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A conjecture in relation to Loewner's conjecture

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Abstract. Let f be a smooth function of two variables x, y and for each positive integer n, let $d^n f$ be a symmetric tensor field of type $(0,n)$ defined by $d^n f := \sum_{i=0}^{n} \binom{n}{i} \left(\partial_x^{n-i}\partial_y^i f\right) dx^{n-i}dy^i$ and $\tilde{\mathscr{D}}_{d^n f}$ a finitely many-valued one-dimensional distribution obtained from $d^n f$: for example, $\tilde{\mathscr{D}}_{d^1 f}$ is the one-dimensional distribution defined by the gradient vector field of f; $\tilde{\mathscr{D}}_{d^2 f}$ consists of two one-dimensional distributions obtained from one-dimensional eigenspaces of Hessian of f. In the present paper, we shall study the behavior of $\tilde{\mathscr{D}}_{d^n f}$ around its isolated singularity in ways which appear in [**1**]–[**4**]. In particular, we shall introduce and study a conjecture which asserts that the index of an isolated singularity with respect to $\tilde{\mathscr{D}}_{d^n f}$ is not more than one.

1. Introduction.

Let f be a smooth function on a domain D of $\boldsymbol{R}^2$ and set $\partial_{\bar{z}} := (\partial/\partial x + \sqrt{-1}\partial/\partial y)/2$. Then *Loewner's conjecture* for a positive integer $n \in \boldsymbol{N}$ asserts that if a vector field $\boldsymbol{V}_f^{(n)} := \mathrm{Re}(\partial_{\bar{z}}^n f)\partial/\partial x + \mathrm{Im}(\partial_{\bar{z}}^n f)\partial/\partial y$ has an isolated zero point, then its index with respect to $\boldsymbol{V}_f^{(n)}$ is not more than n. Loewner's conjecture for $n = 1$ is easily and affirmatively solved; Loewner's conjecture for $n = 2$ is equivalent to a conjecture which asserts that the index of an isolated umbilical point on a surface is not more than one (this conjecture is called the *index conjecture* or the *Local Carathéodory's conjecture*). If the index conjecture is true, then by Hopf-Poincaré's theorem, we may affirmatively solve *Carathéodory's conjecture*, which asserts that there exist at least two umbilical points on a compact, strictly convex surface in $\boldsymbol{R}^3$. We may find [**5**], [**6**], [**9**], [**10**], [**11**] and [**12**] as recent papers in relation to Carathéodory's and Loewner's conjectures.

For each positive integer n, let $d^n f$ be a symmetric tensor field of type $(0,n)$ defined by

$$d^n f := \sum_{i=0}^{n} \binom{n}{i} \frac{\partial^n f}{\partial x^{n-i}\partial y^i} dx^{n-i}dy^i. \tag{1}$$

For a number $\phi \in \boldsymbol{R}$ and a point $p \in D$, we set

$$\boldsymbol{U}_\phi := \cos\phi \frac{\partial}{\partial x} + \sin\phi \frac{\partial}{\partial y}, \qquad (\widehat{d^n f})_p(\phi) := (d^n f)_p(\boldsymbol{U}_\phi, \dots, \boldsymbol{U}_\phi). \tag{2}$$

A one-dimensional subspace L of the tangent plane at $p \in D$ is called a *critical direction* of $d^n f$ at p if there exists a critical point ϕ_0 of $(\widehat{d^n f})_p$ satisfying $\boldsymbol{U}_{\phi_0}(p) \in L$. A point p_0 of D is called an *umbilical point* of $d^n f$ if $(\widehat{d^n f})_{p_0}$ is constant. Let $\tilde{\mathscr{D}}_{d^n f}$ be a finitely many-valued one-dimensional distribution on an open set of non-umbilical points of $d^n f$ such that $\tilde{\mathscr{D}}_{d^n f}$ gives all the critical

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directions of $d^n f$ at each point. For example, $\tilde{\mathscr{D}}_{d^1 f}$ is the one-dimensional distribution defined by the gradient vector field of f; $\tilde{\mathscr{D}}_{d^2 f}$ consists of two one-dimensional distributions obtained from one-dimensional eigenspaces of Hessian of f at each point. The purpose of the present paper is to study the behavior of $\tilde{\mathscr{D}}_{d^n f}$ around an isolated umbilical point of $d^n f$ in ways which appear in [**1**]–[**4**]. In particular, we shall define and study the index of an isolated umbilical point with respect to $\tilde{\mathscr{D}}_{d^n f}$. We shall see that the index is a rational number and not always represented as the half of an integer. We conjecture that *the index of an isolated umbilical point with respect to* $\tilde{\mathscr{D}}_{d^n f}$ *is not more than one*. We shall see that for $n \in \{1,2\}$ (respectively, $n \geqq 3$), this conjecture is equivalent to (respectively, distinct from) Loewner's conjecture. We shall affirmatively solve the former conjecture in the case where f is a homogeneous polynomial. In addition, we shall study this conjecture in the case where f is a real-analytic function.

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2. Many-valued one-dimensional distributions.

Let $\mathscr{D}$ be a continuous one-dimensional distribution on a domain U of a smooth two-dimensional manifold S. In the present paper, a pair $(\mathscr{D}, U)$ is called a *distribution element*. A distribution element $(\mathscr{D}_0, U_0)$ is called a *direct continuation* of $(\mathscr{D}, U)$ if $U_0 \cap U \neq \emptyset$ and if $\mathscr{D}_0 \equiv \mathscr{D}$ on $U_0 \cap U$. A set of distribution elements $\{(\mathscr{D}_i, U_i)\}_{i \in \boldsymbol{N}}$ is called a *continuation* if $(\mathscr{D}_{i+1}, U_{i+1})$ is a direct continuation of $(\mathscr{D}_i, U_i)$ for any $i \in \boldsymbol{N}$.

For a point $p \in S$, let X_p be the set of the distribution elements such that each $(\mathscr{D}, U) \in X_p$ satisfies $p \in U$. We introduce an equivalence relation $\sim$ into X_p: for two $(\mathscr{D}_1, U_1), (\mathscr{D}_2, U_2) \in X_p$, we write $(\mathscr{D}_1, U_1) \sim (\mathscr{D}_2, U_2)$ if there exists a neighborhood U_0 of p in $U_1 \cap U_2$ satisfying $\mathscr{D}_1 \equiv \mathscr{D}_2$ on U_0. We denote by $\tilde{X}_p$ the set of the equivalence classes in relation to the equivalence relation $\sim$.

Let D be a domain of S. A correspondence $\tilde{\mathscr{D}}$ of each $p \in D$ to a subset $\tilde{\mathscr{D}}(p)$ of $\tilde{X}_p$ is called a *many-valued one-dimensional distribution* on D. For a many-valued one-dimensional distribution $\tilde{\mathscr{D}}$ on D and a distribution element $(\mathscr{D}, U)$, we write $(\mathscr{D}, U) \subset (\tilde{\mathscr{D}}, D)$ if $U \subset D$ and if $(\mathscr{D}, U)$ represents an element of $\tilde{\mathscr{D}}(q)$ for any $q \in U$. A many-valued one-dimensional distribution $\tilde{\mathscr{D}}$ is called *continuous* if for each $p \in D$ and each $\omega \in \tilde{\mathscr{D}}(p)$, there exists a distribution element $(\mathscr{D}, U) \in \omega$ satisfying $(\mathscr{D}, U) \subset (\tilde{\mathscr{D}}, D)$; a many-valued one-dimensional distribution $\tilde{\mathscr{D}}$ is called *complete* if the following holds: if a convergent sequence $\{p_i\}_{i \in \boldsymbol{N}}$ in D and a continuation $\{(\mathscr{D}_i, U_i)\}_{i \in \boldsymbol{N}}$ satisfy $p_i \in U_i$ and $(\mathscr{D}_i, U_i) \subset (\tilde{\mathscr{D}}, D)$ for any $i \in \boldsymbol{N}$, then there exists a distribution element $(\mathscr{D}_0, U_0)$ satisfying $\lim_{i \to \infty} p_i \in U_0$, $(\mathscr{D}_0, U_0) \subset (\tilde{\mathscr{D}}, D)$ and the condition that there exists a number $i_0 \in \boldsymbol{N}$ such that $(\mathscr{D}_0, U_0)$ is a direct continuation of $(\mathscr{D}_i, U_i)$ for any $i \geqq i_0$; a many-valued one-dimensional distribution $\tilde{\mathscr{D}}$ is called *separated* if distinct two distribution elements $(\mathscr{D}_1, U), (\mathscr{D}_2, U) \subset (\tilde{\mathscr{D}}, D)$ represent distinct elements of $\tilde{\mathscr{D}}(q)$ for any $q \in U$; a many-valued one-dimensional distribution $\tilde{\mathscr{D}}$ is called *pointwise separated* if $\mathscr{D}_1(q) \neq \mathscr{D}_2(q)$ for distinct two distribution elements $(\mathscr{D}_1, U), (\mathscr{D}_2, U) \subset (\tilde{\mathscr{D}}, D)$ and any $q \in U$; a many-valued one-dimensional distribution $\tilde{\mathscr{D}}$ is called *pointwise separable* if $\tilde{\mathscr{D}}$ is separated and if the following holds: if two distribution elements $(\mathscr{D}_1, U), (\mathscr{D}_2, U) \subset (\tilde{\mathscr{D}}, D)$ satisfy $\mathscr{D}_1(q_0) = \mathscr{D}_2(q_0)$ for some $q_0 \in U$, then there exist a neighborhood O_{q_0} of q_0 in U and continuous functions ϕ_1, ϕ_2 on O_{q_0} satisfying the following:

(a) $\phi_1(q_0) = \phi_2(q_0)$;

(b) $\boldsymbol{U}_{\phi_i} = (\cos\phi_i)\partial/\partial x + (\sin\phi_i)\partial/\partial y$ represents $(\mathscr{D}_i, O_{q_0})$ for $i \in \{1,2\}$;

(c) there exists a nonzero number $c \neq 0$ satisfying $c(\phi_1 - \phi_2) \geqq 0$ on O_{q_0},

where (x,y) are local coordinates on O_{q_0}.

Let $\tilde{\mathscr{D}}$ be a continuous, complete, separated many-valued one-dimensional distribution on D. Then $\tilde{\mathscr{D}}$ is called *connected* if there do not exist two continuous, complete, separated many-valued one-dimensional distributions $\tilde{\mathscr{D}}_1$, $\tilde{\mathscr{D}}_2$ on D satisfying $\tilde{\mathscr{D}}(p) = \tilde{\mathscr{D}}_1(p) \cup \tilde{\mathscr{D}}_2(p)$ and $\tilde{\mathscr{D}}_1(p) \cap \tilde{\mathscr{D}}_2(p) = \emptyset$ for any $p \in D$. If $\tilde{\mathscr{D}}$ is not connected, then there exists a set of connected, continuous, complete, separated many-valued one-dimensional distributions $\{\tilde{\mathscr{D}}_\lambda\}_{\lambda\in\Lambda}$ satisfying $\tilde{\mathscr{D}}(p) = \cup_{\lambda\in\Lambda}\tilde{\mathscr{D}}_\lambda(p)$ and $\tilde{\mathscr{D}}_{\lambda_1}(p) \cap \tilde{\mathscr{D}}_{\lambda_2}(p) = \emptyset$ for arbitrary distinct two $\lambda_1, \lambda_2 \in \Lambda$ and any $p \in D$. Each $\tilde{\mathscr{D}}_\lambda$ is called a *connected component* of $\tilde{\mathscr{D}}$.

Let $\tilde{\mathscr{D}}$ be a continuous, complete, separated many-valued one-dimensional distribution on D. Then we see that if there exists a positive integer $n_0 \in \boldsymbol{N}$ satisfying $\sharp\tilde{\mathscr{D}}(p_0) = n_0$ for some $p_0 \in D$, then $\sharp\tilde{\mathscr{D}}(p) = n_0$ for any $p \in D$. If such a positive integer exists, then $\tilde{\mathscr{D}}$ is in particular called *n_0-valued* or *finitely many-valued*. We see that if $\tilde{\mathscr{D}}$ is n_0-valued and pointwise separable, then there exists a divisor $n_{\tilde{\mathscr{D}}}$ of n_0 such that any connected component of $\tilde{\mathscr{D}}$ is $n_{\tilde{\mathscr{D}}}$-valued.

Let $\tilde{\mathscr{D}}$ be a continuous, complete, pointwise separable n_0-valued one-dimensional distribution on a domain D for some $n_0 \in \boldsymbol{N}$ and suppose that there exists an isolated complement p_0 of D for S, i.e., p_0 is a point of $S \setminus D$ such that a punctured neighborhood of p_0 in S is contained in D. Then p_0 may be an isolated singularity of $\tilde{\mathscr{D}}$, i.e., it is possible that $\tilde{\mathscr{D}}$ may not be completely extended to p_0. Let (x,y) be local coordinates on a neighborhood of p_0 such that p_0 corresponds to $(0,0)$ and r_0 a positive number satisfying $\{0 < x^2+y^2 < r_0^2\} \subset D$. Let $\boldsymbol{\Phi}_{\tilde{\mathscr{D}};p_0}$ denote the set of the continuous functions on $(0,r_0)\times\boldsymbol{R}$ such that for each $\phi_{\tilde{\mathscr{D}};p_0} \in \boldsymbol{\Phi}_{\tilde{\mathscr{D}};p_0}$ and each $(r,\theta) \in (0,r_0)\times\boldsymbol{R}$, there exists a distribution element $(\mathscr{D},U) \subset (\tilde{\mathscr{D}},D)$ satisfying $(r\cos\theta, r\sin\theta) \in U$ and the condition that for any $(r',\theta') \in (0,r_0)\times(\theta-\pi/2, \theta+\pi/2)$ satisfying $(r'\cos\theta', r'\sin\theta') \in U$,

$$\boldsymbol{U}_{\phi_{\tilde{\mathscr{D}};p_0}(r',\theta')} = \cos\phi_{\tilde{\mathscr{D}};p_0}(r',\theta')\frac{\partial}{\partial x} + \sin\phi_{\tilde{\mathscr{D}};p_0}(r',\theta')\frac{\partial}{\partial y} \in \mathscr{D}$$

holds at $(r'\cos\theta', r'\sin\theta')$. We see that there exists an integer $m_0 \in \boldsymbol{Z}$ satisfying

$$m_0 = \frac{\phi_{\tilde{\mathscr{D}};p_0}(r,\theta+2n_0\pi) - \phi_{\tilde{\mathscr{D}};p_0}(r,\theta)}{\pi}$$

for any $\phi_{\tilde{\mathscr{D}};p_0} \in \boldsymbol{\Phi}_{\tilde{\mathscr{D}};p_0}$ and any $(r,\theta) \in (0,r_0)\times\boldsymbol{R}$. Since $\tilde{\mathscr{D}}$ is pointwise separable, we see that the integer m_0 is uniquely determined. The number

$$\mathrm{ind}_{p_0}(\tilde{\mathscr{D}}) := \frac{m_0}{2n_0}$$

is called the *index* of p_0 with respect to $\tilde{\mathscr{D}}$.

REMARK. The definition of $\mathrm{ind}_{p_0}(\tilde{\mathscr{D}})$ does not depend on the choice of local coordinates (x,y).

REMARK. If $n_0 = 1$, then we see that $\tilde{\mathscr{D}}$ may be considered as a continuous one-dimensional distribution in the usual sense and that $\mathrm{ind}_{p_0}(\tilde{\mathscr{D}})$ is equal to the index of p_0 with respect to $\tilde{\mathscr{D}}$ also in the usual sense.

REMARK. We set

$$m_{\tilde{\mathscr{D}}} := \frac{\phi_{\tilde{\mathscr{D}};p_0}(r,\theta+2n_{\tilde{\mathscr{D}}}\pi)-\phi_{\tilde{\mathscr{D}};p_0}(r,\theta)}{\pi}$$

for $\phi_{\tilde{\mathscr{D}};p_0} \in \boldsymbol{\Phi}_{\tilde{\mathscr{D}};p_0}$ and $(r,\theta) \in (0,r_0)\times\boldsymbol{R}$. Then $m_{\tilde{\mathscr{D}}}$ is an integer such that $m_{\tilde{\mathscr{D}}}$ and $n_{\tilde{\mathscr{D}}}$ are relatively prime. The number $m_{\tilde{\mathscr{D}}}/2n_{\tilde{\mathscr{D}}}$ is the index of p_0 with respect to any connected component of $\tilde{\mathscr{D}}$ and equal to $\mathrm{ind}_{p_0}(\tilde{\mathscr{D}})$.

REMARK. If we adopt the above definition of the index of an isolated singularity, then referring to [**7**, pp. 112–113], we may obtain an analogue of Hopf-Poincaré's theorem for a continuous, complete, pointwise separable finitely many-valued one-dimensional distribution.

3. Symmetric tensor fields.

Let n be a positive integer and T a smooth, symmetric tensor field of type $(0,n)$ on a domain D of $\boldsymbol{R}^2$. Then T is represented as follows:

$$\mathrm{T} = \sum_{i=0}^{n} \binom{n}{i} \mathrm{T}_i dx^{n-i}dy^i,$$

where T_i is a smooth function on D. For a number $\phi \in \boldsymbol{R}$ and a point $p \in D$, we set

$$\hat{\mathrm{T}}_p(\phi) := \mathrm{T}_p(\boldsymbol{U}_\phi,\dots,\boldsymbol{U}_\phi).$$

Then

$$\hat{\mathrm{T}}_p(\phi) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{T}_i(p)\cos^{n-i}\phi\sin^i\phi.$$

A one-dimensional subspace L of the tangent plane at $p \in D$ is called a *critical direction* of T at p if there exists a critical point ϕ_0 of $\hat{\mathrm{T}}_p$ satisfying $\boldsymbol{U}_{\phi_0}(p) \in L$. A tensor field T is called *umbilical* at p or p is called an *umbilical point* of T if $\hat{\mathrm{T}}_p$ is constant, i.e., if any one-dimensional subspace of the tangent plane at p is a critical direction of T. The set of the umbilical points of T is denoted by $\mathrm{Umb}(\mathrm{T})$. An umbilical point p_0 of T is called *isolated* if p_0 is an isolated complement of $D\setminus\mathrm{Umb}(\mathrm{T})$. There exists a continuous, complete, pointwise separable, finitely many-valued one-dimensional distribution $\tilde{\mathscr{D}}_{\mathrm{T}}$ on a neighborhood U of each point of $D\setminus\mathrm{Umb}(\mathrm{T})$ formed by critical directions of T at each $p \in U$. If $n=1$ or 2, then $\tilde{\mathscr{D}}_{\mathrm{T}}$ is always well-defined on $D\setminus\mathrm{Umb}(\mathrm{T})$ and consists of one or two continuous one-dimensional distributions on $D\setminus\mathrm{Umb}(\mathrm{T})$ and we see that if $\sharp\tilde{\mathscr{D}}_{\mathrm{T}}=2$, then the two one-dimensional distributions are perpendicular to each other at any point with respect to the Euclidean metric on $D\setminus\mathrm{Umb}(\mathrm{T})$. On the other hand, if $n \geqq 3$, then it is possible that $\tilde{\mathscr{D}}_{\mathrm{T}}$ may not be well-defined on $D\setminus\mathrm{Umb}(\mathrm{T})$.

For a smooth function f on D and each positive integer n, we have defined a symmetric tensor field $d^n f$ of type $(0,n)$ as in (1). The following are examples of $\tilde{\mathscr{D}}_{d^n f}$.

EXAMPLE. We see that $\tilde{\mathscr{D}}_{d^1 f}$ is just the continuous one-dimensional distribution given by the gradient vector field of f and that $\tilde{\mathscr{D}}_{d^2 f}$ consists of one or two continuous one-dimensional distributions obtained from one-dimensional eigenspaces of Hessian of f at each point.

EXAMPLE. Let f be a harmonic function on D, i.e., let f satisfy $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 \equiv 0$ on D. Then noticing

$$(\widehat{d^n f})(\phi) = \frac{\partial^n f}{\partial x^n}\cos n\phi + \frac{\partial^n f}{\partial x^{n-1}\partial y}\sin n\phi,$$

we see that for each $p \in D \setminus \mathrm{Umb}(d^n f)$, there exists a number $\alpha_p \in \boldsymbol{R}$ such that each critical point of $(\widehat{d^n f})_p$ is represented by $\alpha_p + m\pi/n$ for some integer $m \in \mathbf{Z}$. Therefore we see that there exists a continuous, complete, pointwise separated n-valued one-dimensional distribution $\tilde{\mathscr{D}}_{d^n f}$ on $D \setminus \mathrm{Umb}(d^n f)$. Suppose that f is a spherical harmonic function of degree $k > n$. Then we may suppose $D = \boldsymbol{R}^2$ and we see that $(0,0)$ is the only umbilical point of $d^n f$ on $\boldsymbol{R}^2$. In Section 4, we shall see that the index $\mathrm{ind}_{(0,0)}(\tilde{\mathscr{D}}_{d^n f})$ of $(0,0)$ with respect to $\tilde{\mathscr{D}}_{d^n f}$ is equal to $1 - k/n$. Therefore we see that $n_{\tilde{\mathscr{D}}_{d^n f}}$ is equal to $n/(2k,n)$, where $(2k,n)$ is the greatest common divisor of $2k$ and n. In particular, we see that if $2k/n$ is not any integer, then $\tilde{\mathscr{D}}_{d^n f}$ does not consist of n continuous one-dimensional distributions on $\boldsymbol{R}^2 \setminus \{(0,0)\}$ and that if $2k$ and n are relatively prime, then $\tilde{\mathscr{D}}_{d^n f}$ is connected.

EXAMPLE. We set $f := x^4 + y^4$. Then for any $(x,y) \in \boldsymbol{R}^2$, we obtain

$$\frac{1}{24}(\widehat{d^3 f})_{(x,y)}(\phi) = x\cos^3\phi + y\sin^3\phi.$$

Therefore we obtain

$$\frac{1}{72}\frac{d(\widehat{d^3 f})_{(\cos\theta,\sin\theta)}}{d\phi}(\phi) = -\cos\phi\sin\phi\cos(\theta+\phi).$$

We see that $(0,0)$ is the only umbilical point of $d^3 f$ on $\boldsymbol{R}^2$ and that there exists a connected, continuous, complete, pointwise separable (but not pointwise separated) 3-valued one-dimensional distribution $\tilde{\mathscr{D}}_{d^3 f}$ on $\boldsymbol{R}^2 \setminus \{(0,0)\}$ such that the index $\mathrm{ind}_{(0,0)}(\tilde{\mathscr{D}}_{d^3 f})$ of $(0,0)$ with respect to $\tilde{\mathscr{D}}_{d^3 f}$ is equal to $-1/3$.

REMARK. We set $f := x^4 + 18x^2y^2 + 2y^4$. Then we may suppose $D = \boldsymbol{R}^2$. For any $(x,y) \in \boldsymbol{R}^2$, we obtain

$$\frac{1}{24}(\widehat{d^3 f})_{(x,y)}(\phi) = x\cos^3\phi + 3y\cos^2\phi\sin\phi + 3x\cos\phi\sin^2\phi + 2y\sin^3\phi.$$

Therefore we obtain

$$\frac{1}{72}\frac{d(\widehat{d^3 f})_{(\cos\theta,\sin\theta)}}{d\phi}(\phi) = \cos\theta\sin\phi(\cos^2\phi - \sin^2\phi) + \sin\theta\cos^3\phi.$$

We see that $(0,0)$ is the only umbilical point of $d^3 f$ on $\boldsymbol{R}^2$. We shall show that $\tilde{\mathscr{D}}_{d^3 f}$ may not be well-defined on $\boldsymbol{R}^2 \setminus \{(0,0)\}$. We see that there exist

(a) a number $\theta_0 \in (0, \pi/2)$,
(b) a continuous increasing function η_1 on $\bar{I}_1 := [-\pi/2, \theta_0]$,
(c) a continuous decreasing function η_2 on $\bar{I}_2 := [-\theta_0, \theta_0]$, and
(d) a continuous increasing function η_3 on $\bar{I}_3 := [-\theta_0, \pi/2]$

satisfying

$$\frac{d(\widehat{d^3 f})_{(\cos\theta,\sin\theta)}}{d\phi}(\eta_i(\theta)) = 0$$

for any $\theta \in \bar{I}_i$ and

$$\eta_1(-\pi/2) = -\pi/2, \qquad \eta_1(\theta_0) = \eta_2(\theta_0) \in (-\pi/2, 0),$$
$$\eta_3(\pi/2) = \pi/2, \qquad \eta_2(-\theta_0) = \eta_3(-\theta_0) \in (0, \pi/2).$$

In addition, we see that if a number $\phi_0 \in [-\pi/2, \pi/2)$ satisfies

$$\frac{d(\widehat{d^3 f})_{(\cos\theta,\sin\theta)}}{d\phi}(\phi_0) = 0$$

for some $\theta \in [-\pi/2, \pi/2)$, then $\phi_0 = \eta_i(\theta)$ for some $i \in \{1,2,3\}$. Therefore we see that $\tilde{\mathscr{D}}_{d^3 f}$ may not be well-defined on $\boldsymbol{R}^2 \setminus \{(0,0)\}$.

Let f be a smooth function on a domain D of $\boldsymbol{R}^2$ and p_0 an isolated umbilical point of $d^n f$ such that there exists a neighborhood U of p_0 in D satisfying $U \cap \mathrm{Umb}(d^n f) = \{p_0\}$ and the condition that there exists a continuous, complete, pointwise separable, finitely many-valued one-dimensional distribution $\tilde{\mathscr{D}}_{d^n f}$ on $U \setminus \{p_0\}$ formed by all the critical directions of $d^n f$ at each point of $U \setminus \{p_0\}$ (for example, if the sum of the multiplicities of the critical points of $(\widehat{d^n f})_p$ in $[0, \pi)$ does not depend on the choice of $p \in U \setminus \{p_0\}$ and if f is real-analytic, then this condition is satisfied). In the following sections, we shall study the behavior of $\tilde{\mathscr{D}}_{d^n f}$ around p_0 and

CONJECTURE 3.1. *The index* $\mathrm{ind}_{p_0}(\tilde{\mathscr{D}}_{d^n f})$ *of* p_0 *with respect to* $\tilde{\mathscr{D}}_{d^n f}$ *is not more than one*.

REMARK. We set $\boldsymbol{V}_f^{(n)} := \mathrm{Re}(\partial_{\bar{z}}^n f)\partial/\partial x + \mathrm{Im}(\partial_{\bar{z}}^n f)\partial/\partial y$ as in Section 1. We obtain

$$\boldsymbol{V}_f^{(1)} = \frac{1}{2}\left\{\frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y}\right\}.$$

We see that $\boldsymbol{V}_f^{(1)}$ is the half of the gradient vector field of f. Therefore Conjecture 3.1 for $n = 1$ is equivalent to Loewner's conjecture for $n = 1$. The following holds:

$$\boldsymbol{V}_f^{(2)} = \frac{1}{4}\left\{\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)\frac{\partial}{\partial x} + 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial}{\partial y}\right\}.$$

Then we see that for a point $p \in D$, the following are mutually equivalent:

(a) p is a zero point of $\boldsymbol{V}_f^{(2)}$;
(b) at p, Hessian Hess_f of f is represented by the unit matrix up to a constant;
(c) p is an umbilical point of $d^2 f$.

In addition, noticing that for any $\phi \in \boldsymbol{R}$,

$$\begin{aligned}-\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)\sin\phi + 2\frac{\partial^2 f}{\partial x \partial y}\cos\phi &= 2\left\langle \mathrm{Hess}_f \begin{pmatrix}\cos(\phi/2)\\ \sin(\phi/2)\end{pmatrix}, \begin{pmatrix}-\sin(\phi/2)\\ \cos(\phi/2)\end{pmatrix}\right\rangle \\ &= \frac{d(\widehat{d^2 f})}{d\phi}(\phi/2)\end{aligned}$$

(where $\langle\ ,\ \rangle$ is the scalar product in $\boldsymbol{R}^2$), we see that for a number $\phi \in \boldsymbol{R}$ and at a point of D, the following are mutually equivalent:

(a) $\boldsymbol{V}_f^{(2)}$ is represented by $\boldsymbol{U}_\phi$ up to a constant;
(b) ${}^t(\cos(\phi/2), \sin(\phi/2))$ is an eigenvector of Hess_f;
(c) $\boldsymbol{U}_{\phi/2}$ is in a critical direction of $d^2 f$.

In particular, we see that the index of an isolated zero point p_0 of $\boldsymbol{V}_f^{(2)}$ is twice the index of an isolated umbilical point p_0 of $d^2 f$. Hence we see that Conjecture 3.1 for $n = 2$ is equivalent to Loewner's conjecture for $n = 2$. However, if $n \geqq 3$, then $\mathrm{Re}(\partial_{\bar{z}}^n f) = \mathrm{Im}(\partial_{\bar{z}}^n f) = 0$ at a point do not always imply that $d^n f$ is umbilical at the same point: if n is even, then for a polynomial

$$f(x,y) := x^n(1+x) + x^{n-1}y - (-1)^{(n-2)/2}xy^{n-1}(1+y) - (-1)^{n/2}y^n,$$

we obtain

$$\boldsymbol{V}_f^{(n)} = \frac{n!}{2^n}\left((n+1)x\frac{\partial}{\partial x} - ny\frac{\partial}{\partial y}\right),$$

which implies that $(0,0)$ is a (unique) zero point of $\boldsymbol{V}_f^{(n)}$, while there exists no umbilical point of $d^n f$; if n is odd, then for a polynomial

$$f(x,y) := x^n(1+x) + x^{n-1}y - (-1)^{(n-1)/2}xy^{n-1} - (-1)^{(n-1)/2}y^n(1+y),$$

we obtain the same conclusion. In addition, if $n \geqq 3$, then an isolated umbilical point of $d^n f$ is not always an isolated zero point of $\boldsymbol{V}_f^{(n)}$: if we set $f(x,y) := \left(x^2+y^2\right)^l$, where $l := [n/2]+1$, then $(0,0)$ is a unique umbilical point of $d^n f$ and $\tilde{\mathscr{D}}_{d^n f}$ is well-defined on $\boldsymbol{R}^2 \setminus \{(0,0)\}$, while $\boldsymbol{V}_f^{(n)}$ is identically zero. Hence we see that the solution of one of Conjecture 3.1 and Loewner's conjecture for $n \geqq 3$ does not give any solution of the other.

In the next section, we shall study and affirmatively solve Conjecture 3.1 in the case where f is a homogeneous polynomial. The following lemma shall be useful in the next section.

LEMMA 3.2. *Let ϕ_0, a, b be real numbers and (x', y') orthogonal coordinates on $\boldsymbol{R}^2$ satisfying*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 \\ \sin\phi_0 & \cos\phi_0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

at any point of $\boldsymbol{R}^2$. Then for any $\phi \in \boldsymbol{R}$,

$$\begin{aligned} &\sum_{i=0}^{n}\binom{n}{i}\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(x,y)\cos^{n-i}\phi\sin^i\phi \\ &\quad = \sum_{i=0}^{n}\binom{n}{i}\frac{\partial^n f}{\partial (x')^{n-i}\partial (y')^i}(x',y')\cos^{n-i}(\phi+\phi_0)\sin^i(\phi+\phi_0). \end{aligned}$$

We may prove Lemma 3.2 by induction with respect to $n \in \boldsymbol{N}$.

4. Homogeneous polynomials.

4.1. Results.

Let n be a positive integer and g a homogeneous polynomial of degree $k > n$ such that there exists a continuous, complete, pointwise separable, finitely many-valued one-dimensional distribution $\tilde{\mathscr{D}}_{d^n g}$ on $\boldsymbol{R}^2 \setminus \{(0,0)\}$ formed by all the critical directions of $d^n g$ at each point of $\boldsymbol{R}^2 \setminus \mathrm{Umb}(d^n g)$. In order to grasp the behavior of $\tilde{\mathscr{D}}_{d^n g}$ around $(0,0)$, we shall first notice a point at which the "position vector field" $x\partial/\partial x + y\partial/\partial y$ is in a critical direction of $d^n g$.

For each $\theta \in \boldsymbol{R}$, set $\tilde{g}(\theta) := g(\cos\theta, \sin\theta)$. Then by Euler's identity, we obtain

LEMMA 4.1. *For any $\theta \in \boldsymbol{R}$,*

$$(\widehat{d^n g})_{(\cos\theta,\sin\theta)}(\theta) = \left\{ \prod_{i=0}^{n-1} (k-i) \right\} \tilde{g}(\theta), \tag{3}$$

$$\frac{d(\widehat{d^n g})_{(\cos\theta,\sin\theta)}}{d\phi}(\theta) = \left\{ \frac{n}{k} \prod_{i=0}^{n-1} (k-i) \right\} \frac{d\tilde{g}}{d\theta}(\theta). \tag{4}$$

By Lemma 4.1, we see that for a number θ_0, the position vector field is in a critical direction of $d^n g$ at $(\cos\theta_0, \sin\theta_0)$ if and only if θ_0 satisfies $(d\tilde{g}/d\theta)(\theta_0) = 0$. We denote by R_g the set of the numbers at which $d\tilde{g}/d\theta = 0$. Let η be a continuous function on $\boldsymbol{R}$ such that for any $\theta \in \boldsymbol{R}$, $\boldsymbol{U}_{\eta(\theta)}$ is in a critical direction of $d^n g$ at $(\cos\theta, \sin\theta)$ and $E_{d^n g}$ the set of such continuous functions as η. Let $R(d^n g)$ be the set of the numbers θ_0 such that there exists an element $\eta_{\theta_0} \in E_{d^n g}$ satisfying $\theta_0 = \eta_{\theta_0}(\theta_0)$. Then $R(d^n g) \subset R_g$ holds. We are interested in the relation between the function θ (of one variable θ) and η_{θ_0} around $\theta_0 \in R(d^n g)$.

Suppose $R_g = \boldsymbol{R}$. Then k is even and g is represented by $(x^2+y^2)^{k/2}$ up to a constant. We obtain $\theta \in E_{d^n g}$, i.e., $R(d^n g) = \boldsymbol{R}$. In addition, by Lemma 3.2, we see that $\tilde{\mathscr{D}}_{d^n g}$ is pointwise separated. Therefore we obtain $\mathrm{ind}_{(0,0)}\big(\tilde{\mathscr{D}}_{d^n g}\big) = 1$.

In the following, suppose $R_g \neq \boldsymbol{R}$. Then for each $\theta_0 \in R_g$, there exists a positive integer μ satisfying $(d^{\mu+1}\tilde{g}/d\theta^{\mu+1})(\theta_0) \neq 0$. The minimum of such integers is denoted by $\mu_g(\theta_0)$. An element $\theta_0 \in R_g$ is said to be

(a) *related* if θ_0 satisfies $\tilde{g}(\theta_0) = 0$ or if $\mu_g(\theta_0)$ is odd;

(b) *non-related* if θ_0 satisfies $\tilde{g}(\theta_0) \neq 0$ and if $\mu_g(\theta_0)$ is even.

In the next subsection, we shall prove

LEMMA 4.2. *Let θ_0 be an element of $R(d^n g)$ and I_{θ_0} an open interval satisfying $I_{\theta_0} \cap R(d^n g) = \{\theta_0\}$. Then the following hold*:

(a) *if θ_0 is related, then there exists a nonzero number $c_g^{(n)}(\theta_0)$ satisfying*

$$c_g^{(n)}(\theta_0)(\theta - \eta_{\theta_0}(\theta))(\theta - \theta_0) > 0$$

for any $\theta \in I_{\theta_0} \setminus \{\theta_0\}$ and any $\eta_{\theta_0} \in E_{d^n g}$ satisfying $\eta_{\theta_0}(\theta_0) = \theta_0$;

(b) *if θ_0 is non-related, then there exists a nonzero number $\tilde{c}_g^{(n)}(\theta_0)$ satisfying*

$$\tilde{c}_g^{(n)}(\theta_0)(\theta - \eta_{\theta_0}(\theta)) > 0$$

for any $\theta \in I_{\theta_0} \setminus \{\theta_0\}$ and $\eta_{\theta_0} \in E_{d^n g}$ satisfying $\eta_{\theta_0}(\theta_0) = \theta_0$.

For a related element $\theta_0 \in R(d^n g)$, the sign of $c_g^{(n)}(\theta_0)$ in (a) of Lemma 4.2 is called the *sign* of θ_0 and denoted by $\mathrm{sign}_g^{(n)}(\theta_0)$.

For each element $\theta_0 \in R(d^n g)$ and the interval I_{θ_0}, we may suppose that if η_1, η_2 are elements of $E_{d^n g}$ satisfying $\eta_1 = \eta_2$ at some point θ of $I_{\theta_0} \setminus \{\theta_0\}$, then $\eta_1 \equiv \eta_2$ on the connected component of $I_{\theta_0} \setminus \{\theta_0\}$ containing θ. Then there exists a positive integer $N_g^{(n)}(\theta_0) \in \boldsymbol{N}$ such that $N_g^{(n)}(\theta_0)^2$ is the number of the elements $\eta \in E_{d^n g}$ restricted on I_{θ_0} satisfying $\eta(\theta_0) = \theta_0$.

Let $R_+(d^n g)$(respectively, $R_-(d^n g)$) be the set of the related elements of $R(d^n g)$ with positive (respectively, negative) sign and for $\varepsilon \in \{+,-\}$, we set

$$N_\varepsilon(d^n g) := \sum_{\theta_0 \in R_\varepsilon(d^n g) \cap [\theta, \theta+\pi)} N_g^{(n)}(\theta_0).$$

In the next subsection, we shall prove the following:

THEOREM 4.3. *The index* $\mathrm{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^n g}\right)$ *is represented as follows*:

$$\mathrm{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^n g}\right) = 1 - \frac{N_+(d^n g) - N_-(d^n g)}{N_{d^n g}},$$

where $N_{d^n g}$ is a positive integer such that $\tilde{\mathscr{D}}_{d^n g}$ is $N_{d^n g}$-valued.

In addition, we shall prove

LEMMA 4.4. $N_+(d^n g) \geqq N_-(d^n g)$.

REMARK. In [**1**], we may find the prototypes of Lemma 4.2, Theorem 4.3 and Lemma 4.4, respectively. In [**4**], we proved Lemma 4.2 for $n = 2$.

By Theorem 4.3 together with Lemma 4.4, we obtain

$$\mathrm{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^n g}\right) \leqq 1. \tag{5}$$

From (5), we obtain the affirmative answer to Conjecture 3.1 in the case where f is a homogeneous polynomial. Indeed, (5) is a reason why we have reached Conjecture 3.1.

4.2. Proofs.

Let n, g be as in the previous subsection. For numbers $\theta, \phi \in \boldsymbol{R}$, we set

$$\tilde{D}_{d^n g}(\theta, \phi) := \frac{1}{n} \frac{d\widehat{(d^n g)}_{(\cos\theta, \sin\theta)}}{d\phi}(\phi). \tag{6}$$

Then for any $\eta \in E_{d^n g}$ and any $\theta \in \boldsymbol{R}$, $\tilde{D}_{d^n g}(\theta, \eta(\theta)) = 0$. In the following, suppose $R_g \neq \boldsymbol{R}$.

Suppose that for $\theta_0 \in R_g$, $d^n g$ is not umbilical at $(\cos\theta_0, \sin\theta_0)$. Then there exists a positive integer ν satisfying $\left(\partial^\nu \tilde{D}_{d^n g}/\partial\phi^\nu\right)(\theta_0, \theta_0) \neq 0$. The minimum of such integers is denoted by $\nu_g^{(n)}(\theta_0)$. Suppose that for $\theta_0 \in R_g$, $d^n g$ is umbilical at $(\cos\theta_0, \sin\theta_0)$. Then we write $\nu_g^{(n)}(\theta_0) = \infty$. We obtain a map $\nu_g^{(n)}$ from R_g into $\boldsymbol{N} \cup \{\infty\}$. We immediately obtain

LEMMA 4.5. *For $\theta_0 \in R_g$, the following are mutually equivalent*:
(a) $\theta_0 \in R_g \setminus R(d^1 g)$;
(b) $\tilde{g}(\theta_0) = 0$;
(c) $\nu_g^{(1)}(\theta_0) = \infty$.

For a related element $\theta_0 \in R_g$, it is said that the *critical sign* of θ_0 is positive (respectively, negative) if the following holds:

$$\tilde{g}(\theta_0)\frac{d^{\mu_g(\theta_0)+1}\tilde{g}}{d\theta^{\mu_g(\theta_0)+1}}(\theta_0) \leqq 0 \text{ (respectively}, > 0).$$

The critical sign of θ_0 is denoted by $\text{c-sign}_g(\theta_0)$. We shall prove

LEMMA 4.6. *Suppose $n \geqq 2$ and let θ_0 be an element of R_g satisfying $\tilde{g}(\theta_0) \neq 0$. Then*
(a) *$\theta_0 \in R(d^n g)$ holds if and only if $\nu_g^{(n)}(\theta_0)$ is an odd integer;*
(b) *if $\theta_0 \in R_g \setminus R(d^n g)$, then θ_0 is related and satisfies $\text{c-sign}_g(\theta_0) = -$ and $\nu_g^{(n)}(\theta_0) = \infty$.*

PROOF. By (4), (6) and the implicit function theorem, we obtain $\theta_0 \in R(d^n g)$ for an element θ_0 of R_g satisfying $\nu_g^{(n)}(\theta_0) = 1$.

We shall prove $\nu_g^{(n)}(\theta_0) = 1$ for an element θ_0 of R_g satisfying $\tilde{g}(\theta_0) \neq 0$ and $\mu_g(\theta_0) \geqq 2$. Noticing Lemma 3.2, we may suppose $\theta_0 = 0$. If we represent g as $g = \sum_{i=0}^{k} a_i x^{k-i} y^i$, then we obtain $a_0 \neq 0$ by $\tilde{g}(0) \neq 0$, and we obtain $a_1 = 0$ by $0 \in R_g$. In addition, by

$$\frac{d^2\tilde{g}}{d\theta^2}(0) = 2a_2 - ka_0 \tag{7}$$

together with $\mu_g(0) \geqq 2$, we obtain

$$a_2 = \frac{k}{2}a_0. \tag{8}$$

The following hold:

$$\frac{\partial \tilde{D}_{d^n g}}{\partial \phi}(0,0) = -\frac{\partial^n g}{\partial x^n}(1,0) + (n-1)\frac{\partial^n g}{\partial x^{n-2}\partial y^2}(1,0), \tag{9}$$

$$\frac{\partial^n g}{\partial x^n}(1,0) = \left\{\prod_{i=0}^{n-1}(k-i)\right\}a_0, \tag{10}$$

$$\frac{\partial^n g}{\partial x^{n-2}\partial y^2}(1,0) = \left\{\frac{2}{k(k-1)}\prod_{i=0}^{n-1}(k-i)\right\}a_2. \tag{11}$$

Applying (10) and (11) to (9), we obtain

$$\frac{\partial \tilde{D}_{d^n g}}{\partial \phi}(0,0) = \left\{\prod_{i=0}^{n-1}(k-i)\right\}\left\{-a_0 + \frac{2(n-1)}{k(k-1)}a_2\right\}. \tag{12}$$

By (8) together with (12), we obtain

$$\frac{\partial \tilde{D}_{d^n g}}{\partial \phi}(0,0) = -\left\{\frac{1}{k-1}\prod_{i=0}^{n}(k-i)\right\}a_0.$$

Since $a_0 \neq 0$, we obtain $\nu_g^{(n)}(0) = 1$.

We shall prove $\nu_g^{(n)}(0) = 1$ if 0 is a related element of R_g satisfying $\tilde{g}(0) \neq 0$ and $\text{c-sign}_g(0) = +$. By (7) together with $\text{c-sign}_g(0) = +$, we obtain

$$\frac{a_2}{a_0} \leqq \frac{k}{2}. \tag{13}$$

By (12), (13) and $n < k$, we obtain $\left(\partial\tilde{D}_{d^n g}/\partial\phi\right)(0,0) \neq 0$, i.e., $\nu_g^{(n)}(0) = 1$.

We shall prove $0 \notin R(d^n g)$ if 0 is a related element of R_g satisfying c-sign$_g(0) = -$ and $\nu_g^{(n)}(0) = \infty$. We see that n is even and we obtain

$$a_i = \begin{cases} 0, & \text{if } i \in \{1,3,\dots,n-1\}, \\ C(n,k,i)a_0, & \text{if } i \in \{0,2,\dots,n\}, \end{cases}$$

where

$$C(n,k,i) := \binom{n/2}{i/2}\binom{k}{i} \Big/ \binom{n}{i}.$$

Therefore we obtain

$$\begin{aligned} &\widehat{(d^n g)}_{(\cos\theta,\sin\theta)}(\phi) \\ &\quad = \left\{\prod_{i=0}^{n-1}(k-i)\right\} a_0 \cos^{k-n}\theta \\ &\qquad + \left\{A_2 \cos^{n-1}\phi \sin\phi + \alpha(\phi)\sin^2\phi\right\}\cos^{k-n-1}\theta\sin\theta + \beta(\theta,\phi)\sin^2\theta, \end{aligned}$$

where $A_2 \in \boldsymbol{R}\setminus\{0\}$ and α, β are smooth functions. In addition, we obtain

$$n\tilde{D}_{d^n g}(\theta,\phi) = \left\{(A_2\cos^n\phi + \hat{\alpha}(\phi)\sin\phi)\cos^{k-n-1}\theta + \frac{\partial\beta}{\partial\phi}(\theta,\phi)\sin\theta\right\}\sin\theta,$$

where $\hat{\alpha}$ is a smooth function. Hence we obtain $0 \notin R(d^n g)$.

Let 0 be a related element of R_g satisfying c-sign$_g(0) = -$ and $\nu_g^{(n)}(0) \in \boldsymbol{N}\setminus\{1\}$. Then we obtain

$$a_i = \begin{cases} 0, & \text{if } i \in \left\{1,3,\dots,2\left[\left(\nu_g^{(n)}(0)+1\right)/2\right]-1\right\}, \\ C(n,k,i)a_0, & \text{if } i \in \left\{0,2,\dots,2\left[\nu_g^{(n)}(0)/2\right]\right\} \end{cases}$$

and

$$a_{\nu_g^{(n)}(0)+1} \neq \begin{cases} 0, & \text{if } \nu_g^{(n)}(0) \text{ is even}, \\ C\left(n,k,\nu_g^{(n)}(0)+1\right)a_0, & \text{if } \nu_g^{(n)}(0) \text{ is odd}. \end{cases}$$

Then we may represent $\tilde{D}_{d^n g}(\theta,\phi)$ as

$$\tilde{D}_{d^n g}(\theta,\phi) = \sum_{i,j\geqq 0} B_{ij}\theta^i\phi^j,$$

where $B_{10} \neq 0$, $B_{0j} = 0$ for $j \in \left\{0,1,\dots,\nu_g^{(n)}(0)-1\right\}$ and $B_{0\nu_g^{(n)}(0)} \neq 0$. Therefore we obtain $\left(\partial\tilde{D}_{d^n g}/\partial\theta\right)(0,0) \neq 0$. By the implicit function theorem, we see that there exist a positive number $\varepsilon > 0$ and a smooth function γ of one variable satisfying

$$\gamma(\phi) = -\frac{B_{0\nu_g^{(n)}(0)}}{B_{10}}\phi^{\nu_g^{(n)}(0)} + o\big(\phi^{\nu_g^{(n)}(0)}\big) \tag{14}$$

and

$$\big\{(\theta,\phi)\in(-\varepsilon,\varepsilon)^2\,;\,\tilde{D}_{d^n g}(\theta,\phi)=0\big\} = \{(\gamma(\phi),\phi)\,;\,\phi\in(-\varepsilon,\varepsilon)\}.$$

Therefore if $\nu_g^{(n)}(0)$ is odd, then 0 is an element of $R(d^n g)$; if $\nu_g^{(n)}(0)$ is even, then there does not exist any distribution as $\tilde{\mathscr{D}}_{d^n g}$ on $\boldsymbol{R}^2\setminus\{(0,0)\}$.

Hence we obtain Lemma 4.6. □

REMARK. In [**3**], we may find the prototype of Lemma 4.6. In [**4**], we proved that for an element θ_0 of R_g, $\theta_0\in R_g\setminus R(d^2 g)$ holds if and only if $\tilde{g}(\theta_0)\neq 0$ and $\nu_g^{(2)}(\theta_0)=\infty$ hold.

PROOF OF LEMMA 4.2. Let θ_0 be an element of $R(d^n g)$ satisfying $\nu_g^{(n)}(\theta_0)=1$. Then by the implicit function theorem, we see that if η_{θ_0} is an element of $E_{d^n g}$ satisfying $\eta_{\theta_0}(\theta_0)=\theta_0$, then η_{θ_0} is smooth at θ_0 and satisfies

$$\frac{d^\mu(\theta-\eta_{\theta_0})}{d\theta^\mu}(\theta_0) = \left\{\frac{1}{k}\prod_{i=0}^{n-1}(k-i)\right\}\frac{d^{\mu+1}\tilde{g}}{d\theta^{\mu+1}}(\theta_0)\Big/\frac{\partial\tilde{D}_{d^n g}}{\partial\phi}(\theta_0,\theta_0) \tag{15}$$

for any $\mu\in\{0,1,\dots,\mu_g(\theta_0)\}$. Therefore we obtain Lemma 4.2.

Let 0 be an element of $R(d^n g)$ satisfying $\tilde{g}(0)\neq 0$ and $\nu_g^{(n)}(0)\geqq 2$. Then 0 is related and $\nu_g^{(n)}(0)$ is odd. Noticing (14), we obtain

$$\frac{B_{0\nu_g^{(n)}(0)}}{B_{10}}(\theta-\eta_0(\theta))\theta > 0 \tag{16}$$

for any $\theta\in I_0\setminus\{0\}$ and $\eta_0\in E_{d^n g}$ satisfying $\eta_0(0)=0$. Therefore we obtain Lemma 4.2.

Let 0 be an element of $R(d^n g)$ satisfying $\tilde{g}(0)=0$ and $\nu_g^{(n)}(0)=\infty$. Then we see that there exists an integer $i_0>n$ satisfying $a_i=0$ for $i\in\{0,1,\dots,i_0-1\}$ and $a_{i_0}\neq 0$. Therefore we may represent $\tilde{D}_{d^n g}$ as

$$\tilde{D}_{d^n g}(\theta,\phi) = \theta^{i_0-n}\sum_{i\geqq n-1}\tilde{D}^{(i)}_{d^n g}(\theta,\phi),$$

where $\tilde{D}^{(i)}_{d^n g}$ is a homogeneous polynomial of degree i in two variables θ, ϕ. We obtain $\tilde{D}^{(n-1)}_{d^n g}\not\equiv 0$. If we represent $\tilde{D}^{(i)}_{d^n g}$ as

$$\tilde{D}^{(i)}_{d^n g}(\theta,\phi) = \sum_{j=0}^{i}\tilde{D}^{(i,j)}_{d^n g}\theta^{i-j}\phi^j,$$

then we obtain $\tilde{D}^{(n-1,j_1)}_{d^n g}\tilde{D}^{(n-1,j_2)}_{d^n g}\geqq 0$ for arbitrary two j_1, $j_2\in\{0,1,\dots,n-1\}$. Then we obtain $(\theta-\eta_0(\theta))\theta>0$ for any $\theta\in I_0\setminus\{0\}$ and any $\eta_0\in E_{d^n g}$ satisfying $\eta_0(0)=0$. Similarly, we see that if 0 is an element of $R(d^n g)$ satisfying $\tilde{g}(0)=0$ and $\nu_g^{(n)}(0)\in\boldsymbol{N}$, then $(\theta-\eta_0(\theta))\theta>0$ for any $\theta\in I_0\setminus\{0\}$ and any $\eta_0\in E_{d^n g}$ satisfying $\eta_0(0)=0$. Hence we obtain Lemma 4.2. □

We shall prove

PROPOSITION 4.7. *Let θ_0 be a related element of $R(d^n g)$.*
(a) *If $\tilde{g}(\theta_0) \neq 0$, then the sign of the nonzero number*

$$\delta_g^{(n)}(\theta_0) := \frac{d^{\mu_g(\theta_0)+1}\tilde{g}}{d\theta^{\mu_g(\theta_0)+1}}(\theta_0)\frac{\partial^{\nu_g^{(n)}(\theta_0)}\tilde{D}_{d^n g}}{\partial\phi^{\nu_g^{(n)}(\theta_0)}}(\theta_0,\theta_0)$$

gives the sign of θ_0;
(b) *if $\tilde{g}(\theta_0) = 0$, then the sign of θ_0 is positive.*

PROOF. Let θ_0 be a related element of $R(d^n g)$ satisfying $\tilde{g}(\theta_0) \neq 0$ and $\nu_g^{(n)}(\theta_0) = 1$. Then by (15), we obtain (a). Let 0 be a related element of $R(d^n g)$ satisfying $\tilde{g}(0) = 0$. Then in the proof of Lemma 4.2, we have proved $\mathrm{sign}_g^{(n)}(0) = +$. Let 0 be a related element of $R(d^n g)$ satisfying $\tilde{g}(0) \neq 0$ and $\nu_g^{(n)}(0) \geqq 2$. Then noticing (16), we see that the sign of the nonzero number $B_{0\nu_g^{(n)}(0)}B_{10}$ gives the sign of 0. We obtain

$$B_{0\nu_g^{(n)}(0)} = \frac{1}{\nu_g^{(n)}(0)!}\frac{\partial^{\nu_g^{(n)}(0)}\tilde{D}_{d^n g}}{\partial\phi^{\nu_g^{(n)}(0)}}(0,0), \qquad B_{10}\tilde{g}(0) > 0.$$

Since $\text{c-sign}_g(0) = -$, we see that the sign of $\delta_g^{(n)}(0)$ gives the sign of 0. Hence we obtain Proposition 4.7. □

REMARK. In [**1**], we may find the prototype of Proposition 4.7. In [**4**], we proved Proposition 4.7 for $n = 2$.

We shall prove

PROPOSITION 4.8. *Let θ_0 be a related element of $R(d^n g)$ satisfying* $\text{c-sign}_g(\theta_0) = +$. *Then* $\mathrm{sign}_g^{(n)}(\theta_0) = +$.

PROOF. Let θ_0 be a related element of $R(d^n g)$ with $\text{c-sign}_g(\theta_0) = +$. Suppose $n = 1$. Then we obtain

$$\frac{\partial\tilde{D}_{d^1 g}}{\partial\phi}(\theta_0,\theta_0) = -k\tilde{g}(\theta_0).$$

Since $\text{c-sign}_g(\theta_0) = +$, we obtain $\delta_g^{(1)}(\theta_0) > 0$. Therefore from Proposition 4.7, we obtain $\mathrm{sign}_g^{(1)}(\theta_0) = +$. In the following, suppose $n \geqq 2$. In addition, noticing (b) of Proposition 4.7, we may suppose $\tilde{g}(\theta_0) \neq 0$. Then since $\nu_g^{(n)}(\theta_0) = 1$, we may represent $\delta_g^{(n)}(\theta_0)$ as

$$\delta_g^{(n)}(\theta_0) = \left(\tilde{g}(\theta_0)\frac{d^{\mu_g(\theta_0)+1}\tilde{g}}{d\theta^{\mu_g(\theta_0)+1}}(\theta_0)\right)\left(\frac{1}{\tilde{g}(\theta_0)}\frac{\partial\tilde{D}_{d^n g}}{\partial\phi}(\theta_0,\theta_0)\right). \tag{17}$$

We obtain

$$(n-1)\frac{1}{\tilde{g}(\theta_0)}\frac{d^2\tilde{g}}{d\theta^2}(\theta_0) = \frac{k(k-1)}{\left\{\prod_{i=0}^{n-1}(k-i)\right\}}\left(\frac{1}{\tilde{g}(\theta_0)}\frac{\partial\tilde{D}_{d^n g}}{\partial\phi}(\theta_0,\theta_0)\right) + k(k-n). \tag{18}$$

Since $\text{c-sign}_g(\theta_0) = +$, we obtain

$$\frac{1}{\tilde{g}(\theta_0)}\frac{\partial \tilde{D}_{d^n g}}{\partial \phi}(\theta_0, \theta_0) < 0,$$

and $\delta_g^{(n)}(\theta_0) > 0$. Therefore from Proposition 4.7, we obtain Proposition 4.8. □

By (17) together with (18), we obtain

PROPOSITION 4.9. *Let θ_0 be a related element of $R(d^n g)$ satisfying* $\text{c-sign}_g(\theta_0) = -$ *and*

$$(n-1)\frac{d^2\tilde{g}}{d\theta^2}(\theta_0) \neq (k(k-n))\tilde{g}(\theta_0).$$

Then $\text{sign}_g^{(n)}(\theta_0) = +$ (*respectively,* $-$) *is equivalent to*

$$(n-1)\frac{d^2\tilde{g}}{d\theta^2}(\theta_0)\Big/\tilde{g}(\theta_0) \in (k(k-n), \infty) \ (\textit{respectively}, \ [0, k(k-n))).$$

REMARK. Let θ_0 be a related element of R_g satisfying $\text{c-sign}_g(\theta_0) = -$. Then from Lemma 4.5, we obtain $\theta_0 \in R(d^1 g)$ and from Proposition 4.9, we obtain $\text{sign}_g^{(1)}(\theta_0) = -$.

REMARK. Let θ_0 be a related element of $R(d^n g)$ satisfying $\text{c-sign}_g(\theta_0) = -$. We see by (18) that

$$(n-1)\frac{d^2\tilde{g}}{d\theta^2}(\theta_0)\Big/\tilde{g}(\theta_0) = k(k-n)$$

is equivalent to $\nu_g^{(n)}(\theta_0) \geqq 2$. If $\nu_g^{(n)}(\theta_0) \geqq 2$, then both $\text{sign}_g^{(n)}(\theta_0) = +$ and $\text{sign}_g^{(n)}(\theta_0) = -$ may happen and we may grasp the sign of θ_0 by (a) of Proposition 4.7.

REMARK. In [**1**], we may find the prototype of Proposition 4.8; in [**2**], we may find the prototype of Proposition 4.9. In [**4**], we proved Proposition 4.8 for $n = 2$.

We shall prove

LEMMA 4.10. *For an element $\theta_0 \in R(d^n g)$ satisfying $\tilde{g}(\theta_0) \neq 0$, $N_g^{(n)}(\theta_0) = 1$ holds.*

PROOF. If $\nu_g^{(n)}(\theta_0) = 1$, then by the implicit function theorem, we obtain $N_g^{(n)}(\theta_0) = 1$. Suppose $\nu_g^{(n)}(\theta_0) \geqq 2$. Then we obtain $n \geqq 2$ and referring to the proof of Lemma 4.6, we obtain $N_g^{(n)}(\theta_0) = 1$. □

REMARK. For any element $\theta_0 \in R(d^2 g)$, $N_g^{(2)}(\theta_0) = 1$ (see [**4**]).

PROOF OF LEMMA 4.4. Let θ_1, θ_2 be two related elements of $R(d^n g)$ satisfying $\theta_2 > \theta_1$ and the condition that in (θ_1, θ_2), there exists no related element of $R(d^n g)$. Then either $\text{c-sign}_g(\theta_1) = +$ or $\text{c-sign}_g(\theta_2) = +$ holds. Therefore from Proposition 4.8, we see that either $\text{sign}_g(\theta_1) = +$ or $\text{sign}_g(\theta_2) = +$ holds. Noticing (b) of Proposition 4.7 and Lemma 4.10, we obtain Lemma 4.4. □

PROOF OF THEOREM 4.3. We first suppose that $\tilde{\mathscr{D}}_{d^n g}$ is pointwise separated. Let $N(d^n g)$ be the number of the related elements of $R(d^n g)$ in $[0,\pi)$ and $\theta_1, \theta_2, \dots, \theta_{N(d^n g)}$ related elements of $R(d^n g)$ satisfying

$$0 \leqq \theta_1 < \theta_2 < \cdots < \theta_{N(d^n g)} < \pi.$$

In addition, for $i \in \{1, 2, \dots, N(d^n g)\}$ and $j \in \boldsymbol{Z}$, set $\theta_{i+jN(d^n g)} := \theta_i + j\pi$. Then for $i \in \boldsymbol{Z}$, we see that in (θ_{i-1}, θ_i), there exists no related element of $R(d^n g)$. Let $\phi_{d^n g}$ be an element of $\boldsymbol{\Phi}_{\tilde{\mathscr{D}}_{d^n g};(0,0)}$ satisfying $\phi_{d^n g}(r, \theta_1) = \theta_1$ for any $r > 0$. Then we see that if both $\mathrm{sign}_g^{(n)}(\theta_1) = +$ and $\mathrm{sign}_g^{(n)}(\theta_2) = +$ hold, then $\phi_{d^n g}(r, \theta_2) < \theta_2$ and that if just one of $\mathrm{sign}_g^{(n)}(\theta_1) = +$ and $\mathrm{sign}_g^{(n)}(\theta_2) = +$ holds, then $\phi_{d^n g}(r, \theta_2) = \theta_2$. We suppose $\mathrm{sign}_g^{(n)}(\theta_1) = +$. For $i_0 \in \boldsymbol{N}$, suppose that the sign of θ_{i_0} is positive and that the number of the related elements of $R(d^n g)$ in $[\theta_1, \theta_{i_0})$ with positive sign minus the number of the related elements of $R(d^n g)$ in $[\theta_1, \theta_{i_0})$ with negative sign is equal to $l_0 N_{d^n g}$ for some $l_0 \in \boldsymbol{N} \cup \{0\}$. Then for any $r > 0$, we obtain

$$\theta_{i_0} - \phi_{d^n g}(r, \theta_{i_0}) = l_0 \pi.$$

We see that $2N_{d^n g} N(d^n g) + 1$ is such a positive integer as i_0 and that the corresponding integer l_0 is equal to $2(N_+(d^n g) - N_-(d^n g))$. Therefore we obtain

$$\theta_{2N_{d^n g} N(d^n g)+1} - \phi_{d^n g}\big(r, \theta_{2N_{d^n g} N(d^n g)+1}\big) = 2(N_+(d^n g) - N_-(d^n g))\pi$$

for any $r > 0$. This implies

$$\frac{\phi_{d^n g}(r, \theta_1 + 2N_{d^n g}\pi) - \phi_{d^n g}(r, \theta_1)}{2N_{d^n g}\pi} = 1 - \frac{N_+(d^n g) - N_-(d^n g)}{N_{d^n g}}.$$

Hence we obtain Theorem 4.3.

We suppose that $\tilde{\mathscr{D}}_{d^n g}$ is not always pointwise separated. Let $\theta_1 \in R(d^n g)$ satisfy $\tilde{g}(\theta_1) \neq 0$. Then $N_g^{(n)}(\theta_1) = 1$. Let $\phi_{d^n g}^{(1)}$ be an element of $\boldsymbol{\Phi}_{\tilde{\mathscr{D}}_{d^n g};(0,0)}$ satisfying $\phi_{d^n g}^{(1)}(r, \theta_1) = \theta_1$ for any $r > 0$. For each integer $i \geqq 2$, let $\phi_{d^n g}^{(i)}$ be an element of $\boldsymbol{\Phi}_{\tilde{\mathscr{D}}_{d^n g};(0,0)}$ such that for any $(r, \theta) \in (0, \infty) \times \boldsymbol{R}$ and any $i \in \boldsymbol{N}$, the following hold:

(a) $\phi_{d^n g}^{(i+1)}(r, \theta) \geqq \phi_{d^n g}^{(i)}(r, \theta)$;

(b) the following give all the critical directions of $d^n g$ at $(r\cos\theta, r\sin\theta)$:

$$\phi_{d^n g}^{(i)}(r, \theta),\ \phi_{d^n g}^{(i+1)}(r, \theta),\ \phi_{d^n g}^{(i+2)}(r, \theta),\ \dots,\ \phi_{d^n g}^{(i+N_{d^n g}-1)}(r, \theta);$$

(c) $\phi_{d^n g}^{(i+N_{d^n g})}(r, \theta) = \phi_{d^n g}^{(i)}(r, \theta) + \pi$.

Then we obtain

$$\phi_{d^n g}^{(2l(N_+(d^n g) - N_-(d^n g))+1)}(r, \theta_1 + 2l\pi) = \theta_1 + 2l\pi$$

for any $l \in \{1, 2, \dots, N_{d^n g}\}$. In particular, we obtain

$$\phi_{d^n g}^{(1)}(r, \theta_1 + 2N_{d^n g}\pi) + 2(N_+(d^n g) - N_-(d^n g))\pi = \phi_{d^n g}^{(1)}(r, \theta_1) + 2N_{d^n g}\pi,$$

i.e.,

$$\frac{\phi^{(1)}_{d^n g}(r,\theta_1+2N_{d^n g}\pi)-\phi^{(1)}_{d^n g}(r,\theta_1)}{2N_{d^n g}\pi}=1-\frac{N_+(d^n g)-N_-(d^n g)}{N_{d^n g}}.$$

Hence we obtain Theorem 4.3. □

EXAMPLE. Let g be a spherical harmonic function of degree k. We shall compute the index of $(0,0)$ with respect to $\tilde{\mathscr{D}}_{d^n g}$. We see that any $\theta_0\in R_g$ is related and satisfies $\tilde{g}(\theta_0)\neq 0$ and c-sign$_g(\theta_0)=+$. Therefore from Lemma 4.6, we obtain $R(d^n g)=R_g$ and by Proposition 4.8 together with Lemma 4.10, we obtain $\left(N_+(d^n g),N_-(d^n g)\right)=(k,0)$. Since $N_{d^n g}=n$, we obtain $\mathrm{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^n g}\right)=1-k/n$.

5. Real-analytic functions.

Let n be a positive integer and r_0 a positive number. Let F be a real-analytic function on a neighborhood $U:=\left\{x^2+y^2<r_0^2\right\}$ of $(0,0)$ in $\boldsymbol{R}^2$ satisfying the following:

(a) $(0,0)$ is an umbilical point of d^nF;

(b) F is represented as $F:=\sum_{i\geqq n}F^{(i)}$, where $F^{(i)}$ is a homogeneous polynomial of degree i.

We see that if n is odd, then $F^{(n)}$ is identically zero. Suppose that $(0,0)$ is the only umbilical point of d^nF on U and that there exists a continuous, complete, pointwise separable, finitely many-valued one-dimensional distribution $\tilde{\mathscr{D}}_{d^nF}$ on $U\setminus\{(0,0)\}$ formed by all the critical directions of d^nF at each point of $U\setminus\{(0,0)\}$. We set

$$m_F:=\min\left\{i>n\,;\,F^{(i)}\not\equiv 0\right\},\qquad g_F:=F^{(m_F)}.$$

Let ϕ_{d^nF} be an element of $\boldsymbol{\Phi}_{\tilde{\mathscr{D}}_{d^nF};(0,0)}$. We shall prove

PROPOSITION 5.1. *For each number* $\theta_0\in\boldsymbol{R}$,

(a) *there exists a number* $\phi_{d^nF,o}(\theta_0)$ *satisfying*

$$\lim_{r\to 0}\phi_{d^nF}(r,\theta_0)=\phi_{d^nF,o}(\theta_0),$$

and $\phi_{d^nF,o}(\theta_0)$ *is a critical point of* $(\widehat{d^n g_F})_{(\cos\theta_0,\sin\theta_0)}$;

(b) *there exist numbers* $\phi_{d^nF,o}(\theta_0+0)$, $\phi_{d^nF,o}(\theta_0-0)$ *satisfying*

$$\lim_{\theta\to\theta_0\pm 0}\phi_{d^nF,o}(\theta)=\phi_{d^nF,o}(\theta_0\pm 0).$$

Let $S(d^n g_F)$ denote the set of the numbers θ_0 such that $d^n g_F$ is umbilical at $(\cos\theta_0,\sin\theta_0)$. Then $S(d^n g_F)\subset R_{g_F}$. In the following, suppose the following:

(a) each critical point of $(\widehat{d^n g_F})_{(\cos\theta_0,\sin\theta_0)}$ for each $\theta_0\in\boldsymbol{R}\setminus S(d^n g_F)$ is obtained as in (a) of Proposition 5.1 from some $\phi_{d^nF}\in\boldsymbol{\Phi}_{\tilde{\mathscr{D}}_{d^nF};(0,0)}$;

(b) there exists a continuous, complete, pointwise separable, finitely many-valued one-dimensional distribution $\tilde{\mathscr{D}}_{d^n g_F}$ on $\boldsymbol{R}^2\setminus\{(0,0)\}$ formed by all the critical directions of $d^n g_F$ at each point of $\boldsymbol{R}^2\setminus\mathrm{Umb}(d^n g_F)$;

(c) $\tilde{\mathscr{D}}_{d^nF}$ is $N_{d^n g_F}$-valued.

REMARK. If $n\in\{1,2\}$, then conditions (a)–(c) are always satisfied.

For each $\theta_0 \in \boldsymbol{R}$, we set

$$\Gamma_{d^nF,o}(\theta_0) := \phi_{d^nF,o}(\theta_0+0) - \phi_{d^nF,o}(\theta_0-0).$$

We shall prove

PROPOSITION 5.2. (a) *If* $\theta_0 \in \boldsymbol{R}$ *satisfies* $\Gamma_{d^nF,o}(\theta_0) \neq 0$, *then* $\theta_0 \in S(d^n g_F)$;
(b) $\mathrm{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^nF}\right)$ *is represented as follows*:

$$\begin{aligned} &\mathrm{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^nF}\right) \\ &\quad = \mathrm{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^n g_F}\right) + \frac{1}{2N_{d^n g_F}\pi} \sum_{\theta_0 \in S(d^n g_F) \cap [\theta, \theta+2N_{d^n g_F}\pi)} \Gamma_{d^nF,o}(\theta_0). \end{aligned}$$

PROOF OF PROPOSITION 5.1. We represent d^nF as

$$d^nF = \sum_{i \geqq n} d^n F^{(i)}.$$

Then we obtain

$$(\widehat{d^nF})_{(r\cos\theta_0, r\sin\theta_0)} = \sum_{i \geqq n} r^{i-n} (\widehat{d^nF^{(i)}})_{(\cos\theta_0, \sin\theta_0)}$$

for any $r \in (0, r_0)$ and any $\theta_0 \in \boldsymbol{R}$. Therefore we see that for an arbitrary positive number $\varepsilon > 0$, there exists a positive number $r_0 > 0$ such that for any $r \in (0, r_0)$ and any $\phi \in \boldsymbol{R}$,

$$\left| \frac{1}{r^{m_F - n}} \frac{d(\widehat{d^nF})_{(r\cos\theta_0, r\sin\theta_0)}}{d\phi}(\phi) - n\tilde{D}_{d^n g_F}(\theta_0, \phi) \right| < \varepsilon.$$

In particular, we obtain

$$n\left|\tilde{D}_{d^n g_F}(\theta_0, \phi_{d^nF}(r, \theta_0))\right| < \varepsilon \tag{19}$$

for any $r \in (0, r_0)$. If $\theta_0 \in \boldsymbol{R} \setminus S(d^n g_F)$, then each critical point of $(\widehat{d^n g_F})_{(\cos\theta_0, \sin\theta_0)}$ is isolated. Therefore by (19), we obtain (a) of Proposition 5.1 in the case where $\theta_0 \in \boldsymbol{R} \setminus S(d^n g_F)$. Let θ_0 be an element of $S(d^n g_F)$. Since $(0,0)$ is an isolated umbilical point of d^nF, we see that there exists an integer $m_F(\theta_0) > m_F$ satisfying the following:

(a) for any integer i satisfying $m_F \leqq i \leqq m_F(\theta_0) - 1$, $d^nF^{(i)}$ is umbilical at $(\cos\theta_0, \sin\theta_0)$;
(b) $d^nF^{(m_F(\theta_0))}$ is not umbilical at $(\cos\theta_0, \sin\theta_0)$.

Then we see that for an arbitrary positive number $\varepsilon > 0$, there exists a positive number $r_0 > 0$ such that for any $r \in (0, r_0)$,

$$\left|\tilde{D}_{d^nF^{(m_F(\theta_0))}}(\theta_0, \phi_{d^nF}(r, \theta_0))\right| < \varepsilon.$$

Since $d^n g_F$ is umbilical at $(\cos\theta_0, \sin\theta_0)$, we obtain (a) of Proposition 5.1 in the case where $\theta_0 \in S(d^n g_F)$. In addition, by (a) of Proposition 5.1, we obtain (b) of Proposition 5.1. □

PROOF OF PROPOSITION 5.2. If $\theta_0 \in \boldsymbol{R} \setminus S(d^n g_F)$, then noticing Proposition 5.1, we obtain $\Gamma_{d^nF,o}(\theta_0) = 0$. Hence we obtain (a) of Proposition 5.2. For $\theta \in \boldsymbol{R}$, the following holds:

$$\operatorname{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^nF}\right)=\frac{\phi_{d^nF,o}\left(\theta+2N_{d^ng_F}\pi\right)-\phi_{d^nF,o}(\theta)}{2N_{d^ng_F}\pi}. \tag{20}$$

In addition, for any $r>0$, the following holds:

$$\begin{aligned}&\phi_{d^nF,o}\left(\theta+2N_{d^ng_F}\pi\right)-\phi_{d^nF,o}(\theta)\\&\quad=\phi_{d^ng_F}\left(r,\theta+2N_{d^ng_F}\pi\right)-\phi_{d^ng_F}(r,\theta)+\sum_{\theta_0\in S(d^ng_F)\cap[\theta,\theta+2N_{d^ng_F}\pi)}\varGamma_{d^nF,o}(\theta_0).\end{aligned} \tag{21}$$

From (20) and (21), we obtain (b) of Proposition 5.2. □

REMARK. In [**4**], we proved the prototypes of Propositions 5.1 and 5.2 for $n=2$, respectively.

By Theorem 4.3, Lemma 4.4 and Proposition 5.2, we see that if F satisfies $S(d^ng_F)=\emptyset$, then $\operatorname{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^nF}\right)\leqq 1$.

We shall prove

THEOREM 5.3. *Suppose*

$$\sum_{i=0}^{N_{d^ng_F}-1}\varGamma_{d^nF,o}(\theta_0+2i\pi)\leqq\pi \tag{22}$$

for any $\theta_0\in S(d^ng_F)$. *Then* $\operatorname{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^nF}\right)\leqq 1$.

PROOF. By Theorem 4.3, Lemma 4.5, Lemma 4.6 and Proposition 4.8, we obtain

$$\operatorname{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^ng_F}\right)\leqq 1-N_s(d^ng_F)\big/N_{d^ng_F}, \tag{23}$$

where $N_s(d^ng_F):=\sharp\{S(d^ng_F)\cap[\theta,\theta+\pi)\}$. If (22) holds for any $\theta_0\in S(d^ng_F)$, then by (b) of Proposition 5.2 together with (23), we obtain $\operatorname{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^nF}\right)\leqq 1$. Hence we obtain Theorem 5.3. □

REMARK. We see that (22) is always true for $n=1$.

REMARK. In [**4**], we proved the prototype of Theorem 5.3 for $n=2$ on condition that the right hand side of (22) is equal to 2π.

We shall prove

THEOREM 5.4. *Suppose that* $\tilde{g}_F(\theta_0)\neq 0$ *for any* $\theta_0\in S(d^ng_F)$ *and that* $\tilde{\mathscr{D}}_{d^nF}$ *is pointwise separated. Then* $\operatorname{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^nF}\right)\leqq 1$.

In order to prove Theorem 5.4, we need a lemma.

For $n\geqq 2$, we set

$$\begin{aligned}\varpi_{d^nF}:=\frac{1}{n}\sum_{i=0}^{n}\binom{n}{i}\frac{\partial^nF}{\partial x^{n-i}\partial y^i}\Bigg\{&i\left(\frac{\partial F}{\partial x}\right)^{n-i+1}\left(\frac{\partial F}{\partial y}\right)^{i-1}\\&-(n-i)\left(\frac{\partial F}{\partial x}\right)^{n-i-1}\left(\frac{\partial F}{\partial y}\right)^{i+1}\Bigg\}.\end{aligned}$$

We see that for a point $p \in U$, $\varpi_{d^nF}(p) = 0$ holds if and only if the gradient vector field $(\partial F/\partial x)\partial/\partial x + (\partial F/\partial y)\partial/\partial y$ of F is in a critical direction of d^nF at p. We set

$$\tilde{\varpi}_{d^nF}(r,\theta) := \varpi_{d^nF}(r\cos\theta, r\sin\theta)$$

and

$$m_{d^nF} := \begin{cases} (n+1)m_F - 2n, & \text{if } F^{(n)} \equiv 0, \\ m_F + n(n-2), & \text{if } F^{(n)} \not\equiv 0. \end{cases}$$

Then we see that $\tilde{\varpi}_{d^nF}/r^{m_{d^nF}}$ may be continuously extended to $\{r=0\}$. By the implicit function theorem, we obtain

LEMMA 5.5. *Let θ_0 be an element of $S(d^n g_F)$ satisfying $\tilde{g}_F(\theta_0) \neq 0$. Then there exist a neighborhood V_{θ_0} of $(0,\theta_0)$ in $\boldsymbol{R}^2$ and a real-analytic curve C_{θ_0} in V_{θ_0} through $(0,\theta_0)$ satisfying*

(a) $C_{\theta_0} = \left\{(r,\theta) \in V_{\theta_0}\ ;\ \tilde{\varpi}_{d^nF}(r,\theta)/r^{m_{d^nF}} = 0\right\}$;

(b) C_{θ_0} *is not tangent to the θ-axis at $(0,\theta_0)$.*

REMARK. In [**4**], we proved Lemma 5.5 for $n=2$.

PROOF OF THEOREM 5.4. Suppose $n \geqq 2$. Then noticing Lemma 5.5 and that $\tilde{\mathscr{D}}_{d^nF}$ is pointwise separated, we see that there exists a nonzero number $c_{d^nF,o}(\theta_0)$ satisfying

$$c_{d^nF,o}(\theta_0)\Gamma_{d^nF,o}(\theta_0 + 2i\pi) \geqq 0$$

for any $i \in \mathbf{Z}$ and

$$\sum_{i=0}^{N_{d^n g_F}-1} \Gamma_{d^nF,o}(\theta_0 + 2i\pi) \in \{-\pi, 0, \pi\}.$$

Therefore from Theorem 5.3, we obtain $\operatorname{ind}_{(0,0)}\left(\tilde{\mathscr{D}}_{d^nF}\right) \leqq 1$. Suppose $n=1$. Then Lemma 4.5 says that for $\theta_0 \in R_{g_F}$, $\tilde{g}_F(\theta_0) = 0$ is equivalent to $\theta_0 \in S(d^1 g_F)$. This implies that the first assumption in Theorem 5.4 is always false for $n=1$. Hence we obtain Theorem 5.4. $\square$

REMARK. In [**4**], we proved the prototype of Theorem 5.4 for $n=2$.

References

[1] N. Ando, An isolated umbilical point of the graph of a homogeneous polynomial, Geom. Dedicata, **82** (2000), 115–137.

[2] N. Ando, The behavior of the principal distributions around an isolated umbilical point, J. Math. Soc. Japan, **53** (2001), 237–260.

[3] N. Ando, The behavior of the principal distributions on the graph of a homogeneous polynomial, Tohoku Math. J., **54** (2002), 163–177.

[4] N. Ando, The behavior of the principal distributions on a real-analytic surface, J. Math. Soc. Japan, **56** (2004), 201–214.

[5] L. Bates, A weak counterexample to the Carathéodory's conjecture, Differential Geom. Appl., **15** (2001), 79–80.

[6] C. Gutierrez and F. Sanchez-Bringas, Planer vector field versions of Carathéodory's and Loewner's conjectures, Publ. Mat., **41** (1997), no. 1, 169–179.

[7] H. Hopf, Differential geometry in the large, Lecture Notes in Math., vol. 1000, Springer, Berlin-NewYork, 1989.

[8] T. Klotz, On G. Bol's proof of Carathéodory's conjecture, Comm. Pure Appl. Math., **12** (1959), 277–311.

[9] H. Scherbel, A new proof of Hamburger's index theorem on umbilical points, Dissertation, ETH, Zürich, No. 10281, 1994.

[10] B. Smyth and F. Xavier, A sharp geometric estimate for the index of an umbilic on a smooth surface, Bull. London Math. Soc., **24** (1992), 176–180.

[11] B. Smyth and F. Xavier, Real solvability of the equation $\partial_{\bar{z}}^2 \omega = \rho g$ and the topology of isolated umbilics, J. Geom. Anal., **8** (1998), 655–671.

[12] B. Smyth and F. Xavier, Eigenvalue estimates and the index of Hessian fields, Bull. London Math. Soc., **33** (2001), 109–112.

[13] C. J. Titus, A proof of a conjecture of Loewner and of the conjecture of Carathéodory on umbilic points, Acta Math., **131** (1973), 43–77.

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