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Spherical means and Riesz decomposition for superbiharmonic functions

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Abstract. The aim in this note is to discuss the behavior at infinity for superbiharmonic functions on $\boldsymbol{R}^n$ by use of spherical means.

1. Introduction.

A function u on an open set $\Omega \subset \boldsymbol{R}^n$, $n \geq 2$, is called biharmonic if $\Delta^2 u = 0$ on Ω, where Δ^2 denotes the Laplace operator iterated two times. We say that a locally integrable function u on Ω is superbiharmonic in Ω if $\Delta^2 u$ is a nonnegative measure on Ω, that is,

$$\int_\Omega u(x)\Delta^2\varphi(x)\, dx \geq 0 \qquad \text{for all nonnegative } \varphi \in C_0^\infty(\Omega).$$

We denote by $\mathscr{H}(\Omega)$ and $\mathscr{H}^2(\Omega)$ the space of harmonic functions on Ω and the space of biharmonic functions on Ω, respectively. Further, we denote by $\mathscr{S}\mathscr{H}(\Omega)$ and $\mathscr{S}\mathscr{H}^2(\Omega)$ the space of superharmonic functions on Ω and the space of superbiharmonic functions on Ω, respectively.

For a multi-index $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and a point $x = (x_1, x_2, \dots, x_n)$, we set

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$
$$\lambda! = \lambda_1!\lambda_2!\cdots\lambda_n!,$$
$$x^\lambda = {x_1}^{\lambda_1}{x_2}^{\lambda_2}\cdots{x_n}^{\lambda_n}$$

and

$$D^\lambda = \left(\frac{\partial}{\partial x}\right)^\lambda = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1}\left(\frac{\partial}{\partial x_2}\right)^{\lambda_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\lambda_n}.$$

Consider the Riesz kernel of order $2m$ defined by

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$$R_{2m}(x) = \begin{cases} |x|^{2m-n} & \text{if } 2m < n \text{ or } 2m-n \text{ is a positive odd integer} \\ |x|^{2m-n}\log(1/|x|) & \text{if } 2m \geq n \text{ and } n \text{ is even} \end{cases}$$

and its remainder term of Taylor's expansion

$$R_{2m,L}(x,y) = \begin{cases} R_{2m}(x-y) - \sum_{|\lambda| \leq L} \frac{x^\lambda}{\lambda!}(D^\lambda R_{2m})(-y) & \text{if } |y| \geq 1, \\ R_{2m}(x-y) & \text{if } |y| < 1, \end{cases}$$

where L is a nonnegative integer (cf. Hayman-Kennedy [**3**] and the second author [**4**]). Here note that $R_4 \in \mathscr{H}^2(\boldsymbol{R}^n \setminus \{0\})$ and

$$\Delta^2 R_4 = c_n^{-1}\delta_0$$

with the Dirac measure δ_x at x and

$$c_n^{-1} = \sigma_n \times \begin{cases} -4 & \text{when } n = 2, \\ -2 & \text{when } n = 3, \\ 4 & \text{when } n = 4, \\ 2(4-n)(2-n) & \text{when } n \geq 5, \end{cases}$$

where σ_n denotes the area of the unit sphere $S(0,1)$.

Let $u \in \mathscr{S}\mathscr{H}^2(\boldsymbol{R}^n)$ and $\mu = \Delta^2 u$. Then we see that for every $r > 0$, u is of the form

$$u(x) = c_n \int_{B(0,r)} R_4(x-y)\, d\mu(y) + h_r(x) \tag{1.1}$$

on $B(0,r)$, where $h_r \in \mathscr{H}^2(B(0,r))$. This implies that $u(x)/c_n$ is considered to be lower semicontinuous on $\boldsymbol{R}^n$.

We denote by $B(x,r)$ the open ball centered at x of radius r, whose boundary is written as $S(x,r)$. For a Borel measurable function u, we define the spherical mean

$$M(r,u) = \frac{1}{\sigma_n r^{n-1}} \int_{S(0,r)} u(x)\, dS.$$

Recently Premalatha [**5**] has proved that for a superharmonic function u on $\boldsymbol{R}^2$, $M(r^2,u) - 2M(r,u)$ is bounded when $r > 1$ if and only if u is the sum of a logarithmic potential and a harmonic function. Our aim in this note is to extend his result to superbiharmonic functions. Before giving our results, we note from Almansi expansion (see [**1**] and [**4**]) that if u is biharmonic in $\boldsymbol{R}^n$, then

$$M(r,u) = ar^2 + b \tag{1.2}$$

for some constants a and b, so that

$$M(2r,u) - 4M(r,u) = -3b = -3u(0).$$

Further, in view of (1.1), if $u \in \mathscr{S}\mathscr{H}^2(\boldsymbol{R}^n)$, then $M(r,u)$ can be defined and will be shown soon to be finite.

Now we show our results.

THEOREM 1.1. *Let $n \leq 4$, $u \in \mathscr{S}\mathscr{H}^2(\boldsymbol{R}^n)$ and $\mu = \Delta^2 u$. Then $M(2r,u) - 4M(r,u)$ is bounded when $r > 1$ if and only if $u \in \mathscr{H}^2(\boldsymbol{R}^n)$.*

THEOREM 1.2. *Let $n \geq 5$, $u \in \mathscr{S}\mathscr{H}^2(\boldsymbol{R}^n)$ and $\mu = \Delta^2 u$. Then $M(2r,u) - 4M(r,u)$ is bounded when $r > 1$ if and only if u is of the form*

$$u(x) = c_n \int R_4(x-y)\, d\mu(y) + h(x),$$

where $h \in \mathscr{H}^2(\boldsymbol{R}^n)$ and

$$\int (1+|y|)^{4-n}\, d\mu(y) < \infty. \tag{1.3}$$

REMARK 1.3. Note that (1.3) is equivalent to

$$R_4\mu(x) = \int R_4(x-y)\, d\mu(y) \not\equiv \infty$$

(see e.g. [**3**] or [**4**]).

Finally, by applications of the methods used in the proofs of our theorems, we discuss the Riesz decomposition theorem for superharmonic functions, as an extension of Premalatha [**5**].

2. Fundamental properties on spherical means.

Let $u \in \mathscr{S}\mathscr{H}^2(\boldsymbol{R}^n)$ and $\mu = \Delta^2 u \geq 0$. Then we see that for every $r > 0$, u is of the form

$$u(x) = c_n \int_B R_{4,2}(x,y)\, d\mu(y) + h_r(x) \tag{2.1}$$

on $B = B(0,r)$, where $h_r \in \mathscr{H}^2(B)$. Note further that

$$\Delta R_{4,2} = C_n R_{2,0}, \tag{2.2}$$

where $C_n = 2(4-n)$ when $n \neq 4$ and $C_n = -2$ when $n = 4$.

LEMMA 2.1. *Let $u \in \mathscr{S}\mathscr{H}^2(\boldsymbol{R}^n)$ and $\mu = \Delta^2 u$. Then for $r > 1$,*

$$M(r,u) = \int_{B(0,r)} f(r,y)\ d\mu(y) + ar^2 + b,$$

where a, b are constants independent of r and

$$f(r,y) = c_n \begin{cases} R_4(r) + (2n)^{-1}\Delta R_4(r)|y|^2 & \text{if } |y| < 1, \\ R_4(r) + (2n)^{-1}\Delta R_4(r)|y|^2 - R_4(y) - (2n)^{-1}\Delta R_4(y) r^2 & \text{if } 1 \leq |y| < r, \\ 0 & \text{if } |y| \geq r, \end{cases}$$

where we set $R_m(r) = R_m(x)$ when $r = |x|$.

PROOF. Let $r_2 > r_1 > 0$. Write u as in (2.1) as follows:

$$u(x) = c_n \int_{B(0,r_i)} R_{4,2}(x,y)\ d\mu(y) + h_{r_i}(x)$$

for $x \in B(0,r_1)$, where h_{r_i} is biharmonic in $B(0,r_i)$ for each $i = 1,2$. Then we have by Fubini's theorem and Almansi expansion

$$M(r,u) = c_n \int_{B(0,r_i)} M\big(r, R_{4,2}(\cdot,y)\big)\ d\mu(y) + a_i r^2 + b_i$$

when $0 < r < r_1 < r_2$. Since $c_n M(r, R_{4,2}(\cdot,y)) = f(r,y)$ by [**2**, Lemma 4.1], we see that

$$M(r,u) = \int_{B(0,r)} f(r,y)\ d\mu(y) + a_i r^2 + b_i.$$

Hence it follows that

$$a_1 r^2 + b_1 = a_2 r^2 + b_2 \qquad \text{for } 0 < r < r_1 < r_2,$$

which implies that $a_1 = a_2$ $(= a)$ and $b_1 = b_2$ $(= b)$. Consequently,

$$M(r,u) = \int_{B(0,r)} f(r,y)\ d\mu(y) + ar^2 + b,$$

as required. □

COROLLARY 2.2. *Let $u \in \mathscr{S}\mathscr{H}^2(\boldsymbol{R}^n)$ and $\mu = \Delta^2 u$. Then there exists a constant b such that*

$$
\begin{aligned}
M(2r,u)-4M(r,u) = &\int_{B(0,1)} \left\{ f(2r,y)-4f(r,y) \right\}\, d\mu(y) \\
&+ \int_{B(0,r)\setminus B(0,1)} \left\{ f(2r,y)-4f(r,y) \right\}\, d\mu(y) \\
&+ \int_{B(0,2r)\setminus B(0,r)} f(2r,y)\, d\mu(y) - 3b
\end{aligned}
$$

for all $r>1$.

3. Proof of Theorem 1.1.

In this section, we give a proof of Theorem 1.1.

3.1. The case $n=2$.

In case $n=2$, $R_4(x)=|x|^2\log(1/|x|)$ and $\Delta R_4(x)=-4(\log|x|+1)$. Hence, by Lemma 2.1, we see that for $|y|<1$,

$$f(r,y)=4\sigma_2\left\{r^2\log r+|y|^2(\log r+1)\right\},$$

so that

$$f(2r,y)-4f(r,y)=4\sigma_2\left\{4r^2\log 2-|y|^2(3-\log 2+3\log r)\right\}>0 \tag{3.1}$$

when $r>1$.

If $1\le |y|<r$, then

$$f(r,y)=4\sigma_2\left\{r^2\log r+|y|^2(\log r+1)-|y|^2\log|y|-r^2(\log|y|+1)\right\}.$$

If we set $|y|=tr$ with $0<t<1$, then

$$f(r,y)=4\sigma_2 r^2\left(t^2-t^2\log t-\log t-1\right)>0;$$

especially, if $r\le |y|<2r$, then

$$f(2r,y)>0. \tag{3.2}$$

Further we have

$$
\begin{aligned}
f(2r,y)-4f(r,y) &= 4\sigma_2\left\{4r^2\log 2-|y|^2(3\log r-\log 2)-3|y|^2+3|y|^2\log|y|\right\} \\
&= 4\sigma_2 r^2\left(4\log 2+t^2\log 2-3t^2+3t^2\log t\right),
\end{aligned}
$$

when $1\le |y|=tr<r$, so that

$$f(2r, y) - 4f(r, y) > cr^2 \tag{3.3}$$

with $c = 4\sigma_2(5\log 2 - 3) > 0$.

Here we prove the following result, which completes the proof in the case $n = 2$.

Lemma 3.1.1. *If $M(2r, u) - 4M(r, u)$ is bounded when $r > 1$, then $\mu = 0$.*

Proof. Suppose $M(2r, u) - 4M(r, u)$ is bounded when $r > 1$. Then we see from (3.1), (3.2) and (3.3) that

$$\int_{B(0,r)\setminus B(0,1)} \{f(2r, y) - 4f(r, y)\}\, d\mu(y)$$

is bounded for $r > 1$. In view of (3.3), we insist that $r^2\mu(B(0, r) \setminus B(0, 1))$ is bounded. In the same way, we see from (3.1) that $r^2\mu(B(0, 1))$ is bounded. Hence it follows that $\mu(\boldsymbol{R}^2) = 0$, as required. $\square$

3.2. The case $n = 3$.

When $n = 3$, $R_4(x) = |x|$ and $\Delta R_4(x) = 2|x|^{-1}$. By lemma 2.1, we see that if $y \in B(0, 1)$, then

$$f(r, y) = -2\sigma_3\big(r + 3^{-1}r^{-1}|y|^2\big),$$

so that

$$f(2r, y) - 4f(r, y) = 2\sigma_3\left(2r + \frac{7}{6}r^{-1}|y|^2\right) > 0 \tag{3.4}$$

for $r > 1$. If $1 \le |y| < r$, then

$$f(r, y) = -2\sigma_3\big(r + 3^{-1}r^{-1}|y|^2 - |y| - 3^{-1}|y|^{-1}r^2\big)$$

and

$$f(2r, y) - 4f(r, y) = 2\sigma_3\left(2r + \frac{7}{6}r^{-1}|y|^2 - 3|y|\right) > \frac{\sigma_3}{3}r. \tag{3.5}$$

If $r \le |y| < 2r$, then, by the above consideration, we have

$$f(2r, y) > 0. \tag{3.6}$$

Lemma 3.2.1. *If $M(2r, u) - 4M(r, u)$ is bounded when $r > 1$, then $\mu = 0$.*

Proof. Suppose $M(2r, u) - 4M(r, u)$ is bounded when $r > 1$. Then we see from (3.4), (3.5) and (3.6) that

$$\int_{B(0,r)\setminus B(0,1)} \{f(2r,y) - 4f(r,y)\}\, d\mu(y)$$

is bounded. It follows from (3.4) and (3.5) that $r\mu(B(0,r))$ is bounded, which implies that $\mu(\boldsymbol{R}^3) = 0$. □

3.3. The case $n = 4$.

In case $n = 4$, $R_4(x) = \log(1/|x|)$ and $\Delta R_4(x) = -2|x|^{-2}$. By lemma 2.1, we see that

$$f(r,y) = 2\sigma_4\Big(-\log r - \frac{1}{4} r^{-2}|y|^2 + \log|y| + \frac{1}{4}|y|^{-2}r^2 \Big) > 0$$

for $1 \le |y| < r$. Here we also obtain

$$f(2r,y) - 4f(r,y) = 2\sigma_4\Big(3\log(r/|y|) + \frac{15}{16} r^{-2}|y|^2 - \log 2\Big) > \frac{9\sigma_4}{4}\log\frac{r}{|y|} > 0 \qquad (3.7)$$

for $1 \le |y| < r$; moreover,

$$f(2r,y) > 0 \qquad (3.8)$$

when $r \le |y| < 2r$.

If $|y| < 1$, then

$$f(2r,y) - 4f(r,y) = 2\sigma_4\Big(\log(r^3/2) + \frac{15}{16} r^{-2}|y|^2\Big) > 0 \qquad (3.9)$$

for $r > \sqrt[3]{2}$.

Lemma 3.3.1. *If $M(2r,u) - 4M(r,u)$ is bounded when $r > 1$, then $\mu = 0$.*

Proof. We note from (3.7), (3.8) and (3.9) that

$$\int_{B(0,r)\setminus B(0,1)} \log(r/|y|)\, d\mu(y)$$

is bounded. Since $\log(r/|y|) \ge \log\sqrt{r}$ when $|y| \le \sqrt{r}$, it follows with the aid of (3.9) that

$$(\log\sqrt{r})\mu\big(B(0,\sqrt{r})\big)$$

is bounded when $r > 1$. This implies that $\mu(\boldsymbol{R}^4) = 0$. □

3.4. Proof of Theorem 1.1.

Now we are ready to prove Theorem 1.1.

Let $2 \le n \le 4$, $u \in \mathscr{S}\mathscr{H}^2(\boldsymbol{R}^n)$ and $\mu = \Delta^2 u$. If $M(2r,u) - 4M(r,u)$ is bounded when $r > 1$, then it follows from Lemmas 3.1.1, 3.2.1 and 3.3.1 that $\mu = 0$. This implies that u is biharmonic in $\boldsymbol{R}^n$.

Conversely, if u is biharmonic in $\boldsymbol{R}^n$, then $M(2r, u) - 4M(r, u)$ is equal to a constant by (1.2).

Thus the proof is completed. $\square$

4. Proof of Theorem 1.2.

Let $n > 4$, $u \in \mathscr{S}\mathscr{H}^2(\boldsymbol{R}^n)$ and $\mu = \Delta^2 u$. In this case, $R_4(x) = |x|^{4-n}$ and $\Delta R_4(x) = 2(4-n)|x|^{2-n}$.

By lemma 2.1, we see that

$$f(r, y) = 2(4-n)(2-n)\sigma_n \big\{ r^{4-n} + n^{-1}(4-n) r^{2-n}|y|^2 - |y|^{4-n} - n^{-1}(4-n)|y|^{2-n} r^2 \big\} > 0$$

when $1 \le |y| < r$. Hence we have

$$\begin{aligned} & f(2r, y) - 4f(r, y) \\ & \quad = 2(4-n)(2-n)\sigma_n \big\{ (2^{4-n} - 4) r^{4-n} + (2^{2-n} - 4)(4-n) n^{-1} r^{2-n} |y|^2 + 3|y|^{4-n} \big\}, \end{aligned}$$

so that

$$f(2r, y) - 4f(r, y) > c|y|^{4-n}, \tag{4.1}$$

where $c = 2(4-n)(2-n)\sigma_n \{3 - 2n^{-1}(4 - 2^{2-n})\} > 0$; if $r \le |y| < 2r$, then

$$f(2r, y) > 0. \tag{4.2}$$

If $|y| < 1$, then

$$\begin{aligned} & f(2r, y) - 4f(r, y) \\ & \quad = 2(4-n)(2-n)\sigma_n \big\{ (2^{4-n} - 4) r^{4-n} + (2^{2-n} - 4)(4-n) n^{-1} r^{2-n} |y|^2 \big\}, \end{aligned}$$

so that we can find $c > 0$ such that

$$\left| \int_{B(0,1)} \{ f(2r, y) - 4f(r, y) \} \, d\mu(y) \right| \le c r^{4-n} \mu(B(0,1)),$$

which tends to zero as $r \to \infty$.

Lemma 4.1. *If $M(2r, u) - 4M(r, u)$ is bounded when $r > 1$, then* (1.3) *holds.*

Proof. Suppose $M(2r, u) - 4M(r, u)$ is bounded when $r > 1$. Then we see from (4.1) and (4.2) that

$$\int_{B(0,r)\setminus B(0,1)} |y|^{4-n}\ d\mu(y)$$

is bounded when $r > 1$, which yields (1.3). □

PROOF OF THEOREM 1.2. Let $n > 4$, $u \in \mathscr{S}\mathscr{H}^2(\boldsymbol{R}^n)$ and $\mu = \Delta^2 u$. If $M(2r,u) - 4M(r,u)$ is bounded when $r > 1$, then we see from Lemma 4.1 that

$$R_4\mu(x) = \int_{\boldsymbol{R}^n} |x-y|^{4-n}\ d\mu(y)$$

is superbiharmonic in $\boldsymbol{R}^n$ and $u(x) - c_n R_4\mu(x)$ is biharmonic in $\boldsymbol{R}^n$.

Conversely, suppose u is of the form

$$u(x) = c_n R_4\mu(x) + h(x),$$

where h is biharmonic in $\boldsymbol{R}^n$ and μ satisfies (1.3). Then

$$\begin{aligned} M(r, c_n R_4\mu) = c_n &\int_{B(0,r)} \left\{r^{4-n} + n^{-1}(4-n)r^{2-n}|y|^2\right\}\ d\mu(y) \\ &+ c_n \int_{\boldsymbol{R}^n\setminus B(0,r)} \left\{|y|^{4-n} + n^{-1}(4-n)|y|^{2-n}r^2\right\}\ d\mu(y) \end{aligned}$$

for $r > 1$. Applying Lebesgue's dominated convergence theorem, we deduce from (1.3) that

$$\lim_{r\to\infty} M(r, R_4\mu) = 0.$$

Thus the proof is completed. □

REMARK 4.2. As was seen above, if $n \geq 5$ and μ is a nonnegative measure on $\boldsymbol{R}^n$ satisfying (1.3), then we have

$$\lim_{r\to\infty} M(r, R_4\mu) = 0.$$

Hence we see that when $n \geq 5$ and $u \in \mathscr{S}\mathscr{H}(\boldsymbol{R}^n)$, $M(2r,u) - 4M(r,u)$ is bounded when $r > 1$ if and only if $M(r,u) - ar^2$ is bounded when $r > 1$ for some constant a.

5. Superharmonic functions.

Let $R_2(x) = \log(1/|x|)$ when $n = 2$ and $R_2(x) = |x|^{2-n}$ when $n > 2$. Recall that

$$R_{2,0}(x,y) = \begin{cases} R_2(x-y) - R_2(-y) & \text{if } |y| \geq 1, \\ R_2(x-y) & \text{if } |y| < 1. \end{cases}$$

Let u be superharmonic in $\boldsymbol{R}^n$ and $\mu = -\Delta u$, where c_n' is chosen so that $\Delta R_2 = (c_n')^{-1}\delta_0$; in fact,

$$(c_n')^{-1} = -\begin{cases} \sigma_2 & \text{when } n = 2, \\ (n-2)\sigma_n & \text{when } n \geq 3. \end{cases}$$

Then we see that for $r > 0$, u is of the form

$$u(x) = -c_n' \int_B R_{2,0}(x,y)\, d\mu(y) + h_r(x) \tag{5.1}$$

on $B = B(0,r)$, where h_r is harmonic in B.

As in Lemma 2.1, we find a constant a such that

$$M(r,u) = -c_n' \int_{B(0,r)} M\big(r, R_{2,0}(\cdot,y)\big)\, d\mu(y) + a \tag{5.2}$$

for $r > 1$.

We here give another proof of Premalatha [**5**].

Theorem 5.1. *Let $u \in \mathscr{S}\mathscr{H}(\boldsymbol{R}^2)$ and $\mu = -\Delta u$. Then $M(r^2,u) - 2M(r,u)$ is bounded when $r > 1$ if and only if u is of the form*

$$u(x) = -c_2' \int \log(1/|x-y|)\, d\mu(y) + h(x),$$

where $h \in \mathscr{H}(\boldsymbol{R}^2)$ and μ satisfies

$$\int_{\boldsymbol{R}^2} \big(\log(1+|y|)\big)\, d\mu(y) < \infty. \tag{5.3}$$

Proof. Let $u \in \mathscr{S}\mathscr{H}(\boldsymbol{R}^2)$ and $\mu = -\Delta u$. If $r > 1$, then (5.2) gives

$$M(r,u) = -c_2'\big(\log(1/r)\big)\mu\big(B(0,1)\big) - c_2' \int_{B(0,r)\setminus B(0,1)} \big(\log(|y|/r)\big)\, d\mu(y) + a$$

for some constant a. Hence we have

$$\begin{aligned} M(r^2,u) - 2M(r,u) = {}& c_2' \int_{B(0,r)\setminus B(0,1)} (\log|y|)\, d\mu(y) \\ & - c_2' \int_{B(0,r^2)\setminus B(0,r)} \big(\log(|y|/r^2)\big)\, d\mu(y) - a. \end{aligned}$$

If $M(r^2,u) - 2M(r,u)$ is bounded when $1 < r < \infty$, then

$$\int_{B(0,r)\setminus B(0,1)} (\log|y|)\ d\mu(y) \qquad \text{is bounded,}$$

so that (5.3) holds. Thus we see that

$$L\mu(x) = \int \log(1/|x-y|)\ d\mu(y) \qquad \text{is superharmonic in } \boldsymbol{R}^2, \tag{5.4}$$

which implies that $u(x) + c_2' L\mu(x)$ is harmonic in $\boldsymbol{R}^2$.

Conversely, if $h(x) = u(x) + c_2' L\mu(x)$ is harmonic in $\boldsymbol{R}^2$, then we have for $r > 1$

$$M(r,u) = -c_2'\big(\log(1/r)\big)\mu\big(B(0,r)\big) - c_2' \int_{\boldsymbol{R}^2\setminus B(0,r)} \big(\log(1/|y|)\big)\ d\mu(y) + h(0),$$

which gives

$$\begin{aligned} M(r^2,u) - 2M(r,u) = &-2c_2' \int_{B(0,r^2)\setminus B(0,r)} \big(\log(|y|/r)\big)\ d\mu(y) \\ &- c_2' \int_{\boldsymbol{R}^2\setminus B(0,r^2)} (\log|y|)\ d\mu(y) - h(0). \end{aligned}$$

Thus it follows from (5.3) that $M(r^2,u) - 2M(r,u)$ tends to $-h(0)$ as $r \to \infty$ by Lebesgue's dominated convergence theorem. □

Remark 5.2. If $L\mu(x)$ is superharmonic in $\boldsymbol{R}^2$, then

$$\lim_{r\to\infty} \big\{M(r^2, L\mu) - 2M(r, L\mu)\big\} = 0.$$

Theorem 5.3. *Let $n > 2$, $u \in \mathscr{S}\mathscr{H}(\boldsymbol{R}^n)$ and $\mu = -\Delta u$. Then $M(2r,u) - 2^{2-n}M(r,u)$ is bounded when $r > 1$ if and only if u is of the form*

$$u(x) = -c_n' R_2\mu(x) + h(x),$$

where $R_2\mu(x) = \int |x-y|^{2-n}\ d\mu(y)$, $h \in \mathscr{H}(\boldsymbol{R}^n)$ and μ satisfies

$$\int_{\boldsymbol{R}^n} (1+|y|)^{2-n}\ d\mu(y) < \infty. \tag{5.5}$$

Proof. Let $n > 2$, $u \in \mathscr{S}\mathscr{H}(\boldsymbol{R}^n)$ and $\mu = -\Delta u$. If $r > 1$, then (5.2) yields

$$M(r,u) = -c_n' r^{2-n}\mu\big(B(0,1)\big) - c_n' \int_{B(0,r)\setminus B(0,1)} (r^{2-n} - |y|^{2-n})\ d\mu(y) + a$$

for some constant a. Hence we find

$$M(2r,u) - 2^{2-n}M(r,u) = (1-2^{2-n})c'_n \int_{B(0,r)\setminus B(0,1)} |y|^{2-n}\ d\mu(y)$$
$$- c'_n \int_{B(0,2r)\setminus B(0,r)} \left((2r)^{2-n} - |y|^{2-n}\right)\ d\mu(y) + (1-2^{2-n})a.$$

If $M(2r,u) - 2^{2-n}M(r,u)$ is bounded when $r > 1$, then it follows that

$$\int_{B(0,r)\setminus B(0,1)} |y|^{2-n}\ d\mu(y) \qquad \text{is bounded,}$$

which implies (5.5). Consequently, we see that $R_2\mu(x)$ is superharmonic in $\boldsymbol{R}^n$ and $u(x) + c'_n R_2\mu(x)$ is harmonic in $\boldsymbol{R}^n$.

Conversely, if $h(x) = u(x) + c'_n R_2\mu(x)$ is harmonic in $\boldsymbol{R}^n$, then

$$M(r,u) = -c'_n \int_{B(0,r)} r^{2-n}\ d\mu(y) - c'_n \int_{\boldsymbol{R}^n\setminus B(0,r)} |y|^{2-n}\ d\mu(y) + h(0).$$

It follows from (5.5) that $M(r,u)$ tends to $h(0)$ as $r \to \infty$ by Lebesgue's dominated convergence theorem. □

Remark 5.4. If $n > 2$ and $R_2\mu$ is superharmonic in $\boldsymbol{R}^n$, then

$$\lim_{r\to\infty} M(r, R_2\mu) = 0.$$

Hence we see that when $n > 2$ and $u \in \mathscr{S}\mathscr{H}(\boldsymbol{R}^n)$, $M(r,u)$ is bounded when $r > 1$ if and only if $M(2r,u) - 2^{2-n}M(r,u)$ is bounded when $r > 1$.

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