$$
$$- \int_0^T (J\partial_t u, D_N(\zeta D_N v))\,dt - N\int_0^T (HJ, D_N(\zeta D_N v))\,dt$$
$$\le \frac{1}{2}\int_0^T \left\langle \frac{\zeta}{\sqrt{1+|Eu|^2}}(D_N J \cdot m_{ij} + D_N(Jm_{ij})), D_i u \cdot D_j u \right\rangle dt$$
$$+ C\left[1 + C_2 + M^2 + (1+M)\int_0^T \left(\zeta + |D\zeta|, \frac{|DD_N u|}{\sqrt{1+|Eu|^2}}\right) dt\right]$$
$$+ C(1+M)\int_0^T (\zeta + |D\zeta| + |D|D\zeta|\,| + \zeta^{-1}|D\zeta|^2, \sqrt{1+|Eu|^2})\,dt.$$

By Cauchy's inequality we have for $\delta > 0$

$$\int_0^T \left(\zeta + |D\zeta|, \frac{|DD_N|}{\sqrt{1+|Eu|^2}}\right) dt \le \delta\int_0^T \left(\zeta J, \frac{|DD_N u|^2}{(1+|Eu|^2)^{3/2}}\right) dt$$
$$+ C(\delta)\int_0^T (\zeta + \zeta^{-1}|D\zeta|^2, J^{-1}\sqrt{1+|Eu|^2})\,dt$$

(from (4.3))

$$\le \delta\int_0^T \left(\zeta J, \frac{|DD_N u|^2}{(1+|Eu|^2)^{3/2}}\right) dt + C(1+C_2).$$

And on $\{\xi_N = 0\}$

$$(D_N J \cdot m_{ij} + D_N(Jm_{ij}))D_i u \cdot D_j u = (D_N J \cdot m_{NN} + D_N(Jm_{NN}))(D_N u)^2$$
$$+ \sum_{i,j \ne N} (D_N J \cdot m_{ij} + D_N(Jm_{ij}))D_i\phi \cdot D_j\phi.$$

So from (5.25) we have completed the proof of Proposition 5.1. □

6. Main estimate (II).

Continuing the previous section, we proceed. The assumptions are the same as in Section 5. We write $\|H\|_{1,\infty} = \sup_{|\alpha|\le 1}\sup_{\Omega\times(0,T)}|D_x^\alpha H|$. Our aim is to prove

PROPOSITION 6.1. *Under the assumptions in Proposition* 5.1, *it holds that*

$$\frac{\alpha}{4}\int_0^T \left(\frac{J\zeta}{(1+|Eu_v|^2)^{3/2}}, |DD_N u_v|^2\right) dt$$
$$\le \frac{1}{2}\int_0^T \left\langle \frac{\zeta}{\sqrt{1+|Eu_v|^2}}(D_N J \cdot m_{NN} + D_N(Jm_{NN})), (D_N u_v)^2 \right\rangle dt$$
$$- \int_0^T \langle J(NH + \partial_t\phi), \zeta D_N u_v\rangle\,dt + C[1 + M^4 + C_1^2 + C_2^2 + (\|H\|_{1,\infty})^2].$$

We prove Proposition 6.1 as follows. Similarly as in the previous section we denote u_ν, v_ν and ε_ν with u, v and ε, respectively. We remember the estimate in Proposition 5.1.

First we estimate the right-hand side of (5.2). We write

$$-(HJ, D_N(\zeta D_N v)) = -(HJ, D_N(\zeta D_N u)) + (HJ, D_N(\zeta D_N \phi)).$$

By integration by parts

$$-(HJ, D_N(\zeta D_N u)) = (D_N(HJ), \zeta D_N u) + \langle HJ, \zeta D_N u\rangle.$$

Using (4.2), we have

$$-N\int_0^T (HJ, D_N(\zeta D_N v))\,dt \geq N\int_0^T \langle HJ, \zeta D_N u\rangle\,dt - C\|H\|_{1,\infty}(M + C_2). \tag{6.1}$$

Next we estimate the third term on the left-hand side of (5.2). By integration by parts

$$-(J\partial_t u, D_N(\zeta D_N v)) = (JD_N\partial_t u, \zeta D_N v) + (D_N J\cdot\partial_t u, \zeta D_N v) + \langle J\partial_t u, \zeta D_N v\rangle.$$

Since $\partial_t u = \partial_t \phi$ on $\{\xi_N = 0\}$,

$$\langle J\partial_t u, \zeta D_N v\rangle = \langle J\partial_t\phi, \zeta D_N u\rangle - \langle J\partial_t\phi, \zeta D_N\phi\rangle.$$

Hence we have

$$\begin{aligned} -(J\partial_t u, D_N(\zeta D_N v)) &\geq (JD_N\partial_t u, \zeta D_N v) + (D_N J\cdot\partial_t u, \zeta D_N v) \\ &\quad + \langle J\partial_t\phi, \zeta D_N u\rangle - CM^2. \end{aligned} \tag{6.2}$$

We write

$$(JD_N\partial_t u, \zeta D_N v) = \frac{1}{2}(\zeta J, \partial_t(D_N u)^2) - (JD_N\partial_t u, \zeta D_N\phi).$$

By integration by parts

$$\int_0^T (JD_N\partial_t u, \zeta D_N v)\,dt = \frac{1}{2}[(\zeta J, (D_N u)^2)]_{t=0}^T - \int_0^T (JD_N\partial_t u, \zeta D_N\phi)\,dt$$

and

$$-(JD_N\partial_t u, \zeta D_N\phi) = (J\partial_t u, \zeta D_N^2\phi) + (D_N(J\zeta)\cdot\partial_t u, D_N\phi) + \langle J\partial_t\phi, \zeta D_N\phi\rangle.$$

So we have

$$\begin{aligned} \int_0^T (JD_N\partial_t u, \zeta D_N v)\,dt &\geq -\frac{1}{2}(\zeta J, (D_N\phi)(\ \,,0)^2) + \int_0^T (J\partial_t u, \zeta D_N^2\phi)\,dt \\ &\quad + \int_0^T (D_N(J\zeta)\cdot\partial_t u, D_N\phi)\,dt + \int_0^T \langle J\partial_t\phi, \zeta D_N\phi\rangle\,dt. \end{aligned}$$

This implies that

$$\int_0^T (JD_N\partial_t u, \zeta D_N v)\,dt \geq \int_0^T (J\partial_t u, \zeta D_N^2\phi)\,dt + \int_0^T (D_N(J\zeta)\cdot\partial_t u, D_N\phi)\,dt - CM^2.$$

Combining this with (6.2), we obtain

$$-\int_0^T (J\partial_t u, D_N(\zeta D_N v))\,dt \ge \int_0^T (J\partial_t u, \zeta D_N^2\phi)\,dt + \int_0^T (D_N(\zeta J)\cdot\partial_t u, D_N\phi)\,dt + \int_0^T (D_N J\cdot\partial_t u, \zeta D_N u)\,dt - \int_0^T (D_N J\cdot\partial_t u, \zeta D_N\phi)\,dt + \int_0^T \langle J\partial_t\phi, \zeta D_N u\rangle\,dt - CM^2. \tag{6.3}$$

We calculate each term on the right-hand side of (6.3). We see that

$$\int_0^T (J\partial_t u, \zeta D_N^2\phi)\,dt = -\int_0^T (J\zeta u, D_N^2\partial_t\phi)\,dt + [(Ju, \zeta D_N^2\phi)]_{t=0}^T,$$

$$\int_0^T (D_N(J\zeta)\cdot\partial_t u, D_N\phi)\,dt = -\int_0^T (D_N(J\zeta)\cdot u, D_N\partial_t\phi)\,dt + [(D_N(J\zeta)\cdot u, D_N\phi)]_{t=0}^T$$

and

$$\int_0^T (D_N J\cdot\partial_t u, \zeta D_N\phi)\,dt = -\int_0^T (\zeta D_N J\cdot u, D_N\partial_t\phi)\,dt + [(\zeta D_N J\cdot u, D_N\phi)]_{t=0}^T.$$

Hence

$$\left|\int_0^T (J\partial_t u, \zeta D_N^2\phi)\,dt\right|, \left|\int_0^T (D_N(J\zeta)\cdot\partial_t u, D_N\phi)\,dt\right|,$$

$$\left|\int_0^T (D_N J\cdot\partial_t u, \zeta D_N\phi)\,dt\right| \le CC_1 M.$$

Therefore it follows from (6.3) that

$$-\int_0^T (J\partial_t u, D_N(\zeta D_N v))\,dt \ge \int_0^T (D_N J\cdot\partial_t u, \zeta D_N u)\,dt + \int_0^T \langle J\partial_t\phi, \zeta D_N u\rangle\,dt - CM(C_1+M). \tag{6.4}$$

We estimate the first term on the right-hand side of (6.4). From (5.1) we have

$$(D_N J\cdot\partial_t u, \zeta D_N u) = \varepsilon(J^{-1}D_N J\cdot D_j(Jm_{ij}D_i u), \zeta D_N u) + \left(J^{-1}D_N J\cdot D_j\left(\frac{Jm_{ij}}{\sqrt{1+|Eu|^2}}D_i u\right), \zeta D_N u\right) - N(D_N J\cdot H, \zeta D_N u). \tag{6.5}$$

We write

$$(J^{-1}D_N J\cdot D_j(Jm_{ij}D_i u), \zeta D_N u) = (D_N J\cdot m_{ij}D_j D_i u, \zeta D_N u) + (J^{-1}D_N J\cdot D_j(Jm_{ij})\cdot D_i u, \zeta D_N u).$$

If i or $j = N$, we use

$$|(D_N J \cdot m_{ij} D_j D_i u, \zeta D_N u)| \le \delta(\zeta, |DD_N u|^2) + C(\delta)(\zeta, |Du|^2), \quad \delta > 0.$$

If $i, j \ne N$, we have

$$(D_N J \cdot m_{ij} D_j D_i u, \zeta D_N u) = -(D_N J \cdot m_{ij} D_i u, \zeta D_j D_N u) - (D_j(\zeta D_N J \cdot m_{ij}) \cdot D_i u, D_N u).$$

Thus in any case

$$|(D_N J \cdot m_{ij} D_j D_i u, \zeta D_N u)| \le \delta(\zeta, |DD_N u|^2) + C(\delta)(\zeta + |D\zeta|, |Du|^2).$$

From the above and (4.2) we obtain

$$\int_0^T |(J^{-1} D_N J \cdot D_j(J m_{ij} \cdot D_i u), \zeta D_N u)| \, dt \le \delta \int_0^T (\zeta, |DD_N u|^2) + C C_2 \varepsilon^{-1}. \tag{6.6}$$

Next by integration by parts

$$\begin{aligned}
&\left(J^{-1} D_N J \cdot D_j\left(\frac{J m_{ij}}{\sqrt{1+|Eu|^2}} D_i u\right), \zeta D_N u\right) \\
&= -\left(\zeta D_N J \cdot \frac{m_{ij}}{\sqrt{1+|Eu|^2}} D_i u, D_j D_N u\right) \\
&\quad - \left(D_j(\zeta J^{-1} D_N J), \frac{J m_{ij}}{\sqrt{1+|Eu|^2}} D_i u \cdot D_N u\right) \\
&\quad - \left\langle \zeta D_N J \frac{m_{iN}}{\sqrt{1+|Eu|^2}} D_i u \cdot D_N u \right\rangle.
\end{aligned} \tag{6.7}$$

From the assumption on $\{m_{ij}\}$ the last term on the right-hand side equals

$$-\left\langle \zeta D_N J, \frac{m_{NN}}{\sqrt{1+|Eu|^2}}, (D_N u)^2 \right\rangle.$$

Since

$$D_N \sqrt{1+|Eu|^2} = \frac{m_{ij}}{\sqrt{1+|Eu|^2}} D_i u \cdot D_N D_j u + \frac{1}{2} \frac{D_N m_{ij}}{\sqrt{1+|Eu|^2}} D_i u \cdot D_j u,$$

we have

$$-\left(\zeta D_N J \cdot \frac{m_{ij}}{\sqrt{1+|Eu|^2}} D_i u, D_N D_j u\right)$$

$$= -(\zeta D_N J, D_N \sqrt{1+|Eu|^2}) + \frac{1}{2}\left(\zeta D_N J, \frac{D_N m_{ij}}{\sqrt{1+|Eu|^2}} D_i u \cdot D_j u\right).$$

By integration by parts

$$-(\zeta D_N J, D_N \sqrt{1+|Eu|^2}) = (D_N(\zeta D_N J), \sqrt{1+|Eu|^2}) + \langle \zeta D_N J, \sqrt{1+|Eu|^2}\rangle.$$

This implies

$$-\left(\zeta D_N J \cdot \frac{m_{ij}}{\sqrt{1+|Eu|^2}} D_i u, D_N D_j u\right) \geq \langle \zeta D_N J, \sqrt{1+|Eu|^2}\rangle - C[1 + (\zeta + |D\zeta|, |Du|)].$$

Hence from (6.7) we obtain

$$\left(J^{-1} D_N J \cdot D_j\left(\frac{J m_{ij}}{\sqrt{1+|Eu|^2}} D_i u\right), \zeta D_N u\right)$$

$$\geq \left\langle \zeta D_N J, \sqrt{1+|Eu|^2} - \frac{m_{NN}}{\sqrt{1+|Eu|^2}} (D_N u)^2 \right\rangle$$

$$- C[1 + (\zeta + |D\zeta|, |Du|)].$$

Here we note that for $\xi_N = 0$

$$\left|\sqrt{1+|Eu|^2} - \frac{m_{NN}}{\sqrt{1+|Eu|^2}} (D_N u)^2\right|$$

$$= \frac{1}{\sqrt{1+|Eu|^2}} \left|1 + \sum_{i,j \neq N} m_{ij} D_i u \cdot D_j u\right|$$

$$\leq C(1+M).$$

Hence from (4.2) again we have

$$\int_0^T \left(J^{-1} D_N J \cdot D_j\left(\frac{J m_{ij}}{\sqrt{1+|Eu|^2}} D_i u\right), \zeta D_N u\right) dt \geq -C(1 + M + C_2). \tag{6.8}$$

The last term on the right-hand side of (6.5) becomes

$$-N \int_0^T (D_N J \cdot H, \zeta D_N u)\, dt \geq -C C_2 \|H\|_\infty. \tag{6.9}$$

Combining (6.6), (6.8), (6.9) with (6.5), we obtain

$$\int_0^T (D_N J \cdot \partial_t u, \zeta D_N u)\, dt \geq -\varepsilon\delta \int_0^T (\zeta, |DD_N u|^2)\, dt - C(1 + M + C_2) - CC_2 \|H\|_\infty.$$

Therefore it follows from (6.4) that

$$-\int_0^T (J\partial_t u, D_N(\zeta D_N v))\, dt \geq -\varepsilon\delta \int_0^T (\zeta, |DD_N u|^2)\, dt + \int_0^T \langle J\partial_t \phi, \zeta D_N u\rangle\, dt - C(1 + M^2 + MC_1 + C_2 + C_2\|H\|_\infty).$$

Here we use the inequality $|DD_N u|^2 \leq 2(|DD_N v|^2 + |DD_N \phi|^2)$. Then we obtain

$$-\int_0^T (J\partial_t u, D_N(\zeta D_N v))\, dt \geq -2\varepsilon\delta \int_0^T (\zeta, |DD_N v|^2)\, dt + \int_0^T \langle J\partial_t \phi, \zeta D_N u\rangle\, dt - C(1 + C_1^2 + C_2^2 + M^2 + (\|H\|_{1,\infty})^2). \tag{6.10}$$

Combining (6.1) and (6.10) with Proposition 5.1, we complete the proof of Proposition 6.1. □

We take the positive constant d_0 in (2.3). We see that

$$\frac{1}{\sqrt{1 + |Eu_v|^2}} (D_N u_v)^2 \leq \frac{1}{\sqrt{m_{NN}}} D_N u_v \quad \text{on } \{\xi_N = 0\}, \tag{6.11}$$

because $D_N u_v \geq 0$ on $\{\xi_N = 0\}$. Hence it holds that

$$\begin{aligned}
&\frac{1}{2}\left\langle \frac{\zeta}{\sqrt{1 + |Eu_v|^2}} (D_N J \cdot m_{NN} + D_N(J m_{NN})), (D_N u_v)^2 \right\rangle - \langle J(NH + \partial_t \phi), \zeta D_N u_v\rangle \\
&\quad \leq \left\langle \frac{\zeta}{\sqrt{1 + |Eu_v|^2}} \left(\frac{1}{2}(D_N J \cdot m_{NN} + D_N(J m_{NN}))\right), (D_N u_v)^2 \right\rangle \\
&\qquad - \left\langle \frac{\zeta J}{\sqrt{1 + |Eu_v|^2}} \sqrt{m_{NN}} (NH + \partial_t \phi), (D_N u_v)^2 \right\rangle.
\end{aligned}$$

Here we use the assumption

$$NH + \partial_t \phi \geq N d_0 \quad \text{on } \Gamma_1 \times (0, T).$$

Then from (2.3′) and Proposition 6.1 we obtain

Proposition 6.2. *Suppose the assumptions in our theorem. Then it holds that*

$$\int_0^T \left(\frac{J\zeta}{(1 + |Eu|^2)^{3/2}}, |DD_N u_v|^2 \right) dt \leq C_3,$$

where C_3 is a positive constant independent of v.

On the left-hand side of the estimate in Proposition 5.1, we replace $|DD_N u|^2$ with $|DD_i u|^2$, $i \neq N$. Then we see more easily that the boundary integral on the calculation vanishes. Hence we have also

PROPOSITION 6.3. *Under the assumptions in our theorem it holds that for* $1 \leq i \leq N$

$$\int_0^T \left(\frac{\zeta}{(1+|Eu|^2)^{3/2}}, |DD_i u_v|^2 \right) dt \leq C_3.$$

7. Barriers.

Let u be the function in (2.8). In this section we show that $u \in C(Q_T \cup (\Gamma_2 \times (0,T)))$ and $u = \phi$ on $\Gamma_2 \times (0,T)$ under the assumptions in our theorem. Its proof is analogous to [**11**], where it was assumed that $H = 0$ and $\phi(x,t) = \phi(x)$. But in order to make sure we describe the proof. As stated in Section 2, the method is to construct the upper and lower barriers. Let ϕ_v be the function in Section 2 for any fixed v.

Let x^0 be any fixed point in Γ_2. Let us take $\rho > 0$ as sufficiently small and fix it. We consider the following in $(\Omega \cap B_\rho(x^0)) \times (0,T)$. For a positive number K we set

$$\phi_v^{\pm}(x,t) = \phi_v(x,t) \pm K|x - x^0|^2,$$

which satisfies

$$\phi_v^{\pm} \in C^2(\overline{Q_T}), \quad \phi_v^{\pm}(x^0,t) = \phi_v(x^0,t) \quad 0 < t < T.$$

Taking K as sufficiently large, we have

$$\phi_v^+ \geq C_1 \quad \text{and} \quad \phi_v^- \leq -C_1 \quad \text{on } (\Omega \cap \partial B_\rho(x^0)) \times (0,T),$$

where C_1 is the constant in (4.1).

In Sections 7 and 8 write D_{x_i} with D_i. For each v we define an operator Q_v as follows;

$$\begin{aligned} Q_v(\eta) = {} & \varepsilon_v(1+|D\eta|^2)^{3/2} \triangle\eta + (1+|D\eta|^2)\triangle\eta - D_i\eta \cdot D_j\eta \cdot D_iD_j\eta \\ & - (1+|D\eta|^2)^{3/2}(\partial_t\eta + NH). \end{aligned}$$

When we remove the first term on the right-hand side of the above definition, the new quantity is denoted by $Q_0(\eta)$.

As well-known a function $v_v(w_v)$ is called a upper(lower) barrier relative to Q_v, H and ϕ_v at $x^0 \in \Gamma_2$, respectively, if $v_v(w_v)$ satisfies

(i) $v_v(w_v) \in H^{2,1}\{\overline{(\Omega \cap B_\rho(x^0)) \times (0,T)}\}$,

(ii) $v_v = \phi_v^+ (w_v = \phi_v^-)$ on $(\partial\Omega \cap B_\rho(x^0)) \times (0,T)$,

(iii) $v_v \geq \phi_v^+ (w_v \leq \phi_v^-)$ on $((\Omega \cap \partial B_\rho(x^0)) \times (0,T)) \cup ((\Omega \cap B_\rho(x^0)) \times \{t=0\})$,

(iv) $Q_v(v_v) \leq 0$ $(Q_v(w_v) \geq 0)$ in $(\Omega \cap B_\rho(x^0)) \times (0,T)$.

PROPOSITION 7.1. *There is an upper(lower) barrier $v_v(w_v)$, respectively, under the assumptions in our theorem.*

Proof. Let c be a positive number determined later. We define $\psi(s) = (1/c)\log(1+s)$ for $s \geq 0$. We see that $\psi' \geq 0$ and $\psi'' \leq 0$. We set $d(x) = \inf_{y\in\partial\Omega}|x-y|$ and $\Gamma_\mu = \{x \in \Omega \mid d(x) < \mu\}$ for $\mu > 0$. As well-known $d(x)$ belongs to $C^2(\overline{\Gamma_\mu})$ and $|Dd| = 1$ in Γ_μ. Thus $DD_i d \cdot Dd = 0$ in Γ_μ for $i = 1, \dots, N$. From now on we often denote $D_i f$ by f_i simply.

We set $v_\nu = \phi_\nu^+ + \psi(d)$ and $w_\nu = \phi_\nu^- - \psi(d)$. Then (i)–(iii) is trivial. We prove the property (iv) only for v_ν. The case for w_ν is similar. Thus it is sufficient to prove that

$$Q_0(v_\nu) < 0 \quad \text{in } \overline{(\Omega \cap B_\rho(x^0)) \times (0,T)}. \tag{7.1}$$

An easy computation shows that

$$\begin{aligned} Q_0(v_\nu) = {} & (1+|D\phi_\nu^+|^2)\triangle\phi_\nu^+ - (\phi_\nu^+)_i(\phi_\nu^+)_j(\phi_\nu^+)_{ij} + (1+|D\phi_\nu^+|^2 - (\phi_\nu^+)_i(\phi_\nu^+)_j d_i d_j)\psi'' \\ & + ((1+|D\phi_\nu^+|^2)\triangle d - (\phi_\nu^+)_i(\phi_\nu^+)_j d_{ij} - (\phi_\nu^+)_{ij}(d_i(\phi_\nu^+)_j + d_j(\phi_\nu^+)_i) \\ & + 2(D\phi_\nu^+ \cdot Dd)\triangle\phi_\nu^+)\psi' + ((\triangle\phi_\nu^+ - d_{ij}(d_i(\phi_\nu^+)_j + d_j(\phi_\nu^+)_i) + 2(D\phi_\nu^+)\cdot Dd)\triangle d \\ & - d_i d_j(\phi_\nu^+)_{ij})\psi'^2 + (\triangle d - d_i d_j d_{ij})\psi'^3 + (2(D\phi_\nu^+ \cdot Dd) - d_i d_j(d_i(\phi_\nu^+)_j \\ & + d_j(\phi_\nu^+)_i))\psi'\psi'' - (1+|D\phi_\nu^+|^2 + \psi'^2 + 2(D\phi_\nu^+ \cdot Dd)\psi')^{3/2}(\partial_t\phi_\nu^+ + NH). \end{aligned}$$

Since

$$(\phi_\nu^+)_i(\phi_\nu^+)_j d_i d_j \leq |D\phi_\nu^+|^2, \quad d_i d_j d_{ij} = 0$$

and

$$d_i d_j(d_i(\phi_\nu^+)_j + d_j(\phi_\nu^+)_i) = 2(D\phi_\nu^+ \cdot Dd),$$

we have

$$\begin{aligned} Q_0(v_\nu) \leq {} & (1+|D\phi_\nu^+|^2)\triangle\phi_\nu^+ - (\phi_\nu^+)_i(\phi_\nu^+)_j(\phi_\nu^+)_{ij} + \psi'' \\ & + ((1+|D\phi_\nu^+|^2)\triangle d - (\phi_\nu^+)_i(\phi_\nu^+)_j d_{ij} - (\phi_\nu^+)_{ij}(d_i(\phi_\nu^+)_j + d_j(\phi_\nu^+)_i) \\ & + 2(D\phi_\nu^+ \cdot Dd)\triangle\phi_\nu^+)\psi' + (\triangle\phi_\nu^+ - d_{ij}(d_i(\phi_\nu^+)_j + d_j(\phi_\nu^+)_i) \\ & + 2(D\phi_\nu^+ \cdot Dd)\triangle d - d_i d_j(\phi_\nu^+)_{ij})\psi'^2 + \triangle d \cdot \psi'^3 \\ & + (1+|D\phi_\nu^+|^2 + \psi'^2 + 2(D\phi_\nu^+ \cdot Dd)\psi')^{3/2}(N|H| + |\partial_t\phi_\nu^+|). \end{aligned} \tag{7.2}$$

Generally speaking, the following is known: If $|G| \leq ((N-1)/N)\Lambda$ on γ, then $\triangle d + N|G| \leq 0$ on γ, where γ is a portion of $\partial\Omega$ (see e.g., [**4**]). Thus we see that

$$\triangle d + N(1+\delta)|H| \leq 0 \quad \text{on } \Gamma_2 \times (0,T)$$

from our assumption. And we can take a positive number κ such that $\kappa \leq |H|$ on $\Gamma_2 \times (0,T)$. Then we have

$$\triangle d + N\left(1 + \frac{\delta}{2}\right)|H| \leq -\frac{N}{2}\delta\kappa \quad \text{on } \Gamma_2 \times (0,T).$$

If $|\partial_t\phi_\nu^+| \le (2N/3)(\delta\kappa/(2+\delta))$ on $\Gamma_2\times(0,T)$, then $(1+(\delta/2))|\partial_t\phi_\nu^+| \le (N/3)\delta\kappa$ and

$$\triangle d + \left(1+\frac{\delta}{2}\right)(N|H| + |\partial_t\phi_\nu^+|) \le -\frac{N}{6}\delta\kappa \quad \text{on } \Gamma_2\times(0,T).$$

Hence

$$\triangle d\cdot\psi'^3 + (1+|D\phi_\nu^+|^2 + \psi'^2 + 2(D\phi_\nu^+\cdot Dd)\psi')^{3/2}(N|H| + |\partial_t\phi_\nu^+|)$$
$$\le C + C\psi'^2 - \frac{N}{6}\delta\kappa\psi'^3 \quad \text{on } \Gamma_2\times(0,T).$$

More precisely, we have used the inequality

$$(c_1 + c_2 a + a^2)^{3/2} \le \left(1+\frac{1}{2}\delta\right)a^3 + C(\delta)(1+a^2)$$

for $a \ge 1$ and $\delta > 0$. From (7.2) this means that

$$Q_0(v) \le C + C\psi'^2 - c_0\psi'^3 \quad \text{in } (\Omega\times B_\rho(x^0))\times(0,T)$$

for some $c_0 > 0$. The right-hand side of this inequality is negative if $1 \ll 1/c(1+\rho)$. Thus we have finished the proof of Proposition 7.1. □

We set

$$\phi^\pm(x,t) = \phi(x,t) \pm K|x-x^0|^2, \quad v = \phi^+ + \psi(d) \quad \text{and} \quad w = \phi^- - \psi(d).$$

Then the following proposition is easily seen from the proof of Proposition 7.1.

Proposition 7.2. *It holds that*

$$v_\nu \rightrightarrows v \text{ and } w_\nu \rightrightarrows w \text{ in } \overline{\Omega\cap B_\rho(x^0)}\times[0,T] \quad (\nu\to\infty).$$

Lastly we have

Proposition 7.3. *Suppose the assumptions in our theorem. Let u be the solution of* (2.9) *satisfying* (2.8). *Let us set $\delta' = (2N/3)(\delta\kappa/(2+\delta))$. Then it holds that u belongs to $C(Q_T\cup(\Gamma_2\times(0,T)))$ and $u=\phi$ on $\Gamma_2\times(0,T)$.*

Proof. Let $(x^0,t^0)\in\Gamma_2\times(0,T)$. Let $\{\varepsilon_\mu\}$ be the sequence in (2.8). Let $v_\mu(w_\mu)$ be the upper(lower) barrier in Proposition 7.1. Then $v_\mu(w_\mu)$ does not depend on μ, respectively. Let u_ν be the solution of (2.7). We see that

$$w_\mu \le u_\mu \le v_\mu \quad \text{on } \partial_p((\Omega\cap B_\rho(x^0))\times(0,T))$$

and

$$Q_\mu(v_\mu) \le Q_\mu(u_\mu) \le Q_\mu(w_\mu) \quad \text{in } (\Omega\cap B_\rho(x^0))\times(0,T).$$

Then by the comparison theorem it holds that

$$w_\mu \le u_\mu \le v_\mu \quad \text{in } (\Omega\cap B_\rho(x^0))\times(0,T).$$

Hence

$$|u_\mu(x,t) - \phi_\mu(x^0,t^0)| \le \max\{|v_\mu(x,t)-\phi_\mu(x^0,t^0)|, |w_\mu(x,t)-\phi_\mu(x^0,t^0)|\}$$

for $(x,t) \in (\Omega \cap B_\rho(x^0)) \times (0,T)$. Letting $\mu \to \infty$, we have

$$|u - \phi(x^0,t^0)| \le \max\{|v - \phi(x^0,t^0)|, |w - \phi(x^0,t^0)|\}$$

in $(\Omega \cap B_\rho(x^0)) \times (0,T)$, from Proposition 7.2. If we take $(x,t) \to (x^0,t^0)$, then $u(x,t) \to \phi(x^0,t^0)$. This completes the proof of Proposition 7.3. □

8. Proof of our theorem.

As stated at the end of Section 2, it is enough to prove that the trace of $u - \phi$ equals 0 on $\Gamma_1 \times (0,T)$, where u is the function in (2.8). We denote by the same $\{\mu\}$ any subsequence of $\{\nu\}$.

First we prepare the following

PROPOSITION 8.1. *Let u_ν be the solution of the approximating problem* (2.7). *Then there is a positive sequence $\{\alpha_\mu\}$ with $\alpha_\mu \to 0$ $(\mu \to \infty)$ such that for $1 \le i \le N$*

$$\frac{D_i u_\mu}{(1+|Du_\mu|^2)^{\alpha_\mu}} \to D_i u \text{ in } L^1(\Omega \times (0,T)) \quad (\mu \to \infty).$$

PROOF. We determine $\{\alpha_\mu\}$ later. By the convergence theorem

$$\frac{D_i u}{(1+|Du_\mu|^2)^{\alpha_\mu}} \to D_i u \text{ in } L^1(\Omega \times (0,T)) \quad (\mu \to \infty).$$

So it is enough to prove that

$$\frac{D_i(u_\mu - u)}{(1+|Du_\mu|^2)^{\alpha_\mu}} \to 0 \text{ in } L^1(\Omega \times (0,T)) \quad (\mu \to \infty). \tag{8.1}$$

From (2.8) we can take a sequence $\{O_k\}$, subdomains of Q_T such that $\bar{O}_k \subset Q_T$, $O_k \uparrow Q_T$ $(k \to \infty)$ and for each k

$$D_i u_\mu \rightrightarrows D_i u \text{ in } O_k \quad (\mu \to \infty).$$

Retaking some subsequence of $\{u_\mu\}$, we may assume that

$$|D_i u_\mu - D_i u| < \frac{1}{\mu} \quad \text{in } O_\mu.$$

Let us take a positive sequence $\{\alpha_\mu\}$ such that $\alpha_\mu \to 0$ $(\mu \to \infty)$ and

$$|Q_T - O_\mu|^{2\alpha_\mu} \to 0 \quad (\mu \to \infty). \tag{8.2}$$

In fact it is possible, because (8.2) is equivalent to the following

$$\alpha_\mu \log|Q_T - O_\mu| \to -\infty \quad (\mu \to \infty).$$

We write

$$\int_0^T \int_\Omega \frac{|D_i(u_\mu - u)|}{(1+|Du_\mu|^2)^{\alpha_\mu}} dxdt = \int_{O_\mu} + \int_{\Omega \times (0,T) - O_\mu}$$
$$\equiv I_\mu + J_\mu, \quad \text{say.}$$

Easily $I_\mu \to 0$ $(\mu \to \infty)$. And

$$|J_\mu| \le \int_{Q_T - O_\mu} |D_i u_\mu|^{1-2\alpha_\mu}\,dxdt + \int_{Q_T - O_\mu} |D_i u|\,dxdt.$$

Since $D_i u \in L^1(Q_T)$,

$$\int_{Q_T - O_\mu} |D_i u|\,dxdt \to 0 \quad (\mu \to \infty).$$

By Hölder's inequality

$$\int_{Q_T - O_\mu} |D_i u_\mu|^{1-2\alpha_\mu}\,dxdt \le |Q_T - O_\mu|^{2\alpha_\mu}\left(\int_{Q_T} |Du_\mu|\,dxdt\right)^{1-2\alpha_\mu}.$$

Hence $J_\mu \to 0$ $(\mu \to \infty)$ from (4.3) and (8.2). This means (8.1). Thus we have finished the proof. □

Finally we prove our theorem.

Let P be any fixed point on Γ_1 and $B_\delta(P)$ be the ball in the definition of Property (A). Let ψ be any function in $C_0^\infty(B_\delta(P) \times (0,T))$. It is sufficient to prove that

$$\int_0^T \int_{B_\delta(P)\cap\Omega} (u - \phi) D_i \psi\,dxdt = -\int_0^T \int_{B_\delta(P)\cap\Omega} D_i(u-\phi)\cdot\psi\,dxdt \tag{8.3}$$

for any fixed i with $1 \le i \le N$.

We take the sequence $\{\alpha_\mu\}$ in Proposition 8.1. We have

$$D_i\left(\frac{u_\mu - \phi}{(1+|Du_\mu|^2)^{\alpha_\mu}}\right) - \frac{D_i(u_\mu - \phi)}{(1+|Du_\mu|^2)^{\alpha_\mu}} = -2\alpha_\mu(u_\mu - \phi)\frac{Du_\mu \cdot DD_i u_\mu}{(1+|Du_\mu|^2)^{\alpha_\mu+1}}.$$

Hence from (4.1)

$$\left|D_i\left(\frac{u_\mu - \phi}{(1+|Du_\mu|^2)^{\alpha_\mu}}\right) - \frac{D_i(u_\mu - \phi)}{(1+|Du_\mu|^2)^{\alpha_\mu}}\right| \le C\alpha_\mu \frac{|DD_i u_\mu|}{(1+|Du_\mu|^2)^{(2\alpha_\mu+1)/2}}.$$

And from Proposition 6.3

$$\int_0^T \int_{B_\delta(P)\cap\Omega} \frac{|DD_i u_\mu|^2}{(1+|Du_\mu|^2)^{3/2}}\,dxdt \le C. \tag{8.4}$$

By Schwarz inequality

$$\begin{aligned}
&\int_0^T \int_{B_\delta(P)\cap\Omega} \frac{|DD_i u_\mu|}{(1+|Du_\mu|^2)^{(2\alpha_\mu+1)/2}}\,dxdt \\
&\quad \le \left(\int_{Q_T} \sqrt{1+|Du_\mu|^2}\,dxdt\right)^{1/2}\left(\int_0^T \int_{B_\delta(P)\cap\Omega} \frac{|DD_i u_\mu|^2}{(1+|Du_\mu|^2)^{3/2}}\,dxdt\right)^{1/2}.
\end{aligned}$$

So the left-hand side is uniformly bounded from (4.3) and (8.4).

From the above we see that

$$\int_0^T \int_{B_\delta(P)\cap\Omega} D_i\left(\frac{u_\mu - \phi}{(1+|Du_\mu|^2)^{\alpha_\mu}}\right)\cdot\psi\, dxdt - \int_0^T \int_{B_\delta(P)\cap\Omega} \frac{D_i(u_\mu - \phi)}{(1+|Du_\mu|^2)^{\alpha_\mu}}\psi\, dxdt \to 0 \quad (\mu\to\infty). \tag{8.5}$$

On the other hand

$$\begin{aligned}\int_0^T \int_{B_\delta(P)\cap\Omega} D_i\left(\frac{u_\mu - \phi}{(1+|Du_\mu|^2)^{\alpha_\mu}}\right)\cdot\psi\, dxdt &= \int_0^T \int_{B_\delta(P)\cap\Omega} D_i\left(\frac{u_\mu - \phi_\mu}{(1+|Du_\mu|^2)^{\alpha_\mu}}\right)\cdot\psi\, dxdt \\ &\quad + \int_0^T \int_{B_\delta(P)\cap\Omega} D_i\left(\frac{\phi_\mu - \phi}{(1+|Du_\mu|^2)^{\alpha_\mu}}\right)\cdot\psi\, dxdt,\end{aligned}$$

$$\int_0^T \int_{B_\delta(P)\cap\Omega} D_i\left(\frac{u_\mu - \phi_\mu}{(1+|Du_\mu|^2)^{\alpha_\mu}}\right)\cdot\psi\, dxdt = -\int_0^T \int_{B_\delta(P)\cap\Omega} \frac{u_\mu - \phi_\mu}{(1+|Du_\mu|^2)^{\alpha_\mu}} D_i\psi\, dxdt$$

and

$$\int_0^T \int_{B_\delta(P)\cap\Omega} D_i\left(\frac{\phi_\mu - \phi}{(1+|Du_\mu|^2)^{\alpha_\mu}}\right)\cdot\psi\, dxdt = -\int_0^T \int_{B_\delta\cap\Omega} \frac{\phi_\mu - \phi}{(1+|Du_\mu|^2)^{\alpha_\mu}} D_i\psi\, dxdt + K_\mu.$$

Hence we have

$$\int_0^T \int_{B_\delta(P)\cap\Omega} D_i\left(\frac{u_\mu - \phi}{(1+|Du_\mu|^2)^{\alpha_\mu}}\right)\cdot\psi\, dxdt = -\int_0^T \int_{B_\delta\cap\Omega} \frac{u_\mu - \phi}{(1+|Du_\mu|^2)^{\alpha_\mu}} D_i\psi\, dxdt + K'_\mu.$$

Here $K_\mu, K'_\mu \to 0$ $(\mu\to\infty)$. By (2.8) the right-hand side tends to

$$-\int_0^T \int_{B_\delta(P)\cap\Omega} (u-\phi) D_i\psi\, dxdt$$

as $\mu\to\infty$. And from Proposition 8.1 we have

$$\int_0^T \int_{B_\delta(P)\cap\Omega} \frac{D_i(u_\mu - \phi)}{(1+|Du_\mu|^2)^{\alpha_\mu}} \psi\, dxdt$$
$$\to \int_0^T \int_{B_\delta(P)\cap\Omega} D_i(u-\phi)\cdot\psi\, dxdt \quad (\mu\to\infty).$$

Therefore, by using (8.5) we finally conclude (8.3). It complets the proof of our theorem.

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Kazuya Hayasida
Fukui University of Technology
Gakuen 3-6-1, Fukui
910-8505
Japan

Yoshiaki Ikeda
Aichi University of Education
Igaya-cho, Hirosawa
Kariya, 448-8542
Japan