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On the fundamental solutions of linear Fuchsian partial differential equations

By Toshiaki SAITO

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Abstract. Without any assumption on the characteristic exponents, we give fundamental solutions of linear Fuchsian partial differential equations.

1. Introduction and Main result.

Let $\boldsymbol{C}$ be the set of complex numbers, $t \in \boldsymbol{C}$, $x = (x_1, \dots, x_n) \in \boldsymbol{C}^n$, $\boldsymbol{N} = \{0, 1, \dots\}$, $m \in \boldsymbol{N}^* = \boldsymbol{N} - \{0\}$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{N}^n$. Let Δ be a polydisc centered at the origin of $\boldsymbol{C}_t \times \boldsymbol{C}_x^n$ and set $\Delta_0 = \Delta \cap \{t = 0\}$. Let $a_{j,\alpha}(t,x)$ $(j + |\alpha| \leq m, j < m)$ be holomorphic functions defined on Δ satisfying

$$a_{j,\alpha}(0,x) \equiv 0 \quad \text{on } \Delta_0 \text{ if } |\alpha| > 0. \tag{1.1}$$

We consider the Fuchsian partial differential operator

$$P = \left(t\frac{\partial}{\partial t}\right)^m + \sum_{\substack{j+|\alpha| \leq m \\ j<m}} a_{j,\alpha}(t,x)\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha \tag{1.2}$$

and the linear partial differential equation

$$Pu = 0. \tag{1.3}$$

The operator P in (1.2) was introduced by M. S. Baouendi and C. Goulaouic [**1**] and they proved a Cauchy-Kowalevsky type theorem and a Holmgren type theorem. Also, H. Tahara [**2**] investigated the structure of singular solutions of $Pu = 0$.

We now introduce some notations. We let

i) $\mathfrak{R}(\boldsymbol{C} \backslash \{0\})$ be the universal covering space of $\boldsymbol{C} \backslash \{0\}$,
ii) $S(\varepsilon) = \{t \in \mathfrak{R}(\boldsymbol{C} \backslash \{0\}); 0 < |t| < \varepsilon\}$,
iii) $D_L = \{x \in \boldsymbol{C}^n; |x_i| < L, i = 1, \dots, n\}$,
iv) $\tilde{\mathcal{O}}$ be the set of functions $u(t,x)$ that are holomorphic on $S(\varepsilon) \times D_L$ for some $\varepsilon > 0$ and $L > 0$,
v) $\mathcal{O}_0$ be the set of germs of holomorphic functions at $x = 0$, which is the same as the ring $\boldsymbol{C}\{x\}$ of convergent power series in x,
vi) $K[\varphi]$ be the polynomial algebra in φ with coefficients in a ring K,
vii) $\mathcal{O}(D)$ be the set of holomorphic functions on D.

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We set

$$C(\lambda, x) = \lambda^m + \sum_{j<m} a_{j,0}(0, x)\lambda^j.$$

This polynomial in λ is called the characteristic polynomial of P. The roots of the equation

$$C(\lambda, x) = 0$$

will be denoted by $\lambda_1(x), \dots, \lambda_m(x)$ and will be referred to as the characteristic exponent functions of P. Now, let us recall the result of H. Tahara [**2**].

THEOREM 1.1 ([H. Tahara (1979)]). *If the condition*

$$\lambda_i(0) - \lambda_j(0) \notin \boldsymbol{Z} - \{0\} \quad \text{for } 1 \le i \ne j \le m \tag{1.4}$$

is satisfied, there are holomorphic functions $E_i(t, x, y)$ $(i = 1, \dots, m)$ *on*

$$\Omega = \{(t, x, y) \in S(\varepsilon) \times D_L \times D_L; |t| < M|x_i - y_i|^m, i = 1, \dots, n\}$$

for some $\varepsilon > 0$, $L > 0$, *and* $M > 0$ *which satisfy the following properties*:

(I) *For any* $\varphi_i(x) \in \mathcal{O}_0$ $(i = 1, \dots, m)$, *the function* $u(t, x)$ *defined by*

$$u(t, x) = \sum_{i=1}^{m} \oint E_i(t, x, y)\varphi_i(y)\, dy \tag{1.5}$$

is an $\tilde{\mathcal{O}}$-*solution of* $Pu = 0$.

(II) *Conversely, if* $u(t, x)$ *is an* $\tilde{\mathcal{O}}$-*solution of* $Pu = 0$, *then* $u(t, x)$ *is expressed in the form* (1.5) *for some* $\varphi_i(x) \in \mathcal{O}_0$ $(i = 1, \dots, m)$.

The meaning of the integration in (1.5) is as follows:

$$\oint E_i(t, x, y)\varphi_i(y)\, dy = \int_{\Gamma_1} \cdots \int_{\Gamma_n} E_i(t, x, y)\varphi_i(y)\, dy_1 \cdots dy_n,$$

where for $i = 1, \dots, n$, Γ_i denotes the circle

$$\{y_i \in \boldsymbol{C}; |y_i - x_i| = s_i\}$$

in the y_i-plane with a counter-clockwise orientation. Let $\varphi_i(x)$ be a holomorphic function on D_L. Since $E_i(t, x, y)$ is holomorphic with respect to y_i on

$$\left\{ y_i \in \boldsymbol{C}; \left(\frac{|t|}{M}\right)^{1/m} < |x_i - y_i|, |y_i| < L \right\},$$

we take the radius s_i so that

$$\left(\frac{|t|}{M}\right)^{1/m} < s_i < L.$$

H. Tahara called the functions $E_i(t,x,y)$ $(i=1,\dots,m)$ a fundamental system of solutions (or fundamental solutions) of (1.3) in $\tilde{\mathcal{O}}$. It should be noted that if we denote by S the set of all $\tilde{\mathcal{O}}$-solutions of (1.3), then the map defined by

$$
\begin{array}{rccc}
(1.6) & \Phi : (\mathcal{O}_0)^m & \longrightarrow & S \\
 & \Psi & & \Psi \\
 & (\varphi_1,\dots,\varphi_m) & \longmapsto & \displaystyle\sum_{i=1}^{m} \oint E_i(t,x,y)\varphi_i(y)\,dy
\end{array}
$$

is an isomorphism. For the case when the condition (1.4) is not satisfied, the construction of fundamental solutions of (1.3) in $\tilde{\mathcal{O}}$ seemed to be very complicated and it remained an unsolved problem. About two decades later, T. Mandai [**3**] proved the following theorem without any assumption on the characteristic exponents of P.

Theorem 1.2 ([T. Mandai (2000)]). *Without any assumption on the characteristic exponents of P, we can construct an isomorphism*

$$
\begin{array}{rccc}
(1.7) & \Psi : (\mathcal{O}_0)^m & \longrightarrow & S \\
 & \Psi & & \Psi \\
 & (\varphi_1,\dots,\varphi_m) & \longmapsto & \displaystyle\sum_{i=1}^{m} K_i[\varphi_i].
\end{array}
$$

T. Mandai called this map the solution map of (1.3) in $\tilde{\mathcal{O}}$. The construction of $K_i[\varphi_i]$ is very elegant, but still the construction of fundamental solutions as in (1.6) has remained as unsolved problem. We will solve this problem in this paper. The following is our main theorem.

Theorem 1.3 (Main result). *Without any assumption on the characteristic exponents, we can construct holomorphic functions $E_i(t,x,y)$ $(i=1,\dots,m)$ on*

$$\Omega = \{(t,x,y) \in S(\varepsilon) \times D_L \times D_L;\, |t| < M|x_i - y_i|^m, i = 1,\dots,n\}$$

for some $\varepsilon > 0$, $L > 0$, and $M > 0$ such that each $K_i[\varphi_i]$ $(i=1,\dots,m)$ in Theorem 1.2 are expressed in the form

$$K_i[\varphi_i] = \int_{\Gamma_1} \cdots \int_{\Gamma_n} E_i(t,x,y)\varphi_i(y)\,dy_1\cdots dy_n$$

for any $\varphi_i(x) \in \mathcal{O}_0$.

2. Proof of Main Theorem.

We begin by introducing some notations and definitions that will be used throughout this work. We define the indicial polynomial of P by

$$C(\mu) = \mu^m + \sum_{j<m} a_{j,0}(0,0)\mu^j.$$

A characteristic exponent of P is a root of the equation $C(\mu) = 0$. Let $\mu_1, \dots, \mu_d$ be the distinct characteristic exponents of P, and let r_j $(j = 1, \dots, d)$ be the multiplicity of μ_j. Then for $j = 1, \dots, d$, we can take a domain S_j in $\boldsymbol{C}$ enclosed by a simple closed curve γ_j such that

$$\mu_j \in S_j \quad (1 \le j \le d),$$

$$\bar{S}_i \cap \bar{S}_j = \varnothing \quad \text{if } i \neq j$$

and

$$C(\lambda + v, 0) \neq 0 \quad \text{for every } \lambda \in \left(\bigcup_{j=1}^{d} (\bar{S}_j \backslash \{\mu_j\})\right) \text{ and } v \in \boldsymbol{N}.$$

Here $\bar{S}$ denotes the closure of S. Thus, for some sufficiently small $L > 0$, we have

$$C(\lambda + v, x) \neq 0 \quad \text{for every } x \in D_L,\ \lambda \in \left(\bigcup_{j=1}^{d} \gamma_j\right), \text{ and } v \in \boldsymbol{N}.$$

For every $x \in D_L$, above condition implies that the number of the roots of $C(\lambda, x) = 0$ in S_j is r_j. Therefore there exist monic polynomials $B_j(\lambda, x)$ such that

$$C(\lambda, x) = \prod_{j=1}^{d} B_j(\lambda, x),$$

where $B_1(\lambda, x) = (\lambda - \lambda_1(x)) \cdots (\lambda - \lambda_{r_1}(x))$, $B_2(\lambda, x) = (\lambda - \lambda_{r_1+1}(x)) \cdots (\lambda - \lambda_{r_1+r_2}(x)), \dots,$ $B_j(\lambda, x) = (\lambda - \lambda_{r_1+\cdots+r_{j-1}+1}(x)) \cdots (\lambda - \lambda_{r_1+\cdots+r_j}(x))$ and $B_j(\lambda, x) \in \mathcal{O}(D_L)[\lambda]$ $(1 \le j \le d)$. For $0 < L < 1$ we set

$$\Omega_L = \{(x, y) \in \boldsymbol{C}^n \times \boldsymbol{C}^n; |x_i| < L, |y_i| < L, x_i \neq y_i, i = 1, \dots, n\}$$

and for $(x, y) \in \boldsymbol{C}^n \times \boldsymbol{C}^n$ we define

$$\psi_L(x, y) = \min\{L - |x_i|, |x_i - y_i|, i = 1, \dots, n\}.$$

Note that

$$0 < \psi_L(x, y) < 1 \quad \text{for any } (x, y) \in \Omega_L.$$

Let us now review the result of T. Mandai [**3**].

Theorem 2.1. *For any $\varphi_{j,k}(x) \in \mathcal{O}_0$ $(1 \le j \le d, 1 \le k \le r_j)$, there exists a unique solution $K_{j,k}(t, x, \lambda) \in \mathcal{O}(\{t = 0\} \times D_L \times (\bigcup_{j=1}^{d} \gamma_j))$ of the equation*

$$P(K_{j,k}(t, x, \lambda) t^{\lambda}) = \frac{C(\lambda, x) \cdot \partial_{\lambda}^{k} B_j(\lambda, x) \cdot \varphi_{j,k}(x)}{B_j(\lambda, x)} t^{\lambda}.$$

Moreover, the function

$$K_{j,k}[\varphi_{j,k}] = \frac{1}{2\pi i} \int_{\gamma_j} K_{j,k}(t, x, \lambda) t^{\lambda}\, d\lambda$$

is an $\tilde{\mathcal{O}}$-solution of $Pu = 0$.

We then have the linear isomorphism

$$\begin{array}{ccc} \Psi : (\mathcal{O}_0)^m & \longrightarrow & S \\ \Downarrow & & \Downarrow \\ (\varphi_{j,k})_{\substack{1 \le j \le d \\ 1 \le k \le r_j}} & \longmapsto & \displaystyle\sum_{j=1}^{d} \sum_{k=1}^{r_j} K_{j,k}[\varphi_{j,k}]. \end{array}$$

This result will be useful later. For now, let us consider the following partial differential equation:

$$P(F_{j,k}(t,x,y,\lambda)t^{\lambda}) = \frac{\partial_{\lambda}^{k} B_j(\lambda, y) \cdot C(\lambda, x) t^{\lambda}}{(2\pi i)^n B_j(\lambda, y)(y_1 - x_1) \cdots (y_n - x_n)}. \tag{2.1}$$

The above equation is the essence of our construction. Using the function $F_{j,k}(t,x,y,\lambda)$ above, we define

$$E_{j,k}(t,x,y) = \frac{1}{2\pi i} \int_{\gamma_j} F_{j,k}(t,x,y,\lambda) t^{\lambda}\, d\lambda. \tag{2.2}$$

As for (2.1), we have the following result:

Proposition 2.2. *For $1 \le j \le d$ and $1 \le k \le r_j$, equation* (2.1) *has a unique holomorphic solution $F_{j,k}(t,x,y,\lambda)$ defined in*

$$\Omega' = \left\{ (t,x,y,\lambda); (x,y) \in \Omega_L, \lambda \in \left(\bigcup_{j=1}^{d} \gamma_j \right) \text{ and } \frac{|t|}{\psi_L(x,y)^m} < M \right\}$$

for some $L > 0$ and $M > 0$.

The proof of this proposition will be given in the next section. Let us now prove Theorem 1.3 using this proposition.

Proof of Theorem 1.3. By Proposition 2.2, we may take any $\varphi_{j,k}(x) \in \mathcal{O}(D_L)$ and multiply the left and right sides of (2.1) by $\varphi_{j,k}(y)$. If we integrate both sides with respect to y, then an application of Cauchy's integral formula shows that

$$\begin{aligned} &P\left[\left(\oint F_{j,k}(t,x,y,\lambda) \varphi_{j,k}(y)\, dy \right) t^{\lambda} \right] \\ &\qquad = \oint \frac{\partial_{\lambda}^{k} B_j(\lambda, y) \cdot C(\lambda, x) \varphi_{j,k}(y) t^{\lambda}}{(2\pi i)^n B_j(\lambda, y)(y_1 - x_1) \cdots (y_n - x_n)} dy \\ &\qquad = \frac{\partial_{\lambda}^{k} B_j(\lambda, x) \cdot C(\lambda, x) \varphi_{j,k}(x) t^{\lambda}}{B_j(\lambda, x)}. \end{aligned}$$

By Theorem 2.1, we must have

$$K_{j,k}(t,x,\lambda) = \oint F_{j,k}(t,x,y,\lambda) \varphi_{j,k}(y)\, dy.$$

Now applying (2.2) we see that

$$
\begin{aligned}
&K_{j,k}[\varphi_{j,k}] \\
&\quad = \frac{1}{2\pi i}\int_{\gamma_j} K_{j,k}(t,x,\lambda)t^{\lambda}\,d\lambda = \frac{1}{2\pi i}\int_{\gamma_j}\left(\oint F_{j,k}(t,x,y,\lambda)\varphi_{j,k}(y)\,dy\right)t^{\lambda}\,d\lambda \\
&\quad = \oint E_{j,k}(t,x,y)\varphi_{j,k}(y)\,dy.
\end{aligned}
$$

Since, by Proposition 2.2, the function $F_{j,k}(t,x,y,\lambda)$ is holomorphic on Ω', we see that $E_{j,k}(t,x,y)$ is holomorphic on Ω. This proves the theorem. $\square$

3. Proof of Proposition 2.2.

We now prove Proposition 2.2. To avoid confusion, we write β instead of j in (1.2). By expanding $a_{\beta,\alpha}(t,x)$ into a Taylor series in t and using (1.1), we can reduce equation (2.1) into

$$
\begin{aligned}
&C\left(t\frac{\partial}{\partial t},x\right)(F_{j,k}(t,x,y,\lambda)t^{\lambda}) \\
&\quad = -\sum_{\substack{\beta+|\alpha|\le m \\ \beta<m}}\sum_{p\ge 1} a_{\beta,\alpha,p}(x)t^{p}\left(t\frac{\partial}{\partial t}\right)^{\beta}\left(\frac{\partial}{\partial x}\right)^{\alpha}F_{j,k}(t,x,y,\lambda)t^{\lambda} \\
&\qquad + \frac{\partial_{\lambda}^{k}B_j(\lambda,y)\cdot C(\lambda,x)t^{\lambda}}{(2\pi i)^n B_j(\lambda,y)(y_1-x_1)\cdots(y_n-x_n)},
\end{aligned}
\tag{3.1}
$$

where $a_{\beta,\alpha,p}(x)\in\mathcal{O}(D_L)$ for some $L>0$. Let us find a formal solution of (3.1) that is of the form

$$
F_{j,k}(t,x,y,\lambda)=\sum_{\nu=0}^{\infty}F_{j,k,\nu}(x,y,\lambda)t^{\nu}.
$$

Substituting this into (3.1) gives us the following recursive formula:

$$
\begin{aligned}
&C(\lambda+\nu,x)F_{j,k,\nu}(x,y,\lambda) \\
&\quad = -\sum_{\substack{\beta+|\alpha|\le m \\ \beta<m}}\sum_{\substack{p+q=\nu \\ p\ge 1}} a_{\beta,\alpha,p}(x)(\lambda+q)^{\beta}\left(\frac{\partial}{\partial x}\right)^{\alpha}F_{j,k,q}(x,y,\lambda)
\end{aligned}
\tag{3.2}
$$

for $\nu=1,2,\ldots,$ with

$$
F_{j,k,0}(x,y,\lambda)=\frac{\partial_{\lambda}^{k}B_j(\lambda,y)}{(2\pi i)^n B_j(\lambda,y)(y_1-x_1)\cdots(y_n-x_n)}.
\tag{3.3}
$$

It follows from (3.2) and (3.3) that equation (3.1) has a unique formal solution $F_{j,k}(t,x,y,\lambda)=\sum_{\nu=0}^{\infty}F_{j,k,\nu}(x,y,\lambda)t^{\lambda}$.

From now on, we will investigate the domain of convergence of $F_{j,k}(t,x,y,\lambda)$. By using (3.3) we have

$$|F_{j,k,0}(x,y,\lambda)| \le \frac{A}{\psi_L(x,y)^n} \quad \text{on } \Omega_L \times \left(\bigcup_{j=1}^{d} \gamma_j\right) \tag{3.4}$$

for $1 \le j \le d$, $1 \le k \le r_j$, for some $A > 0$. The following lemma will play an important role later.

LEMMA 3.1. *Let $F(x,y)$ be holomorphic in Ω_L. If for some $A \ge 0$ and $\zeta > 0$ we have*

$$|F(x,y)| \le \frac{A}{\psi_L(x,y)^{\zeta}} \quad \text{on } \Omega_L,$$

then for $i = 1, \dots, n$,

$$\left|\frac{\partial F}{\partial x_i}(x,y)\right| \le \frac{A(1+\zeta)e}{\psi_L(x,y)^{\zeta+1}} \quad \text{on } \Omega_L.$$

PROOF OF LEMMA 3.1. We have only to show the case when $i = 1$. By Cauchy's integral formula, we have

$$\frac{\partial F}{\partial x_1}(x,y) = \frac{1}{2\pi i}\int_{|z-x_1|=c} \frac{F(z,x_2,\dots,x_n,y_1,\dots,y_n)}{(z-x_1)^2}\,dz.$$

Now take any $(x,y) \in \Omega_L$ and fix it. Set

$$c = \frac{1}{1+\zeta}\psi_L(x,y).$$

Note that the following hold:

(a) $L - |z| = L - |x_1 + z - x_1| \ge (L - |x_1|) - |z - x_1| \ge \psi_L(x,y) - c$

$$= \psi_L(x,y) - \frac{1}{1+\zeta}\psi_L(x,y) = \frac{\zeta}{1+\zeta}\psi_L(x,y);$$

(b) $L - |x_i| \ge \psi_L(x,y) \ge \dfrac{\zeta}{1+\zeta}\psi_L(x,y), \quad i = 2, \dots, n;$

(c) $|z - y_1| = |x_1 - y_1 - (x_1 - z)| \ge |x_1 - y_1| - |x_1 - z| \ge \psi_L(x,y) - c$

$$= \frac{\zeta}{1+\zeta}\psi_L(x,y);$$

(d) $|x_i - y_i| \ge \psi_L(x,y) \ge \dfrac{\zeta}{1+\zeta}\psi_L(x,y), \quad i = 2, \dots, n.$

These imply that

$$\psi_L(z,x_2,\dots,x_n,y_1,\dots,y_n) \ge \frac{\zeta}{1+\zeta}\psi_L(x,y).$$

Consequently, we have

$$\begin{aligned}\left|\frac{\partial F}{\partial x_1}(x,y)\right| &\le \frac{1}{2\pi c^2}\int_{|z-x_1|=c}\frac{A}{\psi_L(z,x_2,\dots,x_n,y_1,\dots,y_n)^{\zeta}}|dz| \\ &\le \frac{1}{2\pi c^2}\cdot 2\pi c\cdot\frac{A}{((\zeta/(1+\zeta))\psi_L(x,y))^{\zeta}} \\ &= \frac{A(1+\zeta)}{\psi_L(x,y)^{\zeta+1}}\left(1+\frac{1}{\zeta}\right)^{\zeta} \le \frac{A(1+\zeta)e}{\psi_L(x,y)^{\zeta+1}}.\end{aligned}$$ □

Applying this lemma to (3.4), we get

(3.5) $$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}F_{j,k,0}(x,y,\lambda)\right| \le \frac{B}{\psi_L(x,y)^{n+m}}$$

on $\Omega_L\times(\bigcup_{j=1}^{d}\gamma_j)$ for any $|\alpha|\le m$, for some $B>0$. Now, we may assume the following:

(e) $|a_{\beta,\alpha,p}(x)|\le b_{\beta,\alpha,p}$ on D_L for any (β,α,p);

(f) $\sum_{p\ge1} b_{\beta,\alpha,p}t^p\in \boldsymbol{C}\{t\}$ for any (β,α);

(g) There is a positive constant k_0 such that

$$|C(\lambda+v,x)|\ge k_0(v+1)^m \quad\text{on}\quad \left(\bigcup_{j=1}^{d}\gamma_j\right)\times D_L \text{ for } v=0,1,2,\dots.$$

Let

$$J=\max_{\lambda\in(\bigcup_{j=1}^{d}\gamma_j)}|\lambda|.$$

For any fixed $(x,y)\in\Omega_L$, we consider the following linear equation with respect to $G=G(t,x,y)$:

(3.6) $$\begin{aligned}k_0G &= \frac{k_0B}{\psi_L(x,y)^{n+m}} \\ &\quad+\frac{1}{\psi_L(x,y)^m}\sum_{\substack{\beta+|\alpha|\le m\\ \beta<m}}\sum_{p\ge1}\frac{b_{\beta,\alpha,p}}{\psi_L(x,y)^{m(p-1)}}(J+1)^m t^p(e(n+m))^m G,\end{aligned}$$

where B is the positive constant in (3.5). It is obvious that the equation (3.6) has a unique holomorphic solution

$$G=\sum_{l=0}^{\infty}G_l(x,y)t^l\in\boldsymbol{C}\{t\}$$

and that the coefficients $G_l(x,y)$ $(l=0,1,2,\dots)$ are calculated by the following recursive formula:

$$G_0(x,y) = \frac{B}{\psi_L(x,y)^{n+m}},$$

$$G_l(x,y) = \frac{1}{k_0 \psi_L(x,y)^m} \sum_{\substack{\beta+|\alpha|\le m \\ \beta<m}} \sum_{\substack{p+q=l \\ p\ge 1}} \frac{b_{\beta,\alpha,p}}{\psi_L(x,y)^{m(p-1)}} \times (J+1)^m (e(n+m))^m G_q$$

for $l = 1,2,\ldots.$ Moreover, by induction on l we can show that for any $l = 0,1,2,\ldots$ we have

$$G_l(x,y) = \frac{\varepsilon_l}{\psi_L(x,y)^{n+(l+1)m}} \tag{3.7}$$

for some $\varepsilon_l \ge 0$.

From now on, we will prove that $(e(n+m))^m G$ is a majorant series of $F_{j,k}(t,x,y,\lambda)$. To do so, we will need the following proposition:

Proposition 3.2. *For any $|\alpha| \le m$, $1 \le j \le d$, and $1 \le k \le r_j$, the following inequality holds*:

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} F_{j,k,\nu}(x,y,\lambda)\right| \le (\nu+1)^{|\alpha|}(e(n+m))^m G_\nu(x,y) \tag{3.8}$$

on $\Omega_L \times (\bigcup_{j=1}^d \gamma_j)$ for $\nu = 0,1,2,\ldots.$

Proof of Proposition 3.2. We prove this proposition by induction on ν. From (3.5) and (3.7), we see that (3.8) is valid when $\nu = 0$. Next we suppose that (3.8) is true for $\nu = 0,1,\ldots,\mu-1$. Then, we may estimate as follows:

$$\begin{aligned}
&|F_{j,k,\mu}(x,y,\lambda)| \\
&\quad\le \frac{1}{k_0(\mu+1)^m} \sum_{\substack{\beta+|\alpha|\le m \\ \beta<m}} \sum_{\substack{p+q=\mu \\ p\ge 1}} b_{\beta,\alpha,p}(J+q)^\beta (q+1)^{|\alpha|}(e(n+m))^m G_q \\
&\quad\le \frac{1}{k_0} \sum_{\substack{\beta+|\alpha|\le m \\ \beta<m}} \sum_{\substack{p+q=\mu \\ p\ge 1}} b_{\beta,\alpha,p}(J+1)^m (e(n+m))^m \frac{1}{\psi_L(x,y)^{m(p-1)}} G_q \\
&\quad= \psi_L(x,y)^m G_\mu(x,y) = \psi_L(x,y)^m \frac{\varepsilon_\mu}{\psi_L(x,y)^{n+(\mu+1)m}} \\
&\quad= \frac{\varepsilon_\mu}{\psi_L(x,y)^{n+\mu m}}.
\end{aligned}$$

Applying Lemma 3.1 and the estimate:

$$\begin{aligned}
(n+\mu m+1)\cdots(n+\mu m+|\alpha|) &\le (n+\mu m+|\alpha|)^{|\alpha|} \\
&\le \{(\mu+1)(n+m)\}^{|\alpha|} = (\mu+1)^{|\alpha|}(n+m)^{|\alpha|},
\end{aligned}$$

we see that

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} F_{j,k,\mu}(x,y,\lambda)\right| \leq \frac{\varepsilon_\mu (n+\mu m+1)e \cdots (n+\mu m+|\alpha|)e}{\psi_L(x,y)^{n+\mu m+|\alpha|}}$$
$$\leq \frac{\varepsilon_\mu e^{|\alpha|}(\mu+1)^{|\alpha|}(n+m)^{|\alpha|}}{\psi_L(x,y)^{n+\mu m+m}}$$
$$\leq (\mu+1)^{|\alpha|}(e(n+m))^m G_\mu(x,y)$$

on $\Omega_L \times (\bigcup_{j=1}^d \gamma_j)$ for any $|\alpha| \leq m$. Thus, the induction process is completed. $\square$

This proposition implies that $(e(n+m))^m G$ is a majorant series of $F_{j,k}(t,x,y,\lambda)$ and therefore the domain of convergence of $F_{j,k}(t,x,y,\lambda)$ follows from the domain of convergence of G.

Now define $H = H(\eta)$ as follows:

$$H(\eta) = \sum_{l=0}^{\infty} \varepsilon_l \eta^l.$$

Using this and (3.7), we have

$$G = \sum_{l=0}^{\infty} \frac{\varepsilon_l}{\psi_L(x,y)^{n+m+ml}} t^l = \frac{1}{\psi_L(x,y)^{n+m}} H\left(\frac{t}{\psi_L(x,y)^m}\right).$$

We may now rewrite equation (3.6) into the following linear equation with respect to H:

$$k_0 H = k_0 B + \sum_{\substack{\beta+|\alpha|\leq m \\ \beta<m}} \sum_{p\geq 1} b_{\beta,\alpha,p}(J+1)^m \eta^p (e(n+m))^m H$$

which implies $H \in \boldsymbol{C}\{\eta\}$ from the assumption (f). Therefore, the domain of convergence of G includes

$$\Omega' = \left\{(t,x,y,\lambda); (x,y) \in \Omega_L, \lambda \in \left(\bigcup_{j=1}^d \gamma_j\right) \text{ and } \frac{|t|}{\psi_L(x,y)^m} < M\right\}$$

for some $L > 0$ and $M > 0$. Consequently, $F_{j,k}(t,x,y,\lambda)$ is holomorphic on Ω'. This completes the proof of Proposition 2.2.

4. Additional remarks.

Under the condition that $\lambda_i(0) - \lambda_j(0) \notin \boldsymbol{Z}$ holds for $1 \leq i \neq j \leq m$, H. Tahara constructed in [**2**] the fundamental solutions $E_j(t,x,y) = K_j(t,x,y)t^{\lambda_j(y)}$ $(1 \leq j \leq m)$ using the partial differential equations

$$P(K_j(t,x,y)t^{\lambda_j(y)}) = \frac{C(\lambda_j(y),x)t^{\lambda_j(y)}}{(2\pi i)^n (y_1-x_1)\cdots(y_n-x_n)} \tag{4.1}$$

for $1 \leq j \leq m$. Here, we will investigate the relationship between Theorem 1.1 and Theorem 1.3 to the fundamental solutions. To be precise, we wish to prove the following proposition:

PROPOSITION 4.1. *If the characteristic exponents of P do not differ by integer, then the fundamental solutions* $E_j(t,x,y)$ $(1 \le j \le m)$ *in Theorem* 1.3 *coincide with the ones in Theorem* 1.1.

Before proving this proposition we state the following result in [**2**].

LEMMA 4.2. *Under the same assumptions as in Proposition* 4.1, *equation* (4.1) *has a unique holomorphic solution* $K_j(t,x,y)$ *defined in*

$$\{(t,x,y) \in \boldsymbol{C} \times \boldsymbol{C}^n \times \boldsymbol{C}^n; |t| < \varepsilon, |x_i| < L, |y_i| < L, |t| < M|x_i - y_i|^m, i = 1, \dots, n\}$$

for some $\varepsilon > 0$, $L > 0$, *and* $M > 0$.

We now prove Proposition 4.1 using this lemma.

PROOF OF PROPOSITION 4.1. If the characteristic exponents do not differ by an integer, then (2.1) becomes

$$P(F_{j,1}(t,x,y,\lambda)t^{\lambda}) = \frac{C(\lambda,x)t^{\lambda}}{(2\pi i)^n B_j(\lambda,y)(y_1 - x_1)\cdots(y_n - x_n)} \tag{4.2}$$

for $1 \le j \le m$. If we set

$$E_{j,1}(t,x,y) = \frac{1}{2\pi i}\int_{\gamma_j} F_{j,1}(t,x,y,\lambda)t^{\lambda}\,d\lambda,$$

then we have

$$P(E_{j,1}(t,x,y)) = \frac{C(\lambda_j(y),x)t^{\lambda_j(y)}}{(2\pi i)^n (y_1 - x_1)\cdots(y_n - x_n)}.$$

Therefore, by the uniqueness of the solution stated in Lemma 4.2, it follows that our fundamental solutions and the ones in [**2**] are the same. □

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References

[1] M. S. Baouendi and C. Goulaouic, Cauchy problems with characteristic initial hypersurface, Comm. Pure Appl. Math., **26** (1973), 455–475.
[2] H. Tahara, Fuchsian type equations and Fuchsian hyperbolic equations, Japan. J. Math. (N.S.), **5** (1979), 245–347.
[3] T. Mandai, The method of Frobenius to Fuchsian partial differential equations, J. Math. Soc. Japan, **52** (2000), 645–672.

Toshiaki SAITO
Department of Mathematics
Sophia University
7-1 Kioicho, Chiyoda-ku
Tokyo 102-8554, Japan
E-mail: toshi@mm.sophia.ac.jp