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# Approaching points by continuous selections

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**Abstract.** Some further results about special Vietoris continuous selections and totally disconnected spaces are obtained, also several applications are demonstrated. In particular, it is demonstrated that a homogeneous separable metrizable space has a continuous selection for its Vietoris hyperspace if and only if it is discrete, or a discrete sum of copies of the Cantor set, or is the irrational numbers.

## 1. Introduction.

Let $X$ be a topological space, and let $\mathscr{F}(X)$ be the set of all non-empty closed subsets of $X$. A map $f : \mathscr{F}(X) \to X$ is a *selection* for $\mathscr{F}(X)$ if $f(S) \in S$ for every $S \in \mathscr{F}(X)$. A selection $f : \mathscr{F}(X) \to X$ is *continuous* if it is continuous with respect to the Vietoris topology $\tau_V$ on $\mathscr{F}(X)$. Let us recall that $\tau_V$ is generated by all collections of the form

$$\langle \mathscr{V} \rangle = \Big\{ S \in \mathscr{F}(X) : S \subset \bigcup \mathscr{V} \ \text{ and } \ S \cap V \neq \varnothing, \ \text{whenever } V \in \mathscr{V} \Big\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of $X$. Sometimes, for reasons of convenience, we shall say that $f$ is *Vietoris continuous* to stress the attention that $f$ is continuous with respect to the topology $\tau_V$. Finally, for a space $X$, we let $\mathscr{S}e\ell(X)$ to be the set of all Vietoris continuous selections for $\mathscr{F}(X)$.

In the sequel, all spaces are assumed to be at least Hausdorff. This paper was inspired by the following two results about Vietoris continuous selections and disconnectedness-like properties.

Theorem 1.1 ([**7**]). *If $X$ is a first countable space, with $\mathscr{S}e\ell(X) \neq \varnothing$, then it is zero-dimensional if and only if for every point $x \in X$ there exists an $f_x \in \mathscr{S}e\ell(X)$ such that $f_x^{-1}(x) = \{S \in \mathscr{F}(X) : x \in S\}$.*

Theorem 1.2 ([**7**]). *If $X$ is a space such that $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$, then $X$ is totally disconnected.*

Here, $X$ is *zero-dimensional* if it has a base of clopen sets (i.e., if $\operatorname{ind}(X) = 0$), and $X$ is *totally disconnected* if any two points of $X$ can be separated by clopen sets.

Theorem 1.1 was naturally generalized in [**9**]. On the other hand, it is still an open

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question if "totally disconnected" in the conclusion of Theorem 1.2 can be strengthen to "zero-dimensional", see [**7**], [**8**].

The purpose of this paper is to establish some further results about spaces $X$ for which $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$. In the first place, it is shown that $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$ if and only if $X$ has a clopen $\pi$-base and $\mathscr{S}e\ell(X) \neq \varnothing$, see Theorem 2.1. Several applications follow by this characterization. For instance, if $X$ is metrizable and $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$, then the set of all points at which $X$ is zero-dimensional must be also dense in $X$, see Corollary 3.1. Another natural situation to apply this characterization is for homogeneous spaces. Namely, a metrizable homogeneous space $X$ which has a continuous selection for $\mathscr{F}(X)$ must be zero-dimensional, see Corollary 3.2. Finally, we characterize all separable metrizable homogeneous spaces which have a Vietoris continuous selection (Corollary 3.3), also all locally compact topological groups with this property (Corollary 3.4).

In the second place, it is provided another characterization of spaces $X$ with the property that $D = \{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$. It is based on special members of the set $D$, which are called *countably-approachable* points, see Section 4. The idea of this is somehow related to Theorem 1.1, and demonstrates that if there is a "countable approach" to a point $p \in X$, then one can construct a continuous selection $f \in \mathscr{S}e\ell(X)$, with $f(X) = p$, see Theorem 4.1. Such countably-approachable points can be useful to show, for instance, that, for a regular first countable space $X$, the set $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$ if and only if it coincides with $X$, see Corollary 4.5.

## 2. Selections and $\pi$-bases.

A *$\pi$-base* for a space $X$ is a collection $\mathscr{P}$ of open subsets such that every non-empty open subset $U \subset X$ contains some non-empty $V \in \mathscr{P}$.

Theorem 2.1. *If $X$ is a space, with $\mathscr{S}e\ell(X) \neq \varnothing$, then $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$ if and only if $X$ has a clopen $\pi$-base.*

A part of the proof of Theorem 2.1 is based on the following simple observation. This property was used for other hyperspace topologies in [**3**], [**5**], [**6**].

Proposition 2.2. *Let $X$ be a space, $S \in \mathscr{F}(X)$, $f$ be a continuous selection for $\mathscr{F}(X)$, and let $G$ be a clopen neighbourhood of $f(S)$. Then, there exists a clopen subset $H \subset X$, with $S \subset H$ and $f(H) \in G$.*

Proof. Let $\mathscr{H} \subset \mathscr{F}(X)$ be a chain in $f^{-1}(G)$ which contains $S$, and which is maximal with respect to the usual set-theoretical inclusion. Since $f^{-1}(G)$ is a $\tau_V$-closed set, by [**5**, Lemma 2.2] (see, also, [**3**], [**6**]), there exists $H \in f^{-1}(G)$, with $\bigcup \mathscr{H} \subset H$, and, therefore, $H = \max \mathscr{H}$. Since $f^{-1}(G)$ is also $\tau_V$-open, there exists a finite family $\mathscr{W}$ of non-empty open subsets of $X$ such that $H \in \langle \mathscr{W} \rangle \subset f^{-1}(G)$. Since $H = \max \mathscr{H}$, it now follows that $H = \bigcup \mathscr{W}$, which completes the proof. □

Proof of Theorem 2.1. Suppose that $g \in \mathscr{S}e\ell(X)$, and that $\mathscr{P}$ is a clopen $\pi$-base for $X$. Next, take a non-empty open subset $U \subset X$. Then, there exists a non-empty clopen set $V \in \mathscr{P}$ such that $V \subset U$. Now, we can repeat some of the arguments in the proof of [**7**, Lemma 2.1]. Namely, we set

$$\mathscr{V}_0 = \{S \in \mathscr{F}(X) : S \cap V = \varnothing\},$$

and

$$\mathscr{V}_1 = \{S \in \mathscr{F}(X) : S \cap V \neq \varnothing\}.$$

Thus, we get a $\tau_V$-clopen partition $\{\mathscr{V}_0, \mathscr{V}_1\}$ of $\mathscr{F}(X)$. Then, we can define a selection $f$ for $\mathscr{F}(X)$ by letting $f(S) = g(S)$ if $S \in \mathscr{V}_0$, and $f(S) = g(S \cap V)$ otherwise. Clearly, $f$ is $\tau_V$-continuous and $f(X) = g(X \cap V) \in V \subset U$. So, $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$.

To prove the converse, suppose that $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$, and take an open subset $U \subset X$ such that $U \neq \varnothing \neq X \setminus U$. Then, by hypothesis, there exists a selection $f \in \mathscr{S}e\ell(X)$, with $f(X) \in U$. Let $S = X \setminus U$, and let us observe that $f(S) \neq f(X)$ because $f(S) \in S = X \setminus U$. However, by Theorem 1.2, $X$ is totally disconnected. Hence, there exists a clopen set $G \subset X$ such that $f(S) \in G$ and $f(X) \notin G$. According to Proposition 2.2, there now exists a clopen set $H \subset X$ such that $S \subset H$ and $f(H) \in G$. Then, $H \neq X$ because $f(X) \notin G$, which implies that $V = X \setminus H$ is a non-empty clopen subset of $X$. This completes the proof because $V = X \setminus H \subset X \setminus S = U$. $\square$

It should be mentioned that the proof of Theorem 2.1 relies on Theorem 1.2, hence on the total-disconnectedness of $X$. Nevertheless, it seems justifiable to mention the following immediate consequence of Theorems 2.1 and 1.2.

Corollary 2.3. *Let $X$ be a space with a clopen $\pi$-base and $\mathscr{S}e\ell(X) \neq \varnothing$. Then, $X$ is totally-disconnected.*

## 3. Many selections and metrizable spaces.

Let us recall that a regular space $X$ is *Moore* if there is a sequence $\{\mathscr{W}_k : k < \omega\}$ of open covers of $X$ such that $\{\mathrm{St}(x, \mathscr{W}_k) : k < \omega\}$ is a local base at $x$ for every $x \in X$. Here, $\mathrm{St}(x, \mathscr{W}_k) = \bigcup\{W \in \mathscr{W}_k : x \in W\}$.

Also, for a space $X$ and $x \in X$, we will write that $\mathrm{ind}_x(X) = 0$ if $X$ has a clopen base at $x$, i.e. if it is *zero-dimensional* at $x$.

Corollary 3.1. *If $X$ is a Moore space, with $\mathscr{S}e\ell(X) \neq \varnothing$, then the set $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$ if and only if the set $\{x \in X : \mathrm{ind}_x(X) = 0\}$ is dense in $X$.*

Proof. According to Theorem 2.1, we have to show that $\{x \in X : \mathrm{ind}_x(X) = 0\}$ is dense in $X$ if and only if $X$ has a clopen $\pi$-base. So, suppose that $\mathscr{P}$ is a clopen $\pi$-base for $X$, and let $\{\mathscr{W}_k : k < \omega\}$ be as in the definition of a Moore space. Then, for every $k < \omega$ there exists a pairwise disjoint family $\mathscr{P}_k \subset \mathscr{P}$ such that $\mathscr{P}_k$ refines $\mathscr{W}_k$, and $G_k = \bigcup \mathscr{P}_k$ is dense in $X$. By a result of [**10**], $X$ is a Baire space because it is regular and has a continuous selection. Then, $G = \bigcap\{G_k : k < \omega\}$ is a dense $G_\delta$-subset of $X$, and clearly $\mathrm{ind}_x(X) = 0$ for every $x \in G$. Thus $\{x \in X : \mathrm{ind}_x(X) = 0\}$ is dense in $X$ because it contains $G$. To show the converse, for every $x \in X$, with $\mathrm{ind}_x(X) = 0$, take a clopen base $\mathscr{P}_x$ at $x$. Then, $\mathscr{P} = \bigcup\{\mathscr{P}_x : \operatorname{ind}_x(X) = 0\}$ is a clopen $\pi$-base for $X$ because $\{x \in X : \operatorname{ind}_x(X) = 0\}$ is dense in $X$. □

COROLLARY 3.2. *If $X$ is a homogeneous Moore space, with $\mathscr{S}e\ell(X) \neq \varnothing$, then it is zero-dimensional.*

PROOF. Since $X$ is homogeneous, $X = \{f(X) : f \in \mathscr{S}e\ell(X)\}$. Hence, by Corollary 3.1, $\{x \in X : \operatorname{ind}_x(X) = 0\}$ is dense in $X$, and, in particular, non-empty. So, $X$ has at least one point at which it is zero-dimensional, hence it is zero-dimensional because it is homogeneous. □

COROLLARY 3.3. *Let $X$ be a homogeneous separable metrizable space such that $\mathscr{S}e\ell(X) \neq \varnothing$. Then, one of the following holds*:

(a) *$X$ is a discrete space,*
(b) *$X$ is a discrete sum of copies of the Cantor set,*
(c) *$X$ is the irrational line.*

PROOF. If $X$ contains an isolated point, then all points of $X$ must be isolated because it is homogeneous. Hence, in this case, $X$ is discrete. Suppose that $X$ contains a non-isolated point, then $X$ must be dense in itself because it is homogeneous. We distinguish the following two cases. If $X$ contains a clopen compact subset, then it has a discrete cover $\mathscr{C}$ of clopen compact sets. Note that $X$ is separable, hence its covering dimension is zero because, by Corollary 3.2, it is zero-dimensional. Further, let us observe that each $C \in \mathscr{C}$ is a zero-dimensional compact metric space, which is dense in itself because $X$ is dense in itself. Hence, by a result of [**4**], $C$ is homeomorphic to the Cantor set. Thus, (b) holds in this case. Finally, let us suppose that $X$ doesn't contain any non-empty compact open subset. According to a result of [**11**], $X$ must be completely metrizable because it has a Vietoris continuous selection, while, by Corollary 3.2, it is zero-dimensional. So, by a result of [**1**], $X$ is homeomorphic to the irrational numbers. □

A space $X$ is *orderable* (or, *linearly orderable*) if there exists a linear order $\leq$ on $X$ such that the sets $\{y \in Y : x < y\}$ and $\{y \in X : y < x\}$ constitute a subbase for the topology of $X$. A topological group $G$ is called *topologically orderable* if it is an orderable topological space (no relation between the group operations and the order is assumed).

COROLLARY 3.4. *A locally compact topological group $G$ is totally disconnected and topologically orderable if and only if $\mathscr{S}e\ell(G) \neq \varnothing$.*

PROOF. Suppose that $G$ is totally disconnected and orderable. Then, by [**14**, Theorem 5.5] (see, also, [**13**, Theorem 9]), $G$ is either discrete or it contains a clopen subgroup homeomorphic to the Cantor set. In both cases, $G$ is represented as a discrete sum of spaces $G_\alpha$, with $\mathscr{S}e\ell(G_\alpha) \neq \varnothing$. Hence, $G$ itself has a continuous selection $f : \mathscr{F}(G) \to G$.

Suppose now that $\mathscr{S}e\ell(G) \neq \varnothing$. Then, $G = \{f(G) : f \in \mathscr{S}e\ell(G)\}$ because $G$ is homogeneous, so, by Theorem 1.2, $G$ is totally disconnected. Then, $G$ has a basis of neigbourhoods at the identity consisting of compact open subgroups, see [**12**]. Hence, $G$

contains a compact open subgroup $H$. Then, either $H$ is finite or it is infinite. Also, let us observe that $\mathscr{S}e\ell(H) \neq \varnothing$ because $H \in \mathscr{F}(G)$. In case $H$ is infinite, by [**9**, Corollary 5.6] (see, also, [**2**, Corollary 1.27]), $H$ is homeomorphic to the Cantor set. Thus, $H$ is either finite or homeomorphic to the Cantor set. Hence, by [**14**, Theorem 5.5], $G$ is topologically orderable. $\square$

## 4. Countably-approachable points.

In the present section we provide a possible variant of Corollary 3.1 for arbitrary spaces. To this end, let us say that a point $p \in X$ is 0-*approachable* if $p$ is an isolated point of $X$, and that $p$ is $\omega$-*approachable* if there exists an open subset $U \subset X \setminus \{p\}$ such that $\overline{U} = U \cup \{p\}$, and $p$ has a countable clopen base in $\overline{U}$. Let us observe that if $p$ is $\omega$-*approachable*, and if $\{W_n : n < \omega\}$ is a strictly decreasing clopen base at $p$ in $\overline{U}$, then $S_n = W_n \setminus W_{n+1}$, $n < \omega$, is a disjoint family of non-empty clopen subset of $X$ such that $p \notin S_n$ for every $n < \omega$, and $\{S_n : n < \omega\}$ is $\tau_V$-convergent to $p$. One can easily see that the converse is also true, so, in the sequel, we will mainly rely on this characterization of $\omega$-approachable points.

In what follows, we say that $p \in X$ is *countably-approachable* if $p$ is either 0-approachable or it is $\omega$-approachable.

THEOREM 4.1. *For a space $X$, with $\mathscr{S}e\ell(X) \neq \varnothing$, the following are equivalent*:

(a) *The set $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$.*
(b) *The set of all countably-approachable points of $X$ is dense in $X$.*

The proof of Theorem 4.1 consists of the following separate observations.

LEMMA 4.2. *Let $X$ be a space, with $\mathscr{S}e\ell(X) \neq \varnothing$, and let $p \in X$ be a countably-approachable point of $X$. Then, there exists an $f \in \mathscr{S}e\ell(X)$, with $f(X) = p$.*

PROOF. If $p$ is an isolated point of $X$, then this follows by [**7**, Lemma 2.1]. So, suppose that $p$ is an $\omega$-approachable point of $X$, and let $\{S_n : n < \omega\}$ be a disjoint family of non-empty clopen subsets of $X$ such that $p \notin S_n$ for every $n < \omega$, and $\{S_n : n < \omega\}$ is $\tau_V$-convergent to $p$. Following [**7**, Lemma 2.3], we let

$$\mathscr{V}_0 = \{F \in \mathscr{F}(X) : F \cap S_0 = \varnothing\},$$

and

$$\mathscr{V}_1 = \{F \in \mathscr{F}(X) : F \cap S_0 \neq \varnothing\}.$$

Thus, we get a $\tau_V$-clopen partition of $\mathscr{F}(X)$ because $S_0$ is a clopen set. Then, consider the sets

$$\mathscr{V}_1^0 = \{F \in \mathscr{V}_1 : S_n \cap F = \varnothing \text{ for some } n < \omega\},$$

and

$$\mathscr{V}_1^1 = \{F \in \mathscr{V}_1 : S_n \cap F \neq \varnothing \text{ for every } n < \omega\}.$$

For later use, let us observe that $F \in \mathscr{V}_1^1$ implies $p \in F$.

Now, for every $F \in \mathscr{V}_1^0$, let $n(F) = \min\{n < \omega : S_{n+1} \cap F = \varnothing\}$. Next, take a selection $g \in \mathscr{S}e\ell(X)$, and then define another selection $f : \mathscr{F}(X) \to X$ by letting $f \restriction \mathscr{V}_0 = g \restriction \mathscr{V}_0$, while $f(F) = g(S_{n(F)} \cap F)$ if $F \in \mathscr{V}_1^0$, and $f(F) = p$ otherwise. Since $X \in \mathscr{V}_1^1$, we have that $f(X) = p$, hence it only remains to show that $f$ is continuous. Since $\mathscr{V}_0$ is $\tau_V$-clopen and $g$ is continuous, it now suffices to show that $f \restriction \mathscr{V}_1$ is continuous, so take an $F \in \mathscr{V}_1$. We distinguish the following two cases. If $F \in \mathscr{V}_1^0$, then $F \cap S_k \neq \varnothing$ for every $k \leq n(F)$, and $F \cap S_{n(F)+1} = \varnothing$. On the other hand, $f(F) = g(S_{n(F)} \cap F) \in S_{n(F)}$, while $S_{n(F)}$ is clopen. Then, consider the $\tau_V$-clopen set $\langle \mathscr{U} \rangle$, where

$$\mathscr{U} = \{S_k : k \leq n(F)\} \cup \{X \setminus S_{n(F)+1}\}.$$

Note that $F \in \langle \mathscr{U} \rangle \subset \mathscr{V}_1^0$, while the map $\varphi : \langle \mathscr{U} \rangle \to \mathscr{F}(S_{n(F)})$, defined by $\varphi(T) = T \cap S_{n(F)}$, is $\tau_V$-continuous. Also, $T \in \langle \mathscr{U} \rangle$ implies $n(T) = n(F)$, hence $f \restriction \langle \mathscr{U} \rangle = g \circ \varphi$. Thus, $f$ is continuous at $F$ because so are $g$ and $\varphi$. Finally, let us consider the case when $F \in \mathscr{V}_1^1$. By definition, $f(F) = p$, while $F \cap S_k \neq \varnothing$ for every $k < \omega$. Take a neighbourhood $V$ of $p$ in $X$. Then, there exists an $m < \omega$, with $S_n \subset V$ for every $n \geq m$. In this case, let

$$\mathscr{U} = \{S_k : k \leq m\} \cup \{X\}.$$

Thus, we get a $\tau_V$-neighbourhood of $F$ such that $f(\langle \mathscr{U} \rangle) \subset V$. Indeed, take a $T \in \langle \mathscr{U} \rangle$, and then observe that $T \in \mathscr{V}_1$ because $T \cap S_0 \neq \varnothing$. If $T \cap S_k \neq \varnothing$ for every $k < \omega$, then $f(T) = p \in V$. If $T \cap S_k = \varnothing$ for some $k < \omega$, then $n(T) \geq m$, so $f(T) = g(S_{n(T)} \cap T) \in S_{n(T)} \subset V$. $\square$

PROPOSITION 4.3. *Let $p \in X$ be a non-isolated point of $X$, and let $f \in \mathscr{S}e\ell(X)$ be such that $f(X) = p$. Then, for every closed subset $F \subset X$, with $p \notin F$, there exists a closed subset $T \subset X$ such that $F \subset T$, $p \notin T$, and $f(T \cup \{x\}) = x$ for some $x \in X \setminus T$.*

PROOF. Let $F \subset X$ be as in this statement. Then, $U = X \setminus F$ is a neighbourhood of $p$, so there exists a finite open cover $\mathscr{W}$ of $X$ such that $X \in \langle \mathscr{W} \rangle$ and $f(\langle \mathscr{W} \rangle) \subset U$. Since $p$ is a non-isolated point of $U$, there now exists a finite set $S \subset U \setminus \{p\}$ such that $F \cup S \in \langle \mathscr{W} \rangle$. Then, we can take $T = (F \cup S) \setminus \{f(F \cup S)\}$, which works because $f(F \cup S) \in S$. $\square$

We finalize the proof of Theorem 4.4 with the following lemma.

LEMMA 4.4. *Let $X$ be a space such that $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$. Then, every non-empty open subset of $X$ contains a countably-approachable point.*

PROOF. Let $U \subset X$ be a non-empty clopen set. If $U$ contains some isolated point, then clearly $U$ contains a countably-approachable point as well. So, suppose that $U$ has no isolated points, and then set $F_0 = X \setminus U$. Also, take an $f \in \mathscr{S}e\ell(X)$ such that

$p = f(X) \in U$. According to Proposition 4.3, there now exists a closed subset $T_0 \subset X$ such that $F_0 \subset T_0$, $p \notin T_0$, and $f(T_0 \cup \{x\}) = x$ for some $x \notin T_0$. Since $f$ is continuous, we may find an open set $U_0 \subset X \setminus T_0 \subset U$ such that $f(T_0 \cup \{x\}) = x$ for every $x \in U_0$. Then, $|U_0| > 1$ because $U$ does not have any isolated point, so, by Theorem 2.1, $U_0 \setminus \{p\}$ contains a non-empty clopen subset $S_0$. Thus, in fact, we have constructed a closed set $T_0 \subset X$, with $p \notin T_0$, and a non-empty clopen subset $S_0 \subset X$ such that $F_0 \subset T_0 \subset X \setminus S_0$, $p \notin S_0$, and $f(T_0 \cup \{x\}) = x$ for every $x \in S_0$. Now, we can set $F_1 = T_0 \cup S_0$, and we can repeat the same arguments. Hence, by induction, we get an increasing sequence $\{T_n : n < \omega\}$ of closed subsets, and a disjoint family $\{S_n : n < \omega\}$ of non-empty clopen subsets such that, for every $n < \omega$,

(a) $T_n \cup S_n \subset T_{n+1} \subset X \setminus (S_{n+1} \cup \{p\})$,
(b) $f(T_n \cup \{x\}) = x$, for every $x \in S_n$.

Set $T = \bigcup\{T_n : n < \omega\}$, $S = \bigcup\{S_n : n < \omega\}$, and $q = f(\overline{T})$. We are going to show that $q$ is an $\omega$-approachable point. Towards this end, let us observe that if $y_n \in S_n$ for every $n < \omega$, then $q = \lim_{n\to\infty} y_n$. Indeed, in this case, by (a), we have that $T_n \subset T_n \cup \{y_n\} \subset T_{n+1}$, so, by (b),

$$q = \lim_{n\to\infty} f(T_{n+1}) = \lim_{n\to\infty} f(T_n \cup \{y_n\}) = \lim_{n\to\infty} y_n. \tag{4.1}$$

In particular, this implies that $q \notin S$ because $\{S_n : n < \omega\}$ is a disjoint open family. Also, $\{S_n : n < \omega\}$ is $\tau_V$-convergent to $q$. Indeed, if this fails, then there should be some neighbourhood $W$ of $q$ in $X$ so that $S_n \setminus W \neq \varnothing$ for infinitely many $n < \omega$. Hence, we can find a strictly increasing sequence $\{n_k : k < \omega\} \subset \omega$, and a sequence of points $\{y_k : k < \omega\}$ so that $y_k \in S_{n_k} \setminus W$ for every $k < \omega$. According to (4.1), this will imply that $q = \lim_{k\to\infty} y_k \in X \setminus W$, which is clearly impossible. Thus, $q \in \overline{U} = U$ is an $\omega$-approachable point of $X$. Since, by Theorem 2.1, $X$ has a clopen $\pi$-base, this completes the proof. □

The following is now an immediate consequence of Lemma 4.2.

Corollary 4.5. *If $X$ is a regular first countable space and $\{f(X) : f \in \mathscr{S}e\ell(X)\}$ is dense in $X$, then $X = \{f(X) : f \in \mathscr{S}e\ell(X)\}$.*

Proof. Take a non-isolated point $p \in X$, and let $\{W_n : n < \omega\}$ be an open local base at $p$ such that $\overline{W_{n+1}} \subset W_n$ and $C_n = W_n \setminus \overline{W_{n+1}} \neq \varnothing$ for every $n < \omega$. According to Theorem 2.1, every $C_n$ contains a non-empty clopen subset $S_n \subset C_n$. Then, the sequence $\{S_n : n < \omega\}$ is $\tau_V$-convergent to $p$, so $p$ is an $\omega$-approachable point. □

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