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On an integral representation of special values of the zeta function at odd integers

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Abstract. An integral representation of the p-series of odd p is shown;

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} = (-1)^p \frac{(2\pi)^{2p}}{(2p)!} \int_0^1 B_{2p}(t) \log(\sin \pi t) \mathrm{d}t \quad (p = 1, 2, \ldots),$$

where $B_{2p}(t)$ is a Bernoulli polynomial of degree $2p$. As a consequence of this we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} = (-1)^p \frac{(2\pi)^{2p}}{(2p)!} 2 \left[\sum_{k=0}^{p} \binom{2p}{2k} B_{2p-2k}\left(\frac{1}{2}\right) b_{2k} \right],$$

where $b_{2k} = \int_0^{\frac{1}{2}} t^{2k} \log(\cos \pi t) \mathrm{d}t$, $k = 0, 1, \ldots, p$.

Introduction.

We will show a representation of $\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}}$ $(p = 1, 2, \ldots)$ as follows.

Theorem.

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} = (-1)^p \frac{(2\pi)^{2p}}{(2p)!} \int_0^1 B_{2p}(t) \log(\sin \pi t) \mathrm{d}t,$$

where $B_{2p}(t)$ is a Bernoulli polynominal of degree $2p$. As a consequence of this, $\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}}$ can be expressed in terms of the even power moments of $\log(\cos \pi t)$ *over the interval* $[0, 1/2]$. *Namely, let*

$$b_p = \int_0^{\frac{1}{2}} t^p \log(\cos \pi t) \mathrm{d}t \quad (p = 0, 1, 2, \ldots),$$

then our representation is

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} = (-1)^p \frac{(2\pi)^{2p}}{(2p)!} 2 \left[\sum_{k=0}^{p} \binom{2p}{2k} B_{2p-2k}\left(\frac{1}{2}\right) b_{2k} \right].$$

For example, we have for $p = 1, 2, 3$

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$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{(2\pi)^2}{2!}\left[-\frac{1}{12}\log 2-2b_2\right] \tag{1)}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^5}=\frac{(2\pi)^4}{4!}\left[-\frac{7}{240}\log 2-b_2+2b_4\right] \tag{2)}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^7}=\frac{(2\pi)^6}{6!}\left[-\frac{31}{1344}\log 2-\frac{7}{8}b_2+\frac{5}{2}b_4-2b_6\right]. \tag{3)}$$

We will see that 1) above is equivalent to

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{2\pi^2}{7}[-\log 2-8b_1] \tag{1$'$)}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{2\pi^2}{7}\left[\log 2+8\int_0^{\frac{1}{2}}t\log(\sin\pi t)\mathrm{d}t\right], \tag{1$''$)}$$

where the last 1$''$) is essentially the same as the one which can be found on page 150 of Euler's work [**1**]. See also page 233 of [**2**].

1. Bernoulli polynomials.

Let B_p $(p=1,2,\dots)$ be Bernoulli numbers; $B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$, $B_3=\frac{1}{42}$, ... etc. And let $B_p(x)$ $(p=0,1,\dots)$ be Bernoulli polynomials; $B_0(x)=1$, $B_1(x)=x-\frac{1}{2}$, $B_2(x)=x^2-x+\frac{1}{6}$, $B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x$, ... etc. In general, $B_p(x)$ is a polynomial of degree p with rational coefficients involving Bernoulli numbers;

$$B_p(x)=x^p-\frac{p}{2}x^{p-1}+\sum_{k=1}^{\left[\frac{p}{2}\right]}(-1)^{k-1}\binom{p}{2k}B_kx^{p-2k}, \tag{1}$$

where $\left[\frac{p}{2}\right]$ is the integer part of $\frac{p}{2}$. The definitions and the fundamental properties of Bernoulli numbers and Bernoulli polynomials should be referred to any suitable textbook, see [**3**] for instance. Our customary use of notations B_p and $B_p(x)$ is slightly confusing: one is for numbers and the other one is for functions. However, we consistently use a parenthesis ( ) with a variable inside for Bernoulli polynomials.

Fundamental properties of Bernoulli polynomials consist of the following (2), (3) and (4), see [**3**].

$$B_p(1+x)=B_p(x)+px^{p-1}\quad (p=1,2,\dots) \tag{2}$$

$$B_p(1-x)=(-1)^pB_p(x)\quad (p=1,2,\dots) \tag{3}$$

$$B_p'(x)=pB_{p-1}(x),\quad B_p'(x)\text{ is the derivative of }B_p(x). \tag{4}$$

We list all properties of Bernoulli polynomials which will be used for our later arguments. Since these properties (5)∼(8) are easily derived from (1)∼(4) above, their proofs are omitted.

$$B_p(0) = B_p(1) \text{ for all } p \geq 2. \text{ Especially } B_{2p+1}(0) = B_{2p+1}(1) = 0 \text{ and}$$
$$B_{2p}(0) = B_{2p}(1) = (-1)^{p-1} B_p \ (p = 1, 2, \ldots) \tag{5}$$

$B_{2p}(x)$ is an even function and $B_{2p-1}(x)$ is odd with respect to $x = \frac{1}{2}$. More precisely, the expansion of $B_p(x)$ around $x = \frac{1}{2}$ is given by

$$B_{2p}(x) = \sum_{k=0}^{p} \binom{2p}{2k} B_{2p-2k}\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^{2k}$$
$$B_{2p+1}(x) = \sum_{k=0}^{p} \binom{2p+1}{2k+1} B_{2p-2k}\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^{2k+1} \quad (p = 1, 2, \ldots) \tag{6}$$

$$\int_0^1 B_p(x)\mathrm{d}x = 0 \quad (p = 1, 2, \ldots) \tag{7}$$

$$\int_0^1 B_p(x)B_1(x)\mathrm{d}x = \frac{1}{p+1} B_{p+1}(0) \quad (p = 1, 2, \ldots), \quad \text{hence}$$
$$\int_0^1 B_{2p}(x)B_1(x)\mathrm{d}x = 0 \ \text{ and}$$
$$\int_0^1 B_{2p-1}(x)B_1(x)\mathrm{d}x = \frac{(-1)^{p-1}}{2p} B_p \quad (p = 1, 2, \ldots). \tag{8}$$

2. Convolutions and Fourier series.

We restrict $B_p(x)$ onto the interval $[0, 1)$, then extend it over the whole real line with period 1. Thus we have a periodic function on the real line with period 1, which is equal to $B_p(x)$ on the interval $[0, 1)$. We denote this function by $\tilde{B}_p(x)$. It is noticed that $\tilde{B}_p(x)$ for $p \geq 3$ are smooth functions on the real line and $\tilde{B}_1(x)$ and $\tilde{B}_2(x)$ are smooth except for integer points $x = 0, \pm 1, \pm 2, \ldots$.

For any functions $f(x)$ and $g(x)$ of $L^2([0, 1])$, the convolution $f * g(x)$ is defined as usual

$$f * g(x) = \int_0^1 f(x - t)g(t)\mathrm{d}t = \int_0^1 f(t)g(x - t)\mathrm{d}t, \quad 0 \leq x \leq 1.$$

In this integral, $f(x)$ and $g(x)$ are always regarded as periodic functions with period 1 on the real line. About convolutions between Bernoulli polynomials, we have

$$1) \ \ \tilde{B}_p * \tilde{B}_1(x) = \frac{-1}{p+1} \tilde{B}_{p+1}(x) \quad (p = 1, 2, \ldots, \text{ and } 0 \leq x \leq 1)$$

$$2)\quad (\tilde{B}_1 * \underset{p\text{-}times}{\cdots} * \tilde{B}_1)(x) = \frac{(-1)^{p-1}}{p!}\tilde{B}_p(x) \quad (p = 1, 2, \ldots, \text{ and } 0 \le x \le 1). \tag{9}$$

Proof. It is clear that 2) follows from 1) by inductive arguments for p. A proof of 1) goes as follows. Here our notation $\underset{=\!=}{(4)}$ means that (4) implies the equality: =. For any x, $0 \le x \le 1$,

$$\begin{aligned}
\tilde{B}_p * \tilde{B}_1(x) &= \int_0^1 \tilde{B}_p(t)\tilde{B}_1(x-t)dt = \int_0^1 B_p(t)\tilde{B}_1(x-t)\mathrm{d}t \\
&= \int_0^x B_p(t)\Big(x - t - \frac{1}{2}\Big)\mathrm{d}t + \int_x^1 B_p(t)\Big(x - t + \frac{1}{2}\Big)\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\underset{=\!=}{(4)}\ \frac{1}{p+1}B_{p+1}(t)\Big(x - t - \frac{1}{2}\Big)\Big|_{t=0}^{t=x} + \frac{1}{p+1}\int_0^x B_{p+1}(t)\mathrm{d}t \\
&\quad + \frac{1}{p+1}B_{p+1}(t)\Big(x - t + \frac{1}{2}\Big)\Big|_{t=x}^{t=1} + \frac{1}{p+1}\int_x^1 B_{p+1}(t)\mathrm{d}t \\
&\quad = -\frac{1}{p+1}B_{p+1}(x)\cdot\frac{1}{2} - \frac{1}{p+1}B_{p+1}(0)\Big(x - \frac{1}{2}\Big) + \frac{1}{p+1}\int_0^x B_{p+1}(t)\mathrm{d}t \\
&\qquad + \frac{1}{p+1}B_{p+1}(1)\Big(x - \frac{1}{2}\Big) - \frac{1}{p+1}B_{p+1}(x)\cdot\frac{1}{2} + \frac{1}{p+1}\int_x^1 B_{p+1}(t)\mathrm{d}t \\
&\underset{=\!=}{(5)}\ -\frac{1}{p+1}B_{p+1}(x) + \frac{1}{p+1}\int_0^1 B_{p+1}(t)\mathrm{d}t \\
&\underset{=\!=}{(7)}\ -\frac{1}{p+1}B_{p+1}(x).
\end{aligned}$$

This completes the proof. □

A simple calculation shows that the Fourier coefficients of $\tilde{B}_1(x)$ are $\frac{i}{2n\pi}$ $(n = \pm 1, \pm 2, \ldots)$, hence the Fourier series expansion of $\tilde{B}_1(x)$ is given by

$$\tilde{B}_1(x) = \frac{i}{2\pi}\sum_{\substack{-\infty<n<+\infty \\ n\neq 0}} \frac{1}{n}e^{i2n\pi x} \cdots\cdots L^2\text{-convergence on } [0,1].$$

Thus, by applying (9), 2), easily we have Fourier series expansion of $\tilde{B}_p(x)$ as follows.

$$\tilde{B}_p(x) = (-1)^{p-1}p!\Big(\frac{i}{2\pi}\Big)^p \sum_{\substack{-\infty<n<+\infty \\ n\neq 0}} \frac{1}{n^p}e^{i2n\pi x} \quad (p = 1, 2, \ldots). \tag{10}$$

We note that except for $p = 1$, this Fourier series converges uniformly on the interval $[0,1]$, because $\sum_{n-1}^{\infty}\frac{1}{n^p} < +\infty$ for $p > 1$.

3. Analytic parts.

For any function $f(x)$ of $L^2([0,1])$, we define the *analytic part* of $f(x)$, denoted by $f^+(x)$, as follows

$$f^+(x) = \sum_{n=0}^{+\infty} \hat{f}(n) e^{i2n\pi x} \cdots\cdots L^2\text{-convergence on } [0,1],$$

where $\hat{f}(n)$ $(n = 0, 1, 2, \ldots)$ are the n-th Fourier coefficients of $f(x)$. The analytic part of $\tilde{B}_p(x)$ is easily obtained from (10),

$$\tilde{B}_p^+(x) = (-1)^{p-1} p! \left(\frac{i}{2\pi}\right)^p \sum_{n=1}^{\infty} \frac{1}{n^p} e^{i2n\pi x} \quad (p = 1, 2, \ldots). \tag{11}$$

Note again that this series converges uniformly on $[0,1]$ except for $p = 1$, and for $p = 1$ we have only L^2-convergence on $[0,1]$.

In the rest of this section we discuss a more concrete expression of $\tilde{B}_1^+(x)$. Let $\log(z)$ be the principal value of the log-function for complex numbers $z = re^{i\theta}$ of $0 < r$ and $-\pi < \theta < \pi$;

$$\log(z) = \log|z| + i\operatorname{Arg} z = \log r + i\theta.$$

Then the function $\log(1-z)$ is analytic on the whole complex plane except for $z =$ real numbers ≥ 1, and its power series expansion around 0 is given by

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{1}{n} z^n \quad \text{for all } |z| < 1.$$

This expansion holds actually for all $|z| \leq 1$ except for $z = 1$, and one can say a little more. Let D_ε with $0 < \varepsilon < \frac{1}{2}$ be a closed sectorial domain given by $\{z = re^{i2\pi\theta} | 0 \leq r \leq 1 \text{ and } \varepsilon \leq \theta \leq 1 - \varepsilon\}$, then we have

The power series $\sum_{n=1}^{\infty} \frac{1}{n} z^n$ converges uniformly to $-\log(1-z)$ on D_ϵ for all $0 < \varepsilon < \frac{1}{2}$. (12)

This fact is probably well known. Since $\log(1-z) + \sum_{n=1}^{N} \frac{1}{n} z^n = \int_0^z \frac{w^N}{1-w} dw$, a proof can be done simply by estimating $\left| \int_0^z \frac{w^N}{1-w} dw \right|$. We omit its detail.

Now we have a concrete representation of $\tilde{B}_1^+(x)$ as follows:

$$\tilde{B}_1^+(x) = \frac{1}{2}\left(x - \frac{1}{2}\right) - \frac{i}{2\pi} \log(2\sin\pi x) \quad \text{for all} \quad 0 < x < 1. \tag{13}$$

Proof. From (11), our series $\sum_{n=1}^{\infty} \frac{1}{n} e^{i2n\pi x}$ converges to $\frac{2\pi}{i} \tilde{B}_1^+(x)$ in a sense of L^2-convergence on $[0,1]$. On the other hand, from (12) the same series converges uniformly to $-\log(1-e^{i2\pi x})$ on every closed subinterval of the open interval $(0,1)$. Since the latter convergence is stronger than the former convergence, we have

$$\tilde{B}_1^+(x) = \frac{-i}{2\pi}\log(1-e^{i2\pi x}) \quad \text{for all } 0 < x < 1.$$

By the definition of $\log z$, we have

$$\log(1-e^{i2\pi x}) = \log|1-e^{i2\pi x}| + i\,\mathrm{Arg}(1-e^{i2\pi x}) = \log(2\sin\pi x) + i\left(x-\frac{1}{2}\right)\pi.$$

Thus we have $\tilde{B}_1^+(x) = \frac{1}{2}\left(x-\frac{1}{2}\right) - \frac{i}{2\pi}\log(2\sin\pi x)$ for all $0 < x < 1$. This completes the proof. $\square$

4. Integral representations.

Since the Fourier coefficients of convolution $f * g(x)$ are the product of Fourier coefficients of f and g; $\widehat{f*g}(n) = \hat{f}(n)\cdot\hat{g}(n)$ $(n = 0, \pm1, \pm2, \ldots)$, it can be seen easily

$$(f*g)^+(x) = f^+ * g(x) = f * g^+(x).$$

By applying this to (9), 1) we have

$$\tilde{B}_p^+(x) = -p\tilde{B}_{p-1} * \tilde{B}_1^+(x) \quad \text{for all } p \geq 2 \text{ and } 0 \leq x \leq 1. \tag{14}$$

This form can be changed slightly to an equivalent one as follows:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^p}e^{i2n\pi x} &= (-1)^{p-1}\frac{1}{p!}\left(\frac{2\pi}{i}\right)^p\tilde{B}_p^+(x) = i^p\frac{(2\pi)^p}{(p-1)!}\tilde{B}_{p-1} * \tilde{B}_1^+(x)\\ &= i^p\frac{(2\pi)^p}{(p-1)!}\int_0^1\tilde{B}_{p-1}(x-t)\left[\frac{1}{2}\left(t-\frac{1}{2}\right) - \frac{i}{2\pi}\log(2\sin\pi t)\right]\mathrm{d}t.\end{aligned}$$

Note $\int_0^1\tilde{B}_{p-1}(x-t)\mathrm{d}t = \int_0^1 B_{p-1}(t)\mathrm{d}t = 0$, see (7), hence $\log(2\sin\pi t)$ can be replaced by $\log(\sin\pi t)$ in the integration above.

$$= i^p\frac{(2\pi)^p}{(p-1)!}\int_0^1\tilde{B}_{p-1}(x-t)\left[\frac{1}{2}\left(t-\frac{1}{2}\right) - \frac{i}{2\pi}\log(\sin\pi t)\right]\mathrm{d}t.$$

Thus we have the following (15), which is one of our main results.

$$\sum_{n=1}^{\infty}\frac{1}{n^p}e^{i2n\pi x} = i^p\frac{(2\pi)^p}{(p-1)!}\int_0^1\tilde{B}_{p-1}(x-t)\left[\frac{1}{2}\left(t-\frac{1}{2}\right) - \frac{i}{2\pi}\log(\sin\pi t)\right]\mathrm{d}t \tag{15}$$

for all $p \geq 2$ and $0 \leq x \leq 1$. As a consequence of this we have the following which includes the well known Euler's results for even p.

$$1)\quad \sum_{n=1}^{\infty} \frac{1}{n^{2p}} = \frac{(2\pi)^{2p}}{(2p)!}\frac{B_p}{2} \quad (p = 1, 2, \ldots)$$

$$2)\quad \sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} = (-1)^p \frac{(2\pi)^{2p}}{(2p)!} \int_0^1 B_{2p}(t)\log(\sin \pi t)\mathrm{d}t \quad (p = 1, 2, \ldots). \tag{16}$$

Proof. For 1), by setting $x = 0$ in (15) we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p}} = (-1)^p \frac{(2\pi)^{2p}}{(2p-1)!} \int_0^1 \tilde{B}_{2p-1}(-t)\left[\frac{1}{2}\left(t - \frac{1}{2}\right)\right]\mathrm{d}t.$$

Note

$$\begin{aligned}\int_0^1 \tilde{B}_{2p-1}(-t)\left(t - \frac{1}{2}\right)\mathrm{d}t &= \int_0^1 B_{2p-1}(1-t)\left(t - \frac{1}{2}\right)\mathrm{d}t = \int_0^1 B_{2p-1}(x)\left(\frac{1}{2} - x\right)\mathrm{d}x \\ &= -\int_0^1 B_{2p-1}(x)B_1(x)\mathrm{d}x = \frac{(-1)^p}{2p}B_p,\end{aligned}$$

see (8). Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p}} = (-1)^p \frac{(2\pi)^{2p}}{(2p-1)!}\frac{1}{2}\frac{(-1)^p}{2p}B_p = \frac{(2\pi)^{2p}}{(2p)!}\frac{B_p}{2}.$$

For 2), by setting $x = 0$ again in (15)

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} &= (i)^{2p+1}\frac{(2\pi)^{2p+1}}{(2p)!}\frac{-i}{2\pi}\int_0^1 \tilde{B}_{2p}(-t)\log(\sin \pi t)\mathrm{d}t \\ &= (-1)^p \frac{(2\pi)^{2p}}{(2p)!}\int_0^1 \tilde{B}_{2p}(-t)\log(\sin \pi t)\mathrm{d}t.\end{aligned}$$

Note

$$\int_0^1 \tilde{B}_{2p}(-t)\log(\sin \pi t)\mathrm{d}t = \int_0^1 B_{2p}(1-t)\log(\sin \pi t)\mathrm{d}t = \int_0^1 B_{2p}(t)\log(\sin \pi t)\mathrm{d}t.$$

Thus we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} = (-1)^p \frac{(2\pi)^{2p}}{(2p)!}\int_0^1 B_{2p}(t)\log(\sin \pi t)\mathrm{d}t.$$

This completes the proof. $\square$

The integral representation; (16), 2) above enables us to express $\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}}$ in terms of the even power moments of $\log(\cos \pi t)$ over the interval $\left[0, \frac{1}{2}\right]$.

$$B_{2p}(t) = \sum_{k=0}^{p} \binom{2p}{2k} B_{2p-2k}\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right)^{2k}$$

see (6) and

$$\int_0^1 \left(t - \frac{1}{2}\right)^{2k} \log(\sin \pi t)\mathrm{d}t = 2\int_0^{\frac{1}{2}} t^{2k} \log(\cos \pi t)\mathrm{d}t.$$

Thus we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} = (-1)^p \frac{(2\pi)^{2p}}{(2p)!} 2\sum_{k=0}^{p} \binom{2p}{2k} B_{2p-2k}\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} t^{2k} \log(\cos \pi t)\mathrm{d}t.$$

By denoting $b_p = \int_0^{\frac{1}{2}} t^p \log(\cos \pi t)\mathrm{d}t$ $(p = 0, 1, 2, \ldots)$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} = (-1)^p \frac{(2\pi)^{2p}}{(2p)!} 2\sum_{k=0}^{p} \binom{2p}{2k} B_{2p-2k}\left(\frac{1}{2}\right) b_{2k}. \tag{17}$$

For examples of $p = 1, 2, 3$ we have

$$1)\quad \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{(2\pi)^2}{2!}\left[-\frac{1}{12}\log 2 - 2b_2\right]$$

$$2)\quad \sum_{n=1}^{\infty} \frac{1}{n^5} = \frac{(2\pi)^4}{4!}\left[-\frac{7}{240}\log 2 - b_2 + 2b_4\right]$$

$$3)\quad \sum_{n=1}^{\infty} \frac{1}{n^7} = \frac{(2\pi)^6}{6!}\left[-\frac{31}{1344}\log 2 + \frac{7}{8}b_2 + \frac{5}{2}b_4 - 2b_6\right],$$

here we used $B_2\left(\frac{1}{2}\right) = -\frac{1}{12}$, $B_4\left(\frac{1}{2}\right) = \frac{7}{240}$, $B_6\left(\frac{1}{2}\right) = -\frac{31}{1344}$ and $b_0 = -\frac{1}{2}\log 2$, the last one will be shown later, see (19), 1).

5. Power moment sequences.

Denote $a_p = \int_0^{\frac{1}{2}} t^p \log(\sin \pi t)\mathrm{d}t$ and $b_p = \int_0^{\frac{1}{2}} t^p \log(\cos \pi t)\mathrm{d}t$ $(p = 0, 1, 2, \ldots)$.

$$1)\quad a_p = \frac{1}{2^p}\sum_{k=0}^{p} \binom{p}{k}(-1)^k 2^k b_k \quad (p = 1, 2, \ldots)$$

$$2)\quad a_p + b_p = \frac{1}{2^{p+1} - 1}\left[-\frac{1}{p+1}\log 2 + \frac{1}{2^p}\sum_{k=0}^{p-1} \binom{p}{k} 2^k b_k\right] \quad (p = 1, 2, \ldots). \tag{18}$$

PROOF. For 1),

$$a_p = \int_0^{\frac{1}{2}} t^p \log(\sin \pi t)\mathrm{d}t = \int_0^{\frac{1}{2}} \left(\frac{1}{2} - t\right)^p \log(\cos \pi t)\mathrm{d}t$$
$$= \sum_{k=0}^{p} \binom{p}{k} \left(\frac{1}{2}\right)^{p-k} \int_0^{\frac{1}{2}} (-t)^k \log(\cos \pi t)\mathrm{d}t = \frac{1}{2^p} \sum_{k=0}^{p} \binom{p}{k} (-1)^k 2^k b_k.$$

For 2),

$$a_p + b_p = \int_0^{\frac{1}{2}} t^p [\log(\sin \pi t) + \log(\cos \pi t)]\mathrm{d}t = \int_0^{\frac{1}{2}} t^p \log\left(\frac{\sin 2\pi t}{2}\right)\mathrm{d}t$$
$$= \int_0^{\frac{1}{2}} t^p \log(\sin 2\pi t)\mathrm{d}t - \log 2 \int_0^{\frac{1}{2}} t^p \mathrm{d}t$$
$$= \frac{1}{2^{p+1}} \int_0^{1} t^p \log(\sin \pi t)\mathrm{d}t - \frac{1}{p+1}\left(\frac{1}{2}\right)^{p+1} \log 2$$

and

$$\int_0^1 t^p \log(\sin \pi t)\mathrm{d}t = \int_0^{\frac{1}{2}} t^p \log(\sin \pi t)\mathrm{d}t + \int_{\frac{1}{2}}^1 t^p \log(\sin \pi t)\mathrm{d}t$$
$$= a_p + \int_0^{\frac{1}{2}} \left(t + \frac{1}{2}\right)^p \log(\cos \pi t)\mathrm{d}t = a_p + \frac{1}{2^p} \sum_{k=0}^{p} \binom{p}{k} 2^k b_k.$$

Thus

$$a_p + b_p = \left(\frac{1}{2}\right)^{p+1} \left[a_p + b_p + \frac{1}{2^p} \sum_{k=0}^{p-1} \binom{p}{k} 2^k b_k\right] - \frac{1}{p+1}\left(\frac{1}{2}\right)^{p+1} \log 2,$$

and we have

$$\left(1 - \frac{1}{2^{p+1}}\right)(a_p + b_p) = \frac{1}{2^{2p+1}} \sum_{k=0}^{p-1} \binom{p}{k} 2^k b_k - \frac{1}{p+1}\left(\frac{1}{2}\right)^{p+1} \log 2$$

Hence,

$$a_p + b_p = \frac{1}{2^{p+1} - 1}\left[\frac{1}{2^p} \sum_{k=0}^{p-1} \binom{p}{k} 2^k b_k - \frac{1}{p+1} \log 2\right].$$

This completes the proof. □

$$\begin{aligned}&1)\quad a_0=b_0=-\frac{1}{2}\log 2\\ &2)\quad a_1+b_1=-\frac{1}{4}\log 2\\ &3)\quad b_2=\frac{5}{168}\log 2+\frac{4}{7}b_1.\end{aligned}\tag{19}$$

Proof. For 1),

$$\begin{aligned}a_0+b_0&=\int_0^{\frac{1}{2}}\log\left(\frac{\sin 2\pi t}{2}\right)\mathrm{d}t=\frac{1}{2}\int_0^1\log(\sin\pi t)\mathrm{d}t-\frac{1}{2}\log 2\\ &=\frac{1}{2}\cdot 2\int_0^{\frac{1}{2}}\log(\sin\pi t)\mathrm{d}t-\frac{1}{2}\log 2=a_0-\frac{1}{2}\log 2,\end{aligned}$$

hence $b_0=-\frac{1}{2}\log 2$, and

$$a_0=\int_0^{\frac{1}{2}}\log(\sin\pi t)\mathrm{d}t=\int_0^{\frac{1}{2}}\log(\cos\pi t)\mathrm{d}t=b_0.$$

For 2), by setting $p=1$ in (18), 2) we have

$$a_1+b_1=\frac{1}{4-1}\Big(-\frac{1}{2}\log 2+\frac{1}{2}b_0\Big)=\frac{1}{3}\Big(-\frac{1}{2}\log 2-\frac{1}{4}\log 2\Big)=-\frac{1}{4}\log 2.$$

For 3), by setting $p=2$ in (18), 2) we have

$$a_2+b_2=\frac{1}{7}\Big(-\frac{1}{3}\log 2+\frac{1}{4}b_0+b_1\Big)=\frac{1}{7}\Big(-\frac{11}{24}\log 2+b_1\Big).$$

On the other hand, we have from (18), 1)

$$a_2=b_2-\frac{1}{8}\log 2-b_1,\ \text{thus}\ a_2-b_2=-\frac{1}{8}\log 2-b_1.$$

By canceling a_2, we have $b_2=\frac{5}{168}\log 2+\frac{4}{7}b_1$. This completes the proof. □

$\sum_{n=1}^{\infty}\frac{1}{n^3}$ can be expressed in three different ways,

$$\begin{aligned}&1)\quad 2\pi^2\Big(-\frac{1}{12}\log 2-2b_2\Big)\\ &2)\quad \frac{2\pi^2}{7}(-\log 2-8b_1)\\ &3)\quad \frac{2\pi^2}{7}(\log 2+8a_1).\end{aligned}\tag{20}$$

Because 1) was proved in (17), 1), we have 2) by substituting $b_2 = \frac{5}{168}\log 2 + \frac{4}{7}b_1$ into 1). We have 3) by substituting $b_1 = -a_1 - \frac{1}{4}\log 2$ into 2). The last one above, namely

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{2\pi^2}{7}\left[\log 2 + 8\int_0^{\frac{1}{2}} t\log(\sin \pi t)\right] \mathrm{d}t,$$

can be found in Euler's work [**1**]. Euler's expression given on page 150 of [**1**] is slightly different but essentially the same as ours. The author owes this information to a commentary given on page 233 of a book [**2**].

Finally we add a few remarks. We give here only statements without detailed proofs.

1) Every even power moment b_{2p} can be expressed as a linear combination of $\log 2$ and the odd power moments $b_1, b_3, \ldots$ up to b_{2p-1} with rational coefficients. This generalization of (19), 3) is proved in a similar way by using (18), 1) and 2).

In (17), we have seen that $\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}}$ can be expressed in terms of the even power moments. One can have a similar expression with respect to the odd power moments as follows.

2) There are rational numbers $\alpha_{p,k}$, $k = 0, 1, \ldots, p$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p+1}} = \pi^{2p}\left[\alpha_{p,0}\log 2 + \sum_{k=1}^{p} \alpha_{p,k} b_{2k-1}\right] \quad (p = 1, 2, \ldots).$$

A question whether $\log 2$ and the odd power moments, $b_1, b_3, \ldots$ are linearly independent over the rational number field is left open. For the even power moments, one can ask the same question which is also not answered.

References

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[2] S. Kurokawa, M. Wakayama and T. Momotani, Translators' Commentaries, In: W. Dunham, Euler, The Master of Us All, Springer-Verlag, Tokyo, 2004, pp. 231–250.

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