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Inequalities of Noether type for 3-folds of general type

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Abstract. If X is a smooth complex projective 3-fold with ample canonical divisor K, then the inequality $K^3 \geq (2/3)(2p_g - 7)$ holds, where p_g denotes the geometric genus. This inequality is nearly sharp. We also give similar, but more complicated, inequalities for general minimal 3-folds of general type.

Introduction.

Given a minimal surface S of general type, we have two famous inequalities, which play crucial roles in detailed analysis of surfaces. One is the Bogomolov-Miyaoka-Yau inequality $K_S^2 \leq 9\chi(S)$ ([**M1**], [**Y1**], [**Y2**]), while the other is the classical Noether inequality $K_S^2 \geq 2p_g - 4 \geq 2\chi(S) - 6$. The fundamental importance of these inequalities in mind, M. Reid asked in 1980s.

Question 1. *What would be the right analogue of the Noether inequality in dimension three?*

Let X be a minimal threefold. If K_X is Cartier and very ample, then $K_X^3 \geq 2p_g - 6$ by Clifford's theorem applied to the intersection curve cut out by two general members of $|K_X|$. In 1992, Kobayashi [**Kob**] studied Gorenstein canonical 3-folds and obtained an effective, but partial, upper bound of K_X^3 in terms of $p_g(X)$ for such varieties. One of his discoveries is that too naive a generalization of the classical Noether inequality is in general false; there are a series of smooth projective 3-folds X with ample canonical divisor such that

$$K_X^3 = \frac{2}{3}(2p_g(X) - 5), \quad (p_g(X) = 7, 10, 13, \ldots). \tag{0.1}$$

In what follows, we show that Kobayashi's examples indeed attain the minima of K_X^3, provided X is smooth and K_X is ample:

Corollary 2. *If X is a smooth complex projective 3-fold with ample canonical divisor. Then*

$$K_X^3 \geq \frac{2}{3}(2p_g(X) - 7).$$

When X is not necessarily smooth, we have the following

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THEOREM 3. *Let X be a minimal projective* 3*-fold of general type* (*with only $\boldsymbol{Q}$-factorial terminal singularities*). *Assume that $n+1=p_g(X)\geq 2$ and let $\phi_1 : X \to \boldsymbol{P}^n$ be the canonical map. Then we have the following inequalities according to the dimension of $\phi_1(X)$*:

(1) $K_X^3 \geq 2p_g(X)-6$ *if* $\dim \phi_1(X)=3$.

(2) $K_X^3 \geq p_g(X)-2$ *if* $\dim\phi_1(X)=2$ *and* $p_g(X)\geq 6$. *If, in addition, a general fibre of ϕ_1 is a curve of genus* ≥ 3, *then* $K_X^3 \geq 2p_g(X)-4$.

(3) *When $\phi_1(X)$ is a curve, let S be the minimal model of a general irreducible member of the movable part of $|K_X|$ and put $a=K_S^2$, $b=p_g(S)$. Assume $k=[(p_g-2)/2]\geq 4$, where $[x]$ stands for the round down of x. Then we have*

$$K_X^3 \geq \begin{cases} \min\left\{\dfrac{6k^2}{3k^2+8k+4}\cdot\left(p_g(X)-\dfrac{4}{3}\right), \dfrac{6k}{3k+4}\cdot\left(p_g(X)-\dfrac{5}{3}\right)\right\}, & \text{if } (a,b)=(1,1)\\ \dfrac{k^2}{(k+1)^2}\cdot a\cdot(p_g(X)-1), & \text{if } (a,b)\neq(1,1). \end{cases}$$

The intersection numbers between Weil divisors on singular surfaces are not necessarily integers, which causes difficulties to get optimal estimates in case (3).

REMARK 4. We make extra assumptions on $p_g(X)$ in Theorem 3(2), 3(3) simply for getting better inequalities. Our method works also for the case $p_g(X)\geq 2$. Recall that the geometric genus of a surface of general type with $K_S^2=1$ is bounded by 2 from above. Furthermore, the surface in case (3) of the theorem has positive geometric genus. Hence Theorem 3 asserts that $K_X^3\geq 2p_g(X)-6$ unless X is canonically fibred by curves of genus two in case (2) or by surfaces with $a=K_S^2=1$, $b=p_g(S)=2$ in case (3).

When X is Gorenstein, we have the following theorem, which improves the results known so far:

THEOREM 5. *Let X be a minimal projective Gorenstein* 3*-fold of general type with only locally factorial terminal singularities.*

(1) *Assume that X is neither canonically fibred by surfaces S with $c_1(S)^2=1$, $p_g(S)=2$ nor by curves of genus two. Then $K_X^3\geq 2p_g(X)-6$.*

(2) *Assume that X is smooth and that X is not canonically fibred by surfaces S with $c_1(S)^2=1$, $p_g(S)=2$. Then $K_X^3\geq(2/3)(2p_g(X)-5)$.*

(3) *Assume that the canonical model of X is factorial. If $K_X^3<(2/21)\cdot(11p_g(X)-16)$, then X is not smooth and is canonically fibred by curves of genus two.*

These inequalities have a certain interesting application which will be presented in another note.

1. Preliminaries.

1.1. Conventions.

Let X be a normal projective variety of dimension d. We denote by $\mathrm{Div}(X)$ the group of Weil divisors on X. An element $D\in\mathrm{Div}(X)\otimes\boldsymbol{Q}$ is called a *$\boldsymbol{Q}$-divisor*. A $\boldsymbol{Q}$-

divisor D is said to be ***Q**-Cartier* if mD is a Cartier divisor for some positive integer m. For a $\boldsymbol{Q}$-Cartier divisor D and an irreducible curve $C \subset X$, we can define the intersection number $D \cdot C$ in a natural way. A $\boldsymbol{Q}$-Cartier divisor D is called *nef* (or *numerically effective*) if $D \cdot C \geq 0$ for any effective curve $C \subset X$. A nef divisor D is called *big* if $D^d > 0$. We say that X is ***Q**-factorial* if every Weil divisor on X is $\boldsymbol{Q}$-Cartier. For a Weil divisor D on X, denote by $\mathcal{O}_X(D)$ the corresponding reflexive sheaf. Denote by K_X a canonical divisor of X, which is a Weil divisor. X is called *minimal* if K_X is a nef $\boldsymbol{Q}$-Cartier divisor. X is said to be of general type if $\kappa(X) = \dim(X)$. We refer to [**R1**] for definitions of canonical and terminal singularities.

The symbols $\sim, \equiv$ and $=_{\boldsymbol{Q}}$ respectively stands for linear, numerical and $\boldsymbol{Q}$-linear equivalences.

1.2. Vanishing theorem.

Let $D = \sum a_i D_i$ be a $\boldsymbol{Q}$-divisor on X, where the D_i are distinct prime divisors and $a_i \in \boldsymbol{Q}$. We define

the round-down $\llcorner D \lrcorner := \sum \llcorner a_i \lrcorner D_i$, where $\llcorner a_i \lrcorner$ is the integral part of a_i;

the round-up $\ulcorner D \urcorner := -\llcorner -D \lrcorner$;

the fractional part $\{D\} := D - \llcorner D \lrcorner$.

We always use the Kawamata-Viehweg vanishing theorem in the following form.

Vanishing Theorem ([**Ka**] or [**V1**]). *Let X be a smooth complete variety, $D \in \mathrm{Div}(X) \otimes \boldsymbol{Q}$. Assume the following two conditions*:

(i) *D is nef and big*;

(ii) *the fractional part of D has supports with only normal crossings.*

Then $H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$ for all $i > 0$.

Note that, when S is a surface, the above theorem is true without the condition (ii) according to Sakai ([**S**]) or Miyaoka ([**M3**, Proposition 2.3]) (also cited in [**E-L**, (1.2)]).

1.3. Set up for canonical maps.

Let X be a projective minimal 3-fold with only $\boldsymbol{Q}$-factorial terminal singularities. Suppose $p_g(X) \geq 2$. We study the canonical map ϕ_1 which is usually a rational map. Take the birational modification $\pi : X' \to X$, following Hironaka, such that

(1) X' is smooth;

(2) the movable part of $|K_{X'}|$ is base point free;

(3) $\pi^*(K_X)$ is linearly equivalent to a divisor supported by a divisor of normal crossings.

Denote by g the composition $\phi_1 \circ \pi$. So $g : X' \to W' \subseteq \boldsymbol{P}^{p_g(X)-1}$ is a morphism. Let $g : X' \xrightarrow{f} W \xrightarrow{s} W'$ be the Stein factorization of g. We can write

$$K_{X'} =_{\boldsymbol{Q}} \pi^*(K_X) + E =_{\boldsymbol{Q}} S_1 + Z_1,$$

where S_1 is the movable part of $|K_{X'}|$, Z_1 the fixed part and E is an effective $\boldsymbol{Q}$-divisor which is a $\boldsymbol{Q}$-linear combination of distinct exceptional divisors. We can also write

$$\pi^*(K_X) =_{\boldsymbol{Q}} S_1 + E',$$

where $E' = Z_1 - E$ is actually an effective $\boldsymbol{Q}$-divisor and so $\ulcorner \pi^*(K_X) \urcorner$ means $\ulcorner S_1 + E' \urcorner$. We note that $1 \le \dim(W) \le 3$.

If $\dim \phi_1(X) = 2$, we see that a general fiber of f is a smooth projective curve of genus $g \ge 2$. We say that X is *canonically fibred by curves of genus* g.

If $\dim \phi_1(X) = 1$, we see that a general fiber F of f is a smooth projective surface of general type. We say that X is *canonically fibred by surfaces* with invariants $(c_1^2, p_g) := (K_{F_0}^2, p_g(F))$, where F_0 is the minimal model of F.

2. Several simple lemmas.

The following result is a direct application of an inequality on curves proved by Castelnuovo ([**Cas**]) and Beauville ([**Be**]).

Lemma 2.1 ([**Ch1**, Proposition 2.1]). *Let S be a smooth projective algebraic surface and L an effective, nef and prime divisor on S. Suppose $(K_S - L) \cdot L \ge 0$ and $|L|$ defines a birational rational map onto its image. Then*

$$L^2 \ge 3h^0(S, \mathcal{O}_S(L)) - 7.$$

Lemma 2.2. *Let S be a smooth projective surface of general type and L a nef divisor on S. The following holds.*

(i) *Suppose that $|L|$ gives a non-birational, generically finite map onto its image. Then $L^2 \ge 2h^0(S, \mathcal{O}_S(L)) - 4$.*

(ii) *Suppose that there exists a linear subsystem $\Lambda \subset |L|$ such that Λ defines a generically finite map of degree d onto its image. Then $L^2 \ge d[\dim_{\boldsymbol{C}} \Lambda - 1]$ where $\dim_{\boldsymbol{C}} \Lambda$ denotes the projective dimension of Λ.*

Proof. (i) is a special case of (ii).

In order to prove (ii), we take blow-ups $\pi : S' \to S$ such that $\Phi_{\pi^*\Lambda}$ gives a morphism. Let M be the movable part of $\pi^*\Lambda$. Then $h^0(S', M) = \dim_{\boldsymbol{C}} \Lambda + 1$ and

$$M^2 \ge d(h^0(S', M) - 2).$$

Since $M \le \pi^*(L)$, we get the inequality $L^2 \ge M^2 \ge d(\dim_{\boldsymbol{C}} \Lambda - 1)$. □

Lemma 2.3. *Let C be a complete smooth algebraic curve. Suppose D is a divisor on C such that $h^0(C, \mathcal{O}_C(D)) \ge g(C) + 1$. Then* $\deg(D) \ge 2g(C)$.

Proof. This is a direct result by virtue of R-R and Clifford's theorem. □

Lemma 2.4. *Let S be a smooth minimal projective surface of general type. The following holds*:

(i) *$|mK_S|$ is base point free for all $m \ge 4$;*

(ii) *$|3K_S|$ is base point free provided $K_S^2 \ge 2$;*

(iii) *$|3K_S|$ is base point free provided $p_g(S) > 0$ and $p_g(S) \ne 2$;*

(iv) *$|2K_S|$ is base point free provided $p_g(S) > 0$ or $K_S^2 \ge 5$.*

Proof. Both (i) and (ii) can be derived from results of Bombieri ([**Bo**]) and Reider ([**Rr**]).

If $p_g(S) \ge 3$, then $K_S^2 \ge 2$ by Noether inequality. The base point freeness of $|3K_S|$

follows from (ii). If $K_S^2 = 1$ and $p_g(S) = 1$, $|3K_S|$ is base point free by [**Cat**]. If $K_S^2 = 1$ and $p_g(S) = 2$, $|3K_S|$ definitely has base points. So (iii) is true.

(iv) follows from [**Ci**, Theorem 3.1] and Reider's theorem. $\square$

LEMMA 2.5. *Let S be a smooth projective surface of general type. Let $\sigma : S \to S_0$ be the contraction onto the minimal model. Suppose that there is an effective irreducible curve C on S such that $C \le \sigma^*(2K_{S_0})$ and $h^0(S, C) = 2$. If $K_{S_0}^2 = p_g(S) = 1$, then $C \cdot \sigma^*(K_{S_0}) \ge 2$.*

PROOF. We may assume that $|C|$ is a free pencil. Otherwise, we blow-up S at base points of $|C|$. Denote $C_1 := \sigma(C)$. Then $h^0(S_0, C_1) \ge 2$. Suppose $C \cdot \sigma^*(K_{S_0}) = 1$. Then $C_1 \cdot K_{S_0} = 1$. Because $p_a(C_1) \ge 2$, we see that $C_1^2 > 0$. From $K_{S_0}(K_{S_0} - C_1) = 0$, we get $(K_{S_0} - C_1)^2 \le 0$, i.e. $C_1^2 \le 1$. Thus $C_1^2 = 1$ and $K_{S_0} \equiv C_1$. This means $K_{S_0} \sim C_1$ by virtue of [**Cat**], which is impossible because $p_g(S) = 1$. So $C \cdot \sigma^*(K_{S_0}) \ge 2$. $\square$

LEMMA 2.6 ([**Ch4**, Lemma 2.7]). *Let X be a smooth projective variety of dimension ≥ 2. Let D be a divisor on X such that $h^0(X, \mathcal{O}_X(D)) \ge 2$. Let S be a smooth prime divisor on X and assume that S is not contained in the fixed part of $|D|$. Denote by M the movable part of $|D|$ and by N the movable part of $|D|_S|$ on S. If the natural restriction map*

$$H^0(X, \mathcal{O}_X(D)) \xrightarrow{\theta} H^0(S, \mathcal{O}_S(D|_S))$$

is surjective, then $M|_S \ge N$ and, in particular,

$$h^0(S, \mathcal{O}_S(M|_S)) = h^0(S, \mathcal{O}_S(N)) = h^0(S, \mathcal{O}_S(D|_S)).$$

3. Proof of Theorem 3.

We give estimates of K_X^3 according to the dimension of the canonical image $\phi_1(X)$. Let the notation be as in (1.3) throughout this section. Thus S_1 is a general member of the movable part of $|\pi^*(K_X)|$ on a resolution of the indeterminacy of ϕ_1.

The first case is $\dim \phi_1(X) = 3$. Kobayashi ([**Kob**]) proved

PROPOSITION 3.1. *Let X be a projective minimal algebraic 3-fold of general type with only $\mathbf{Q}$-factorial terminal singularities. Suppose $\dim \phi_1(X) = 3$. Then*

$$K_X^3 \ge 2p_g(X) - 6.$$

PROOF. We give a very simple proof of this result in order to keep this note self-contained.

In this situation, a general member $S_1 \in |S_1|$ is a smooth irreducible projective surface of general type. Because K_X is nef and big, we have $K_X^3 = \pi^*(K_X)^3 \ge S_1^3$. Denote $L := S_1|_{S_1}$. Then L is a nef and big divisor on S_1 and $|L|$ defines a generically finite map onto its image. It is obvious that

$$h^0(S_1, L) \ge h^0(X', S_1) - 1 = p_g(X) - 1.$$

Note also that $p_g(X) \ge 4$ under the assumption of this proposition.

If $|L|$ gives a birational map, then, by Lemma 2.1,

$$L^2 \ge 3h^0(S_1, L) - 7 \ge 3p_g(X) - 10 \ge 2p_g(X) - 6.$$

If $|L|$ gives a non-birational rational map, then, by Lemma 2.2,

$$L^2 \geq 2h^0(S_1, L) - 4 \geq 2p_g(X) - 6.$$

Therefore $K_X^3 \geq S_1^3 = L^2 \geq 2p_g(X) - 6$. The proof is complete. $\square$

The second case is $\dim \phi_1(X) = 2$. The general member S_1 is an irreducible smooth surface of general type. The canonical map gives a fibration $f : X' \to W$, and we let C denote its general fiber, which is a smooth curve of genus ≥ 2.

Proposition 3.2. *Let X be a projective minimal algebraic 3-fold of general type with only $\boldsymbol{Q}$-factorial terminal singularities. Suppose $\dim \phi_1(X) = 2$ and $p_g(X) \geq 6$. Then either $g(C) \geq 3$ and $K_X^3 \geq (2/3)g(C)(p_g(X) - 2)$ or C is a curve of genus 2 and $K_X^3 \geq p_g(X) - 2$.*

Proof. We prove the proposition through several steps.

Step 1 (bounding K_X^3 in terms of (L_1, C)). Recall that we have $\pi^*(K_X) =_{\boldsymbol{Q}} S_1 + E'$, where E' is an effective $\boldsymbol{Q}$-divisor. Put $L_1 := \pi^*(K_X)|_{S_1}$ and $L := S_1|_{S_1}$. Then L_1 is a nef and big $\boldsymbol{Q}$-divisor on the surface S_1 and $|L|$ is composed of a free pencil of curves on S_1. It is obvious that $L_1^2 \geq L_1 \cdot L$. We can write

$$L = S_1|_{S_1} \sim \sum_{i=1}^{a} C_i \equiv aC,$$

where $a \geq h^0(S_1, L) - 1 \geq p_g(X) - 2$ and the $C_i's$ are fibers of f contained in the surface S_1. Thus we see that

$$K_X^3 = \pi^*(K_X)^3 \geq L_1^2 \geq L_1 \cdot L \geq (L_1 \cdot C) \cdot (p_g(X) - 2),$$

and we get a lower bound of K_X^3 by giving an estimate of $(L_1 \cdot C)$ from below.

Step 2 (the generic finiteness of the tricanonical map ϕ_3). Look at the sublinear system

$$|K_{X'} + \ulcorner \pi^*(K_X) \urcorner + S_1| \subset |3K_{X'}|.$$

We claim that ϕ_3 is generically finite whenever $p_g(X) \geq 4$. We only have to prove that $\phi_3|_{S_1}$ is generically finite for a general member S_1. By the vanishing theorem, we have

$$\begin{aligned}|K_{X'} + \ulcorner \pi^*(K_X) \urcorner + S_1|\,|_{S_1} &= |K_{S_1} + \ulcorner \pi^*(K_X) \urcorner|_{S_1}| \\ &\supset |K_{S_1} + \ulcorner \pi^*(K_X)|_{S_1} \urcorner|.\end{aligned}$$

We want to prove that $\Phi_{|K_{S_1} + \ulcorner \pi^*(K_X)|_{S_1} \urcorner|}$ is generically finite. Because $K_{S_1} + \ulcorner \pi^*(K_X)|_{S_1} \urcorner \geq L$, we see that $|K_{S_1} + \ulcorner \pi^*(K_X)|_{S_1} \urcorner|$ separates different fibers of $\Phi_{|L|}$. So we only have to verify that $\Phi_{|K_{S_1} + \ulcorner \pi^*(K_X)|_{S_1} \urcorner|}|_C$ is finite for an arbitrary smooth fiber C of f contained in S_1. We have

$$L_1 \equiv L + E_{\boldsymbol{Q}} \equiv aC + E_{\boldsymbol{Q}},$$

where $a \geq p_g(X) - 2 \geq 2$ and $E_{\boldsymbol{Q}} := E'|_{S_1}$ is an effective $\boldsymbol{Q}$-divisor on S_1. Thus

$$L_1 - C - \frac{1}{a}E_{\boldsymbol{Q}} \equiv \left(1 - \frac{1}{a}\right)L_1$$

is a nef and big $\boldsymbol{Q}$-divisor. Using the vanishing theorem again, we get

$$H^1\left(S_1, K_{S_1} + \ulcorner L_1 - \frac{1}{a}E_{\boldsymbol{Q}}\urcorner - C\right) = 0.$$

This means that $|K_{S_1} + \ulcorner L_1 - (1/a)E_{\boldsymbol{Q}}\urcorner|\,|_C = |K_C + D|$, where $D := \ulcorner L_1 - (1/a)E_{\boldsymbol{Q}}\urcorner|_C$ is a divisor on C with positive degree. Because $g(C) \geq 2$, the linear system $|K_C + D|$ gives a finite map, implying the generic finiteness of ϕ_3.

Step 3 (Estimation of $(L_1 \cdot C)$). Since $|3K_{X'}|$ gives a generically finite map, so does $|M_3|_{S_1}|$ on the surface S_1, where M_3 is the movable part of $|3K_{X'}|$. Thus $\Phi_{|M_3|_{S_1}|}$ maps general C of genus ≥ 2 to a curve and hence $M_3|_{S_1} \cdot C \geq 2$. Noting that $3\pi^*(K_X) =_{\boldsymbol{Q}} M_3 + E_3$ where E_3 is an effective $\boldsymbol{Q}$-divisor, we see that

$$3\pi^*(K_X)|_{S_1} \cdot C \geq M_3|_{S_1} \cdot C \geq 2,$$

i.e., $L_1 \cdot C \geq 2/3$. From this crude initial estimate, we derive a better one. To do this, we run a recursive program (the *α-program*) below.

Pick up a positive integer α. We have

$$|K_{X'} + \ulcorner \alpha\pi^*(K_X)\urcorner + S_1| \subset |(\alpha+2)K_{X'}|.$$

The vanishing theorem gives

$$\begin{aligned} |K_{X'} + \ulcorner \alpha\pi^*(K_X)\urcorner + S_1|\,|_{S_1} &= |K_{S_1} + \ulcorner \alpha\pi^*(K_X)\urcorner|_{S_1}| \\ &\supset |K_{S_1} + \ulcorner \alpha L_1\urcorner|. \end{aligned}$$

We see that $\alpha L_1 - C - (1/a)E_{\boldsymbol{Q}} \equiv (\alpha - 1/a)L_1$ is a nef and big $\boldsymbol{Q}$-divisor. Using the vanishing theorem on S_1 again, we get

$$\left|K_{S_1} + \ulcorner \alpha L_1 - \frac{1}{a}E_{\boldsymbol{Q}}\urcorner\right|\,|_C = |K_C + D_\alpha|, \tag{3.1}$$

where $D_\alpha := \ulcorner \alpha L_1 - (1/a)E_{\boldsymbol{Q}}\urcorner|_C$ with $\deg(D_\alpha) \geq \ulcorner(\alpha - 1/a)L_1 \cdot C\urcorner$. We have to use several symbols in order to obtain our result. Let $M_{\alpha+2}$ be the movable part of $|(\alpha+2)K_{X'}|$. Let $M'_{\alpha+2}$ be the movable part of

$$|K_{X'} + \ulcorner \alpha\pi^*(K_X)\urcorner + S_1|.$$

Clearly we have $M'_{\alpha+2} \leq M_{\alpha+2}$. Let N_α be the movable part of $|K_{S_1} + \ulcorner \alpha L_1\urcorner|$. Then it is easy to see $M'_{\alpha+2}|_{S_1} \geq N_\alpha$ by Lemma 2.6. So

$$(\alpha+2)L_1 \geq_{\boldsymbol{Q}} M_{\alpha+2}|_{S_1} \geq M'_{\alpha+2}|_{S_1} \geq N_\alpha.$$

Let N'_α be the movable part of $|K_{S_1} + \ulcorner \alpha L_1 - (1/a)E_{\boldsymbol{Q}}\urcorner|$. Then obviously $N_\alpha \geq N'_\alpha$. From (3.1) and Lemma 2.6, we have $h^0(C, N'_\alpha|_C) = h^0(C, K_C + D_\alpha)$. Thus we see that

$$h^0(C, N_\alpha|_C) \geq h^0(C, N'_\alpha|_C) = h^0(C, K_C + D_\alpha).$$

Now take $\alpha = 2$ and run the α-program. We get $4L_1 \cdot C \geq N_2 \cdot C$. Because $a > 3$ under the assumption, we see that $\deg(D_2) \geq \ulcorner(2 - 1/a)(2/3)\urcorner = 2$. Thus $h^0(C, N_2|_C) \geq g(C) + 1$. By Lemma 2.3, we have $N_2 \cdot C \geq 2g(C)$. If $g(C) = 2$, we get

$L_1 \cdot C \geq 1$ and thus the inequality $K_X^3 \geq p_g(X) - 2$. If $g(C) \geq 3$, we get $L_1 \cdot C \geq 3/2$. This is a better bound than the initial one. However this is not enough to derive our statement. We have to optimize our estimation.

Step 4 (Optimization). As has been seen in the previous step, we have $L_1 \cdot C \geq 3/2$ when $g \geq 3$. We take $\alpha = 1$ now and run the α-program. Since $p_g(X) \geq 6$, we have $a \geq 4$. Thus

$$\deg(D_1) \geq \ulcorner \left(1 - \frac{1}{a}\right)\frac{3}{2} \urcorner = 2.$$

So $h^0(C, N_1|_C) \geq g(C) + 1$. Therefore we get, by Lemma 2.3, that

$$3L_1 \cdot C \geq N_1 \cdot C \geq 2g(C) \geq 6 \quad \text{whenever } g(C) \geq 3.$$

This means $L_1 \cdot C \geq 2$, which is what we want. So we have the inequality

$$K_X^3 \geq \frac{2}{3} \cdot g(C) \cdot (p_g(X) - 2) \tag{3.2}$$

whenever $g(C) \geq 3$. The proof is complete. □

The last case is $\dim \phi_1(X) = 1$. The canonical map gives a fibration $f : X' \to W$ where W is a smooth projective curve. Denote $b := g(W)$. We see that a general fiber F of f is a smooth projective surface of general type. Let $\sigma : F \to F_0$ be the contraction onto the minimal model. Note that we always have $p_g(F) > 0$ in this situation. We also have $S_1 \sim \sum_{i=1}^{b_1} F_i \equiv b_1 F$, where the $F_i's$ are fibers of f and $b_1 \geq p_g(X) - 1$.

PROPOSITION 3.3. *Let X be a projective minimal algebraic 3-fold of general type with only $\boldsymbol{Q}$-factorial terminal singularities. Suppose $\dim \phi_1(X) = 1$. Let $k \geq 4$ be an integer and assume that $p_g(X) \geq 2k + 2$. Then $K_X^3 \geq (k^2/(k+1)^2) \cdot K_{F_0}^2 \cdot (p_g(X) - 1)$.*

PROOF. The proof proceeds through two steps.

Step 1 (bounding K_X^3 in terms of L^2). On the surface F, we denote $L := \pi^*(K_X)|_F$. Then L is an effective nef and big $\boldsymbol{Q}$-divisor. Because $\pi^*(K_X) \equiv b_1 F + E'$ with E' effective, we get

$$K_X^3 = \pi^*(K_X)^3 \geq (\pi^*(K_X)^2 \cdot F) \cdot (p_g(X) - 1) = L^2 \cdot (p_g(X) - 1).$$

So the main point is to estimate L^2 from below in order to prove the proposition.

Step 2 (bounding L^2 from below by studying the $(k+1)$-canonical map ϕ_{k+1}). Let M_{k+1} be the movable part of $|(k+1)K_{X'}|$. Then we may write

$$(k+1)\pi^*(K_X) =_{\boldsymbol{Q}} M_{k+1} + E_{k+1}$$

where E_{k+1} is an effective $\boldsymbol{Q}$-divisor. Therefore we see that $(k+1)L \geq_{\text{num}} M_{k+1}|_F$. Let N_k be the movable part of $|kK_F|$. According to Lemma 2.4, $|kK_{F_0}|$ is base point free. Thus $N_k = \sigma^*(kK_{F_0})$. We claim that $M_{k+1}|_F \geq N_k$. Then $(k+1)L \geq N_k$ and we get

$$L^2 \geq \frac{1}{(k+1)^2} N_k^2 = \frac{k^2}{(k+1)^2} K_{F_0}^2.$$

So we have the inequality

$$K_X^3 \geq \frac{k^2}{(k+1)^2} \cdot K_{F_0}^2 \cdot (p_g(X) - 1). \tag{3.3}$$

Now we prove the claim. In fact, ϕ_1 is a morphism if $b > 0$. In this case, we do not need any modification and $f : X' = X \to W$ is a fibration. A general fiber F is a smooth projective surface of general type, because the singularities on X are isolated. Furthermore F is minimal because K_X is nef. By Kawamata's vanishing theorem for $\boldsymbol{Q}$-Cartier Weil divisor ([**KMM**]), we have $H^1(X, kK_X) = 0$. This means $|kK_X + F|\,|_F = |kK_F|$. Noting that $F \leq K_X$ and using Lemma 2.6, we see that the claim is true in this case.

We then consider the case with $b = 0$. We use the approach in [**Kol**, Corollary 4.8] to prove it. The canonical map gives a fibration $f : X' \to \boldsymbol{P}^1$. Because $p_g(X) \geq 2k + 2$, we see that $\mathcal{O}(2k+1) \hookrightarrow f_*\omega_{X'}$. Thus we have

$$\mathcal{E} := \mathcal{O}(1) \otimes f_*\omega_{X'/\boldsymbol{P}^1}^k = \mathcal{O}(2k+1) \otimes f_*\omega_{X'}^k \hookrightarrow f_*\omega_{X'}^{k+1}.$$

Note that $H^0(\boldsymbol{P}^1, f_*\omega_{X'}^{k+1}) \cong H^0(X', \omega_{X'}^{k+1})$. It is well known that $\mathcal{E}$ is generated by global sections and that $f_*\omega_{X'/\boldsymbol{P}^1}^k$ is a sum of line bundles with non-negative degree (cf. [**F**], [**V2**], [**V3**]). Thus the global sections of $\mathcal{E}$ separates different fibers of f. On the other hand, the local sections of $f_*\omega_{X'}^k$ give the k-canonical map of F and these local sections can be extended to global sections of $\mathcal{E}$. This essentially means $M_{k+1}|_F \geq N_k$. □

Proposition 3.4. *Let X be a projective minimal algebraic 3-fold of general type with only $\boldsymbol{Q}$-factorial terminal singularities. Suppose that* $\dim \phi_1(X) = 1$. *Let $k \geq 3$ be an integer and assume $p_g(X) \geq 2k + 2$. If $(K_{F_0}^2, p_g(F)) = (1, 1)$, then*

$$K_X^3 \geq \min\left\{\frac{6k^2}{3k^2 + 8k + 4} \cdot \left(p_g(X) - \frac{4}{3}\right), \frac{6k}{3k+4} \cdot \left(p_g(X) - \frac{5}{3}\right)\right\}.$$

Proof. From Step 2 in the proof of Proposition 3.3, we have shown that

$$(k+1)\pi^*(K_X)|_F \geq M_{k+1}|_F \geq k\sigma^*(K_{F_0}).$$

(Although we suppose $k \geq 4$ in Proposition 3.3, the case with $k = 3$ can be parallelly treated since $|3K_{F_0}|$ is base point free for a surface with $(K_{F_0}^2, p_g(F)) = (1, 1)$.)

The canonical map derives a fibration $f : X' \to W$. Because $q(F) = 0$, we have

$$q(X) = h^1(\mathcal{O}_{X'}) = b + h^1(W, R^1 f_*\omega_{X'}) = b,$$

$$\begin{aligned} h^2(\mathcal{O}_X) &= h^1(W, f_*\omega_{X'}) + h^0(W, R^1 f_*\omega_{X'}) \\ &= h^1(W, f_*\omega_{X'}) \leq 1. \end{aligned}$$

It is obvious that $h^2(\mathcal{O}_X) = 0$ when $b = 0$, since $f_*\omega_{X'}$ is a line bundle of positive degree. Anyway, we have $q(X) - h^2(\mathcal{O}_X) \geq 0$. Thus we get

$$\chi(\omega_X) = p_g(X) + q(X) - h^2(\mathcal{O}_X) - 1 \geq p_g(X) - 1.$$

By the plurigenus formula of Reid ([**R1**]), we have

$$P_2(X) \geq \frac{1}{2}K_X^3 - 3\chi(\mathcal{O}_X) \geq \frac{1}{2}K_X^3 + 3[p_g(X) - 1]. \tag{3.4}$$

Let M_2 be the movable part of $|2K_{X'}|$. We consider the natural restriction map γ:

$$H^0(X', M_2) \xrightarrow{\gamma} V_2 \subset H^0(F, M_2|_F) \subset H^0(F, 2K_F),$$

where V_2 is the image of γ as a $\boldsymbol{C}$-subspace of $H^0(F, M_2|_F)$. Because $h^0(2K_F) = 3$, we see that $1 \leq \dim_{\boldsymbol{C}} V_2 \leq 3$. Denote by Λ_2 the linear system corresponding to V_2. We have $\dim \Lambda_2 = \dim_{\boldsymbol{C}} V_2 - 1$.

Case 1. $\dim_{\boldsymbol{C}} V_2 = 3$.

Since Λ_2 is a sub-system of $|2K_F|$, we see that the restriction of $\phi_{2,X'}$ to F is exactly the bicanonical map of F. Because $\phi_{2,F}$ is a generically finite morphism of degree 4, $\phi_{2,X'}$ is also a generically finite map of degree 4. Let $S_2 \in |M_2|$ be a general member. We can further modify π such that $|M_2|$ is base point free. Then S_2 is a smooth projective irreducible surface of general type. On the surface S_2, denote $L_2 := S_2|_{S_2}$. L_2 is a nef and big divisor. We have

$$2\pi^*(K_X)|_{S_2} \geq S_2|_{S_2} = L_2.$$

We consider the natural map

$$H^0(X', S_2) \xrightarrow{\gamma'} \overline{V_2} \subset H^0(S_2, L_2),$$

where $\overline{V_2}$ is the image of γ'. Denote by $\overline{\Lambda_2}$ the linear system corresponding to $\overline{V_2}$. Because ϕ_2 is generically finite map of degree 4, we see that $|L_2|$ has a sub-system $\overline{\Lambda_2}$ which gives a generically finite map of degree 4. By Lemma 2.2(ii), we get $L_2^2 \geq 4(\dim_{\boldsymbol{C}} \overline{\Lambda_2} - 1) \geq 4(P_2(X) - 3)$. Therefore we have

$$K_X^3 \geq \frac{1}{8}L_2^2 \geq \frac{1}{2}(P_2(X) - 3) \geq \frac{1}{2}\left(\frac{1}{2}K_X^3 + 3p_g(X) - 6\right).$$

Therefore

$$K_X^3 \geq 2p_g(X) - 4. \tag{3.5}$$

Case 2. $\dim_{\boldsymbol{C}} V_2 = 2$.

In this case, $\dim \phi_2(F) = 1$ and $\dim \phi_2(X) = 2$. We may further modify π such that $|M_2|$ is base point free. Taking the Stein factorization of ϕ_2, we get a derived fibration $f_2 : X' \to W_2$ where W_2 is a surface. Let C be a general fiber of f_2. we see that F is naturally fibred by curves with the same numerical type as C. On the surface F, we have a free pencil $\Lambda_2 \subset |2K_F|$. Let $|C_0|$ be the movable part of Λ_2. Then $h^0(F, C_0) = 2$. Because $q(F) = 0$, we see that $|C_0|$ is a pencil over the rational curve. So a general member of $|C_0|$ is an irreducible curve. According to Lemma 2.5, we have $(C_0 \cdot \sigma^*(K_{F_0}))_F \geq 2$ whence

$$(\pi^*(K_X) \cdot C)_{X'} = (\pi^*(K_X)|_F \cdot C_0)_F \geq \frac{k}{k+1}(\sigma^*(K_{F_0}) \cdot C_0)_F \geq \frac{2k}{k+1}.$$

Now we study on the surface S_2. We may write

$$S_2|_{S_2} \sim \sum_{i=1}^{a_2} C_i \equiv a_2 C,$$

where the $C_i's$ are fibers of f_2 and $a_2 \geq P_2(X) - 2$. Noting that

$$(\pi^*(K_X)|_{S_2} \cdot C)_{S_2} = (\pi^*(K_X) \cdot C)_{X'} \geq \frac{2k}{k+1}$$

and $2\pi^*(K_X)|_{S_2} \geq S_2|_{S_2}$, we get

$$\begin{aligned} 4K_X^3 &\geq 2\pi^*(K_X)^2 \cdot S_2 = 2(\pi^*(K_X)|_{S_2})^2_{S_2} \\ &\geq a_2(\pi^*(K_X)|_{S_2} \cdot C)_{S_2} \geq \frac{2k}{k+1}(P_2(X) - 2) \\ &\geq \frac{2k}{k+1}\left(\frac{1}{2}K_X^3 + 3p_g(X) - 5\right). \end{aligned}$$

Equivalently

$$K_X^3 \geq \frac{6k}{3k+4} p_g(X) - \frac{10k}{3k+4}. \tag{3.6}$$

Case 3. $\dim_{\mathbf{C}} V_2 = 1$.

In this case, $\dim \phi_2(X) = 1$. Because $p_g(X) > 0$, we see that both ϕ_2 and ϕ_1 give the same fibration $f : X' \to W$ after taking the Stein factorization of them. So we may write

$$2\pi^*(K_X) \sim \sum_{i=1}^{a_2'} F_i + E_2' \equiv a_2' F + E_2',$$

where the $F_i's$ are fibers of f, E_2' is an effective $\mathbf{Q}$-divisor, $a_2' \geq P_2(X) - 1$ and F is a surface with $(K_{F_0}^2, p_g(F)) = (1, 1)$. So we get

$$\begin{aligned} 2K_X^3 &\geq a_2'(\pi^*(K_X)|_F)^2_F \geq \frac{k^2}{(k+1)^2}(P_2(X) - 1) \\ &\geq \frac{k^2}{(k+1)^2}\left(\frac{1}{2}K_X^3 + 3p_g(X) - 4\right). \end{aligned}$$

Equivalently

$$K_X^3 \geq \frac{6k^2}{3k^2+8k+4} p_g(X) - \frac{8k^2}{3k^2+8k+4}. \tag{3.7}$$

Comparing (3.5), (3.6) and (3.7), we get the inequality. $\square$

Propositions 3.1, 3.2, 3.3 and 3.4 imply Theorem 3.

4. Inequalities for minimal Gorenstein 3-folds.

This section is devoted to study lower bounds for K_X^3 of Gorenstein 3-folds. Let X be a projective minimal Gorenstein 3-fold of general type with only locally factorial terminal singularities. It is well known that K_X^3 is a positive even integer and $\chi(\mathcal{O}_X) < 0$. We also have the Miyaoka-Yau inequality ([**M2**]): $K_X^3 \leq -72\chi(\mathcal{O}_X)$. Besides, after taking a special birational modification to X according to Reid ([**R2**]) while using a result of Miyaoka ([**M2**]), we get the plurigenus formula as follows.

$$P_m(X) = (2m-1)\left(\frac{m(m-1)}{12}K_X^3 - \chi(\mathcal{O}_X)\right). \tag{4.1}$$

The following theorem improves [**Kob**, Main Theorem], where we use the same notations as in previous sections.

THEOREM 4.1. *Let X be a projective minimal Gorenstein 3-fold of general type with only locally factorial terminal singularities. Then we have*

(i) *If* $\dim \phi_1(X) = 3$, *then* $K_X^3 \geq 2p_g(X) - 6$.

(ii) *If* $\dim \phi_1(X) = 2$, *i.e., X is canonically fibered by curves of genus g, then*

$$K_X^3 \geq \ulcorner \frac{2}{3}(g-1) \urcorner (p_g(X) - 2).$$

(iii) *If* $\dim \phi_1(X) = 1$, *then either* $K_X^3 \geq 2p_g(X) - 4$ *or* $(K_{F_0}^2, p_g(F)) = (1, 2)$.

PROOF. By Proposition 3.1, it is sufficient to study the cases $\dim \phi_1(X) < 3$.

Case 1. $\dim \phi_1(X) = 2$.

The canonical map gives a fibration $f : X' \to W$, where a general fiber C is a smooth curve of genus g. If $g = 2$, our inequality is $K_X^3 \geq p_g(X) - 2$, which is trivially true. Now we assume $g \geq 3$. Denote $L := \pi^*(K_X)|_{S_1}$, which is a nef and big Cartier divisor. Let $S_1 \in |M_1|$ be a general member. Then S_1 is a smooth projective surface of general type. Noting that $|S_1|_{S_1}|$ is composed of a free pencil of curves with the same numerical type as C, we have

$$\pi^*(K_X)|_{S_1} \equiv aC + E_2,$$

where E_2 is effective and $a \geq p_g(X) - 2$, and we immediately see

$$K_X^3 \geq (L \cdot C)(p_g(X) - 2).$$

Thus it is sufficient to bound $(L \cdot C)$ from below.

We run once more a recursive program (the *β-program*) which is essentially similar to the *α-program*. There is, however, a minor difference between them. Pick up a positive integer β. Obviously, we have

$$|K_{X'} + \beta\pi^*(K_X) + S_1| \subset |(\beta+2)K_{X'}|.$$

The vanishing theorem gives

$$|K_{X'} + \beta\pi^*(K_X) + S_1|\,|_{S_1} = |K_{S_1} + \beta L|.$$

We have $L \geq C$. If $\beta > 1$, then we have

$$|K_{S_1} + (\beta-1)L + C|\,|_C = |K_C + D_\beta|,$$

where $D_\beta := (\beta-1)L|_C$. Let $M_{\beta+2}$ be the movable part of $|(\beta+2)K_{X'}|$ and $M'_{\beta+2}$

be the movable part of $|K_{X'} + \beta\pi^*(K_X) + S_1|$. Then $M_{\beta+2} \geq M'_{\beta+2}$. Let N_β be the movable part of $|K_{S_1} + (\beta - 1)L + C|$. Then, by Lemma 2.6, we have

$$(\beta + 2)L \geq M_{\beta+2}|_{S_1} \geq M'_{\beta+2}|_{S_1} \geq N_\beta.$$

Also by Lemma 2.6, we have $h^0(C, N_\beta|_C) = h^0(K_C + D_\beta)$. If $\deg(D_\beta) = (\beta - 1) \cdot (L \cdot C) \geq 2$, then

$$h^0(C, N_\beta|_C) = g - 1 + (\beta - 1)(L \cdot C).$$

Using R-R again and Clifford's theorem, we see that $h^1(C, N_\beta|_C) = 0$ and

$$(\beta + 2)(L \cdot C) \geq N_\beta \cdot C = 2g - 2 + (\beta - 1)(L \cdot C).$$

We get the inequality

$$L \cdot C \geq \frac{2g - 2 + (\beta - 1)(L \cdot C)}{\beta + 2}. \tag{4.2}$$

Now take $\beta = 3$. Then $\deg(D_3) \geq 2$. According to (4.2), we see $L \cdot C > 1$, i.e. $L \cdot C \geq 2$. From now on, we can constantly take $\beta = 2$. We see that $\deg(D_2) \geq 2$. So (4.2) becomes $L \cdot C \geq (2g - 2)/3$. This means $L \cdot C \geq \ulcorner (2/3)(g - 1) \urcorner$.

Case 2. $\dim \phi_1(X) = 1$.

In this case, the canonical map derives a fibration $f : X' \to W$ onto a smooth curve W where a general fiber F of f is a smooth irreducible surface of general type. We have $\pi^*(K_X) = S_1 + E'$ and $S_1 \equiv b_1 F$, where $b_1 \geq p_g(X) - 1$. Denote $\bar{S} = \pi(S_1)$ and $\bar{F} = \pi(F)$. Then $\bar{S} \equiv b_1 \bar{F}$. Because $\bar{F}^2$ is pseudo-effective, $K_X \cdot \bar{F}^2 \geq 0$. Note that $K_X \cdot \bar{F}^2$ is an even integer.

If $K_X \cdot \bar{F}^2 > 0$, then we have $K_X^2 \cdot \bar{F} \geq 2(p_g(X) - 1)$ and thus $K_X^3 \geq 2(p_g(X) - 1)^2$.

If $K_X \cdot \bar{F}^2 = 0$, then $\mathcal{O}_F(\pi^*(K_X)|_F) \cong \mathcal{O}_F(\sigma^*(K_{F_0}))$ by a trivial generalization of [**Ch3**, Lemma 2.3]. Thus we always have

$$\begin{aligned} K_X^3 &= \pi^*(K_X)^3 \geq (\pi^*(K_X)^2 \cdot F)(p_g(X) - 1) \\ &= \sigma^*(K_{F_0})^2 (p_g(X) - 1) \geq 2(p_g(X) - 1) \end{aligned}$$

whenever $K_{F_0}^2 \geq 2$.

When $K_{F_0}^2 = 1$, the only possibility is $1 \leq p_g(F) \leq 2$. We can prove that $K_X^3 \geq 2p_g(X) - 4$ if $(K_{F_0}^2, p_g(F)) = (1, 1)$. In fact, this is the special case of Proposition 3.4 and the estimation here is more exact since X is Gorenstein. The main point is that we have $\pi^*(K_X)|_F \sim \sigma^*(K_{F_0})$. We see from the proof of Proposition 3.4 that (3.5) is still as $K_X^3 \geq 2p_g(X) - 4$, that (3.6) corresponds to $K_X^3 \geq 2p_g(X) - 3(1/3)$ and that (3.7) will be replaced by $K_X^3 \geq 2p_g(X) - 2(2/3)$. □

From Theorem 4.1, one sees that bad cases possibly occur when X is canonically fibered by curves of genus 2 or by surfaces with invariants $(c_1^2, p_g) = (1, 2)$. For technical reasons, we are only able to treat a nonsingular 3-fold. One needs a new method to cover singular 3-folds.

Now suppose that X is a smooth projective 3-fold. Let $\overline{M}$ be a divisor on X such that $h^0(X, \overline{M}) \geq 2$ and that $|\overline{M}|$ has base points but no fixed part. By Hironaka's theorem ([**Hi**]), we may take successive blow-ups

$$\pi : X' = X_n \xrightarrow{\pi_n} X_{n-1} \to \cdots \to X_i \xrightarrow{\pi_i} X_{i-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

such that

(i) π_i is a single blow-up along smooth center W_i on X_{i-1} for all i;

(ii) W_i is contained in the base locus of the movable part of

$$|(\pi_1 \circ \pi_2 \circ \cdots \circ \pi_{i-1})^*(\overline{M})|$$

and thus W_i is a reduced closed point or a smooth projective curve on X_{i-1};

(iii) the movable part of $|\pi^*(\overline{M})|$ has no base points.

It is clear that the resulting 3-fold X' is still smooth. Let E_i be the exceptional divisor on X' corresponding to W_i. Then we may write

$$K_{X'} = \pi^*(K_X) + \sum_{i=1}^{n} a_i E_i, \quad \pi^*(\overline{M}) = M + \sum_{i=1}^{n} e_i E_i,$$

where $a_i, e_i \in \boldsymbol{Z}$, $a_i \geq 0$ and M is the movable part of $|\pi^*(\overline{M})|$. From the definition of π, we see $e_i > 0$ for all i.

Lemma 4.2. *$a_i \leq 2e_i$ for all i.*

Proof. We prove the simple lemma by induction. Denote by M_i the strict transform of $\overline{M}$ in X_i for all i. Let $E_i^{(i)}$ be the exceptional divisor on X_i corresponding to W_i. Let $E_i^{(j)}$ be the strict transform of $E_i^{(i)}$ in X_j for $j > i$.

For $i = 1$, we have

$$K_{X_1} = \pi_1^*(K_X) + a_1^{(1)} E_1^{(1)} \quad \text{and} \quad \pi_1^*(\overline{M}) = M_1 + e_1^{(1)} E_1^{(1)}.$$

From the definition of π_1, we know that $e_1^{(1)} \geq 1$. Note that $a_1^{(1)}$ is computable. In fact, $a_1^{(1)} = 2$ if W_1 is a reduced smooth point of X; $a_1^{(1)} = 1$ if W_1 is a smooth curve on X. Clearly, we have $a_1^{(1)} \leq 2e_1^{(1)}$.

For $i = n-1$, we have

$$K_{X_{n-1}} = (\pi_1 \circ \cdots \circ \pi_{n-1})^*(K_X) + \sum_{i=1}^{n-1} a_i^{(n-1)} E_i^{(n-1)}$$

$$(\pi_1 \circ \cdots \circ \pi_{n-1})^*(\overline{M}) = M_{n-1} + \sum_{i=1}^{n-1} e_i^{(n-1)} E_i^{(n-1)}.$$

Suppose we have already had $a_i^{(n-1)} \leq 2e_i^{(n-1)}$. Then we get

$$\begin{aligned} K_{X_n} &= \pi_n^*(K_{X_{n-1}}) + a_n^{(n)} E_n^{(n)} \\ &= \pi^*(K_X) + \pi_n^* \sum_{i=1}^{n-1} a_i^{(n-1)} E_i^{(n-1)} + a_n^{(n)} E_n^{(n)}. \end{aligned}$$

$$\begin{aligned} \pi^*(\overline{M}) &= \pi_n^*(M_{n-1}) + \pi_n^* \sum_{i=1}^{n-1} e_i^{(n-1)} E_i^{(n-1)} \\ &= M + \pi_n^* \sum_{i=1}^{n-1} e_i^{(n-1)} E_i^{(n-1)} + e_n^{(n)} E_n^{(n)}. \end{aligned}$$

Because π_n is also a single blow-up, we see similarly that $a_n^{(n)} \le 2e_n^{(n)}$. Note that $E_n^{(n)} = E_n$ and

$$\sum_{i=1}^{n} a_i E_i = \pi_n^* \sum_{i=1}^{n-1} a_i^{(n-1)} E_i^{(n-1)} + a_n^{(n)} E_n;$$

$$\sum_{i=1}^{n} e_i E_i = \pi_n^* \sum_{i=1}^{n-1} e_i^{(n-1)} E_i^{(n-1)} + e_n^{(n)} E_n.$$

We see that $a_i \le 2e_i$. The proof is complete. □

Theorem 4.3. *Let X be a projective minimal smooth 3-fold of general type. Suppose $\dim \phi_1(X) = 2$ and X is canonically fibred by curves of genus 2. Then*

$$K_X^3 \ge \frac{1}{3}(4p_g(X) - 10).$$

The inequality is sharp.

Proof. We keep the same notations as in 1.3 and in Case 1 of the proof of Theorem 4.1. Set $K_X \sim \overline{M} + \bar{Z}$, where $\overline{M}$ is the movable part of $|K_X|$ and $\bar{Z}$ is the fixed part. We may take the same successive blow-ups

$$\pi : X' = X_n \xrightarrow{\pi_n} X_{n-1} \to \cdots \to X_i \xrightarrow{\pi_i} X_{i-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

as in the set up for Lemma 4.2.

Let $g = \phi_1 \circ \pi$. Taking the Stein-factorization of g, we get the induced fibration $f : X' \to W$. A general fiber of f is a smooth curve of genus 2 by assumption of the theorem. Let S_1 be the movable part of $|\pi^*(\overline{M})|$. Then we have

$$K_{X'} = \pi^*(K_X) + E = \pi^*(K_X) + \sum_{i=0}^{p} a_i E_i$$

and $\pi^*(\overline{M}) \sim S_1 + \sum_{i=0}^{p} e_i E_i$. We know that $a_i \ge 0$, $e_i > 0$ and both a_i and e_i are integers for all i. We also have

$$\pi^*(K_X) = \pi^*(\overline{M}) + \pi^*(\bar{Z}) = S_1 + \sum_{i=0}^{p} e_i E_i + \pi^*(\bar{Z})$$

$$\sim S_1 + \sum_{i=0}^{p} e_i' E_i + \sum_{j=1}^{q} d_j L_j = S_1 + E',$$

where $e_i' \ge e_i$, $d_j > 0$, $E_i \ne L_j$ and $L_{j_1} \ne L_{j_2}$ provided $j_1 \ne j_2$. On the surface S_1, set $L := \pi^*(K_X)|_{S_1}$. We also have $S_1|_{S_1} \equiv aC$ where $a \ge p_g(X) - 2$ and C is a general fiber of the restricted fibration $f|_{S_1} : S_1 \to f(S_1)$. Note that the above C lies in the same numerical class as that of a general fiber of f. If $L \cdot C \ge 2$, we have already seen in the proof of Theorem 4.1 that $K_X^3 \ge 2p_g(X) - 4$. From now on, we suppose $L \cdot C = 1$. Note that, in this situation, $|\overline{M}|$ definitely has base points. (Otherwise, $\pi =$ identity and

$$L \cdot C = K_X|_{S_1} \cdot C = (K_X + S_1)|_{S_1} \cdot C = K_{S_1} \cdot C = 2$$

which contradicts to the assumption $L \cdot C = 1$.)

Denote $E'|_{S_1} := E'_V + E'_H$, where E'_V is the vertical part, *i.e.*, $\dim f|_{S_1}(E'_V) = 0$, and E'_H is the horizontal part, *i.e.*, $E'_H \cdot C > 0$. Because $E'|_{S_1} \cdot C = L \cdot C = 1$, we see that $E'_H \cdot C = 1$. This means that E'_H is an irreducible curve and is a section of the restricted fibration $f|_{S_1}$. Denote $E|_{S_1} := E_V + E_H$, where E_V is the vertical part and E_H is the horizontal part. From $K_{S_1} \cdot C = 2$, one sees that $E_H \cdot C = E|_{S_1} \cdot C = 1$. This also means that E_H is an irreducible curve and E_H comes from some exceptional divisor E_i with $a_i = 1$. We may suppose that E_H comes from E_0. Then $a_0 = 1$. Because $e'_0 > 0$ and $\pi^*(K_X) \cdot C = 1$, we see that $e'_0 = 1$ and thus E'_H also comes from E_0. Since $E_0|_{S_1}$ has only one horizontal part, E_H and E'_H coincide with a curve G. Now it is quite clear that

$$E_V = \sum_{i=1}^{p} a_i(E_i|_{S_1}) + (E_0|_{S_1} - G),$$

$$E'_V = \sum_{i=1}^{p} e'_i(E_i|_{S_1}) + \sum_{j=1}^{q} d_j(L_j|_{S_1}) + (E_0|_{S_1} - G).$$

We have the following

Claim. $E_V \le 2E'_V$.

This is apparently a direct consequence of Lemma 4.2. In fact, we have $a_i \le 2e_i \le 2e'_i$ by Lemma 4.2 for all $i > 0$. Thus

$$\sum_{i=1}^{p} a_i(E_i|_{S_1}) \le 2\sum_{i=1}^{p} e'_i(E_i|_{S_1}) \le 2\left(\sum_{i=1}^{p} e'_i(E_i|_{S_1}) + \sum_{j=1}^{q} d_j(L_j|_{S_1})\right).$$

On the other hand, it is obvious that $E_0|_{S_1} - G \le 2(E_0|_{S_1} - G)$. Therefore we get

$$\begin{aligned} E_V &= (E_0|_{S_1} - G) + \sum_{i=1}^{p} a_i(E_i|_{S_1}) \\ &\le 2(E_0|_{S_1} - G) + 2\left(\sum_{i=1}^{p} e'_i(E_i|_{S_1}) + \sum_{j=1}^{q} d_j(L_j|_{S_1})\right) = 2E'_V \end{aligned}$$

and the claim is proved.

Since that $2E'_V - E_V$ is effective and vertical, we see that $E_V \cdot G \le 2E'_V \cdot G$. On the surface S_1, we have

$$(K_{S_1} + 2C + G)G = 2p_a(G) - 2 + 2G \cdot C = 2p_a(G) \ge 0.$$

On the other hand, we have

$$\begin{aligned} &(K_{S_1} + 2C + G)G \\ &\quad = ((\pi^*(K_X)|_{S_1} + E_V + G + S_1|_{S_1}) + 2C + G)G \\ &\quad \le (\pi^*(K_X)|_{S_1} + S_1|_{S_1} + G) \cdot G + 2E'_V \cdot G + 2 + G^2 \\ &\quad = 2\pi^*(K_X)|_{S_1} \cdot G + E'_V \cdot G + G^2 + 2. \end{aligned}$$

So we have

$$2\pi^*(K_X)|_{S_1} \cdot G + E'_V \cdot G + G^2 + 2 \geq 0. \tag{4.3}$$

We also have

$$\pi^*(K_X)|_{S_1} \cdot G = S_1|_{S_1} \cdot G + E'_V \cdot G + G^2. \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$3\pi^*(K_X)|_{S_1} \cdot G \geq S_1|_{S_1} \cdot G - 2 \geq p_g(X) - 4.$$

$$\pi^*(K_X) \cdot S_1 \cdot E' \geq \pi^*(K_X)|_{S_1} \cdot G \geq \frac{1}{3}(p_g(X) - 4).$$

Finally, we have

$$\begin{aligned} K_X^3 &= \pi^*(K_X)^3 \geq \pi^*(K_X)^2 \cdot S_1 \\ &= \pi^*(K_X)|_{S_1} \cdot S_1|_{S_1} + \pi^*(K_X)|_{S_1} \cdot E'|_{S_1} \\ &\geq (p_g(X) - 2) + \frac{1}{3}(p_g(X) - 4) = \frac{2}{3}(2p_g(X) - 5). \end{aligned}$$

The inequality is sharp by virtue of (0.1). The proof is complete. □

Remark 4.4. As was pointed out by M. Reid ([**R3**, Remark (0.4)(v)]), the blow-up of a canonical singularity need not be normal and thus it need not be canonical, even if the original canonical point is a hypersurface singularity of multiplicity 2. Because of this reason, we would rather treat a smooth 3-fold in Theorem 4.3, although the method might be all right for Gorenstein 3-folds.

Lemma 4.5. *Let X be a smooth projective 3-fold of general type. Suppose $p_g(X) \geq 3$, $\dim \phi_1(X) = 1$. Keep the same notations as in subsection 1.3. If $(K_{F_0}^2, p_g(F)) = (1, 2)$, then one of the following holds*:

(i) $b = 1$, $q(X) = 1$ *and* $h^2(\mathcal{O}_X) = 0$;

(ii) $b = 0$, $q(X) = 0$ *and* $h^2(\mathcal{O}_X) \leq 1$.

Proof. Replacing X by a birational model, if necessary, we may suppose that ϕ_1 is a morphism. Note that we do not need here the minimality of X. Taking the Stein-factorization of ϕ_1, we get a derived fibration $f : X \to W$. Let F be a general fiber of f. By assumption, $(K_{F_0}^2, p_g(F)) = (1, 2)$ where F_0 is the minimal model of F. According to [**Ch2**, Theorem 1], we see that $b = g(W) \leq 1$ whenever $p_g(X) \geq 3$. Because $q(F) = 0$, we can easily see that $q(X) = b$ and $h^2(\mathcal{O}_X) = h^1(W, f_*\omega_X)$. In order to prove the lemma, it is sufficient to study $h^1(W, f_*\omega_X)$. Since we are in a very special situation, we should be able to obtain much more explicit information.

Let $\mathcal{L}_0$ be the saturated sub-bundle of $f_*\omega_X$ which is generated by $H^0(W, f_*\omega_X)$. Because $|K_X|$ is composed of a pencil of surfaces and ϕ_1 factors through f, we see that $\mathcal{L}_0$ is a line bundle on W. Denote $\mathcal{L}_1 := f_*\omega_X/\mathcal{L}_0$. Then we have the exact sequence:

$$0 \to \mathcal{L}_0 \to f_*\omega_X \to \mathcal{L}_1 \to 0.$$

Noting that $\mathrm{rk}(f_*\omega_X) = 2$, we see that $\mathcal{L}_1$ is also a line bundle. Noting that $H^0(W, \mathcal{L}_0) \cong H^0(W, f_*\omega_X)$, we have $h^1(W, \mathcal{L}_0) \geq h^0(W, \mathcal{L}_1)$. When $b = 1$,

$\deg(\mathscr{L}_0) = p_g(X) \geq 3$. When $b = 0$, $\deg(\mathscr{L}_0) = p_g(X) - 1 \geq 2$. Anyway, we have $h^1(W, \mathscr{L}_0) = 0$. So $h^0(W, \mathscr{L}_1) = 0$. On the other hand, it is well-known that $f_*\omega_{X/W}$ is semi-positive. Thus $\deg(\mathscr{L}_1 \otimes \omega_W^{-1}) \geq 0$. This means $\deg(\mathscr{L}_1) \geq 2(b-1)$. Using the R-R, we may easily deduce that $h^1(\mathscr{L}_1) \leq 1 - b$. So

$$h^1(W, f_*\omega_X) \leq h^1(W, \mathscr{L}_0) + h^1(W, \mathscr{L}_1) \leq 1 - b.$$

So $h^2(\mathscr{O}_X) \leq 1 - b$. The proof is complete. $\square$

Lemma 4.6. *Let X be a smooth projective 3-fold of general type. Suppose $p_g(X) \geq 3$, $\dim \phi_1(X) = 1$ and $(K_{F_0}^2, p_g(F)) = (1, 2)$. Let $f : X \to W$ be a derived fibration of ϕ_1. Suppose F_1 and F_2 are two fixed smooth fibres of f such that $\phi_1(F_1) \neq \phi_1(F_2)$. Then $\dim \Phi_{|K_X+F_1+F_2|}(X) = 2$ and $\Phi_{|K_X+F_1+F_2|}|_F = \Phi_{|K_F|}$ for a general fiber F.*

Proof (i) If $b = 1$, we have $h^2(\mathscr{O}_X) = 0$ by Lemma 4.5. From the exact sequence

$$H^0(X, K_X + F_1 + F_2) \to H^0(F_1, K_{F_1}) \oplus H^0(F_2, K_{F_2}) \to 0,$$

one may easily see that $\dim \Phi_{|K_X+F_1+F_2|}(X) = 2$. Thus, for a general fiber F, $\dim \Phi_{|K_X+F_1+F_2|}(F) = 1$. Since $p_g(F) = 2$, one sees that $\Phi_{|K_X+F_1+F_2|}|_F = \Phi_{|K_F|}$.

(ii) If $b = 0$, we only have to study $|K_X + 2F_1||_F$ for a general fiber F. From the short exact sequence:

$$0 \to \mathscr{O}_X(K_X + F_1 - F) \to \mathscr{O}_X(K_X + F_1) \to \mathscr{O}_F(K_F) \to 0,$$

we have the long exact sequence

$$\cdots \to H^0(X, K_X + F_1) \xrightarrow{\alpha_1} H^0(F, K_F) \xrightarrow{\beta_1} H^1(X, K_X)$$
$$\to H^1(X, K_X + F_1) \to H^1(F, K_F) = 0,$$

If α_1 is surjective for general F, then we see that

$$\dim \Phi_{|K_X+F_1|}(F) = \dim \Phi_{|K_F|}(F) = 1 \quad \text{and} \quad \dim \Phi_{|K_X+F_1|}(X) = 2.$$

So $\dim \Phi_{|K_X+2F_1|}(X) = 2$. We are done. Otherwise, α_1 is not surjective. Because $\alpha_1 \neq 0$, we see that $h^2(\mathscr{O}_X) = h^1(X, K_X) \geq 1$. Because $h^2(\mathscr{O}_X) \leq 1$, $h^2(\mathscr{O}_X) = 1$ and β_1 is surjective. Therefore $H^1(X, K_X + F_1) = 0$. This also means that $H^1(X, K_X + F') = 0$ for any smooth fiber F' since $F' \sim F_1$. So we have $H^1(X, K_X + 2F_1 - F) = 0$, which means $|K_X + 2F_1||_F = |K_F|$. So $\dim \Phi_{|K_X+2F_1|}(X) = 2$. The proof is complete. $\square$

Theorem 4.7. *Let X be a smooth projective 3-fold with ample canonical divisor. Suppose $\dim \phi_1(X) = 1$ and X is canonically fibered by surfaces with invariants $(c_1^2, p_g) = (1, 2)$. Then $K_X^3 \geq (2/3)(2p_g(X) - 7)$.*

Proof. The proof is slightly longer, however with the same flavour as that of Theorem 4.3.

Denote by $\bar{F}$ a generic irreducible element of $|K_X|$. We see that $\bar{F}^2$ is a 1-cycle on X. If the movable part of $|K_X|$ has base points, then $\bar{F}^2$ is a non-trivial effective 1-cycle. So $K_X \cdot \bar{F}^2 \geq 2$. Thus $K_X^3 \geq 2p_g(X) - 2$. Therefore we only have to treat the case when ϕ_1 is a morphism.

We suppose $p_g(X) \geq 3$. We still assume that $f : X \to W$ is a derived fibration of ϕ_1. Note that $b = g(W) \leq 1$. Let $\overline{M}$ be the movable part of $|K_X + F_1 + F_2|$. Also note that F is minimal in this situation and $(K_F^2, p_g(F)) = (1, 2)$. It is well-known that $|K_F|$ has exactly one base point, but no fixed part, and that a general member of $|K_F|$ is a smooth irreducible curve of genus 2. Since $|K_X + F_1 + F_2|\,|_F = |K_F|$ and according to Lemma 2.6, we see that $\overline{M}|_F = K_F$. This means that $|\overline{M}|$ definitely has base points. According to Hironaka, we can take successive blow-ups

$$\pi : X' = X_n \xrightarrow{\pi_n} X_{n-1} \to \cdots \to X_i \xrightarrow{\pi_i} X_{i-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

such that

(i) π_i is a single blow-up along smooth center W_i on X_{i-1} for all i;

(ii) W_i is contained in the base locus of the movable part of

$$|(\pi_1 \circ \pi_2 \circ \cdots \circ \pi_{i-1})^*(\overline{M})|$$

and thus W_i is a reduced closed point or a smooth projective curve on X_{i-1};

(iii) the movable part of $|\pi^*(\overline{M})|$ has no base points.

Denote by E_i the exceptional divisor on X' corresponding to W_i for all i. Note that the resulting 3-fold X' is still smooth. Let M be the movable part of $|\pi^*(\overline{M})|$ and $S \in |M|$ be a general member. Then S is a smooth irreducible projective surface of general type. Denote $f' := f \circ \pi$. Then $f' : X' \to W$ is still a fibration. Let F' be a general fiber of f'. Note that F' has the minimal model F. We may write

$$K_{X'} \sim \pi^*(K_X) + \sum_{i=0}^{p} a_i E_i = \pi^*(K_X) + E$$

and $\pi^*(\overline{M}) = M + \sum_{i=0}^{p} e_i E_i$. According to Lemma 4.2, we have $0 < a_i \leq 2e_i$ for all i. Recall that we have $K_X \sim S_1 + Z = \sum_{i=1}^{b_1} F_i + Z$, where $b_1 \geq p_g(X) - 1$, the $F_i's$ are fibers of f, S_1 is the movable part of $|K_X|$ and Z the fixed part of $|K_X|$. Note that there is an effective divisor $Z_0 \leq Z$ such that $\overline{M} \sim S_1 + F_1 + F_2 + Z_0$. We write

$$\begin{aligned}\pi^*(K_X + F_1 + F_2) &\sim \pi^*(\overline{M} + Z - Z_0) = M + \sum_{i=0}^{p} e_i E_i + \pi^*(Z - Z_0)\\ &= M + \sum_{i=0}^{p} e_i' E_i + \sum_{j=1}^{q} d_j L_j =: M + E',\end{aligned}$$

where $E_i \neq L_j$, $d_j > 0$, $e_i' \geq e_i$ for all i and $L_{j_1} \neq L_{j_2}$ whenever $j_1 \neq j_2$. Note that $\pi^*(\overline{M}) \geq \pi^*(S_1 + F_1 + F_2)$ and that the strict transform of S_1 is a union of b_1 fibers of f', we see that

$$M|_S \geq \sum_{j=1}^{b_1+m} F_j'|_S \equiv (b_1 + m)F'|_S$$

where the $F_j's$ are fibers of f' and $m = 2$. Because $\dim \Phi_{|M|}(X') = 2$, we see $\dim \Phi_{|M|}(S) = 1$ for a general member S. So, on S, the system $|M|_S|$ should be composed of a free pencil of curves since $(M|_S)^2 = M^3 = 0$. On the other hand, we obviously have $H^0(X', K_{X'} - S) = 0$. This instantly gives the inclusion $H^0(X', K_{X'}) \hookrightarrow H^0(S, K_{X'}|_S)$. So $\dim \Phi_{|K_{X'}|}(S) \geq 1$. Because $\dim \phi_1(X) = 1$, we see that $\dim \Phi_{|K_{X'}|}(S) = 1$. Thus it is clear $f'(S) = W$. So we have a surjective morphism $f'|_S : S \to W$. The fiber of $f'|_S$ is exactly $F' \cap S$ or the divisor $F'|_S$. Since $|M|_S|$ is composed of a pencil of curves, $M|_S \geq \sum_{j=1}^{b_1+m} F_j'|_S$ and $|\sum_{j=1}^{b_1+m} F_j'|_S|$ is vertical, we see that $|M|_S|$ is also vertical, i.e. $\dim f'|_S(M|_S) = 0$. This means that the divisor $M|_S$ is vertical with respect to the morphism $f'|_S$. By taking the Stein-factorization of $f'|_S$, one can see that $F'|_S$ is linearly equivalent to a disjoint union of irreducible curves of the same numerical type and $F'|_S \equiv \xi C$ where C is certain irreducible curve and ξ is a positive integer.

Recall that $E' := \sum_{i=0}^p e_i' E_i + \sum_{j=1}^q d_j L_j$. We may write $E'|_S := E_V' + E_H'$ where E_V' is the vertical part and E_H' is the horizontal part with $E_H' \cdot F'|_S > 0$. Noting that $\pi^*(K_X + F_1 + F_2)|_S$ is nef and big and that $M|_S$ is vertical, we see that E_H' is non-trivial. So we have

$$\pi^*(K_X + F_1 + F_2)|_S = M|_S + E'|_S = M|_S + E_V' + E_H'.$$

Also recall that $E := \sum_{i=0}^p a_i E_i$. Denote $E|_S := E_V + E_H$ where E_V is the vertical part and E_H is the horizontal part. We have

$$\begin{aligned}
0 < F'|_S \cdot E_H' &= F'|_S \cdot E'|_S = F'|_S \cdot \pi^*(K_X + F_1 + F_2)|_S \\
&= F' \cdot \pi^*(K_X + F_1 + F_2) \cdot S \\
&\leq F' \cdot \pi^*(K_X + F_1 + F_2) \cdot \pi^*(K_X + F_1 + F_2) = K_X^2 \cdot F = 1.
\end{aligned}$$

This means

$$F'|_S \cdot E_H' = F'|_S \cdot \pi^*(K_X)|_S = 1, \tag{4.5}$$

$$\pi^*(F_1)|_S \cdot F'|_S = 0. \tag{4.6}$$

Thus we see that $\xi = 1$ and thus $f'|_S : S \to W$ is a fibration. This also means that E_H' is irreducible and that it comes from certain irreducible component of E'. For generic S and F', because $S|_{F'}$ is the movable part of $|K_{F'}|$, we see that $S|_{F'}$ is an irreducible curve of genus two. This means $C = S \cap F'$ is a smooth curve of genus 2 on S and $C^2 = (F'|_S)^2 = 0$. Thus $K_S \cdot C = 2$, i.e.

$$(E_V + E_H + \pi^*(K_X)|_S + S|_S) \cdot C = 2.$$

Noting that, from (4.5), $S|_S \cdot C = M|_S \cdot F'|_S = 0$ and $\pi^*(K_X) \cdot C = 1$, we have $E_H \cdot C = 1$. This also says that E_H comes from certain irreducible component E_i in E with $a_i = 1$. For simplicity we may suppose that this component is just E_0. So $a_0 = 1$. Now it is quite clear about the structure of $E'|_S$ and $E|_S$:

$$E_H = E_H' \leq E_0|_S, \quad \sum_{i=1}^p a_i(E_i|_S) + (E_0|_S - E_H) = E_V,$$

$$\sum_{i=1}^p e_i'(E_i|_S) + \sum_{j=1}^q d_j(L_j|_S) + (E_0|_S - E_H') = E_V'.$$

Noting that $E_0|_S$ can have only one horizontal component, we denote it by $G := E_H = E'_H$. Similar to the Claim in the proof of Theorem 4.3, It is easy to see that $E_V \le 2E'_V$.

Now we may perform the computation on the surface S. We have

$$(K_S + G + 2(1-b)F'|_S) \cdot G = 2p_a(G) - 2 + 2(1-b) \ge 0.$$

(One notes that $p_a(G) \ge 1$ if $b = 1$ and $p_a(G) \ge 0$ if $b = 0$.)

$$\begin{aligned} K_S \cdot G &= (E|_S + \pi^*(K_X)|_S + S|_S) \cdot G = E_V \cdot G + G^2 + \pi^*(K_X)|_S \cdot G + S|_S \cdot G \\ &\le 2E'_V \cdot G + G^2 + S|_S \cdot G + \pi^*(K_X)|_S \cdot G \\ &= E'_V \cdot G + \pi^*(K_X + F_1 + F_2)|_S \cdot G + \pi^*(K_X)|_S \cdot G. \end{aligned}$$

So we get

$$E'_V \cdot G + \pi^*(2K_X + F_1 + F_2)|_S \cdot G + G^2 + 2(1-b) \ge 0. \tag{4.7}$$

On the other hand, we have

$$\begin{aligned} \pi^*(K_X + F_1 + F_2)|_S \cdot G &= S|_S \cdot G + E'_V \cdot G + G^2 \\ &\ge (b_1 + m)F'|_S \cdot G + E'_V \cdot G + G^2, \end{aligned} \tag{4.8}$$

where we note that $S|_S$ is vertical and, numerically, $S|_S \ge_{\mathrm{num}} (b_1 + m)F'|_S$ and $F'|_S \cdot G = 1$ by (4.5). Combining (4.7) and (4.8), we get

$$\pi^*(3K_X + 2F_1 + 2F_2)|_S \cdot G \ge (b_1 + m) + 2(b-1).$$

We have

$$\begin{aligned} \pi^*(3K_X + 2F_1 + 2F_2)|_S \cdot G &\le \pi^*(3K_X + 2F_1 + 2F_2)|_S \cdot E'|_S \\ &= \pi^*(3K_X + 2F_1 + 2F_2)|_S \cdot (\pi^*(K_X + F_1 + F_2)|_S - S|_S) \\ &= \pi^*(3K_X + 2F_1 + 2F_2)|_S \cdot \pi^*(K_X + F_1 + F_2)|_S - \pi^*(3K_X + 2F_1 + 2F_2)|_S \cdot S|_S \\ &\le (3K_X + 2F_1 + 2F_2)(K_X + F_1 + F_2)^2 - \pi^*(3K_X + 2F_1 + 2F_2)|_S \cdot S|_S \\ &= 3K_X^3 + 8m - \pi^*(3K_X + 2F_1 + 2F_2)|_S \cdot S|_S. \end{aligned}$$

Thus $3K_X^3 \ge b_1 - 7m + 2(b-1) + \pi^*(3K_X + 2F_1 + 2F_2)|_S \cdot S|_S$. By (4.5) and (4.6), we get

$$\pi^*(3K_X + 2F_1 + 2F_2)|_S \cdot S|_S \ge \pi^*(3K_X + 2F_1 + 2F_2)|_S \cdot (b_1 + m)F'|_S = 3(b_1 + m).$$

So $3K_X^3 \ge 4b_1 - 4m + 2(b-1)$. We obtain

$$K_X^3 \ge \frac{4}{3}b_1 - \frac{4}{3}m + \frac{2}{3}(b-1) \ge \begin{cases} \frac{4}{3}p_g(X) - \frac{8}{3}, & \text{if } b = 1 \\ \frac{4}{3}p_g(X) - \frac{14}{3}, & \text{if } b = 0. \end{cases}$$

Finally, we discuss what happens when $K_X^3 > (4/3)p_g(X) - (10/3)$. Definitely,

$b = 0$ and $3K_X^3 = 4p_g(X) - 11$, $4p_g(X) - 12$, $4p_g(X) - 13$, or $4p_g(X) - 14$. Noting that K_X^3 is an even number, one excludes possibilities $4p_g(X) - 11$ and $4p_g(X) - 13$. The proof is complete. □

Corollary 4.8. *Let X be a smooth projective 3-fold with ample canonical divisor. Then we have the following Noether inequality*

$$K_X^3 \geq \frac{2}{3}(2p_g(X) - 7).$$

Proof. This is a direct result of Theorem 4.1, Theorem 4.3 and Theorem 4.7. □

Corollary 4.8 implies Corollary 2. Theorem 4.1, Theorem 4.3 and Theorem 4.7 imply Theorem 5(1) and Theorem 5(2).

5. An appendix.

We go on proving Theorem 5 in this section.

Proposition 5.1. *Let X be a projective minimal Gorenstein 3-fold of general type with only locally factorial terminal singularities. Suppose X has a locally factorial canonical model. If $\dim \phi_1(X) = 1$ and $(K_{F_0}^2, p_g(F)) = (1, 2)$, then*

$$K_X^3 \geq \frac{2}{21}(11p_g(X) - 16).$$

Proof. If the movable part of $|K_X|$ has base points, then we have $K_X^3 \geq 2p_g(X) - 2$ according to [**Kob**, Case 1, Theorem (4.1)] because X is assumed to have a locally factorial canonical model. So we may suppose $\Phi_{|K_X|}$ is a morphism.

Taking the Stein-factorization of $\Phi_{|K_X|}$, we get the derived fibration $f : X \to W$. Let M_1 be the movable part of $|K_X|$ and $S_1 \in |M_1|$ a general member. We may write $S_1 \sim \sum_{i=1}^{b_1} F_i \equiv b_1 F$, where the $F_i's$ are fibers of f, F is a general fiber of f and $b_1 \geq p_g(X) - 1$. Because X is minimal, F is a minimal surface. Since X has isolated singularities, F is smooth. Note that we have $K_F^2 = 1$ and $p_g(F) = 2$ under the assumption of the proposition. We may also write $K_X \equiv b_1 F + Z$, where Z is the fixed part of $|K_X|$. According to [**Ch2**, Theorem 1], we have $b := g(W) \leq 1$ provided $p_g(X) \geq 3$. From [**L**], we know that $|4K_X|$ is base point free. Let $S_4 \in |4K_X|$ be a general member. Since X has isolated singularities, S_4 is a smooth projective irreducible surface of general type. We see that $f(S_4) = W$. Denote $f_0 := f|_{S_4}$. Then $f_0 : S_4 \to W$ is a proper surjective morphism onto W (f_0 need not be a fibration). Because $f(F)$ is a point, $F|_{S_4}$ is vertical with respect to f_0, *i.e.*, $\dim f_0(F|_{S_4}) = 0$. Now we have $K_X|_{S_4} \equiv b_1 F|_{S_4} + Z|_{S_4}$. Denote $Z|_{S_4} := Z_V + Z_H$, where Z_V is the vertical part and Z_H is the horizontal part. We may write $Z_H := \sum m_i G_i$, where $m_i > 0$ and the $G_i's$ are distinct irreducible curves on S_4. We have

$$\begin{aligned}(F|_{S_4} \cdot Z_H)_{S_4} &= (F|_{S_4} \cdot Z|_{S_4})_{S_4} = (F \cdot S_4 \cdot Z)_X \\ &= (S_4|_F \cdot Z|_F)_F = 4(K_X|_F \cdot K_X|_F)_F = 4K_F^2 = 4.\end{aligned}$$

Thus $m_i \leq 4$ for all i. Denote

$$D := 4K_{S_4} - 8(b-1)F|_{S_4} + Z_V + Z_H.$$

We claim that $D \cdot G_i \geq 0$ for all i. In fact, since $Z_V \cdot G_i \geq 0$ and $G_i \cdot G_j \geq 0$ for $i \neq j$, we have

$$\begin{aligned} D \cdot G_i &\geq 4K_{S_4} \cdot G_i - 8(b-1)F|_{S_4} \cdot G_i + m_i G_i^2 \\ &= (4 - m_i)K_{S_4} \cdot G_i + m_i(K_{S_4} \cdot G_i + G_i^2) - 8(b-1)F|_{S_4} \cdot G_i \\ &= (4 - m_i)K_{S_4} \cdot G_i + m_i(2p_a(G_i) - 2) - 8(b-1)F|_{S_4} \cdot G_i. \end{aligned}$$

Note that both K_{S_4} and $F|_{S_4}$ are nef. When $b = 1$, we have $p_a(G_i) \geq b = 1$. Thus $D \cdot G_i \geq (4 - m_i)K_{S_4} \cdot G_i \geq 0$. When $b = 0$,

$$D \cdot G_i \geq (4 - m_i)K_{S_4} \cdot G_i + (8 - 2m_i)F|_{S_4} \cdot G_i + m_i[2p_a(G_i) - 2 + 2F|_{S_4} \cdot G_i] \geq 0.$$

Therefore we get $D \cdot Z_H \geq 0$. This means

$$4K_{S_4} \cdot Z_H - 8(b-1)F|_{S_4} \cdot Z_H + (Z_V + Z_H)Z_H \geq 0. \tag{5.1}$$

On the other hand, we have

$$K_X|_{S_4} \cdot Z_H = b_1 F|_{S_4} \cdot Z_H + (Z_V + Z_H)Z_H. \tag{5.2}$$

Combining (5.1) and (5.2), we get

$$\begin{aligned} 4K_{S_4} \cdot Z_H + K_X|_{S_4} \cdot Z_H &\geq (b_1 + 8(b-1))F|_{S_4} \cdot Z_H \\ &\geq 4(p_g(X) + 10b - 11). \end{aligned}$$

We also have

$$\begin{aligned} 4K_{S_4} \cdot Z_H + K_X|_{S_4} \cdot Z_H &= 5K_X|_{S_4} \cdot Z_H + 4S_4|_{S_4} \cdot Z_H \\ &\leq 5K_X|_{S_4} \cdot Z|_{S_4} + 4S_4|_{S_4} \cdot Z|_{S_4} = 84K_X^2 \cdot Z. \end{aligned}$$

Thus we obtain

$$K_X^2 \cdot Z \geq \frac{1}{21}(p_g(X) + 10b - 11) = \begin{cases} \frac{1}{21}(p_g(X) - 11), & \text{if } b = 0, \\ \frac{1}{21}(p_g(X) - 1), & \text{if } b = 1. \end{cases}$$

Finally we get

$$K_X^3 \geq b_1 K_X^2 \cdot F + K_X^2 \cdot Z \geq \begin{cases} \frac{2}{21}(11p_g(X) - 16), & \text{if } b = 0, \\ \frac{22}{21}(p_g(X) - 1), & \text{if } b = 1. \end{cases}$$

The proof is complete. $\square$

Section 4 and Proposition 5.1 imply Theorem 5(3).

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References

[Be] A. Beauville, L'application canonique pour les surfaces de type général, Invent. Math., **55** (1979), 121–140.

[Bo] E. Bombieri, Canonical models of surfaces of general type, Publ. Math. Inst. Hautes Études Sci., **42** (1973), 171–219.

[Cas] G. Castelnuovo, Osservazioni intorno alla geometria sopra una superficie, I, II, Rendiconti del R. Istituto Lombardo, s. II, **24** (1891), also in 'Memorie scelte', Zanichelli, 1937, Bologna, 245–265.

[Cat] F. Catanese, Surfaces with $K^2 = p_g = 1$ and their period mapping, Lecture Notes in Math., **732** (1979), pp. 1–29.

[Ch1] M. Chen, An inequality with regard to the canonical birationality for varieties, Comm. Algebra, **23** (1995), 4439–4446.

[Ch2] M. Chen, Complex varieties of general type whose canonical systems are composed with pencils, J. Math. Soc. Japan, **51** (1999), 331–335.

[Ch3] M. Chen, Kawamata-Viehweg vanishing and quint-canonical maps for threefolds of general type, Comm. Algebra, **27** (1999), 5471–5486.

[Ch4] M. Chen, Canonical stability in terms of singularity index for algebraic threefolds, Math. Proc. Cambridge Philos. Soc., **131** (2001), 241–264.

[Ci] C. Ciliberto, The bicanonical map for surfaces of general type, Proc. Sympos. Pure Math., **62** (1997), 57–83.

[E-L] L. Ein and R. Lazarsfeld, Global generation of pluricanonical and adjoint linear systems on smooth projective threefolds, J. Amer. Math. Soc., **6** (1993), 875–903.

[F] T. Fujita, On Kähler fiber spaces over curves, J. Math. Soc. Japan, **30** (1978), 779–794.

[Hrs] J. Harris, A bound on the geometric genus of projective varieties, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **8** (1981), 35–68.

[Hi] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, I, Ann. of Math. (2), **79** (1964), 109–203; II, ibid., 205–326.

[Ka] Y. Kawamata, A generalization of Kodaira-Ramanujam's vanishing theorem, Math. Ann., **261** (1982), 43–46.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem, In: Algebraic Geometry, Sendai, 1985, (ed. T. Oda), Adv. Stud. Pure Math., **10**, Kinokuniya, Tokyo; North-Holland, Amsterdam-New York-Oxford, 1987, 283–360.

[Kob] M. Kobayashi, On Noether's inequality for threefolds, J. Math. Soc. Japan, **44** (1992), 145–156.

[Kol] J. Kollár, Higher direct images of dualizing sheaves, I, Ann. of Math. (2), **123** (1986), 11–42; II, ibid., **124** (1986), 171–202.

[L] S. Lee, Quartic-canonical systems on canonical threefolds of index 1, Comm. Algebra, **28** (2000), 5517–5530.

[M1] Y. Miyaoka, On the chern numbers of surfaces of general type, Invent. Math., **42** (1977), 225–237.

[M2] Y. Miyaoka, The pseudo-effectivity of $3c_2 - c_1^2$ for varieties with numerically effective canonical classes, In: Algebraic Geometry, Sendai, 1985, (ed. T. Oda), Adv. Stud. Pure Math., **10**, Kinokuniya, Tokyo; North-Holland, Amsterdam-New York-Oxford, 1987, 449–476.

[M3] Y. Miyaoka, On the Mumford-Ramanujam vanishing theorem on a surface, In: Journées de Géométrie Algébrique d'Angers, (ed. A. Beauville), Sijthoff and Noordhof, Alphen ann den Rijn, 1980, 239–247.

[N] M. Noether, Zur Theorie des eindeutigen Entsprechens algebraischer Ge-bilde, Math. Ann., **2** (1870), 293–316; **8** (1875), 495–533.

[R1] M. Reid, Young person's guide to canonical singularities, Proc. Sympos. Pure Math., **46** (1987), 345–414.

[R2] M. Reid, Minimal models of canonical 3-folds, In: Algebraic Varieties and Analytic Varieties, Tokyo, 1981, (ed. S. Iitaka), Adv. Stud. Pure Math., **1**, Kinokuniya, Tokyo; North-Holland, Amsterdam-New York-Oxford, 1983, 131–180.

[R3] M. Reid, Canonical 3-folds, In: Journées de Géométrie Algébrique d'Angers, (ed. A. Beauville), Sijthoff and Noordhof, Alphen ann den Rijn, 1980, 273–310.

[Rr] I. Reider, Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. of Math. (2), **127** (1988), 309–316.

[S] F. Sakai, Weil divisors on normal surfaces, Duke Math. J., **51** (1984), 877–887.

[V1] E. Viehweg, Vanishing theorems, J. Reine Angew. Math., **335** (1982), 1–8.

[V2] E. Viehweg, Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces, In: Algebraic Varieties and Analytic Varieties, Tokyo, 1981, (ed. S. Iitaka), Adv. Stud. Pure Math., **1**, Kinokuniya, Tokyo; North-Holland, Amsterdam-New York-Oxford, 1983, 329–353.

[V3] E. Viehweg, Weak positivity and the additivity of the Kodaira dimension, II: The local Torelli map, In: Classification of algebraic and analytic manifolds, Progr. Math., **39**, Birkhäuser Boston, Boston, 1983, 567–589.

[Y1] S. T. Yau, Calabi's conjecture and some new results in algebraic geometry, Proc. Natl. Acad. Sci. USA, **74** (1977), 1798–1799.

[Y2] S. T. Yau, On the Ricci curvature of a complex Kähler manifold and the complex Monge-Ampere equations, Comm. Pure Appl. Math., **31** (1978), 339–411.

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