

The category of reduced orbifolds in local charts

By Anke D. POHL

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Abstract. It is well-known that reduced smooth orbifolds and proper effective foliation Lie groupoids form equivalent categories. However, for certain recent lines of research, equivalence of categories is not sufficient. We propose a notion of maps between reduced smooth orbifolds and a definition of a category in terms of marked proper effective étale Lie groupoids such that the arising category of orbifolds is isomorphic (not only equivalent) to this groupoid category.

1. Introduction.

Given a reduced orbifold (in local charts) and an orbifold atlas representing its orbifold structure it is well known how to construct (in an explicit way) a proper effective foliation groupoid (orbifold groupoid) from these data (see, e. g., Haefliger [4] or the book by Moerdijk and Mrčun [8]). Over the years various authors (in particular, Moerdijk [7], Pronk [10]) used this link to provide a definition of a category of orbifolds by proposing a definition of a category of orbifold groupoids, either as a 2-category or as a bicategory of fractions. Lerman [5] provides a very good discussion of these approaches. These approaches have in common that the morphisms in the orbifold category are only given implicitly.

While the notion of isomorphisms between orbifolds is clear, that of more general maps allows variations. Various propositions for orbifold maps in local charts have been made serving different purposes, e. g., by Borzellino and Brunsden [2] or the Chen–Ruan good maps [3]. Unfortunately, neither of these definitions model the morphisms coming from the groupoid category. We remark that Luperchio and Uribe [6], even though widely believed, do not prove that Chen–Ruan good maps correspond to groupoid homomorphisms (a counterexample to the characterization of groupoid homomorphisms via good maps and thus to [6, Proposition 5.1.7] is provided by the unique map between any orbifold with at least two charts in any orbifold atlas and the 0-dimensional connected reduced orbifold).

All these proposed groupoid categories are just equivalent, not isomorphic, to the orbifold category. This is caused by the fact that the construction mentioned above assigns the same groupoid to different (but isomorphic) orbifolds, and conversely various (Morita equivalent) groupoids to the same orbifold. Moerdijk and Pronk [9] show that isomorphism classes of orbifolds correspond to Morita equivalence classes of orbifold groupoids. For many investigations about orbifolds, an equivalence of categories

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suffices to translate the problem to groupoids. However, one cannot investigate e.g. the diffeomorphism group of an orbifold using any of the groupoid categories.

In this article we provide a correct characterization of groupoid homomorphisms in local charts. We use the arising maps to define a geometrically motivated notion of orbifold maps as certain equivalence classes. This allows us to define a geometrically natural orbifold category (with orbifolds as objects). Characterizing these orbifold maps in terms of groupoid homomorphisms enables us to define a category in terms of marked proper effective étale Lie groupoids (which is not the classical one) which is isomorphic to the orbifold category.

The results in this article are used by Schmeding [13] to show that the diffeomorphism group of a paracompact reduced orbifold can be endowed with the structure of an (infinite-dimensional) Lie group. Moreover, this Lie group is even C^k -regular for any $k \in \mathbb{N}_0 \cup \{\infty\}$ and *a fortiori* regular in the sense of Milnor; we refer to [13] for details.

For convenience we briefly comment on the outline of the paper. We start by reviewing the necessary background material on orbifolds, groupoids, pseudogroups, and the well-known construction of a groupoid from an orbifold and an orbifold atlas representing its orbifold structure. Groupoids which arise in this way will be called *atlas groupoids*. To overcome the problem that different orbifolds are identified with the same atlas groupoid we introduce, in Section 3, a certain marking of atlas groupoids. It consists in attaching to an atlas groupoid a certain topological space and a certain homomorphism between its orbit space and the topological space. The general concept of marking already appeared in [7]. There, however, the relation between a marking of a groupoid and an orbifold atlas (in local charts) is not discussed. The specific marking of an atlas groupoid introduced here allows us to recover the orbifold. There is a natural notion of homomorphisms between marked atlas groupoids. In Section 4 we characterize these homomorphisms in local charts. On the orbifold side, this characterization involves the choice of representatives of the orbifold structures, namely those orbifold atlases which were used to construct the marked atlas groupoids. Hence, at this point we get a notion of orbifold map with fixed representatives of orbifold structures, which we will call *charted orbifold maps*. In Section 5 we introduce a natural definition of composition of charted orbifold maps and a geometrically motivated definition of the identity morphism (a certain class of charted orbifold maps), which allows us to establish a natural equivalence relation on the class of charted orbifold maps. An orbifold map (which does not depend on the choice of orbifold atlases) is then an equivalence class of charted orbifold maps. The leading idea for this equivalence relation is geometric: we consider charted orbifold maps as equivalent if and only if they induce the same charted orbifold map on common refinements of the orbifold atlases. Moreover, using the same idea, we define the composition of orbifold maps. In this way, we construct a category of reduced orbifolds. Finally, in Section 6, we characterize orbifolds as certain equivalence classes of marked atlas groupoids, and orbifold maps as equivalence classes of homomorphisms of marked atlas groupoids. These equivalence relations are natural adaptations of the classical Morita equivalence. In this way, there arises a category of marked atlas groupoids which is isomorphic to the orbifold category. As an additional benefit the isomorphism functor is constructive. As a final step, in Section 7, we show that the category of marked atlas groupoids is isomorphic to an analogously defined category of marked proper effective étale Lie groupoids (by

embedding).

Even though the category of marked proper effective étale Lie groupoids constructed in this article is not the classical one, all the classical groupoid (2- or bi-) categories can be recovered by considering elements in an equivalence class as stand-alone entities and the equivalence-providing maps as 2-morphisms, and by forgetting the marking. The ultimate orbifold category constructed here is the one which is satisfying from a geometric point of view (cf. the results in [13]) and also allows to adapt without any difficulties all results on groupoids that are invariant under Morita equivalence (e.g. the behavior of orbifold vector bundles under pullback).

We expect that all these constructions can be performed in a similar way for orbifolds over other fields, e. g., complex ones, and for orbifolds with additional structures such as Riemannian ones.

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NOTATION AND CONVENTIONS. We use $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ to denote the set of non-negative integers. If not stated otherwise, every manifold is assumed to be real, paracompact, Hausdorff and smooth (C^∞). We also consider second-countable manifolds, and it will always be indicated whether we require a given manifold to be just paracompact or even second-countable. If M is a manifold, then $\text{Diff}(M)$ denotes the group of diffeomorphisms of M . If G is a subgroup of $\text{Diff}(M)$, then

$$G \setminus M := \{Gm \mid m \in M\}$$

denotes the space of G -orbits endowed with the final topology. If A_1, A_2, B are sets (manifolds) and $f_1: A_1 \rightarrow B$, $f_2: A_2 \rightarrow B$ are maps (submersions), then we denote the fibered product of f_1 and f_2 by $A_1 \times_{f_1, f_2} A_2$ and identify it with the set (manifold)

$$A_1 \times_{f_1, f_2} A_2 = \{(a_1, a_2) \in A_1 \times A_2 \mid f_1(a_1) = f_2(a_2)\}.$$

Finally, we say that a family $\mathcal{V} = \{V_i \mid i \in I\}$ is *indexed by* I if $I \rightarrow \mathcal{V}$, $i \mapsto V_i$, is a bijection.

2. Reduced orbifolds, groupoids, and pseudogroups.

In this section we recall the necessary background on reduced orbifolds and groupoids. The notion of (reduced) orbifolds goes back to at least Satake [11], [12], and has been refined since then, see e.g., [1], [4], [8], [14]. Here, we use one of these more modern definitions of reduced orbifolds, and we consider different flavors for the manifolds involved. We recall that throughout any manifold is required to be real, smooth, Hausdorff and paracompact.

2.1. Reduced orbifolds.

Let Q be a topological space, and $n \in \mathbb{N}_0$. A *reduced orbifold chart* of dimension n on Q is a triple (V, G, φ) where V is a connected n -manifold without boundary, G is a finite subgroup of $\text{Diff}(V)$, and $\varphi: V \rightarrow Q$ is a map with open image $\varphi(V)$ that induces a homeomorphism from $G \backslash V$ to $\varphi(V)$. In this case, (V, G, φ) is said to *uniformize* $\varphi(V)$.

One might argue that orbifold charts should better be called orbifold parametrizations. However, orbifold charts are the more established notion and we prefer to stick to it. The orbifold charts are called reduced (or effective) to indicate that the action of G is effective.

Two reduced orbifold charts (V, G, φ) , (W, H, ψ) on Q are called *compatible* if for each pair $(x, y) \in V \times W$ with $\varphi(x) = \psi(y)$ there are open connected neighborhoods \tilde{V} of x and \tilde{W} of y and a diffeomorphism $h: \tilde{V} \rightarrow \tilde{W}$ with $\psi \circ h = \varphi|_{\tilde{V}}$. The map h is called a *change of charts*. The neighborhoods \tilde{V} and \tilde{W} and the diffeomorphism h can always be chosen in such a way that $h(x) = y$. Moreover \tilde{V} may assumed to be open G -stable. This means that \tilde{V} is open and connected, and for each $g \in G$ we either have $g\tilde{V} = \tilde{V}$ or $g\tilde{V} \cap \tilde{V} = \emptyset$. In this case, \tilde{W} is open H -stable by [8, Proposition 2.12(i)].

A *reduced orbifold atlas* of dimension n on Q is a collection of pairwise compatible reduced orbifold charts

$$\mathcal{V} := \{(V_i, G_i, \varphi_i) \mid i \in I\}$$

of dimension n on Q such that $\bigcup_{i \in I} \varphi_i(V_i) = Q$. Two reduced orbifold atlases are *equivalent* if their union is a reduced orbifold atlas. A *reduced orbifold structure* of dimension n on Q is an equivalence class of reduced orbifold atlases of dimension n on Q . A *reduced (paracompact resp. second-countable) orbifold* of dimension n is a pair (Q, \mathcal{U}) where Q is a (paracompact resp. second-countable) Hausdorff space and \mathcal{U} is a reduced orbifold structure of dimension n on Q . We note that since the manifolds V in the orbifold charts (V, G, φ) are assumed to be connected and finite-dimensional, paracompactness of V is equivalent to second-countability of V . However, since the topological space Q is not required to be connected, paracompactness does not necessarily imply second-countability of Q .

Let \mathcal{U} be a reduced orbifold structure on Q . Each reduced orbifold atlas \mathcal{V} in \mathcal{U} is called a *representative* of \mathcal{U} or a *reduced orbifold atlas of* (Q, \mathcal{U}) .

Since we are considering reduced orbifolds only, we omit the term “reduced” from now on. Moreover, when implicitly understood we will omit the terms “paracompact” and “second-countable”.

Let (V, G, φ) , (W, H, ψ) be orbifold charts of identical dimension on the topological space Q . Then an (*open*) *embedding* $\mu: (V, G, \varphi) \rightarrow (W, H, \psi)$ between these two orbifold charts is an (*open*) embedding $\mu: V \rightarrow W$ between manifolds which satisfies $\psi \circ \mu = \varphi$. We remark that any embedding is automatically open by the Theorem on the Invariance of Domain. If, in addition, μ is a diffeomorphism between V and W , then μ is called an *isomorphism* from (V, G, φ) to (W, H, ψ) . Suppose that S is an open G -stable subset of V and let $G_S := \{g \in G \mid gS = S\}$ denote the *isotropy group* of S . Then $(S, G_S, \varphi|_S)$ is an orbifold chart on Q as well, the *restriction* of (V, G, φ) to S .

REMARK 2.1. Suppose that $\mu: (V, G, \varphi) \rightarrow (W, H, \psi)$ is an embedding. By [8, Proposition 2.12(i)], $\mu(V)$ is an open H -stable subset of W , and moreover that there is a unique group isomorphism $\bar{\mu}: G \rightarrow H_{\mu(V)}$ for which $\mu(gx) = \bar{\mu}(g)\mu(x)$ for $g \in G$, $x \in V$.

In the following example we provide two orbifolds with the same underlying topological space. These orbifolds are particularly simple since both orbifold structures have one-chart-representatives. Despite their simplicity they serve as motivating examples for several definitions in this paper.

EXAMPLE 2.2. Let $Q := [0, 1)$ be endowed with the induced topology of \mathbb{R} . The map

$$f: Q \rightarrow Q, \quad x \mapsto x^2$$

is a homeomorphism. Further the map $\text{pr}: (-1, 1) \rightarrow [0, 1)$, $x \mapsto |x|$, induces a homeomorphism $\{\pm \text{id}\} \setminus (-1, 1) \rightarrow Q$. Then

$$V_1 := ((-1, 1), \{\pm \text{id}\}, \text{pr}) \quad \text{and} \quad V_2 := ((-1, 1), \{\pm \text{id}\}, f \circ \text{pr})$$

are two orbifold charts on Q . They are easily seen to be non-compatible.

Let \mathcal{U}_1 be the orbifold structure on Q generated by V_1 , and \mathcal{U}_2 be the one generated by V_2 .

2.2. Groupoids and homomorphisms.

A groupoid is a small category in which each morphism is an isomorphism. In the context of orbifolds this concept is most commonly expressed (equivalently) in terms of sets and maps. The morphisms are then called arrows.

A *groupoid* G is a tuple $G = (G_0, G_1, s, t, m, u, i)$ consisting of the set G_0 of *objects*, or the *base* of G , the set G_1 of *arrows*, and five *structure maps*, namely the *source map* $s: G_1 \rightarrow G_0$, the *target map* $t: G_1 \rightarrow G_0$, the *multiplication* or *composition* $m: G_1 \times_s G_1 \rightarrow G_1$, the *unit map* $u: G_0 \rightarrow G_1$, and the *involution* $i: G_1 \rightarrow G_1$ which satisfy that

- (i) for all $(g, f) \in G_1 \times_s G_1$ we have $s(m(g, f)) = s(f)$ and $t(m(g, f)) = t(g)$,
- (ii) for all $(h, g), (g, f) \in G_1 \times_s G_1$ we have $m(h, m(g, f)) = m(m(h, g), f)$,
- (iii) for all $x \in G_0$ we have $s(u(x)) = x = t(u(x))$,
- (iv) for all $x \in G_0$ and all $(u(x), f), (g, u(x)) \in G_1 \times_s G_1$ it follows $m(u(x), f) = f$ and $m(g, u(x)) = g$,

- (v) for all $g \in G_1$ we have $s(i(g)) = t(g)$ and $t(i(g)) = s(g)$, and $m(g, i(g)) = u(t(g))$ and $m(i(g), g) = u(s(g))$.

We often use the notations $m(g, f) = gf$, $u(x) = 1_x$, $i(g) = g^{-1}$, and $g: x \rightarrow y$ or $x \xrightarrow{g} y$ for an arrow $g \in G_1$ with $s(g) = x$, $t(g) = y$. Moreover, $G(x, y)$ denotes the set of arrows from x to y .

Classically, the base space of a Lie groupoid is required to be a second-countable manifold. However, parts of the theory of Lie groupoids stay valid if the base manifold is just paracompact. Therefore, as with orbifolds, we consider two flavors of Lie groupoids here.

A (*paracompact resp. second-countable*) *Lie groupoid* is a groupoid G for which G_0 is a paracompact resp. second-countable manifold, G_1 is a smooth (possibly non-Hausdorff, possibly non-second-countable) manifold, the structure maps $s, t: G_1 \rightarrow G_0$ are smooth submersions (hence $G_1 \times_s G_1$, the domain of m , is a smooth, possibly non-Hausdorff manifold), and the structure maps m, u and i are smooth. A Lie groupoid is called *étale* if its source and target map are local diffeomorphisms. It is called *proper* if G_1 is Hausdorff and the map $(s, t): G_1 \rightarrow G_0 \times G_0$ is proper. An étale Lie groupoid G is called *effective* if the map

$$g \mapsto \text{germ}_{s(g)}(t \circ (s|_U)^{-1}),$$

where $g \in G_1$ and U is an open neighborhood of g in G_1 such that $t|_U$ and $s|_U$ are injective, is injective.

Let G and H be groupoids. A *homomorphism* from G to H is a functor $\varphi: G \rightarrow H$, i.e., it is a tuple $\varphi = (\varphi_0, \varphi_1)$ of maps $\varphi_0: G_0 \rightarrow H_0$ and $\varphi_1: G_1 \rightarrow H_1$ which commute with all structure maps. If G and H are Lie groupoids, then φ is a homomorphism between them if it is a homomorphism of the abstract groupoids with the additional requirement that φ_0 and φ_1 be smooth maps.

Let G be a groupoid. The *orbit* of $x \in G_0$ is the set

$$Gx := t(s^{-1}(x)) = \left\{ y \in G_0 \mid \exists g \in G_1: x \xrightarrow{g} y \right\}.$$

Two elements $x, y \in G_0$ are called *equivalent*, $x \sim y$, if they are in the same orbit. The quotient space $|G| := G_0/\sim$ is called the *orbit space* of G . The canonical quotient map $G_0 \rightarrow |G|$ is denoted by pr or pr_G , and $[x] := \text{pr}(x)$ for $x \in G_0$.

2.3. Pseudogroups and groupoids.

We recall how to construct a Lie groupoid from an orbifold and a representative of its orbifold structure. This construction is well known, see e.g. the book by Moerdjik and Mrčun [8]. We provide it here for the convenience of the reader and to introduce the notations we will use later on. It is a two-step process in which one first assigns a pseudogroup to the orbifold, which depends on the representative of the orbifold structure. Then one constructs an étale Lie groupoid from the pseudogroup. For reasons of generality and clarity we start with the second step.

DEFINITION 2.3. Let M be a manifold. A *transition* on M is a diffeomorphism $f: U \rightarrow V$ where U, V are open subsets of M . In particular, the empty map $\emptyset \rightarrow \emptyset$ is a

transition on M . The *product* of two transitions $f: U \rightarrow V$, $g: U' \rightarrow V'$ is the transition

$$f \circ g: g^{-1}(U \cap V') \rightarrow f(U \cap V'), \quad x \mapsto f(g(x)).$$

The *inverse* of f is the inverse of f as a function. If $f: U \rightarrow V$ is a transition, we use $\text{dom } f$ to denote its *domain* and $\text{cod } f$ to denote its *codomain*. Further, if $x \in \text{dom } f$, then $\text{germ}_x f$ denotes the germ of f at x .

Let $\mathcal{A}(M)$ be the set of all transitions on M . A *pseudogroup* on M is a subset P of $\mathcal{A}(M)$ which is closed under multiplication and inversion. A pseudogroup P is called *full* if $\text{id}_U \in P$ for each open subset U of M . It is said to be *complete* if it is full and satisfies the following gluing property: Whenever there is a transition $f \in \mathcal{A}(M)$ and an open covering $(U_i)_{i \in I}$ of $\text{dom } f$ such that $f|_{U_i} \in P$ for all $i \in I$, then $f \in P$.

We now recall how to construct an étale Lie groupoid from a full pseudogroup.

CONSTRUCTION 2.4. Let M be a manifold and P a full pseudogroup on M . The *associated groupoid* $\Gamma := \Gamma(P)$ is given by

$$\Gamma_0 := M, \quad \Gamma_1 := \{\text{germ}_x f \mid f \in P, \quad x \in \text{dom } f\},$$

and

$$\Gamma(x, y) := \{\text{germ}_x f \mid f \in P, \quad x \in \text{dom } f, \quad f(x) = y\}.$$

For $f \in P$ define $U_f := \{\text{germ}_x f \mid x \in \text{dom } f\}$. The topology and differential structure of Γ_1 is given by the germ topology and germ differential structure, that is, for each $f \in P$ the bijection

$$\varphi_f: \begin{cases} U_f & \rightarrow \text{dom } f \\ \text{germ}_x f & \mapsto x \end{cases}$$

is required to be a diffeomorphism. The structure maps (s, t, m, u, i) of Γ are the obvious ones, namely

$$\begin{aligned} s(\text{germ}_x f) &:= x \\ t(\text{germ}_x f) &:= f(x) \\ m(\text{germ}_{f(x)} g, \text{germ}_x f) &:= \text{germ}_x(g \circ f) \\ u(x) &:= \text{germ}_x \text{id}_U \text{ for an open neighborhood } U \text{ of } x \\ i(\text{germ}_x f) &:= \text{germ}_{f(x)} f^{-1}. \end{aligned}$$

Obviously, $\Gamma(P)$ is an étale Lie groupoid.

SPECIAL CASE 2.5. Let (Q, \mathcal{U}) be an orbifold, and let

$$\mathcal{V} = \{(V_i, G_i, \pi_i) \mid i \in I\}$$

be a representative of \mathcal{U} indexed by I . We define

$$V := \coprod_{i \in I} V_i \quad \text{and} \quad \pi := \coprod_{i \in I} \pi_i.$$

Then

$$\Psi(\mathcal{V}) := \{f \text{ transition on } V \mid \pi \circ f = \pi|_{\text{dom } f}\}.$$

is a complete pseudogroup on V . The associated groupoid

$$\Gamma(\mathcal{V}) := \Gamma(\Psi(\mathcal{V}))$$

is the proper effective étale Lie groupoid we shall associate to Q and \mathcal{V} . Note that this groupoid depends on the choice of the representative of the orbifold structure \mathcal{U} of Q . A groupoid which arises in this way we call *atlas groupoid*.

If (Q, \mathcal{U}) is a second-countable orbifold, then we can choose \mathcal{V} to be countable and the associated atlas groupoid has a second-countable base. Whenever dealing with second-countable orbifolds in the following, we will assume that the chosen representative of the orbifold structure is countable.

EXAMPLE 2.6. Recall the orbifolds (Q, \mathcal{U}_i) ($i = 1, 2$) from Example 2.2, and consider the representative $\mathcal{V}_i := \{V_i\}$ of \mathcal{U}_i . Proposition 2.12 in [8] implies that

$$\Psi(\mathcal{V}_i) = \{g|_U : U \rightarrow g(U) \mid U \subseteq (-1, 1) \text{ open}, g \in \{\pm \text{id}\}\}.$$

In both cases the associated groupoid $\Gamma := \Gamma(\mathcal{V}_i)$ is

$$\begin{aligned} \Gamma_0 &= (-1, 1) \\ \Gamma(x, y) &= \begin{cases} \{\text{germ}_0 \text{id}, \text{germ}_0(-\text{id})\} & \text{if } x = 0 = y \\ \{\text{germ}_x \text{id}\} & \text{if } x = y \neq 0 \\ \{\text{germ}_x(-\text{id})\} & \text{if } x = -y \neq 0 \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

3. Marked Lie groupoids and their homomorphisms.

In Example 2.6 we have seen that it may happen that the same atlas groupoid is associated to two different orbifolds. The reason for this is that in the definition of the pseudogroup which is needed for the construction of the atlas groupoid one loses information about the projection maps φ of the orbifold charts (V, G, φ) . To be able to distinguish atlas groupoids constructed from different orbifolds, we mark the groupoids with a topological space and a homeomorphism. It will turn out that this marking suffices to identify the orbifold one started with.

A *marked Lie groupoid* is a triple (G, α, X) consisting of a Lie groupoid G , a topological space X , and a homeomorphism $\alpha : |G| \rightarrow X$.

For atlas groupoids there exists a specific marking which is crucial for the isomorphism between the orbifold category and the groupoid category. It is stated in the following lemma of which we omit its straightforward proof.

LEMMA 3.1. Let (Q, \mathcal{U}) be an orbifold and

$$\mathcal{V} = \{(V_i, G_i, \pi_i) \mid i \in I\}$$

a representative of \mathcal{U} indexed by I . Set $V := \coprod_{i \in I} V_i$ and $\pi := \coprod_{i \in I} \pi_i: V \rightarrow Q$. Then the map

$$\alpha: \begin{cases} |\Gamma(\mathcal{V})| \rightarrow Q \\ [x] \mapsto \pi(x) \end{cases}$$

is a homeomorphism.

Let (Q, \mathcal{U}) be an orbifold. To each (countable if (Q, \mathcal{U}) is second-countable) orbifold atlas \mathcal{V} of Q we assign the marked atlas groupoid $(\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q)$ with $\alpha_{\mathcal{V}}$ being the homeomorphism from Lemma 3.1. We often only write $\Gamma(\mathcal{V})$ to refer to this marked groupoid.

EXAMPLE 3.2. Recall from Example 2.6 the orbifolds (Q, \mathcal{U}_i) for $i = 1, 2$, their respective orbifold atlases \mathcal{V}_i , and the associated groupoids $\Gamma = \Gamma(\mathcal{V}_i)$. The orbit of $x \in \Gamma_0$ is $\{x, -x\}$. Hence the homeomorphism associated to (Q, \mathcal{U}_i) is $\alpha_{\mathcal{V}_i}: |\Gamma| \rightarrow Q$ given by $\alpha_{\mathcal{V}_1}([x]) = |x|$ resp. $\alpha_{\mathcal{V}_2}([x]) = x^2$. Thus, the associated marked groupoids $(\Gamma, \alpha_{\mathcal{V}_1}, Q)$ and $(\Gamma, \alpha_{\mathcal{V}_2}, Q)$ are different.

PROPOSITION 3.3. Let (Q, \mathcal{U}) and (Q', \mathcal{U}') be orbifolds. Suppose that \mathcal{V} is a representative of \mathcal{U} , and \mathcal{V}' a representative of \mathcal{U}' . If the associated marked atlas groupoids $(\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q)$ and $(\Gamma(\mathcal{V}'), \alpha_{\mathcal{V}'}, Q')$ are equal, then the orbifolds (Q, \mathcal{U}) and (Q', \mathcal{U}') are equal. More precisely, we even have $\mathcal{V} = \mathcal{V}'$.

PROOF. Clearly, $Q = Q'$. Suppose that

$$\mathcal{V} = \{(V_i, G_i, \pi_i) \mid i \in I\} \quad \text{and} \quad \mathcal{V}' = \{(V'_j, G'_j, \pi'_j) \mid j \in J\},$$

indexed by I resp. by J . From $\Gamma(\mathcal{V}) = \Gamma(\mathcal{V}')$ it follows that

$$\coprod_{i \in I} V_i = \Gamma(\mathcal{V})_0 = \Gamma(\mathcal{V}')_0 = \coprod_{j \in J} V'_j.$$

Since each V_i and each V'_j is connected, there is a bijection between I and J . We may assume $I = J$ and $V_i = V'_i$ for all $i \in I$. Let $x \in V_i$. Then $\pi_i(x) = \alpha_{\mathcal{V}}([x]) = \alpha_{\mathcal{V}'}([x]) = \pi'_i(x)$. Therefore $\pi_i = \pi'_i$ for all $i \in I$. Thus $\Psi(\mathcal{V}) = \Psi(\mathcal{V}')$. Moreover

$$G_i = \{f \in \Psi(\mathcal{V}) \mid \text{dom } f = V_i = \text{cod } f\} = G'_i.$$

To show that the actions G_i and G'_i on V_i are equal, let $g \in G_i$. For each $x \in V_i$ we have $\pi'_i(g(x)) = \pi'_i(x)$. This shows that $g(x) \in G'_i x$ for each $x \in V_i$. By [8, Lemma 2.11] there exists a unique element $g' \in G'_i$ such that $g = g'$. From this it follows that $G_i = G'_i$ as acting groups. Thus, $\mathcal{V} = \mathcal{V}'$. \square

LEMMA 3.4. Let (G, α, X) and (H, β, Y) be marked Lie groupoids and suppose that

$\varphi = (\varphi_0, \varphi_1): G \rightarrow H$ is a homomorphism of Lie groupoids. Then φ induces a unique map ψ such that the diagram

$$\begin{array}{ccccc} G_0 & \xrightarrow{\text{pr}_G} & |G| & \xrightarrow{\alpha} & X \\ \varphi_0 \downarrow & & & & \downarrow \psi \\ H_0 & \xrightarrow{\text{pr}_H} & |H| & \xrightarrow{\beta} & Y \end{array}$$

commutes. Moreover, ψ is continuous.

PROOF. The map φ induces a unique map $|\varphi|: |G| \rightarrow |H|$ such that $|\varphi| \circ \text{pr}_G = \text{pr}_H \circ \varphi_0$, which is continuous. Then $\psi = \beta \circ |\varphi| \circ \alpha^{-1}$. \square

4. Groupoid homomorphisms in local charts.

In this section we characterize homomorphisms between marked atlas groupoids on the orbifold side, i. e., in terms of local charts. We proceed in a two-step process. First we define a concept which we call representatives of orbifold maps. Each representative of an orbifold map gives rise to exactly one homomorphism between the associated marked atlas groupoids. Since, in general, each groupoid homomorphism corresponds to several such representatives, we then impose an equivalence relation on the class of all representatives for fixed orbifold atlases. The equivalence classes, called charted orbifold maps, turn out to be in bijection with the homomorphisms between the marked atlas groupoids. The constructions in this section are subject to a fixed choice of representatives of the orbifold structures. In the following sections we will use this construction as a basic building block for a notion of maps (or morphisms) between orbifolds which is independent of the chosen representatives.

Throughout this section let $(Q, \mathcal{U}), (Q', \mathcal{U}')$ denote two orbifolds.

DEFINITION 4.1. Let $f: Q \rightarrow Q'$ be a continuous map, and suppose that $(V, G, \pi) \in \mathcal{U}, (V', G', \pi') \in \mathcal{U}'$ are orbifold charts. A *local lift* of f with respect to (V, G, π) and (V', G', π') is a smooth map $\tilde{f}: V \rightarrow V'$ such that $\pi' \circ \tilde{f} = f \circ \pi$. In this case, we call \tilde{f} a *local lift of f at q* for each $q \in \pi(V)$.

Recall the pseudogroup $\mathcal{A}(M)$ from Definition 2.3.

DEFINITION 4.2. Let M be a manifold and A a pseudogroup on M which satisfies the gluing property from Definition 2.3 and is closed under restrictions. The latter means that if $f \in A$ and $U \subseteq \text{dom } f$ is open, then the map $f|_U: U \rightarrow f(U)$ is in A . Suppose that B is a subset of $\mathcal{A}(M)$. Then A is said to be *generated* by B if $B \subseteq A$ and for each $f \in A$ and each $x \in \text{dom } f$ there exist some $g \in B$ with $x \in \text{dom } g$ and an open set $U \subseteq \text{dom } f \cap \text{dom } g$ such that $x \in U$ and $f|_U = g|_U$. In this case we say that B generates A .

DEFINITION 4.3. Let M be a manifold. A subset P of $\mathcal{A}(M)$ is called a *quasi-pseudogroup* on M if it satisfies the following two properties:

- (i) If $f \in P$ and $x \in \text{dom } f$, then there exist an open set U with $x \in U \subseteq \text{dom } f$ and $g \in P$ such that there exists an open set V with $f(x) \in V \subseteq \text{dom } g$ and

$$(f|_U)^{-1} = g|_V.$$

- (ii) If $f, g \in P$ and $x \in f^{-1}(\text{cod } f \cap \text{dom } g)$, then there exists $h \in P$ with $x \in \text{dom } h$ such that we find an open set U with

$$x \in U \subseteq f^{-1}(\text{cod } f \cap \text{dom } g) \cap \text{dom } h \quad \text{and} \quad g \circ f|_U = h|_U.$$

A quasi-pseudogroup is designed to work with the germs of its elements. Therefore identities (like inversion and composition) of elements in quasi-pseudogroups are only required to be satisfied locally, whereas for (ordinary) pseudogroups these identities have to be valid globally. One easily proves that each quasi-pseudogroup generates a unique pseudogroup which satisfies the gluing property from Definition 2.3 and is closed under restrictions. Conversely, each generating set for such a pseudogroup is necessarily a quasi-pseudogroup.

In the following definition of a representative of an orbifold map, the underlying continuous map f is the only entity which is stable under change of orbifold atlases or, in other words, under the choice of local lifts. The pair (P, ν) should be considered as one entity. It serves as a transport of changes of charts from one orbifold to another. Here we ask for a quasi-pseudogroup P instead of working with all of $\Psi(\mathcal{V})$ (recall $\Psi(\mathcal{V})$ from Special Case 2.5) for two reasons. In general, P is much smaller than $\Psi(\mathcal{V})$. Sometimes it may even be finite. In Example 4.6 below we see that for some orbifolds, P may consist of only two elements. Moreover, if the orbifold is a connected manifold, P can always be chosen to be $\{\text{id}\}$. The other reason is that it is much easier to construct a quasi-pseudogroup P and a compatible map ν from a given groupoid homomorphism than a map ν defined on all of $\Psi(\mathcal{V})$.

Examples 4.5 and 4.6 below show that the objects requested in the following definition need not exist nor, if they exist, are uniquely determined.

DEFINITION 4.4. A *representative of an orbifold map* from (Q, \mathcal{U}) to (Q', \mathcal{U}') is a tuple

$$\hat{f} := (f, \{\tilde{f}_i\}_{i \in I}, P, \nu)$$

where

(R1) $f: Q \rightarrow Q'$ is a continuous map,

(R2) for each $i \in I$, the map \tilde{f}_i is a local lift of f with respect to some orbifold charts $(V_i, G_i, \pi_i) \in \mathcal{U}$, $(V'_i, G'_i, \pi'_i) \in \mathcal{U}'$ such that

$$\bigcup_{i \in I} \pi_i(V_i) = Q$$

and $(V_i, G_i, \pi_i) \neq (V_j, G_j, \pi_j)$ for $i, j \in I$, $i \neq j$,

(R3) P is a quasi-pseudogroup which consists of changes of charts of the orbifold atlas

$$\mathcal{V} := \{(V_i, G_i, \pi_i) \mid i \in I\}$$

of (Q, \mathcal{U}) and generates $\Psi(\mathcal{V})$.

(R4) Let $\psi := \coprod_{i \in I} \tilde{f}_i$ and let \mathcal{V}' be a representative of \mathcal{U}' which contains

$$\{(V'_i, G'_i, \pi'_i) \mid i \in I\}.$$

Then $\nu: P \rightarrow \Psi(\mathcal{V}')$ is a map which assigns to each $\lambda \in P$ an embedding

$$\nu(\lambda): (W', H', \chi') \rightarrow (V', G', \varphi')$$

between some orbifold charts in \mathcal{V}' such that

- (a) $\psi \circ \lambda = \nu(\lambda) \circ \psi|_{\text{dom } \lambda}$,
- (b) for all $\lambda, \mu \in P$ and all $x \in \text{dom } \lambda \cap \text{dom } \mu$ with $\text{germ}_x \lambda = \text{germ}_x \mu$, we have

$$\text{germ}_{\psi(x)} \nu(\lambda) = \text{germ}_{\psi(x)} \nu(\mu),$$

- (c) for all $\lambda, \mu \in P$, for all $x \in \lambda^{-1}(\text{cod } \lambda \cap \text{dom } \mu)$ we have

$$\text{germ}_{\psi(\lambda(x))} \nu(\mu) \cdot \text{germ}_{\psi(x)} \nu(\lambda) = \text{germ}_{\psi(x)} \nu(h),$$

where h is an element of P with $x \in \text{dom } h$ such that there is an open set U with

$$x \in U \subseteq \lambda^{-1}(\text{cod } \lambda \cap \text{dom } \mu) \cap \text{dom } h$$

and $\mu \circ \lambda|_U = h|_U$,

- (d) for all $\lambda \in P$ and all $x \in \text{dom } \lambda$ such that there exists an open set U with $x \in U \subseteq \text{dom } \lambda$ and $\lambda|_U = \text{id}_U$ we have

$$\text{germ}_{\psi(x)} \nu(\lambda) = \text{germ}_{\psi(x)} \text{id}_{U'}$$

with $U' := \coprod_{i \in I} V'_i$.

The orbifold atlas \mathcal{V} is called the *domain atlas* of the representative \hat{f} , and the set

$$\{(V'_i, G'_i, \pi'_i) \mid i \in I\}$$

is called the *range family* of \hat{f} . The latter set is not necessarily indexed by I .

Condition (R4c) is in fact independent of the choice of h . The technical (and easily satisfied) condition in (R2) that each two orbifold charts in \mathcal{V} be distinct is required because we use I as an index set for \mathcal{V} in (R3) and other places.

Example 4.5 below shows that the continuous map f in (R1) cannot be chosen arbitrarily. It is not even sufficient for f to be any homeomorphism.

EXAMPLE 4.5. Recall the orbifold (Q, \mathcal{U}_1) from Example 2.2. The map

$$f: Q \rightarrow Q, \quad f(x) = \sqrt{x},$$

is a homeomorphism on Q . We show that f has no local lift at 0. Each orbifold chart in \mathcal{U}_1 that uniformizes a neighborhood of 0 is isomorphic to an orbifold chart of the form $(I, \{\pm \text{id}_I\}, \text{pr})$ where $I = (-a, a)$ for some $0 < a < 1$. Seeking a contradiction assume that \tilde{f} is a local lift of f at 0 with domain $I = (-a, a)$. For each $x \in I$, necessarily $\tilde{f}(x) \in \{\pm \sqrt{|x|}\}$. Since \tilde{f} is required to be continuous, there remain four possible candidates for \tilde{f} , namely

$$\begin{aligned} \tilde{f}_1(x) &= \sqrt{|x|}, & \tilde{f}_2 &= -\tilde{f}_1, \\ \tilde{f}_3(x) &= \begin{cases} \sqrt{x} & x \geq 0 \\ -\sqrt{-x} & x \leq 0, \end{cases} & \tilde{f}_4 &= -\tilde{f}_3. \end{aligned}$$

But none of these is differentiable in $x = 0$, hence there is no local lift of f at 0.

The following example shows that the pair (P, ν) is not uniquely determined by the choice of the family of local lifts.

EXAMPLE 4.6. Recall the orbifold (Q, \mathcal{U}_1) and the representative $\mathcal{V}_1 = \{V_1\}$ of \mathcal{U}_1 from Example 2.2. The map $f: Q \rightarrow Q$, $q \mapsto 0$, is clearly continuous and has the local lift

$$\tilde{f}: \begin{cases} (-1, 1) \rightarrow (-1, 1) \\ x \mapsto 0 \end{cases}$$

with respect to V_1 and V_1 . Consider the quasi-pseudogroup $P = \{\pm \text{id}_{(-1,1)}\}$ on $(-1, 1)$. Proposition 2.12 in [8] implies that P generates $\Psi(\mathcal{V}_1)$. The triple (f, \tilde{f}, P) can be completed in the following two different ways to representatives of orbifold maps on (Q, \mathcal{U}_1) :

- (a) $\nu_1(\pm \text{id}_{(-1,1)}) := \text{id}_{(-1,1)}$,
- (b) $\nu_2(\text{id}_{(-1,1)}) := \text{id}_{(-1,1)}$, $\nu_2(-\text{id}_{(-1,1)}) := -\text{id}_{(-1,1)}$.

We will see in Example 4.8 below that (f, \tilde{f}, P, ν_1) and (f, \tilde{f}, P, ν_2) give rise to different groupoid homomorphisms.

PROPOSITION 4.7. Let $\hat{f} = (f, \{\tilde{f}_i\}_{i \in I}, P, \nu)$ be a representative of an orbifold map from (Q, \mathcal{U}) to (Q', \mathcal{U}') . Suppose that $\mathcal{V} = \{(V_i, G_i, \pi_i) \mid i \in I\}$, is the domain atlas of \hat{f} , which is an orbifold atlas of (Q, \mathcal{U}) indexed by I . Let \mathcal{V}' be an orbifold atlas of (Q', \mathcal{U}') which contains the range family $\{(V'_i, G'_i, \pi'_i) \mid i \in I\}$. Define the map $\varphi_0: \Gamma(\mathcal{V})_0 \rightarrow \Gamma(\mathcal{V}')_0$ by

$$\varphi_0 := \coprod_{i \in I} \tilde{f}_i.$$

Suppose that $\varphi_1: \Gamma(\mathcal{V})_1 \rightarrow \Gamma(\mathcal{V}')_1$ is determined by

$$\varphi_1(\text{germ}_x \lambda) := \text{germ}_{\varphi_0(x)} \nu(\lambda)$$

for all $\lambda \in P$, $x \in \text{dom } \lambda$. Then

$$\varphi = (\varphi_0, \varphi_1) : \Gamma(\mathcal{V}) \rightarrow \Gamma(\mathcal{V}')$$

is a homomorphism. Moreover, $\alpha_{\mathcal{V}'} \circ |\varphi| = f \circ \alpha_{\mathcal{V}}$.

PROOF. Obviously, φ_0 is smooth. To show that φ_1 is a well-defined map on all of $\Gamma(\mathcal{V})_1$, let $g \in \Psi(\mathcal{V})$ and $x \in \text{dom } g$. Then there exists $\lambda \in P$ such that $x \in \text{dom } \lambda$ and

$$g|_U = \lambda|_U$$

for some open subset $U \subseteq \text{dom } g \cap \text{dom } \lambda$ with $x \in U$. Hence $\text{germ}_x g = \text{germ}_x \lambda$. Thus

$$\varphi_1(\text{germ}_x g) = \varphi_1(\text{germ}_x \lambda) = \text{germ}_{\varphi_0(x)} \nu(\lambda).$$

If there is $\mu \in P$ such that $x \in \text{dom } \mu$ and $g|_W = \mu|_W$ for some open subset W of $\text{dom } g \cap \text{dom } \mu$ with $x \in W$, then $\text{germ}_x \mu = \text{germ}_x \lambda$. By (R4b), $\text{germ}_{\varphi_0(x)} \nu(\mu) = \text{germ}_{\varphi_0(x)} \nu(\lambda)$ and thus

$$\varphi_1(\text{germ}_x \mu) = \varphi_1(\text{germ}_x \lambda).$$

This shows that φ_1 is indeed well-defined on all of $\Gamma(\mathcal{V})_1$. The properties (R4a), (R4c) and (R4d) yield that φ commutes with the structure maps. It remains to show that φ_1 is smooth. For this, let $\text{germ}_x \lambda \in \Gamma(\mathcal{V})_1$ with $\lambda \in P$. The definition of ν shows that φ_1 maps

$$U := \{\text{germ}_y \lambda \mid y \in \text{dom } \lambda\}$$

to

$$U' := \{\text{germ}_z \nu(\lambda) \mid z \in \text{dom } \nu(\lambda)\}.$$

Now the diagram

$$\begin{array}{ccc} U & \xrightarrow{\varphi_1} & U' \\ s \downarrow & & \downarrow s \\ \text{dom } \lambda & \xrightarrow{\varphi_0} & \text{dom } \nu(\lambda) \end{array} \quad \begin{array}{ccc} \text{germ}_y \lambda & \longmapsto & \text{germ}_{\varphi_0(y)} \nu(\lambda) \\ \downarrow & & \downarrow \\ y & \longmapsto & \varphi_0(y) \end{array}$$

commutes, the vertical maps (restriction of source maps) are diffeomorphisms and φ_0 is smooth, so φ_1 is smooth. Finally, suppose $x \in V_i$. Then

$$(\alpha_{\mathcal{V}'} \circ |\varphi|)([x]) = \alpha_{\mathcal{V}'}([\varphi_0(x)]) = \pi'_i(\tilde{f}_i(x)) = f(\pi_i(x)) = (f \circ \alpha_{\mathcal{V}})([x]).$$

This completes the proof. \square

Example 4.8 below shows that the choice of (P, ν) in Definition 4.4 is not unique, and more elaborated examples would immediately show that not each pair (P, ν) automatically satisfies the compatibility conditions in (R4a). However, for some important types of maps between orbifolds (e.g. isomorphisms) the choice of (P, ν) is canonical and the compatibility conditions are also automatic, which is the essence of Proposition 5.5 below.

EXAMPLE 4.8. Recall the setting of Example 4.6 and the associated groupoid $\Gamma := \Gamma(\mathcal{V}_1)$ from Example 2.6. The homomorphism $\varphi = (\varphi_0, \varphi_1)$ of Γ induced by (f, \tilde{f}, P, ν_1) is $\varphi_0 = \tilde{f}$ and

$$\varphi_1(\text{germ}_x(\pm \text{id}_{(-1,1)})) = \text{germ}_0 \text{id}_{(-1,1)}.$$

The homomorphism $\psi = (\psi_0, \psi_1): \Gamma \rightarrow \Gamma$ induced by (f, \tilde{f}, P, ν_2) is $\psi_0 = \tilde{f}$ and

$$\begin{aligned} \psi_1(\text{germ}_x \text{id}_{(-1,1)}) &= \text{germ}_0 \text{id}_{(-1,1)}, \\ \psi_1(\text{germ}_x(-\text{id}_{(-1,1)})) &= \text{germ}_0(-\text{id}_{(-1,1)}). \end{aligned}$$

The following proposition is the converse to Proposition 4.7. Its proof is constructive. In Section 6 we will use this construction to define the functor between the category of orbifolds and that of marked atlas groupoids.

PROPOSITION 4.9. *Let \mathcal{V} be a representative of \mathcal{U} , \mathcal{V}' a representative of \mathcal{U}' , and*

$$\varphi = (\varphi_0, \varphi_1): \Gamma(\mathcal{V}) \rightarrow \Gamma(\mathcal{V}')$$

a homomorphism. Then φ induces a representative of an orbifold map

$$(f, \{\tilde{f}_i\}_{i \in I}, P, \nu)$$

with domain atlas \mathcal{V} , range family contained in \mathcal{V}' , and

$$\tilde{f}_i = \varphi_0|_{V_i}$$

for all $i \in I$. Moreover, we have $f = \alpha_{\mathcal{V}'} \circ |\varphi| \circ \alpha_{\mathcal{V}}^{-1}$.

PROOF. We start by showing that for each $h \in \Psi(\mathcal{V})$ and each $x \in \text{dom } h$ there exist an element $g \in \Psi(\mathcal{V}')$ and an open neighborhood U of x (which may depend on g) with $U \subseteq \text{dom } h$ such that for each $y \in U$ we have

$$\varphi_1(\text{germ}_y h) = \text{germ}_{\varphi_0(y)} g. \tag{1}$$

So, let $h \in \Psi(\mathcal{V})$ and $x \in \text{dom } h$. By the definitions of $\Gamma(\mathcal{V})_1$ and φ_1 , there exists $g \in \Psi(\mathcal{V}')$ such that

$$\varphi_1(\text{germ}_x h) = \text{germ}_{\varphi_0(x)} g.$$

Since φ_1 is continuous, the preimage of the $\text{germ}_{\varphi_0(x)}$ g -neighborhood

$$U'_g = \{\text{germ}_z g \mid z \in \text{dom } g\}$$

is a neighborhood of $\text{germ}_x h$. Hence there exists an open neighborhood U of x with $U \subseteq \text{dom } h$ such that

$$U_{h|U} = \{\text{germ}_y h \mid y \in U\} \subseteq \varphi_1^{-1}(U'_g).$$

Thus, for all $y \in U$ we have (1) as claimed. We remark that each two possible choices for g coincide on some neighborhood of $\varphi_0(x)$.

For each $h \in \Psi(\mathcal{V})$ and each $x \in \text{dom } h$ we now choose a pair (g, U) where $g \in \Psi(\mathcal{V}')$ is an embedding between some orbifold charts in \mathcal{U}' and U is an open neighborhood of x such that $h|_U$ is a change of charts of \mathcal{V} . Let $P(h, x) := (g, U)$. We adjust choices such that for $h_1, h_2 \in \Psi(\mathcal{V})$ and $x_1 \in \text{dom } h_1, x_2 \in \text{dom } h_2$ the chosen pairs $P(h_1, x_1) = (g_1, U_1)$ resp. $P(h_2, x_2) = (g_2, U_2)$ are either equal or $U_1 \neq U_2$. Let P denote the family of the changes of charts we have chosen in this way:

$$P = \{h|_U : U \rightarrow h(U) \mid h \in \Psi(\mathcal{V}), x \in \text{dom } h, P(h, x) = (g, U)\}.$$

By construction, P is a quasi-pseudogroup which generates $\Psi(\mathcal{V})$. We define the map $\nu : P \rightarrow \Psi(\mathcal{V}')$ by

$$\nu(\lambda) := g$$

where g is the unique element in $\Psi(\mathcal{V}')$ attached to $\lambda \in P$ by our choices. For $\lambda \in P$ and $x \in \text{dom } \lambda$ we clearly have

$$\varphi_1(\text{germ}_x \lambda) = \text{germ}_{\varphi_0(x)} \nu(\lambda). \quad (2)$$

Properties (R4) are easily checked using the compatibility of φ with the structure maps.

It remains to show that the image of $\varphi_0|_{V_i}$ is contained in V'_j for some orbifold chart $(V'_j, G'_j, \pi'_j) \in \mathcal{V}'$. Since V_i is connected, the image $\varphi_0(V_i)$ is connected as well. The connected components of $\Gamma(\mathcal{V}')_0$ are exactly the sets W' with $(W', G', \varphi') \in \mathcal{V}'$. From this the claim follows. \square

Proposition 4.9 guarantees that each homomorphism

$$\varphi = (\varphi_0, \varphi_1) : \Gamma(\mathcal{V}) \rightarrow \Gamma(\mathcal{V}')$$

induces a representative of an orbifold map $(f, \{\tilde{f}_i\}_{i \in I}, P, \nu)$ with domain atlas \mathcal{V} , range family contained in \mathcal{V}' , $\tilde{f}_i = \varphi_0|_{V_i}$, and $f = \alpha_{\mathcal{V}'} \circ |\varphi| \circ \alpha_{\mathcal{V}}^{-1}$. For the pair (P, ν) , Proposition 4.9 allows (in general) a whole bunch of choices. On the other hand, different representatives of an orbifold map may induce the same groupoid homomorphism. In view of Proposition 4.7 and the proof of Proposition 4.9, the relevant information stored by the pair (P, ν) are the germs of the elements in P and the via ν associated germs of elements in $\Psi(\mathcal{V}')$. This observation is the motivation for the equivalence relation in the following definition.

DEFINITION 4.10. Let

$$\hat{f} := (f, \{\tilde{f}_i\}_{i \in I}, P_1, \nu_1) \quad \text{and} \quad \hat{g} := (g, \{\tilde{g}_i\}_{i \in I}, P_2, \nu_2)$$

be two representatives of orbifold maps with the same domain atlas \mathcal{V} representing the orbifold structure \mathcal{U} on Q and both range families being contained in the orbifold atlas \mathcal{V}' of (Q', \mathcal{U}') . Set $\psi := \coprod_{i \in I} \tilde{f}_i$. We say that \hat{f} is *equivalent* to \hat{g} if $f = g$, $\tilde{f}_i = \tilde{g}_i$ for all $i \in I$, and

$$\text{germ}_{\psi(x)} \nu_1(\lambda_1) = \text{germ}_{\psi(x)} \nu_2(\lambda_2)$$

for all $\lambda_1 \in P_1$, $\lambda_2 \in P_2$, $x \in \text{dom } \lambda_1 \cap \text{dom } \lambda_2$ with $\text{germ}_x \lambda_1 = \text{germ}_x \lambda_2$. This defines an equivalence relation. The equivalence class of \hat{f} will be denoted by $[\hat{f}]$ or

$$(f, \{\tilde{f}_i\}_{i \in I}, [P_1, \nu_1]),$$

or even \hat{f} if it is clear that we refer to the equivalence class. It is called an *orbifold map with domain atlas \mathcal{V} and range atlas \mathcal{V}'* , in short *orbifold map with $(\mathcal{V}, \mathcal{V}')$* or, if the specific orbifold atlases are not important, a *charted orbifold map*. The set of all orbifold maps with $(\mathcal{V}, \mathcal{V}')$ is denoted $\text{Orb}(\mathcal{V}, \mathcal{V}')$. For convenience we often denote an element $\hat{h} \in \text{Orb}(\mathcal{V}, \mathcal{V}')$ by

$$\mathcal{V} \xrightarrow{\hat{h}} \mathcal{V}'.$$

REMARK 4.11. Instead of using the pair $[P, \nu]$ in the definition of charted orbifold maps one could also define a variant where $[P, \nu]$ is replaced by a mapping

$$\theta: \text{Germ}(\Psi(\mathcal{V})) \rightarrow \text{Germ}(\Psi(\mathcal{V}')),$$

which maps germs of transitions to germs such that the equivalence conditions in (R4a) are satisfied by θ . This would stress the actual “germ nature” of $[P, \nu]$ and might simplify the compositions of charted orbifold maps as defined in Construction 5.9 below and support the intuition for its definition. Anyhow, for the reasonings in this manuscript (and other concrete situations) it is more convenient to work with the actual transitions and to have the possibility to choose “small” quasi-pseudogroups P , for which reason we decided to use the variant with the pair $[P, \nu]$.

Propositions 4.7 and 4.9 yield the following statement of which we omit the proof.

PROPOSITION 4.12. *Let \mathcal{V} be a representative of \mathcal{U} , and \mathcal{V}' a representative of \mathcal{U}' . Then the set $\text{Orb}(\mathcal{V}, \mathcal{V}')$ of all orbifold maps with $(\mathcal{V}, \mathcal{V}')$ and the set $\text{Hom}(\Gamma(\mathcal{V}), \Gamma(\mathcal{V}'))$ of all homomorphisms from $\Gamma(\mathcal{V})$ to $\Gamma(\mathcal{V}')$ are in bijection. More precisely, the construction in Proposition 4.7 induces a bijection*

$$F_1: \text{Orb}(\mathcal{V}, \mathcal{V}') \rightarrow \text{Hom}(\Gamma(\mathcal{V}), \Gamma(\mathcal{V}')),$$

and the construction in Proposition 4.9 defines a bijection

$$F_2: \text{Hom}(\Gamma(\mathcal{V}), \Gamma(\mathcal{V}')) \rightarrow \text{Orb}(\mathcal{V}, \mathcal{V}'),$$

which is inverse to F_1 .

5. The category of reduced orbifolds.

To define an orbifold category where the objects are orbifolds and the morphisms are equivalence classes of charted orbifold maps we have to answer the following questions:

- (i) When shall two charted orbifold maps be considered as equal? In other words, what shall be the equivalence relation?
- (ii) What shall be the identity morphism of an orbifold?
- (iii) How does one compose $\varphi \in \text{Orb}(\mathcal{V}, \mathcal{V}')$ and $\psi \in \text{Orb}(\mathcal{V}', \mathcal{V}'')$?
- (iv) What is the composition in the category?

The leading idea is that charted orbifold maps are equivalent if and only if they induce the same charted orbifold map on common refinements of the orbifold atlases. Therefore, we will introduce the notion of an induced charted orbifold map.

It turns out that answers to the questions (ii) and (iii) naturally extend to answers of (i) and (iv), and that the arising category has a counterpart in terms of marked atlas groupoids and homomorphisms. We start with the definition of the identity morphism of an orbifold. This definition is based on the idea that the identity morphism of (Q, \mathcal{U}) shall be represented by a collection of local lifts of id_Q which locally induce id_S on some orbifold charts, and that each such collection which satisfies (R2) shall be a representative.

5.1. The identity morphism.

DEFINITION AND REMARK 5.1. Let (Q, \mathcal{U}) and (Q', \mathcal{U}') be orbifolds and let $f: Q \rightarrow Q'$ be a continuous map. Suppose that \tilde{f} is a local lift of f with respect to the orbifold charts $(V, G, \pi) \in \mathcal{U}$ and $(V', G', \pi') \in \mathcal{U}'$. Further suppose that

$$\lambda: (W, K, \chi) \rightarrow (V, G, \pi) \quad \text{and} \quad \mu: (W', K', \chi') \rightarrow (V', G', \pi')$$

are embeddings between orbifold charts in \mathcal{U} resp. in \mathcal{U}' such that

$$\tilde{f}(\lambda(W)) \subseteq \mu(W').$$

Then the map

$$\tilde{g} := \mu^{-1} \circ \tilde{f} \circ \lambda: W \rightarrow W'$$

is a local lift of f with respect to (W, K, χ) and (W', K', χ') . We say that \tilde{f} induces the local lift \tilde{g} with respect to λ and μ , and we call \tilde{g} the induced lift of f with respect to \tilde{f} , λ and μ .

$$\begin{array}{ccccc}
& & V & \xrightarrow{\tilde{f}} & V' \\
& \nearrow \lambda & \uparrow \lambda(W) & \xrightarrow{\tilde{f}|_{\lambda(W)}} & \downarrow \mu(W') \\
W & \xrightarrow{\lambda} & \tilde{g} & \xleftarrow{\mu} & W' \\
& \searrow \lambda & & \swarrow \mu &
\end{array}$$

Suppose that \tilde{f} is a local lift of the identity id_Q for some orbifold (Q, \mathcal{U}) . Proposition 5.3 below shows that \tilde{f} induces the identity on sufficiently small orbifold charts. This means that locally \tilde{f} is related to the identity itself via embeddings. In particular, \tilde{f} is a local diffeomorphism. For its proof we need the following lemma, which is easily shown and crucially depends on the finiteness of G and the Hausdorff property of M . We refer to [13, Lemma B.3] for the details of the construction of the G -stable neighborhood S .

LEMMA 5.2. *Let M be a manifold, G a finite subgroup of $\text{Diff}(M)$, and $x \in M$. There exist arbitrary small open G -stable neighborhoods S of x . Moreover, one can choose S so small that $G_S = G_x$, the isotropy group of x .*

PROPOSITION 5.3. *Let (Q, \mathcal{U}) be an orbifold and suppose that \tilde{f} is a local lift of id_Q with respect to $(V, G, \pi), (V', G', \pi') \in \mathcal{U}$. For each $v \in V$ there exist a restriction $(S, G_S, \pi|_S)$ of (V, G, π) with $v \in S$ and a restriction $(S', (G')_{S'}, \pi'|_{S'})$ of (V', G', π') such that $\tilde{f}|_S$ is an isomorphism from $(S, G_S, \pi|_S)$ to the orbifold chart $(S', (G')_{S'}, \pi'|_{S'})$. In particular, $\tilde{f}|_S$ induces the identity id_S with respect to id_S and $(\tilde{f}|_S)^{-1}$.*

PROOF. Let $v \in V$ and set $v' := \tilde{f}(v)$. Then $\pi(v) = \pi'(v')$. By compatibility of orbifold charts there exist a restriction (W, H, χ) of (V, G, π) with $v \in W$ and an open embedding $\lambda: (W, H, \chi) \rightarrow (V', G', \pi')$ such that $\lambda(v) = v'$. Lemma 5.2 yields an open H -stable neighborhood S of v with $S \subseteq \tilde{f}^{-1}(\lambda(W)) \cap W$. Let

$$\tilde{g} := \lambda^{-1} \circ \tilde{f}|_S: S \rightarrow W$$

denote the induced lift of id_Q . Since $\chi \circ \tilde{g} = \chi$, [8, Lemma 2.11] shows the existence of a unique $h \in H$ such that $\tilde{g} = h|_S$. Thus $\tilde{f}|_S = \lambda \circ h|_S: S \rightarrow \lambda(h(S))$ is a diffeomorphism. In turn

$$\tilde{f}|_S: (S, H_S, \chi|_S) \rightarrow (\tilde{f}(S), G'_{\tilde{f}(S)}, \pi'|_{\tilde{f}(S)})$$

is an isomorphism of orbifold charts. □

Not each local lift of the identity is a global diffeomorphism, as the following example shows.

EXAMPLE 5.4. Let Q be the open annulus in \mathbb{R}^2 with inner radius 1 and outer radius 2 centered at the origin, i.e.,

$$Q = \{w \in \mathbb{C} \mid 1 < |w| < 2\}.$$

The map $\alpha: Q \rightarrow \mathbb{C} \times \mathbb{R}$,

$$\alpha(w) := \left(\frac{w^2}{|w|^2}, |w| - 1 \right)$$

maps Q onto the cylinder $Z := S^1 \times (0, 1)$. Note that $\alpha(Q)$ covers Z twice. Then the map $\beta: Z \rightarrow \mathbb{C}$,

$$\beta(z, s) := \frac{2}{2-s} z$$

is the linear projection of Z from the point $(0, 2) \in \mathbb{C} \times \mathbb{R}$ to the complex plane. The composed map $\tilde{f} = \beta \circ \alpha: Q \rightarrow \mathbb{C}$,

$$\tilde{f}(w) := \frac{2w^2}{(3 - |w|)|w|^2}$$

is smooth (where we consider $\mathbb{C} = \mathbb{R}^2$ as a 2-dimensional real manifold) and maps Q onto Q . Further it induces a homeomorphism between $Q/\{\pm \text{id}\}$ and Q . Hence, if we endow Q with the orbifold atlas

$$\{(Q, \{\pm \text{id}\}, \tilde{f}), (Q, \{\text{id}\}, \text{id})\},$$

then \tilde{f} is a local lift of id_Q with respect to $(Q, \{\pm \text{id}\}, \tilde{f})$ and $(Q, \{\text{id}\}, \text{id})$ but not a global diffeomorphism.

PROPOSITION 5.5. *Let (Q, \mathcal{U}) be an orbifold and $\{\tilde{f}_i\}_{i \in I}$ a family of local lifts of id_Q which satisfies (R2). Then there exists a pair (P, ν) such that $(\text{id}_Q, \{\tilde{f}_i\}_{i \in I}, P, \nu)$ is a representative of an orbifold map on (Q, \mathcal{U}) . The pair (P, ν) is unique up to equivalence of representatives of orbifold maps.*

PROOF. This follows immediately from Proposition 5.3 in combination with (R4a). \square

PROPOSITION 5.6. *Let Q be a topological space and suppose that \mathcal{U} and \mathcal{U}' are orbifold structures on Q . Let*

$$\hat{f} = (f, \{\tilde{f}_i\}_{i \in I}, [P, \nu])$$

be a charted orbifold map for which $f = \text{id}_Q$, the domain atlas \mathcal{V} is a representative of \mathcal{U} , the range family \mathcal{V}' , which here is an orbifold atlas, is a representative of \mathcal{U}' , and for each $i \in I$, the map \tilde{f}_i is a local diffeomorphism. Then $\mathcal{U} = \mathcal{U}'$.

PROOF. Let $(V_i, G_i, \pi_i) \in \mathcal{V}$, $(V'_j, G'_j, \pi'_j) \in \mathcal{V}'$ and $x \in V_i$, $y \in V'_j$ such that $\pi_i(x) = \pi'_j(y)$. Since $\tilde{f}_i: V_i \rightarrow V'_i$ is a local diffeomorphism, there are open neighborhoods U of x in V_i and U' of $\tilde{f}_i(x)$ in V'_i such that $\tilde{f}_i|_U: U \rightarrow U'$ is a diffeomorphism. We have

$$\pi'_i(\tilde{f}_i(x)) = \pi_i(x) = \pi'_j(y).$$

Therefore there exist open neighborhoods W'_1 of $\tilde{f}_i(x)$ in U' and W'_2 of y in V'_j and a diffeomorphism $h: W'_1 \rightarrow W'_2$ satisfying $\pi'_j \circ h = \pi'_i$. Shrinking U shows that (V_i, G_i, π_i) and (V'_j, G'_j, π'_j) are compatible. Thus $\mathcal{U} = \mathcal{U}'$. \square

The following example shows that the requirement in Proposition 5.6 that the local lifts are local diffeomorphisms is essential.

EXAMPLE 5.7. Recall the orbifolds (Q, \mathcal{U}_i) , $i = 1, 2$, from Example 2.2, the representatives $\mathcal{V}_1 := \{V_1\}$ and $\mathcal{V}_2 := \{V_2\}$ of \mathcal{U}_1 resp. \mathcal{U}_2 , and set $g(x) := x^2$ for $x \in (-1, 1)$. Then g is a lift of id_Q with respect to V_2 and V_1 . Further let

$$P := \{\pm \text{id}_{(-1,1)}\} \quad \text{and} \quad \nu(\pm \text{id}_{(-1,1)}) := \text{id}_{(-1,1)}.$$

Then $(\text{id}_Q, \{g\}, [P, \nu])$ is an orbifold map with $(\mathcal{V}_2, \mathcal{V}_1)$ from (Q, \mathcal{U}_2) to (Q, \mathcal{U}_1) , but $\mathcal{U}_1 \neq \mathcal{U}_2$.

Motivated by Propositions 5.5 and 5.6 we make the following definition.

DEFINITION 5.8. Let (Q, \mathcal{U}) be an orbifold and let $\hat{f} = (f, \{\tilde{f}_i\}_{i \in I}, [P, \nu])$ be a charted orbifold map whose domain atlas is a representative of \mathcal{U} . If and only if $f = \text{id}_Q$ and \tilde{f}_i is a local diffeomorphism for each $i \in I$, we call \hat{f} a *lift of the identity* $\text{id}_{(Q, \mathcal{U})}$ or a *representative* of $\text{id}_{(Q, \mathcal{U})}$. The set of all lifts of $\text{id}_{(Q, \mathcal{U})}$ is the *identity morphism* $\text{id}_{(Q, \mathcal{U})}$ of (Q, \mathcal{U}) .

5.2. Composition of charted orbifold maps.

CONSTRUCTION 5.9. Let (Q, \mathcal{U}) , (Q', \mathcal{U}') and (Q'', \mathcal{U}'') be orbifolds, and

$$\mathcal{V} := \{(V_i, G_i, \pi_i) \mid i \in I\}, \quad \mathcal{V}' := \{(V'_j, G'_j, \pi'_j) \mid j \in J\}$$

resp. \mathcal{V}'' be representatives for \mathcal{U} , \mathcal{U}' resp. \mathcal{U}'' , where \mathcal{V} resp. \mathcal{V}' are indexed by I resp. J . Suppose that

$$\hat{f} = (f, \{\tilde{f}_i\}_{i \in I}, [P_f, \nu_f]) \in \text{Orb}(\mathcal{V}, \mathcal{V}')$$

and

$$\hat{g} = (g, \{\tilde{g}_j\}_{j \in J}, [P_g, \nu_g]) \in \text{Orb}(\mathcal{V}', \mathcal{V}'')$$

are charted orbifold maps and that $\alpha: I \rightarrow J$ is the unique map such that for each $i \in I$, \tilde{f}_i is a local lift of f with respect to (V_i, G_i, π_i) and $(V'_{\alpha(i)}, G'_{\alpha(i)}, \pi'_{\alpha(i)})$. The composition

$$\hat{g} \circ \hat{f} := \hat{h} = (h, \{\tilde{h}_i\}_{i \in I}, [P_h, \nu_h]) \in \text{Orb}(\mathcal{V}, \mathcal{V}'')$$

is given by $h := g \circ f$ and $\tilde{h}_i := \tilde{g}_{\alpha(i)} \circ \tilde{f}_i$ for all $i \in I$. To construct a representative (P_h, ν_h) of $[P_h, \nu_h]$ we fix representatives (P_f, ν_f) and (P_g, ν_g) of $[P_f, \nu_f]$ and $[P_g, \nu_g]$, respectively. The leading idea to define (P_h, ν_h) is to take $P_h = P_f$ and $\nu_h = \nu_g \circ \nu_f$.

But since $\nu_f(\lambda)$ is not necessarily in P_g for $\lambda \in P_f$, the composition $\nu_g \circ \nu_f$ might be ill-defined. In the following we refine this idea.

Let $\mu \in P_f$ and suppose that $\text{dom } \mu \subseteq V_i$ and $\text{cod } \mu \subseteq V_j$ for the orbifold charts (V_i, G_i, π_i) and (V_j, G_j, π_j) in \mathcal{V} . By (R4a)

$$\tilde{f}_j \circ \mu = \nu_f(\mu) \circ \tilde{f}_i|_{\text{dom } \mu},$$

where $\nu_f(\mu) \in \Psi(\mathcal{U}')$. By possibly shrinking domains, we may assume that $\nu_f(\mu) \in \Psi(\mathcal{V}')$. For $x \in \text{dom } \mu$ we set $y_x := \tilde{f}_i(x)$, which is an element of $\text{dom } \nu_f(\mu)$. Hence we find (and fix a choice) $\xi_{\mu,x} \in P_g$ with $y_x \in \text{dom } \xi_{\mu,x}$ and an open set $U'_{\mu,x} \subseteq \text{dom } \xi_{\mu,x} \cap \text{dom } \nu_f(\mu)$ such that $y_x \in U'_{\mu,x}$ and

$$\xi_{\mu,x}|_{U'_{\mu,x}} = \nu_f(\mu)|_{U'_{\mu,x}}.$$

Then we find (and fix) an open set $U_{\mu,x} \subseteq \text{dom } \mu$ with $x \in U_{\mu,x}$ such that $\tilde{f}_i(U_{\mu,x}) \subseteq U'_{\mu,x}$. By adjusting choices we achieve that for $\mu_1, \mu_2 \in P_f$ and $x_1 \in \text{dom } \mu_1$, $x_2 \in \text{dom } \mu_2$ we either have

$$\mu_1|_{U_{\mu_1,x_1}} \neq \mu_2|_{U_{\mu_2,x_2}} \quad \text{or} \quad \xi_{\mu_1,x_1} = \xi_{\mu_2,x_2}. \quad (3)$$

Define

$$P_h := \{\mu|_{U_{\mu,x}} \mid \mu \in P_f, x \in \text{dom } \mu\},$$

which obviously is a quasi-pseudogroup generating $\Psi(\mathcal{V})$, and set

$$\nu_h(\mu|_{U_{\mu,x}}) := \nu_g(\xi_{\mu,x})$$

for $\mu|_{U_{\mu,x}} \in P_h$. Property (3) yields that ν_h is a well-defined map from P_h to $\Psi(\mathcal{U}'')$. One easily sees that ν_h satisfies (R4a)–(R4d), and that the equivalence class of (P_h, ν_h) does not depend on the choices we made for the construction of P_h and ν_h .

REMARK 5.10. The construction of the composition of two charted orbifold maps immediately implies that the maps F_1 and F_2 (see Proposition 4.12) are both functorial.

The following lemma provides the definition of induced charted orbifold map and shows its relation to lifts of the identity.

LEMMA AND DEFINITION 5.11. *Let (Q, \mathcal{U}) and (Q', \mathcal{U}') be orbifolds. Further let*

$$\begin{aligned} \mathcal{V} &= \{(V_i, G_i, \pi_i) \mid i \in I\} \text{ be a representative of } \mathcal{U}, \text{ indexed by } I, \\ \mathcal{V}' &= \{(V'_l, G'_l, \pi'_l) \mid l \in L\} \text{ be a representative of } \mathcal{U}', \text{ indexed by } L, \\ \hat{f} &= (f, \{\tilde{f}_i\}_{i \in I}, [P_f, \nu_f]) \in \text{Orb}(\mathcal{V}, \mathcal{V}'), \end{aligned}$$

and let $\beta: I \rightarrow L$ be the unique map such that for each $i \in I$, \tilde{f}_i is a local lift of f with respect to (V_i, G_i, π_i) and $(V'_{\beta(i)}, G'_{\beta(i)}, \pi'_{\beta(i)})$. Suppose that we have

- a representative $\mathcal{W} = \{(W_j, H_j, \psi_j) \mid j \in J\}$ of \mathcal{U} , indexed by J ,

- a subset $\{(W'_j, H'_j, \psi'_j) \mid j \in J\}$ of \mathcal{U}' , indexed by J (not necessarily an orbifold atlas),
- a map $\alpha: J \rightarrow I$,
- for each $j \in J$, an embedding

$$\lambda_j: (W_j, H_j, \psi_j) \rightarrow (V_{\alpha(j)}, G_{\alpha(j)}, \pi_{\alpha(j)}),$$

and an embedding

$$\mu_j: (W'_j, H'_j, \psi'_j) \rightarrow (V'_{\beta(\alpha(j))}, G'_{\beta(\alpha(j))}, \pi'_{\beta(\alpha(j))})$$

such that

$$\tilde{f}_{\alpha(j)}(\lambda_j(W_j)) \subseteq \mu_j(W'_j).$$

For each $j \in J$ set

$$\tilde{h}_j := \mu_j^{-1} \circ \tilde{f}_{\alpha(j)} \circ \lambda_j: W_j \rightarrow W'_j.$$

Then

- (i) $\varepsilon := (\text{id}_Q, \{\lambda_j\}_{j \in J}, [P_\varepsilon, \nu_\varepsilon]) \in \text{Orb}(\mathcal{W}, \mathcal{V})$ (with $[P_\varepsilon, \nu_\varepsilon]$ provided by Proposition 5.5) is a lift of $\text{id}_{(Q, \mathcal{U})}$.
- (ii) The set $\{(W'_j, H'_j, \psi'_j) \mid j \in J\}$ and the family $\{\mu_j\}_{j \in J}$ can be extended to a representative

$$\mathcal{W}' = \{(W'_k, H'_k, \psi'_k) \mid k \in K\}$$

of \mathcal{U}' and a family of embeddings $\{\mu_k\}_{k \in K}$ such that

$$\varepsilon' := (\text{id}_{Q'}, \{\mu_k\}_{k \in K}, [P_{\varepsilon'}, \nu_{\varepsilon'}]) \in \text{Orb}(\mathcal{W}', \mathcal{V}')$$

(with $[P_{\varepsilon'}, \nu_{\varepsilon'}]$ provided by Proposition 5.5) is a lift of the identity $\text{id}_{(Q', \mathcal{U}')}$.

- (iii) There is a uniquely determined equivalence class $[P_h, \nu_h]$ such that

$$\hat{h} := (f, \{\tilde{h}_j\}_{j \in J}, [P_h, \nu_h]) \in \text{Orb}(\mathcal{W}, \mathcal{W}')$$

and such that the diagram

$$\begin{array}{ccccc} & & \mathcal{V} & \xrightarrow{\hat{f}} & \mathcal{V}' \\ & \nearrow \varepsilon & & & \swarrow \varepsilon' \\ \mathcal{W} & \xrightarrow{\hat{h}} & & & \mathcal{W}' \end{array}$$

commutes.

We say that \hat{h} is induced by \hat{f} .

PROOF. (i) is clear by Propositions 5.3 and 5.5. To show that (ii) holds we construct one possible extension: Let

$$y \in Q' \setminus \bigcup_{j \in J} \psi'_j(W'_j).$$

Then there is a chart $(V', G', \pi') \in \mathcal{V}'$ such that $y \in \pi'(V')$. Extend the set

$$\{(W'_j, H'_j, \psi'_j) \mid j \in J\}$$

with (V', G', π') and the family $\{\mu_j\}_{j \in J}$ with $\text{id}_{V'}$. If this is done iteratively, one finally gets an orbifold atlas of Q' as wanted. Then Propositions 5.3 and 5.5 yield the remaining claim of (ii). The following considerations are independent of the specific choices of extensions. Concerning (iii) we remark that each \tilde{h}_j is obviously a local lift of f . Fix a representative (P_f, ν_f) of $[P_f, \nu_f]$. In the following we construct a pair (P_h, ν_h) for which \hat{h} is an orbifold map and the diagram in (iii) commutes. It will be clear from the construction that the equivalence class $[P_h, \nu_h]$ is independent of the choice of (P_f, ν_f) and uniquely determined by the requirement of the commutativity of the diagram. Let $\gamma \in \Psi(\mathcal{W})$ and $x \in \text{dom } \gamma$. Possibly shrinking the domain of γ , we may assume that $\text{dom } \gamma \subseteq W_j$ and $\text{cod } \gamma \subseteq W_k$ for some $j, k \in J$. In the following we further shrink the domain of γ to be able to define ν_h as a composition of ν_f with elements of $\{\mu_j\}_{j \in J}$. Let $y := \lambda_j(x)$. Since

$$\tilde{\gamma} := \lambda_k \circ \gamma \circ (\lambda_j|_{\text{dom } \gamma})^{-1} : \lambda_j(\text{dom } \gamma) \rightarrow \lambda_k(\text{cod } \gamma)$$

is an element of $\Psi(\mathcal{V})$, we find $\beta_\gamma \in P_f$ such that $y \in \text{dom } \beta_\gamma$ and $\text{germ}_y \beta_\gamma = \text{germ}_y \tilde{\gamma}$. Then

$$z := \tilde{f}_{\alpha(j)}(y) \in \text{dom } \nu_f(\beta_\gamma) \cap \mu_j(W'_j).$$

Since

$$\nu_f(\beta_\gamma)(z) = \tilde{f}_{\alpha(k)}(\beta_\gamma(y)) \in \mu_k(W'_k),$$

the set

$$U' := \text{dom } \nu_f(\beta_\gamma) \cap \mu_j(W'_j) \cap \nu_f(\beta_\gamma)^{-1}(\mu_k(W'_k))$$

is an open neighborhood of z . Define

$$U_1 := \{w \in \text{dom } \beta_\gamma \cap \lambda_j(\text{dom } \gamma) \mid \text{germ}_w \beta_\gamma = \text{germ}_w \tilde{\gamma}\},$$

which is an open neighborhood of y . Then also

$$U := U_1 \cap \tilde{f}_{\alpha(j)}^{-1}(U')$$

is an open neighborhood of y . We fix an open neighborhood $U_{\gamma, x}$ of x in $\lambda_j^{-1}(U)$. Further we suppose that for $\gamma_1, \gamma_2 \in \Psi(\mathcal{W})$, $x_1 \in \text{dom } \gamma_1$, $x_2 \in \text{dom } \gamma_2$, we either have

$$\gamma_1|_{U_{\gamma_1, x_1}} \neq \gamma_2|_{U_{\gamma_2, x_2}} \quad \text{or} \quad \nu_f(\beta_{\gamma_1}) = \nu_f(\beta_{\gamma_2}). \quad (4)$$

Then we define

$$P_h := \{\gamma|_{U_{\gamma, x}} \mid \gamma \in \Psi(\mathcal{W}), x \in \text{dom } \gamma\}$$

and set

$$\nu_h(\gamma|_{U_{\gamma, x}}) := \mu_k^{-1} \circ \nu_f(\beta_\gamma) \circ \mu_j$$

for $\gamma|_{U_{\gamma, x}} \in P_h$ with $x \in W_j$ and $\gamma(x) \in W_k$ ($j, k \in J$). The map $\nu_h: P_h \rightarrow \Psi(\mathcal{W}')$ is well-defined by (4). One easily checks that (P_h, ν_h) satisfies all requirements of (iii). \square

We consider two charted orbifold maps as equivalent if they induce the same charted orbifold map on common refinements of the orbifold atlases. The following definition provides a precise specification of this idea.

DEFINITION 5.12. Let (Q, \mathcal{U}) and (Q', \mathcal{U}') be orbifolds. Further let $\mathcal{V}_1, \mathcal{V}_2$ be representatives of \mathcal{U} , and $\mathcal{V}'_1, \mathcal{V}'_2$ be representatives of \mathcal{U}' . Suppose that $\hat{f}_1 \in \text{Orb}(\mathcal{V}_1, \mathcal{V}'_1)$ and $\hat{f}_2 \in \text{Orb}(\mathcal{V}_2, \mathcal{V}'_2)$. We call \hat{f}_1 and \hat{f}_2 *equivalent* ($\hat{f}_1 \sim \hat{f}_2$) if there are a representative \mathcal{W} of \mathcal{U} , a representative \mathcal{W}' of \mathcal{U}' , $\varepsilon_1 \in \text{Orb}(\mathcal{W}, \mathcal{V}_1)$, $\varepsilon_2 \in \text{Orb}(\mathcal{W}, \mathcal{V}_2)$ lifts of $\text{id}_{(Q, \mathcal{U})}$, $\varepsilon'_1 \in \text{Orb}(\mathcal{W}', \mathcal{V}'_1)$, $\varepsilon'_2 \in \text{Orb}(\mathcal{W}', \mathcal{V}'_2)$ lifts of $\text{id}_{(Q', \mathcal{U}')}$, and a map $\hat{h} \in \text{Orb}(\mathcal{W}, \mathcal{W}')$ such that the diagram

$$\begin{array}{ccccc} & & \mathcal{V}_1 & \xrightarrow{\hat{f}_1} & \mathcal{V}'_1 \\ & \nearrow \varepsilon_1 & & & \swarrow \varepsilon'_1 \\ \mathcal{W} & \xrightarrow{\hat{h}} & \mathcal{W}' & & \\ & \searrow \varepsilon_2 & & & \swarrow \varepsilon'_2 \\ & & \mathcal{V}_2 & \xrightarrow{\hat{f}_2} & \mathcal{V}'_2 \end{array}$$

commutes.

Proposition 5.15 below shows that \sim is indeed an equivalence relation. For its proof we need the following two lemmas. The first lemma discusses how local lifts which belong to the same charted orbifold map are related to each other. The second lemma shows that two charted orbifold maps which are induced from the same charted orbifold map induce the same charted orbifold map on common refinements of orbifold atlases. This means that \sim satisfies the so-called diamond property.

LEMMA 5.13. Let (Q, \mathcal{U}) and (Q', \mathcal{U}') be orbifolds and let

$$\hat{f} := (f, \{\tilde{f}_i\}_{i \in I}, [P, \nu]) \in \text{Orb}(\mathcal{V}, \mathcal{V}')$$

be a charted orbifold map where \mathcal{V} is a representative of \mathcal{U} and \mathcal{V}' one of \mathcal{U}' . Suppose that we have orbifold charts $(V_a, G_a, \pi_a), (V_b, G_b, \pi_b) \in \mathcal{V}$ and points $x_a \in V_a, x_b \in V_b$ such

that $\pi_a(x_a) = \pi_b(x_b)$. Then there are arbitrarily small orbifold charts $(W, K, \chi) \in \mathcal{U}$, $(W', K', \chi') \in \mathcal{U}'$ and embeddings

$$\begin{aligned}\lambda: (W, K, \chi) &\rightarrow (V_a, G_a, \pi_a) \\ \lambda': (W', K', \chi') &\rightarrow (V'_a, G'_a, \pi'_a) \\ \mu: (W, K, \chi) &\rightarrow (V_b, G_b, \pi_b) \\ \mu': (W', K', \chi') &\rightarrow (V'_b, G'_b, \pi'_b)\end{aligned}$$

with $x_a \in \lambda(W)$ and $x_b \in \mu(W)$ such that the induced lift \tilde{g} of f with respect to $\tilde{f}_a, \lambda, \lambda'$ coincides with the one induced by \tilde{f}_b, μ, μ' . In other words, the diagram

$$\begin{array}{ccccc} & V_a & \xrightarrow{\tilde{f}_a} & V'_a & \\ \lambda \nearrow & & & & \swarrow \lambda' \\ W & \xrightarrow{\tilde{g}} & W' & & \\ \mu \searrow & & & & \swarrow \mu' \\ & V_b & \xrightarrow{\tilde{f}_b} & V'_b & \end{array}$$

commutes.

PROOF. By compatibility of orbifold charts we find an arbitrarily small restriction (W, K, χ) of (V_a, G_a, π_a) with $x_a \in W$ and an embedding

$$\mu: (W, K, \chi) \rightarrow (V_b, G_b, \pi_b)$$

such that $\mu(x_a) = x_b$. Then $\mu: W \rightarrow \mu(W)$ is an element of $\Psi(\mathcal{V})$. Fix a representative (P, ν) of $[P, \nu]$. Hence there is $\gamma \in P$ with $x_a \in \text{dom } \gamma$ and an open neighborhood U of x_a such that $U \subseteq \text{dom } \gamma \cap W$ and

$$\mu|_U = \gamma|_U.$$

With loss of generality, $\gamma = \mu$. Property (R4a) yields that

$$\nu(\mu) \circ \tilde{f}_a|_W = \tilde{f}_b \circ \mu.$$

By shrinking the domain of $\nu(\mu)$, we can achieve that $\text{cod } \nu(\mu) \subseteq V'_b$ and still $\tilde{f}_a(W) \subseteq \text{dom } \nu(\mu) =: W'$. With $\mu' := \nu(\mu)$ it follows

$$\tilde{f}_b(\mu(W)) = \mu'(\tilde{f}_a(W)) \subseteq \mu'(W')$$

and further

$$\tilde{f}_a|_W = (\mu')^{-1} \circ \tilde{f}_b \circ \mu.$$

This proves the claim. □

LEMMA 5.14. Let (Q, \mathcal{U}) and (Q', \mathcal{U}') be orbifolds, \mathcal{V} a representative of \mathcal{U} , and \mathcal{V}' one of \mathcal{U}' . Further let $\hat{f} \in \text{Orb}(\mathcal{V}, \mathcal{V}')$. Suppose that $\hat{h} \in \text{Orb}(\mathcal{W}_1, \mathcal{W}'_1)$ and $\hat{g} \in \text{Orb}(\mathcal{W}_2, \mathcal{W}'_2)$ are both induced by \hat{f} . Then we find a representative \mathcal{W} of \mathcal{U} and charted orbifold maps $\varepsilon_1 \in \text{Orb}(\mathcal{W}, \mathcal{W}_1)$, $\varepsilon_2 \in \text{Orb}(\mathcal{W}, \mathcal{W}_2)$ which are lifts of $\text{id}_{(Q, \mathcal{U})}$, and a representative \mathcal{W}' of \mathcal{U}' and charted orbifold maps $\varepsilon'_1 \in \text{Orb}(\mathcal{W}', \mathcal{W}'_1)$, $\varepsilon'_2 \in \text{Orb}(\mathcal{W}', \mathcal{W}'_2)$ which are lifts of $\text{id}_{(Q', \mathcal{U}')}$, and a charted orbifold map $\hat{k} \in \text{Orb}(\mathcal{W}, \mathcal{W}')$ such that the diagram

$$\begin{array}{ccccc}
& & \mathcal{W}_1 & \xrightarrow{\hat{h}} & \mathcal{W}'_1 \\
& \nearrow \varepsilon_1 & & & \searrow \varepsilon'_1 \\
\mathcal{W} & \xrightarrow{\hat{k}} & \mathcal{W}' & & \\
& \searrow \varepsilon_2 & & & \swarrow \varepsilon'_2 \\
& & \mathcal{W}_2 & \xrightarrow{\hat{g}} & \mathcal{W}'_2
\end{array}$$

commutes. If the orbifolds are second-countable, we can choose $\mathcal{W}, \mathcal{W}'$ to be countable.

PROOF. Suppose that $\hat{f} = (f, \{\tilde{f}_a\}_{a \in A}, [P_f, \nu_f])$, $\hat{h} = (f, \{\tilde{h}_i\}_{i \in I}, [P_h, \nu_h])$ and $\hat{g} = (f, \{\tilde{g}_j\}_{j \in J}, [P_g, \nu_g])$. Let

$$\begin{aligned}
\mathcal{W}_1 &:= \{(W_{1,i}, H_{1,i}, \psi_{1,i}) \mid i \in I\}, \text{ indexed by } I, \\
\mathcal{W}'_1 &:= \{(W'_{1,k}, H'_{1,k}, \psi'_{1,k}) \mid k \in K\}, \text{ indexed by } K, \\
\mathcal{W}_2 &:= \{(W_{2,j}, H_{2,j}, \psi_{2,j}) \mid j \in J\}, \text{ indexed by } J, \\
\mathcal{W}'_2 &:= \{(W'_{2,l}, H'_{2,l}, \psi'_{2,l}) \mid l \in L\}, \text{ indexed by } L,
\end{aligned}$$

and let $\alpha_1: I \rightarrow K$ resp. $\alpha_2: J \rightarrow L$ be the map such that for each $i \in I$, \tilde{h}_i is a local lift of f with respect to $(W_{1,i}, H_{1,i}, \psi_{1,i})$ and $(W'_{1,\alpha_1(i)}, H'_{1,\alpha_1(i)}, \psi'_{1,\alpha_1(i)})$ resp. for each $j \in J$, \tilde{g}_j is a local lift of f with respect to $(W_{2,j}, H_{2,j}, \psi_{2,j})$ and $(W'_{2,\alpha_2(j)}, H'_{2,\alpha_2(j)}, \psi'_{2,\alpha_2(j)})$. Further let

$$\begin{aligned}
\delta_1 &= (\text{id}_Q, \{\lambda_{1,i}\}_{i \in I}, [R_1, \sigma_1]) \in \text{Orb}(\mathcal{W}_1, \mathcal{V}), \\
\delta_2 &= (\text{id}_Q, \{\lambda_{2,j}\}_{j \in J}, [R_2, \sigma_2]) \in \text{Orb}(\mathcal{W}_2, \mathcal{V})
\end{aligned}$$

be lifts of $\text{id}_{(Q, \mathcal{U})}$ and

$$\begin{aligned}
\delta'_1 &= (\text{id}_{Q'}, \{\mu_{1,k}\}_{k \in K}, [R'_1, \sigma'_1]) \in \text{Orb}(\mathcal{W}'_1, \mathcal{V}'), \\
\delta'_2 &= (\text{id}_{Q'}, \{\mu_{2,l}\}_{l \in L}, [R'_2, \sigma'_2]) \in \text{Orb}(\mathcal{W}'_2, \mathcal{V}')
\end{aligned}$$

be lifts of $\text{id}_{(Q', \mathcal{U}')}$ such that $\hat{f} \circ \delta_1 = \delta'_1 \circ \hat{h}$ and $\hat{f} \circ \delta_2 = \hat{g} \circ \delta'_2$. With loss of generality we assume that all $\lambda_{1,i}$, $\mu_{1,k}$, $\lambda_{2,j}$ and $\mu_{2,l}$ are embeddings. We will use Lemma 5.11 to show the existence of \hat{k} . More precisely, we attach to each $q \in Q$ an orbifold chart $(W_q, H_q, \psi_q) \in \mathcal{U}$ with $q \in \psi_q(W_q)$ and an orbifold chart $(W'_q, H'_q, \psi'_q) \in \mathcal{U}'$ with $f(q) \in \psi'_q(W'_q)$. We consider orbifold charts defined for distinct q to be distinct. In this way, we

get a representative

$$\mathcal{W} := \{(W_q, H_q, \psi_q) \mid q \in Q\} \quad (5)$$

of \mathcal{U} which is indexed by Q , and a subset $\{(W'_q, H'_q, \psi'_q) \mid q \in Q\}$ of \mathcal{U}' , indexed by Q as well. Moreover, we will find maps $\beta_1: Q \rightarrow I$ and $\beta_2: Q \rightarrow J$ and embeddings

$$\begin{aligned} \xi_{1,q}: (W_q, H_q, \psi_q) &\rightarrow (W_{1,\beta_1(q)}, H_{1,\beta_1(q)}, \psi_{1,\beta_1(q)}) \\ \xi_{2,q}: (W_q, H_q, \psi_q) &\rightarrow (W_{2,\beta_2(q)}, H_{2,\beta_2(q)}, \psi_{2,\beta_2(q)}) \\ \chi_{1,q}: (W'_q, H'_q, \psi'_q) &\rightarrow (W'_{1,\alpha_1(\beta_1(q))}, H'_{1,\alpha_1(\beta_1(q))}, \psi'_{1,\alpha_1(\beta_1(q))}) \\ \chi_{2,q}: (W'_q, H'_q, \psi'_q) &\rightarrow (W'_{2,\alpha_2(\beta_2(q))}, H'_{2,\alpha_2(\beta_2(q))}, \psi'_{2,\alpha_2(\beta_2(q))}) \end{aligned}$$

such that for each $q \in Q$ the local lift \tilde{k}_q of f induced by $\tilde{h}_{\beta_1(q)}$, $\xi_{1,q}$ and $\chi_{1,q}$ coincides with the one induced by $\tilde{g}_{\beta_2(q)}$, $\xi_{2,q}$ and $\chi_{2,q}$. Then Lemma 5.11 shows that \hat{h} resp. \hat{g} induces a charted orbifold map $(f, \{\tilde{k}_q\}_{q \in Q}, [P_1, \nu_1])$ resp. $(f, \{\tilde{k}_q\}_{q \in Q}, [P_2, \nu_2])$. It then remains to show that we can choose all the embeddings $\xi_{1,q}, \xi_{2,q}, \chi_{1,q}, \chi_{2,q}$ in a way that $[P_1, \nu_1]$ equals $[P_2, \nu_2]$.

Let $q \in Q$. We fix $i \in I$ such that $q \in \psi_{1,i}(W_{1,i})$ and we pick $w_1 \in W_{1,i}$ with $q = \psi_{1,i}(w_1)$. We set $\beta_1(q) := i$. Further we fix $j \in J$ such that $q \in \psi_{2,j}(W_{2,j})$, and pick an element $w_2 \in W_{2,j}$ with $q = \psi_{2,j}(w_2)$. We set $\beta_2(q) := j$. By Lemma 5.13 we find orbifold charts $(W_q, H_q, \psi_q) \in \mathcal{U}$ with $q \in \psi_q(W_q)$, say $q = \psi_q(w_q)$, and $(W'_q, H'_q, \psi'_q) \in \mathcal{U}'$ with $f(q) \in \psi'_q(W'_q)$ and embeddings $\xi_{1,q}, \xi_{2,q}, \chi_{1,q}, \chi_{2,q}$ with $w_1 = \xi_{1,q}(w_q)$, $w_2 = \xi_{2,q}(w_q)$, and a local lift \tilde{k}_q of f such that the diagram

$$\begin{array}{ccccc} & & \lambda_{1,\beta_1(q)}(W_{1,\beta_1(q)}) & \xrightarrow{\tilde{f}_{\beta_1(q)}} & \mu_{1,\alpha_1(\beta_1(q))}(W'_{1,\alpha_1(\beta_1(q))}) \\ & \lambda_{1,\beta_1(q)} \uparrow & & & \uparrow \mu_{1,\alpha_1(\beta_1(q))} \\ & W_{1,\beta_1(q)} & \xrightarrow{\tilde{h}_{\beta_1(q)}} & W'_{1,\alpha_1(\beta_1(q))} & \\ \xi_{1,q} \nearrow & & & & \chi_{1,q} \swarrow \\ W_q & \xrightarrow{\tilde{k}_q} & W'_q & & \\ \xi_{2,q} \searrow & & & & \chi_{2,q} \swarrow \\ & W_{2,\beta_2(q)} & \xrightarrow{\tilde{g}_{\beta_2(q)}} & W'_{2,\alpha_2(\beta_2(q))} & \\ & \lambda_{2,\beta_2(q)} \downarrow & & \downarrow \mu_{2,\alpha_2(\beta_2(q))} & \\ & \lambda_{2,\beta_2(q)}(W_{2,\beta_2(q)}) & \xrightarrow{\tilde{f}_{\beta_2(q)}} & \mu_{2,\alpha_2(\beta_2(q))}(W'_{2,\alpha_2(\beta_2(q))}) & \end{array}$$

commutes. We may assume that $\xi_{1,q} = \text{id}$ and $\chi_{1,q} = \text{id}$. Now

$$\eta := \lambda_{2,\beta_2(q)} \circ \xi_{2,q} \circ \lambda_{1,\beta_1(q)}^{-1}: \lambda_{1,\beta_1(q)}(W_q) \rightarrow \lambda_{2,\beta_2(q)}(\xi_{2,q}(W_q))$$

is an element of $\Psi(\mathcal{V})$ with $y := \lambda_{1,\beta_1(q)}(\xi_{1,q}(w_q))$ in its domain. We pick a representative

(P_f, ν_f) of $[P_f, \nu_f]$. Then there is an element $\gamma \in P_f$ with $y \in \text{dom } \gamma$ and an open neighborhood U of y such that $U \subseteq \text{dom } \gamma \cap \text{dom } \eta$ and $\eta|_U = \gamma|_U$. By (R4a),

$$\nu_f(\gamma) \circ \tilde{f}_{\beta_1(q)}|_U = \tilde{f}_{\beta_2(q)} \circ \gamma|_U = \tilde{f}_{\beta_2(q)} \circ \eta|_U.$$

The map

$$\mu := \mu_{2,\alpha_2(\beta_2(q))} \circ \chi_{2,q} \circ \mu_{1,\alpha_1(\beta_1(q))}^{-1} : \mu_{1,\alpha_1(\beta_1(q))}(W'_q) \rightarrow \mu_{2,\alpha_2(\beta_2(q))}(\chi_{2,q}(W'_q))$$

is a diffeomorphism as well. Further there exists an open neighborhood V of y such that

$$\tilde{f}_{\beta_2(q)} \circ \eta|_V = \mu \circ \tilde{f}_{\beta_1(q)}|_V.$$

Hence

$$\nu_f(\gamma) \circ \tilde{f}_{\beta_1(q)} = \mu \circ \tilde{f}_{\beta_1(q)}$$

on some neighborhood of y . Therefore, after possibly shrinking W_q , we can redefine W'_q , $\chi_{2,q}$ and \tilde{k}_q such that

$$\chi_{2,q} = \mu_{2,\alpha_2(\beta_2(q))}^{-1} \circ \nu_f(\gamma) \circ \mu_{1,\alpha_1(\beta_1(q))}|_{W'_q}. \quad (6)$$

We remark that this redefinition might be quite serious if $\tilde{f}_{\beta_1(q)}$ and hence $\tilde{h}_{\beta_1(q)}$, $\tilde{g}_{\beta_2(q)}$ and $\tilde{f}_{\beta_2(q)}$ are highly non-injective. But since these maps all behave in the same way, we may perform the changes without running into problems. Let \mathcal{W} be defined by (5). Lemma 5.11, more precisely its proof, shows that \hat{h} resp. \hat{g} induces the orbifold maps

$$\hat{k}_1 = (f, \{\tilde{k}_q\}_{q \in Q}, [P_1, \nu_1]) \quad \text{resp.} \quad \hat{k}_2 = (f, \{\tilde{k}_q\}_{q \in Q}, [P_2, \nu_2])$$

with $(\mathcal{W}, \mathcal{W}')$, where \mathcal{W}' is a representative of \mathcal{U}' which contains the set

$$\{(W'_q, H'_q, \psi'_q) \mid q \in Q\}$$

(the proof of Lemma 5.11 shows that we can indeed have the same \mathcal{W}' for \hat{k}_1 and \hat{k}_2).

It remains to show that $[P_1, \nu_1] = [P_2, \nu_2]$. Recall from Lemma 5.11 that $[P_1, \nu_1]$ is uniquely determined by \hat{h} , $\{\xi_{1,q}\}_{q \in Q}$ and $\{\chi_{1,q}\}_{q \in Q}$, and analogously for $[P_2, \nu_2]$. Alternatively, we may consider \hat{k}_1 and \hat{k}_2 to be induced by \hat{f} . Thus, $[P_1, \nu_1]$ is uniquely determined by \hat{f} , $\{\lambda_{1,\beta_1(q)} \circ \xi_{1,q}\}_{q \in Q}$ and $\{\mu_{1,\alpha_1(\beta_1(q))} \circ \chi_{1,q}\}_{q \in Q}$, and $[P_2, \nu_2]$ is uniquely determined by \hat{f} , $\{\lambda_{2,\beta_2(q)} \circ \xi_{2,q}\}_{q \in Q}$ and $\{\mu_{2,\alpha_2(\beta_2(q))} \circ \chi_{2,q}\}_{q \in Q}$. We fix a representative (P_f, ν_f) of $[P_f, \nu_f]$. Let γ be a change of charts in $\Psi(\mathcal{W})$ and $x \in \text{dom } \gamma$. Suppose $\text{dom } \gamma \subseteq W_p$ and $\text{cod } \gamma \subseteq W_q$. Using the same arguments and notation as in the proof of Lemma 5.11 (without discussing the necessary shrinking of domains, since we are only interested in equality in a neighborhood of x) we have

$$\begin{aligned} \beta_h &= \lambda_{1,\beta_1(q)} \circ \gamma \circ \lambda_{1,\beta_1(p)}^{-1}, \\ \beta_g &= \lambda_{2,\beta_2(q)} \circ \xi_{2,q} \circ \gamma \circ \xi_{2,p}^{-1} \circ \lambda_{2,\beta_2(p)}^{-1}, \end{aligned}$$

$$\begin{aligned}\nu_1(\gamma) &= \mu_{1,\alpha_1(\beta_1(q))}^{-1} \circ \nu_f(\beta_h) \circ \mu_{1,\alpha_1(\beta_1(p))}, \\ \nu_2(\gamma) &= \chi_{2,q}^{-1} \circ \mu_{2,\alpha_2(\beta_2(q))}^{-1} \circ \nu_f(\beta_g) \circ \mu_{2,\alpha_2(\beta_2(p))} \circ \chi_{2,p}.\end{aligned}$$

Hence

$$\beta_g = \lambda_{2,\beta_2(q)} \circ \xi_{2,q} \circ \lambda_{1,\beta_1(q)}^{-1} \circ \beta_h \circ \lambda_{1,\beta_1(p)} \circ \xi_{2,p}^{-1} \circ \lambda_{2,\beta_2(p)}^{-1}.$$

Definition (6) shows that

$$\nu_f(\lambda_{2,\beta_2(q)} \circ \xi_{2,q} \circ \xi_{1,q}^{-1} \circ \lambda_{1,\beta_1(q)}^{-1}) = \mu_{2,\alpha_2(\beta_2(q))} \circ \chi_{2,q} \circ \mu_{1,\alpha_1(\beta_1(q))}^{-1}.$$

Then

$$\nu_2(\gamma) = \mu_{1,\alpha_1(\beta_1(q))}^{-1} \circ \nu_f(\beta_h) \circ \mu_{1,\alpha_1(\beta_1(p))} = \nu_1(\gamma).$$

Hence the induced equivalence classes $[P_1, \nu_1]$ and $[P_2, \nu_2]$ indeed coincide. The lift ε_1 of $\text{id}_{(Q, \mathcal{U})}$ is given by the family $\{\xi_{1,q}\}_{q \in Q}$, the lift ε_2 by $\{\xi_{2,q}\}_{q \in Q}$, the lift ε'_1 of $\text{id}_{(Q', \mathcal{U}')}$ is any extension of $\{\chi_{1,q}\}_{q \in Q}$, and the lift ε'_2 is any extension of $\{\chi_{2,q}\}_{q \in Q}$.

If the orbifolds are second-countable, we find countable subfamilies of \mathcal{W} and \mathcal{W}' which are themselves orbifold atlases and restricted to which the orbifold map \hat{f} still satisfies the statement of the lemma. Alternatively, we could have started the construction with an appropriate countable subset of Q . \square

The following proposition now follows immediately.

PROPOSITION 5.15. *The relation \sim from Definition 5.12 is an equivalence relation.*

The equivalence class of a charted orbifold map \hat{f} with respect to the equivalence from Definition 5.12 is denoted by $[\hat{f}]$. It will always be clear from context whether \hat{f} is a charted orbifold map and $[\hat{f}]$ denotes an equivalence class of charted orbifold maps, or \hat{f} is a representative of an orbifold map and $[\hat{f}]$ denotes an equivalence class of representatives, that is, a charted orbifold map (cf. Definition 4.10).

5.3. The orbifold category.

Now we can define the category of reduced orbifolds.

DEFINITION 5.16. The category Orb of reduced (paracompact resp. second-countable) orbifolds is defined as follows: In both cases, the class of objects is the class of orbifolds. For two paracompact orbifolds (Q, \mathcal{U}) and (Q', \mathcal{U}') , the morphisms (*orbifold maps*) from (Q, \mathcal{U}) to (Q', \mathcal{U}') are the equivalence classes $[\hat{f}]$ of all charted orbifold maps $\hat{f} \in \text{Orb}(\mathcal{V}, \mathcal{V}')$ where \mathcal{V} is any representative of \mathcal{U} , and \mathcal{V}' is any representative of \mathcal{U}' , that is

$$\text{Morph } ((Q, \mathcal{U}), (Q', \mathcal{U}')) := \left\{ [\hat{f}] \mid \begin{array}{l} \hat{f} \in \text{Orb}(\mathcal{V}, \mathcal{V}'), \mathcal{V} \text{ representative of } \mathcal{U}, \\ \mathcal{V}' \text{ representative of } \mathcal{U}' \end{array} \right\}.$$

For second-countable orbifolds we restrict to countable representatives of the orbifold structures. We now describe the composition in Orb . For this let

$$[\hat{f}] \in \text{Morph}((Q, \mathcal{U}), (Q', \mathcal{U}')) \quad \text{and} \quad [\hat{g}] \in \text{Morph}((Q', \mathcal{U}'), (Q'', \mathcal{U}''))$$

be orbifold maps. Choose representatives $\hat{f} \in \text{Orb}(\mathcal{V}, \mathcal{V}')$ of $[\hat{f}]$ and $\hat{g} \in \text{Orb}(\mathcal{W}', \mathcal{W}'')$ of $[\hat{g}]$. Then find (countable, if the orbifolds are second-countable) representatives \mathcal{K} , \mathcal{K}' , \mathcal{K}'' of \mathcal{U} , \mathcal{U}' , \mathcal{U}'' , resp., and lifts of identity $\varepsilon \in \text{Orb}(\mathcal{K}, \mathcal{V})$, $\varepsilon'_1 \in \text{Orb}(\mathcal{K}', \mathcal{V}')$, $\varepsilon'_2 \in \text{Orb}(\mathcal{K}', \mathcal{W}')$, $\varepsilon'' \in \text{Orb}(\mathcal{K}'', \mathcal{W}'')$ and two charted orbifold maps $\hat{h} \in \text{Orb}(\mathcal{K}, \mathcal{K}')$, $\hat{k} \in \text{Orb}(\mathcal{K}', \mathcal{K}'')$ such that the diagram

$$\begin{array}{ccccc} & \mathcal{V} & \xrightarrow{\hat{f}} & \mathcal{V}' & \\ \varepsilon \nearrow & & & \swarrow \varepsilon'_1 & \\ \mathcal{K} & \xrightarrow{\hat{h}} & \mathcal{K}' & \xrightarrow{\hat{k}} & \mathcal{K}'' \\ & \mathcal{W}' & \xrightarrow{\hat{g}} & \mathcal{W}'' & \\ \varepsilon'_2 \nearrow & & & \swarrow \varepsilon'' & \end{array}$$

commutes. The composition of $[\hat{g}]$ and $[\hat{f}]$ is defined to be

$$[\hat{g}] \circ [\hat{f}] := [\hat{k} \circ \hat{h}].$$

The following lemma shows that this composition is always possible, Proposition 5.18 below that it is well-defined.

LEMMA 5.17. *Let (Q, \mathcal{U}) , (Q', \mathcal{U}') and (Q'', \mathcal{U}'') be orbifolds. Further let \mathcal{V} be a representative of \mathcal{U} , \mathcal{V}' and \mathcal{W}' be representatives of \mathcal{U}' , and \mathcal{W}'' a representative of \mathcal{U}'' . Suppose that $\hat{f} \in \text{Orb}(\mathcal{V}, \mathcal{V}')$ and $\hat{g} \in \text{Orb}(\mathcal{W}', \mathcal{W}'')$. Then there exist representatives \mathcal{K} of \mathcal{U} , \mathcal{K}' of \mathcal{U}' , \mathcal{K}'' of \mathcal{U}'' , lifts of the respective identities $\varepsilon \in \text{Orb}(\mathcal{K}, \mathcal{V})$, $\eta_1 \in \text{Orb}(\mathcal{K}', \mathcal{V}')$, $\eta_2 \in \text{Orb}(\mathcal{K}', \mathcal{W}')$, $\delta \in \text{Orb}(\mathcal{K}'', \mathcal{W}'')$, and charted orbifold maps $\hat{h} \in \text{Orb}(\mathcal{K}, \mathcal{K}')$, $\hat{k} \in \text{Orb}(\mathcal{K}', \mathcal{K}'')$ such that the diagram*

$$\begin{array}{ccccc} & \mathcal{V} & \xrightarrow{\hat{f}} & \mathcal{V}' & \\ \varepsilon \nearrow & & & \swarrow \eta_1 & \\ \mathcal{K} & \xrightarrow{\hat{h}} & \mathcal{K}' & \xrightarrow{\hat{k}} & \mathcal{K}'' \\ & \mathcal{W}' & \xrightarrow{\hat{g}} & \mathcal{W}'' & \\ \eta_2 \nearrow & & & \swarrow \delta & \end{array}$$

commutes. For second-countable orbifolds we can choose \mathcal{K} , \mathcal{K}' and \mathcal{K}'' to be countable.

PROOF. Let $\hat{f} = (f, \{\tilde{f}_i\}_{i \in I}, [P_f, \nu_f])$ and $\hat{g} = (g, \{\tilde{g}_j\}_{j \in J}, [P_g, \nu_g])$. Suppose that

$$\mathcal{V} = \{(V_i, G_i, \pi_i) \mid i \in I\}, \text{ indexed by } I,$$

$$\mathcal{V}' = \{(V'_c, G'_c, \pi'_c) \mid c \in C\}, \text{ indexed by } C,$$

$$\mathcal{W}' = \{(W'_j, H'_j, \psi'_j) \mid j \in J\}, \text{ indexed by } J,$$

$$\mathcal{W}'' = \{(W''_d, H''_d, \psi''_d) \mid d \in D\}, \text{ indexed by } D.$$

Let $\tau: I \rightarrow C$ be the map such that for each $i \in I$, \tilde{f}_i is a local lift of f with respect to (V_i, G_i, π_i) and $(V'_{\tau(i)}, G'_{\tau(i)}, \pi'_{\tau(i)})$, and $\nu: J \rightarrow D$ the map such that for each $j \in J$, \tilde{g}_j is a local lift of g with respect to (W'_j, H'_j, ψ'_j) and $(W''_{\nu(j)}, H''_{\nu(j)}, \psi''_{\nu(j)})$. By Lemma 5.11 it suffices to find

- a representative $\mathcal{K} = \{(K_a, L_a, \chi_a) \mid a \in A\}$ of \mathcal{U} , indexed by A ,
- a representative $\mathcal{K}' = \{(K'_b, L'_b, \chi'_b) \mid b \in B\}$ of \mathcal{U}' , indexed by B ,
- a subset $\{(K''_b, L''_b, \chi''_b) \mid b \in B\}$ of \mathcal{U}'' , indexed by (the same set) B ,
- a map $\alpha: A \rightarrow I$,
- an injective map $\beta: A \rightarrow B$,
- for each $a \in A$, an embedding

$$\lambda_a: (K_a, L_a, \chi_a) \rightarrow (V_{\alpha(a)}, G_{\alpha(a)}, \pi_{\alpha(a)})$$

and an embedding

$$\mu_a: (K'_{\beta(a)}, L'_{\beta(a)}, \chi'_{\beta(a)}) \rightarrow (V'_{\tau(\alpha(a))}, G'_{\tau(\alpha(a))}, \pi'_{\tau(\alpha(a))})$$

such that

$$\tilde{f}_{\alpha(a)}(\lambda_a(K_a)) \subseteq \mu_a(K'_{\beta(a)}),$$

- a map $\gamma: B \rightarrow J$,
- for each $b \in B$, an embedding

$$\varrho_b: (K'_b, L'_b, \chi'_b) \rightarrow (W'_{\gamma(b)}, H'_{\gamma(b)}, \psi'_{\gamma(b)})$$

and an embedding

$$\sigma_b: (K''_b, L''_b, \chi''_b) \rightarrow (W''_{\nu(\gamma(b))}, H''_{\nu(\gamma(b))}, \psi''_{\nu(\gamma(b))})$$

such that

$$\tilde{g}_{\gamma(b)}(\varrho_b(K'_b)) \subseteq \sigma_b(K''_b).$$

Let $q \in Q$ and set $r := f(q)$. We fix $i \in I$ and $j \in J$ such that $q \in \pi_i(V_i)$ and $r \in \psi'_j(W'_j)$. Further we choose $v' \in V'_{\tau(i)}$ and $w' \in W'_j$ such that $\pi'_{\tau(i)}(v') = r = \psi'_j(w')$. By compatibility of orbifold charts we find a restriction (K'_q, L'_q, χ'_q) of $(V'_{\tau(i)}, G'_{\tau(i)}, \pi'_{\tau(i)})$ with $v' \in K'_q$ and an embedding

$$\varrho_q: (K'_q, L'_q, \chi'_q) \rightarrow (W'_j, H'_j, \psi'_j).$$

Since \tilde{f}_i is continuous, there is a restriction (K_q, L_q, χ_q) of (V_i, G_i, π_i) such that $q \in \chi_q(K_q)$ and $\tilde{f}_i(K_q) \subseteq K'_q$. We set

$$(K''_q, L''_q, \chi''_q) := (W''_j, H''_j, \psi''_j).$$

We consider orbifold charts constructed for distinct q to be distinct. Then we set

$$A := Q, \quad \alpha(q) := i, \quad \lambda_q := \text{id}, \quad \mu_q := \text{id},$$

$$B := Q \sqcup Q' \setminus f(Q), \quad \beta(q) := q, \quad \gamma(q) := j, \quad \sigma_q := \text{id} \quad \text{for } q \in Q.$$

For $q' \in Q' \setminus f(Q)$ we fix $j \in J$ with $q' \in \psi'_j(W'_j)$ and set $\gamma(q') := j$. Further we set

$$(K'_{q'}, L'_{q'}, \chi'_{q'}) := (W'_j, H'_j, \psi'_j) \quad \text{and} \quad (K''_{q'}, L''_{q'}, \chi''_{q'}) := (W''_j, H''_j, \psi''_j).$$

Again we consider orbifold charts built for distinct q' to be distinct and to be distinct from all those defined for some $q \in Q$, and we define $\varrho_{q'} := \text{id}$ and $\sigma_{q'} := \text{id}$. Then all requirements are satisfied. If the orbifolds are second-countable, then we see as in Lemma 5.14 that we may restrict this construction to countably many points of Q and Q' resp. countable sets A and B to achieve that $\mathcal{K}, \mathcal{K}'$ and \mathcal{K}'' are countable. \square

PROPOSITION 5.18. *The composition in Orb is well-defined.*

PROOF. We use the notation from the definition of the composition. We have to show that the composition of $[\hat{f}]$ and $[\hat{g}]$ neither depends on the choice of the induced orbifold maps \hat{h} and \hat{k} nor on the choice of the representatives of $[\hat{f}]$ and $[\hat{g}]$. To prove independence of the choice of \hat{h} and \hat{k} , suppose that we have two pairs (\hat{h}_j, \hat{k}_j) of induced orbifold maps $\hat{h}_j \in \text{Orb}(\mathcal{K}_j, \mathcal{K}'_j)$, $\hat{k}_j \in \text{Orb}(\mathcal{K}'_j, \mathcal{K}''_j)$ ($j = 1, 2$) such that the diagram

$$\begin{array}{ccccccc} \mathcal{K}_1 & \xrightarrow{\hat{h}_1} & \mathcal{K}'_1 & \xrightarrow{\hat{k}_1} & \mathcal{K}''_1 \\ \searrow & & \downarrow & & \swarrow \\ & \mathcal{V} & \xrightarrow{\hat{f}} & \mathcal{V}' & & & \\ \nearrow & & \uparrow & & & & \\ \mathcal{K}_2 & \xrightarrow{\hat{h}_2} & \mathcal{K}'_2 & \xrightarrow{\hat{k}_2} & \mathcal{K}''_2 & & \end{array}$$

commutes. The non-horizontal maps are lifts of identity. Lemma 5.14 shows the existence of representatives \mathcal{H} of \mathcal{U} , \mathcal{H}' , \mathcal{I}' of \mathcal{U}' , \mathcal{I}'' of \mathcal{U}'' , and charted orbifold maps $\hat{h}_3 \in \text{Orb}(\mathcal{H}, \mathcal{H}')$, $\hat{k}_3 \in \text{Orb}(\mathcal{I}', \mathcal{I}'')$, and appropriate lifts of identity such that the diagrams

$$\begin{array}{ccc} \mathcal{K}_1 & \xrightarrow{\hat{h}_1} & \mathcal{K}'_1 \\ \nearrow & & \swarrow \\ \mathcal{H} & \xrightarrow{\hat{h}_3} & \mathcal{H}' \\ \searrow & & \swarrow \\ & \mathcal{K}_2 & \xrightarrow{\hat{h}_2} & \mathcal{K}'_2 \end{array} \quad \begin{array}{ccc} \mathcal{K}'_1 & \xrightarrow{\hat{k}_1} & \mathcal{K}''_1 \\ \nearrow & & \swarrow \\ \mathcal{I}' & \xrightarrow{\hat{k}_3} & \mathcal{I}'' \\ \searrow & & \swarrow \\ & \mathcal{K}'_2 & \xrightarrow{\hat{k}_2} & \mathcal{K}''_2 \end{array}$$

commute. By Lemma 5.17 we find representatives $\mathcal{K}, \mathcal{K}', \mathcal{K}''$ of $\mathcal{U}, \mathcal{U}', \mathcal{U}''$, resp., charted orbifold maps $\hat{h} \in \text{Orb}(\mathcal{K}, \mathcal{K}')$, $\hat{k} \in \text{Orb}(\mathcal{K}', \mathcal{K}'')$, and appropriate lifts of identity such that

$$\begin{array}{ccccc}
& \mathcal{H} & \xrightarrow{\hat{h}_3} & \mathcal{H}' & \\
\swarrow & & & \searrow & \\
\mathcal{K} & \xrightarrow{\hat{h}} & \mathcal{K}' & \xrightarrow{\hat{k}} & \mathcal{K}'' \\
& \nearrow & & \nearrow & \\
& \mathcal{I}' & \xrightarrow{\hat{k}_3} & \mathcal{I}'' &
\end{array}$$

commutes. Hence, altogether we have the commutative diagram

$$\begin{array}{ccccc}
& \mathcal{K}_1 & \xrightarrow{\hat{h}_1} & \mathcal{K}'_1 & \xrightarrow{\hat{k}_1} \mathcal{K}''_1 \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{K} & \xrightarrow{\hat{h}} & \mathcal{K}' & \xrightarrow{\hat{k}} & \mathcal{K}'' \\
\downarrow & & \downarrow & & \downarrow \\
& \mathcal{K}_2 & \xrightarrow{\hat{h}_2} & \mathcal{K}'_2 & \xrightarrow{\hat{k}_2} \mathcal{K}''_2
\end{array} \tag{7}$$

which shows that $\hat{k}_1 \circ \hat{h}_1$ and $\hat{k}_2 \circ \hat{h}_2$ are equivalent (see Remark 5.19 below for some more details).

For the proof of the independence of the choices of the representatives of $[\hat{f}]$ and $[\hat{g}]$, let $\hat{f}_1 \in \text{Orb}(\mathcal{V}_1, \mathcal{V}'_1)$, $\hat{f}_2 \in \text{Orb}(\mathcal{V}_2, \mathcal{V}'_2)$ be representatives of $[\hat{f}]$, and $\hat{g}_1 \in \text{Orb}(\mathcal{W}'_1, \mathcal{W}''_1)$, $\hat{g}_2 \in \text{Orb}(\mathcal{W}'_2, \mathcal{W}''_2)$ be representatives of $[\hat{g}]$. Further, for $j = 1, 2$, let $\hat{h}_j \in \text{Orb}(\mathcal{K}_j, \mathcal{K}'_j)$ be induced by \hat{f}_j , and $\hat{k}_j \in \text{Orb}(\mathcal{K}'_j, \mathcal{K}''_j)$ be induced by \hat{g}_j . Since \hat{f}_1 and \hat{f}_2 are equivalent, we find representatives $\mathcal{V}, \mathcal{V}'$ of $\mathcal{U}, \mathcal{U}'$, resp., a charted orbifold map $\hat{f} \in \text{Orb}(\mathcal{V}, \mathcal{V}')$ and appropriate lifts of identities, and analogously for \hat{g}_1 and \hat{g}_2 , such that the diagrams

$$\begin{array}{ccc}
\begin{array}{ccc}
\mathcal{V}_1 & \xrightarrow{\hat{f}_1} & \mathcal{V}'_1 \\
\swarrow & & \searrow \\
\mathcal{V} & \xrightarrow{\hat{f}} & \mathcal{V}' \\
\searrow & & \swarrow \\
\mathcal{V}_2 & \xrightarrow{\hat{f}_2} & \mathcal{V}'_2
\end{array} & \quad &
\begin{array}{ccc}
\mathcal{W}'_1 & \xrightarrow{\hat{g}_1} & \mathcal{W}''_1 \\
\swarrow & & \searrow \\
\mathcal{W}' & \xrightarrow{\hat{g}} & \mathcal{W}'' \\
\searrow & & \swarrow \\
\mathcal{W}'_2 & \xrightarrow{\hat{g}_2} & \mathcal{W}''_2
\end{array}
\end{array}$$

commute. Lemma 5.17 yields the existence of $\hat{h} \in \text{Orb}(\mathcal{K}, \mathcal{K}')$ and $\hat{k} \in \text{Orb}(\mathcal{K}', \mathcal{K}'')$ and appropriate lifts of identities such that

$$\begin{array}{ccccc}
& \mathcal{V} & \xrightarrow{\hat{f}} & \mathcal{V}' & \\
\swarrow & & \searrow & & \\
\mathcal{K} & \xrightarrow{\hat{h}} & \mathcal{K}' & \xrightarrow{\hat{k}} & \mathcal{K}'' \\
& \nearrow & & \nearrow &
\end{array}$$

commutes. Since \hat{h} is induced by \hat{f}_1 and by \hat{f}_2 , and likewise, \hat{k} is induced by \hat{g}_1 and by \hat{g}_2 , we conclude as above that $\hat{k}_1 \circ \hat{h}_1$ and $\hat{k}_2 \circ \hat{h}_2$ are both equivalent to $\hat{k} \circ \hat{h}$. This yields that the composition map is well-defined. \square

REMARK 5.19. We elaborate on a subtle issue in the proof of Proposition 5.18. By the reasoning in the proof, the diagram (7) *a priori* is of the form

$$\begin{array}{ccccc}
& & \mathcal{K}_1 \xrightarrow{\hat{h}_1} \mathcal{K}'_1 & & \\
& & \uparrow & & \uparrow \hat{k}_1 \\
& & \mathcal{H} \xrightarrow{\hat{h}_3} \mathcal{H}' & & \mathcal{K}'_1 \xrightarrow{\hat{k}_3} \mathcal{K}''_1 \\
& & \uparrow & & \uparrow \\
& & \mathcal{K} \xrightarrow{\hat{h}} \mathcal{K}' & & \mathcal{I}' \xrightarrow{\hat{k}_3} \mathcal{I}'' \\
& & \downarrow & & \downarrow \\
& & \mathcal{H} \xrightarrow{\hat{h}_3} \mathcal{H}' & & \mathcal{I}' \xrightarrow{\hat{k}_3} \mathcal{I}'' \\
& & \downarrow & & \downarrow \\
& & \mathcal{K}_2 \xrightarrow{\hat{h}_2} \mathcal{K}'_2 & & \mathcal{K}''_2
\end{array}$$

and we implicitly claim that the two maps from \mathcal{K}' to \mathcal{K}'_1 as well as from \mathcal{K}' to \mathcal{K}'_2 are identical. This might lead to the suspicion that the composition of charted orbifold maps as defined above is not well-defined.

However, since $\hat{h}, \hat{h}_1, \hat{h}_2$ are all induced by \hat{f} , and $\hat{k}, \hat{k}_1, \hat{k}_2$ are all induced by \hat{g} , the arising maps $\mathcal{K}' \rightarrow \mathcal{K}'_1$ are indeed identical as well as the arising maps $\mathcal{K}' \rightarrow \mathcal{K}'_2$. The diagram in Figure 1 shows the maps between the local charts, which proves the latter statement as well as that diagram (7) is indeed commutative. The notation in this diagram is intuitive, e.g. K is an orbifold chart in \mathcal{K} , all greek letters refer to diffeomorphisms. To improve readability we may assume without loss of generality that $K' \subseteq K'_1$, $K'_1 \subseteq V' \cap W'$ and $K'_2 \subseteq V' \cap W'$. The latter two choices implicitly mean that we assume that the local diffeomorphism between V' and W' is the identity, which we indeed may. Otherwise we would just need to add the same local diffeomorphism to several maps in the diagram. The local diffeomorphism between K' and K'_2 is denoted by ξ (note that we are not allowed to suppose that $\xi = \text{id}$ in general). Note that the top and bottom maps in the diagram are identical.

We end this section with a discussion of the equivalence class represented by a lift of identity. The following proposition shows that it is precisely the class of all lifts of identity of the considered orbifold. This justifies the notion “identity morphism” in Definition 5.8.

PROPOSITION 5.20. *Let (Q, \mathcal{U}) be an orbifold and ε a lift of $\text{id}_{(Q, \mathcal{U})}$. Then the equivalence class $[\varepsilon]$ of ε consists precisely of all lifts of $\text{id}_{(Q, \mathcal{U})}$.*

PROOF. Let $\varepsilon_1 \in \text{Orb}(\mathcal{V}_1, \mathcal{W}_1)$ and $\varepsilon_2 \in \text{Orb}(\mathcal{V}_2, \mathcal{W}_2)$ be two lifts of $\text{id}_{(Q, \mathcal{U})}$. Propositions 5.3 and 5.5 imply that there is a representative \mathcal{V} of \mathcal{U} such that ε_1 and ε_2 both induce the orbifold map

$$\widehat{\text{id}}_Q := (\text{id}_Q, \{\text{id}_{V_i}\}_{i \in I}, [R, \sigma])$$

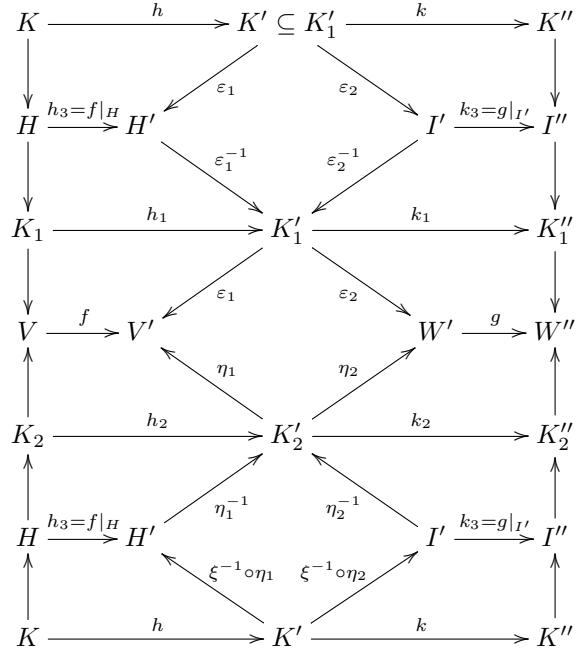


Figure 1. Maps in local charts.

with $(\mathcal{V}, \mathcal{V})$. Thus, each two lifts of $\text{id}_{(Q, \mathcal{U})}$ are equivalent.

Let f be a charted orbifold map which is equivalent to ε . With loss of generality we may assume that $\varepsilon = \text{id}_Q$. To fix notation let

$$\begin{aligned}\mathcal{V} &= \{(V_i, G_i, \pi_i) \mid i \in I\}, \text{ indexed by } I, \\ \mathcal{K}_1 &= \{(K_{1,a}, L_{1,a}, \chi_{1,a}) \mid a \in A\}, \text{ indexed by } A, \\ \mathcal{K}_2 &= \{(K_{2,b}, L_{2,b}, \chi_{2,b}) \mid b \in B\}, \text{ indexed by } B, \\ \mathcal{W}_1 &= \{(W_{1,j}, H_{1,j}, \psi_{1,j}) \mid j \in J\}, \text{ indexed by } J, \\ \mathcal{W}_2 &= \{(W_{2,k}, H_{2,k}, \psi_{2,k}) \mid k \in K\}, \text{ indexed by } K,\end{aligned}$$

be representatives of \mathcal{U} . Let

$$\hat{f} = (f, \{\tilde{f}_j\}_{j \in J}, [P_f, \nu_f]) \in \text{Orb}(\mathcal{W}_1, \mathcal{W}_2).$$

Suppose that

$$\hat{g} = (g, \{\tilde{g}_a\}_{a \in A}, [P_g, \nu_g]) \in \text{Orb}(\mathcal{K}_1, \mathcal{K}_2)$$

is a charted orbifold map and

$$\varepsilon_1 = (\text{id}_Q, \{\lambda_{1,a}\}_{a \in A}, [P_1, \nu_1]) \in \text{Orb}(\mathcal{K}_1, \mathcal{V})$$

$$\begin{aligned}\varepsilon_2 &= (\text{id}_Q, \{\lambda_{2,a}\}_{a \in A}, [P_2, \nu_2]) \in \text{Orb}(\mathcal{K}_1, \mathcal{W}_1) \\ \delta_1 &= (\text{id}_Q, \{\mu_{1,b}\}_{b \in B}, [R_1, \sigma_1]) \in \text{Orb}(\mathcal{K}_2, \mathcal{V}) \\ \delta_2 &= (\text{id}_Q, \{\mu_{2,b}\}_{b \in B}, [R_2, \sigma_2]) \in \text{Orb}(\mathcal{K}_2, \mathcal{W}_2)\end{aligned}$$

are lifts of $\text{id}_{(Q,\mathcal{U})}$ such that the diagram (which shows that \hat{f} and id_Q are equivalent)

$$\begin{array}{ccccc} & & \mathcal{V} & & \\ & \swarrow \varepsilon_1 & \xrightarrow{\text{id}_Q} & \searrow \delta_1 & \\ \mathcal{K}_1 & \xrightarrow{\hat{g}} & \mathcal{K}_2 & & \\ & \searrow \varepsilon_2 & & \swarrow \delta_2 & \\ & \mathcal{W}_1 & \xrightarrow{\hat{f}} & \mathcal{W}_2 & \end{array}$$

commutes. Clearly, $g = \text{id}_Q$ and hence $f = \text{id}_Q$. Let $\alpha: A \rightarrow I$, $\beta: A \rightarrow J$, $\gamma: A \rightarrow B$, $\delta: B \rightarrow I$, $\eta: B \rightarrow K$ and $\zeta: J \rightarrow K$ be the induced maps on the index sets as, e.g., in Construction 5.9. For each $a \in A$, we have

$$\text{id}_{V_{\alpha(a)}} \circ \lambda_{1,a} = \mu_{1,\gamma(a)} \circ \tilde{g}_a.$$

Since $\text{id}_{V_{\alpha(a)}}$, $\lambda_{1,a}$ and $\mu_{1,\gamma(a)}$ are local diffeomorphisms, so is \tilde{g}_a . Now

$$\tilde{f}_{\beta(a)} \circ \lambda_{2,a} = \mu_{2,\gamma(a)} \circ \tilde{g}_a$$

for each $a \in A$. Hence $\tilde{f}_{\beta(a)}$ is a local diffeomorphism. Lemma 5.13 implies that \tilde{f}_j is a local diffeomorphism for each $j \in J$. Therefore, \hat{f} is a lift of $\text{id}_{(Q,\mathcal{U})}$. \square

6. The orbifold category in terms of marked atlas groupoids.

Proposition 4.12 and Remark 5.10 show that charted orbifold maps and their composition correspond to homomorphisms between marked atlas groupoids and their composition. By characterizing lifts of identity and equivalence of charted orbifold maps in terms of marked atlas groupoids and their homomorphisms, we construct a category for marked atlas groupoids which is isomorphic to the one of reduced orbifolds. To that end we first show that lifts of identity correspond to unit weak equivalences, a notion we define below. Throughout this section let pr_1 denote the projection onto the first component.

A homomorphism $\varphi = (\varphi_0, \varphi_1): G \rightarrow H$ between Lie groupoids is called a *weak equivalence* if

- (i) the map

$$t \circ \text{pr}_1: H_1 \times_{\varphi_0} G_0 \rightarrow H_0$$

is a surjective submersion, and

(ii) the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi_1} & H_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ G_0 \times G_0 & \xrightarrow{\varphi_0 \times \varphi_0} & H_0 \times H_0 \end{array}$$

is a fibered product.

Two Lie groupoids G, H are called *Morita equivalent* if there is a Lie groupoid K and weak equivalences

$$G \xleftarrow{\varphi} K \xrightarrow{\psi} H.$$

DEFINITION 6.1. Let (G_1, α_1, X_1) and (G_2, α_2, X_2) be marked atlas groupoids. A homomorphism

$$\varphi = (\varphi_0, \varphi_1): (G_1, \alpha_1, X_1) \rightarrow (G_2, \alpha_2, X_2)$$

is called a *unit weak equivalence* if $\varphi: G_1 \rightarrow G_2$ is a weak equivalence and $\alpha_2 \circ |\varphi| \circ \alpha_1^{-1} = \text{id}_{X_1}$. Necessarily we have $X_1 = X_2 =: X$. A *unit Morita equivalence* between (G_1, α_1, X) and (G_2, α_2, X) is a pair (ψ_1, ψ_2) of unit weak equivalences

$$\psi_j: (G, \alpha, X) \rightarrow (G_j, \alpha_j, X)$$

where (G, α, X) is some marked atlas groupoid. If such a unit Morita equivalence exists, then the marked atlas groupoids (G_1, α_1, X) and (G_2, α_2, X) are called *unit Morita equivalent*.

In contrast to Morita equivalence of Lie groupoids, unit Morita equivalence of marked atlas groupoids requires the third (marked) Lie groupoid to be an atlas groupoid. In Proposition 6.3 below we will show that unit Morita equivalence of marked atlas groupoids is indeed an equivalence relation.

The following proposition identifies lifts of identity with unit weak equivalences.

PROPOSITION 6.2. Let \mathcal{U} and \mathcal{U}' be orbifold structures on the topological space Q . Further let \mathcal{V} resp. \mathcal{W}' be a representative of \mathcal{U} resp. of \mathcal{U}' .

- (i) Suppose that $\mathcal{U} = \mathcal{U}'$. If $\hat{f} \in \text{Orb}(\mathcal{V}, \mathcal{W}')$ is a lift of $\text{id}_{(Q, \mathcal{U})}$, then $F_1(\hat{f})$ is a unit weak equivalence.
- (ii) Let $\varepsilon \in \text{Hom}(\Gamma(\mathcal{V}), \Gamma(\mathcal{W}'))$ be a unit weak equivalence. Then $\mathcal{U} = \mathcal{U}'$, and $F_2(\varepsilon)$ is a lift of $\text{id}_{(Q, \mathcal{U})}$.

PROOF. Let

$$\mathcal{V} = \{(V_i, G_i, \pi_i) \mid i \in I\} \quad \text{resp.} \quad \mathcal{W}' = \{(W'_j, H'_j, \psi'_j) \mid j \in J\},$$

indexed by I resp. by J , and let $G := \Gamma(\mathcal{V})$ and $H := \Gamma(\mathcal{W}')$. We will first prove (i). Suppose $\hat{f} = (\text{id}_Q, \{\tilde{f}_i\}_{i \in I}, [P, \nu])$. By Proposition 4.7 it suffices to show that $\varepsilon = (\varepsilon_0, \varepsilon_1) := F_1(\hat{f})$ is a weak equivalence. We first show that

$$t \circ \text{pr}_1 : \begin{cases} H_{1,s \times \varepsilon_0} G_0 \rightarrow H_0 \\ (h, x) \mapsto t(h) \end{cases}$$

is a submersion. Let $(h, x) \in H_{1,s \times \varepsilon_0} G_0$. Recall from Proposition 5.3 that ε_0 is a local diffeomorphism, and from Special Case 2.5 that G and H are étale groupoids. Choose open neighborhoods U_x of x in G_0 and U_h of h in H_1 such that $\varepsilon_0|_{U_x}$ and $s|_{U_h}$ are embeddings with $s(U_h) = \varepsilon_0(U_x)$. Then $U_h s \times_{\varepsilon_0} U_x$ is open in $H_{1,s \times \varepsilon_0} G_0$. Further

$$\begin{aligned} U_h s \times_{\varepsilon_0} U_x &= \{(k, y) \in U_h \times U_x \mid s(k) = \varepsilon_0(y)\} \\ &= \{(k, \varepsilon_0^{-1}(s(k))) \mid k \in U_h\}. \end{aligned}$$

Therefore,

$$\text{pr}_1 : U_h s \times_{\varepsilon_0} U_x \rightarrow U_h$$

is a diffeomorphism. Since t is a local diffeomorphism, $t \circ \text{pr}_1$ is a submersion.

Now we prove that $t \circ \text{pr}_1$ is surjective. Let $y \in H_0$, say $y \in W'_j$, and set $\psi'_j(y) =: q \in Q$. Then there is an orbifold chart $(V_i, G_i, \pi_i) \in \mathcal{V}$ such that $q \in \pi_i(V_i)$, say $q = \pi_i(x)$.

$$\begin{array}{ccc} V_i & \xrightarrow{\tilde{f}_i} & W'_i \\ \pi_i \searrow & & \swarrow \psi'_i \\ & Q & \end{array}$$

Set $z := \tilde{f}_i(x)$, hence $\psi'_i(z) = q = \psi'_j(y)$. Hence, there are a restriction (S', K', χ') of (W'_i, H'_i, ψ'_i) with $z \in S'$ and an embedding

$$\lambda : (S', K', \chi') \rightarrow (W'_j, H'_j, \psi'_j)$$

such that $\lambda(z) = y$. Then $\lambda \in \Psi(\mathcal{W}')$ and $(\text{germ}_z \lambda, x) \in H_{1,s \times \varepsilon_0} G_0$ with

$$t \circ \text{pr}_1(\text{germ}_z \lambda, x) = t(\text{germ}_z \lambda) = y.$$

This means that $t \circ \text{pr}_1$ is surjective.

Set

$$K := (G_0 \times G_0)_{(\varepsilon_0, \varepsilon_0)} \times_{(s,t)} H_1.$$

It remains to show that the map

$$\beta : \begin{cases} G_1 & \rightarrow K \\ \text{germ}_x g \mapsto (x, g(x), \varepsilon_1(\text{germ}_x g)) \end{cases}$$

is a diffeomorphism. Note that $\beta = (s, t, \varepsilon_1)$. Let $(x, y, \text{germ}_{\varepsilon_0(x)} h)$ be in K , hence $\text{germ}_{\varepsilon_0(x)} h : \varepsilon_0(x) \rightarrow \varepsilon_0(y)$. By the definition of H_1 there are open neighborhoods U'_1 of $\varepsilon_0(x)$ and U'_2 of $\varepsilon_0(y)$ in $W' := \coprod_{j \in J} W'_j$ such that $h : U'_1 \rightarrow U'_2$ is an element of $\Psi(W')$. Since ε_0 is a local diffeomorphism, there are open neighborhoods U_1 of x and U_2 of y in $V := \coprod_{i \in I} V_i$ such that $\varepsilon_0|_{U_k}$ is an embedding with $\varepsilon_0(U_k) \subseteq U'_k$ ($k = 1, 2$). After shrinking U'_k we can assume that $\varepsilon_0(U_k) = U'_k$. Let $\gamma_k := \varepsilon_0|_{U_k}$. Then

$$g := \gamma_2^{-1} \circ h \circ \gamma_1 : U_1 \rightarrow U_2$$

is a diffeomorphism, hence $g \in \Psi(\mathcal{V})$. By Proposition 5.5 (or rather its proof), we have $\nu(g) = h$ and hence (see (R4a) and recall Proposition 4.7)

$$\varepsilon_1(\text{germ}_x g) = \text{germ}_{\varepsilon_0(x)} h.$$

Finally, we see

$$\beta(\text{germ}_x g) = (x, g(x), \varepsilon_1(\text{germ}_x g)) = (x, y, \text{germ}_{\varepsilon_0(x)} h).$$

Therefore β is surjective. Since $\text{germ}_x g$ does not depend on the choice of U_k and U'_k , the map β is also injective. Finally, we will show that β is a local diffeomorphism. Since s and t are local diffeomorphisms, we only need to prove that ε_1 is one as well. Let $\text{germ}_x f \in G_1$. Choose an open neighborhood U of x such that $U \subseteq \text{dom } f$ and $\varepsilon_0|_U : U \rightarrow \varepsilon_0(U)$ is a diffeomorphism. By the germ topology, the set

$$\tilde{U} := \{\text{germ}_y f \mid y \in U\}$$

is open in G_1 , and the set

$$\tilde{V} := \{\text{germ}_z \nu(f) \mid z \in \varepsilon_0(U)\}$$

is open in H_1 . Further the diagrams

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\varepsilon_1} & \tilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\varepsilon_0} & \varepsilon_0(U) \end{array} \quad \begin{array}{ccc} \text{germ}_y f & \xrightarrow{\varepsilon_1} & \text{germ}_{\varepsilon_0(y)} \nu(f) \\ \downarrow & & \downarrow \\ y & \xrightarrow{\varepsilon_0} & \varepsilon_0(y) \end{array}$$

commute. Since the vertical arrows are diffeomorphisms by definition,

$$\varepsilon_1|_{\tilde{U}} : \tilde{U} \rightarrow \tilde{V}$$

is a diffeomorphism as well. This completes the proof of (i).

We will now prove (ii). Proposition 3.3 shows that the orbifold atlases \mathcal{V} and \mathcal{W}' are determined completely by the marked atlas groupoids $\Gamma(\mathcal{V})$ and $\Gamma(\mathcal{W}')$, resp. Hence we can apply Proposition 4.9, which shows that $F_2(\varepsilon)$ is well-defined. Suppose that

$$F_2(\varepsilon) = (f, \{\tilde{f}_i\}_{i \in I}, [P, \nu]).$$

Proposition 4.9 yields $f = \text{id}_Q$. By [8, Exercises 5.16(4)] ε_0 is a local diffeomorphism. Thus, Proposition 4.9 implies that each \tilde{f}_i is a local diffeomorphism. The domain atlas of $F_2(\varepsilon)$ is \mathcal{V} , its range family is \mathcal{W}' . From Proposition 5.6 it follows that $\mathcal{U} = \mathcal{U}'$. By Definition 5.8 $F_2(\varepsilon)$ is a lift of $\text{id}_{(Q, \mathcal{U})}$. \square

The combination of Propositions 3.3, 6.2 and Remark 5.10 now allows to identify each step in the construction of the category of reduced orbifolds and each intermediate object in terms of marked atlas groupoids. For an orbifold (Q, \mathcal{U}) we define

$$\Gamma(Q, \mathcal{U}) := \{(\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q) \mid \mathcal{V} \text{ is a (countable) representative of } \mathcal{U}\}.$$

We recall from Special Case 2.5 the convention that for second-countable orbifolds we always restrict here to countable representatives of the orbifold structure. For non-second-countable, we also consider non-countable representatives.

Then Proposition 5.20 gives rise to the following proposition.

PROPOSITION 6.3. *Unit Morita equivalence of marked atlas groupoids is an equivalence relation. Further, if \mathcal{V} is a (countable) representative of the orbifold structure \mathcal{U} of the orbifold (Q, \mathcal{U}) , then the unit Morita equivalence class of $(\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q)$ is $\Gamma(Q, \mathcal{U})$.*

Equivalence of charted orbifold maps translates to marked atlas groupoids as follows.

DEFINITION 6.4. Let (G_1, α_1, X) , (G_2, α_2, X) , as well as (H_1, β_1, Y) and (H_2, β_2, Y) be marked atlas groupoids. For $j = 1, 2$ let

$$\psi_j: (G_j, \alpha_j, X) \rightarrow (H_j, \beta_j, Y)$$

be a homomorphism of marked Lie groupoids. We call ψ_1 and ψ_2 *unit Morita equivalent* if there exist marked atlas groupoids (G, α, X) and (H, β, Y) , a homomorphism $\chi: (G, \alpha, X) \rightarrow (H, \beta, Y)$, and unit weak equivalences $\varepsilon_j: (G, \alpha, X) \rightarrow (G_j, \alpha_j, X)$, $\delta_j: (H, \beta, Y) \rightarrow (H_j, \beta_j, Y)$ such that the diagram

$$\begin{array}{ccccc} & & (G_1, \alpha_1, X) & \xrightarrow{\psi_1} & (H_1, \beta_1, Y) \\ & \swarrow \varepsilon_1 & & & \nwarrow \delta_1 \\ (G, \alpha, X) & \xrightarrow{\chi} & & & (H, \beta, Y) \\ & \searrow \varepsilon_2 & & & \swarrow \delta_2 \\ & & (G_2, \alpha_2, X) & \xrightarrow{\psi_2} & (H_2, \beta_2, Y) \end{array}$$

commutes.

Proposition 5.15 in terms of atlas groupoids means the following.

PROPOSITION 6.5. *Unit Morita equivalence of homomorphisms between marked atlas groupoids is an equivalence relation.*

We define the category Agr of marked atlas groupoids as follows: Its class of objects consists of all $\Gamma(Q, \mathcal{U})$. The morphisms from $\Gamma(Q, \mathcal{U})$ to $\Gamma(Q', \mathcal{U}')$ are the unit Morita equivalence classes $[\varphi]$ of homomorphisms $\varphi: (G, \alpha, Q) \rightarrow (G', \alpha', Q')$ where (G, α, Q) is any representative of $\Gamma(Q, \mathcal{U})$ and (G', α', Q') is any representative of $\Gamma(Q', \mathcal{U}')$.

The composition of two morphisms $[\varphi] \in \text{Morph}(\Gamma(Q, \mathcal{U}), \Gamma(Q', \mathcal{U}'))$ and $[\psi] \in \text{Morph}(\Gamma(Q', \mathcal{U}'), \Gamma(Q'', \mathcal{U}''))$ is defined as follows: Choose representatives

$$\varphi: (G, \alpha, Q) \rightarrow (G', \alpha', Q') \quad \text{of } [\varphi]$$

and

$$\psi: (H', \beta', Q') \rightarrow (H'', \beta'', Q'') \quad \text{of } [\psi].$$

Then find representatives (K, γ, Q) , (K', γ', Q') , (K'', γ'', Q'') of the classes $\Gamma(Q, \mathcal{U})$, $\Gamma(Q', \mathcal{U}')$, $\Gamma(Q'', \mathcal{U}'')$, resp., and unit Morita equivalences

$$\begin{aligned} \varepsilon: (K, \gamma, Q) &\rightarrow (G, \alpha, Q), \\ \varepsilon'_1: (K', \gamma', Q') &\rightarrow (G', \alpha', Q'), \\ \varepsilon'_2: (K', \gamma', Q') &\rightarrow (H', \beta', Q'), \\ \varepsilon'': (K'', \gamma'', Q'') &\rightarrow (H'', \beta'', Q''), \end{aligned}$$

and homomorphisms of marked Lie groupoids

$$\begin{aligned} \chi: (K, \gamma, Q) &\rightarrow (K', \gamma', Q'), \\ \kappa: (K', \gamma', Q') &\rightarrow (K'', \gamma'', Q'') \end{aligned}$$

such that the diagram

$$\begin{array}{ccccc} (G, \alpha, Q) & \xrightarrow{\varphi} & (G', \alpha', Q') & & (H', \beta', Q') & \xrightarrow{\psi} & (H'', \beta'', Q'') \\ \varepsilon \uparrow & & \searrow \varepsilon'_1 & & \nearrow \varepsilon'_2 & & \uparrow \varepsilon'' \\ (K, \gamma, Q) & \xrightarrow{\chi} & (K', \gamma', Q') & \xrightarrow{\kappa} & (K'', \gamma'', Q'') \end{array}$$

commutes. Then the composition of $[\varphi]$ and $[\psi]$ is defined as

$$[\psi] \circ [\varphi] := [\kappa \circ \chi].$$

Invoking Lemmas 5.11, 5.17 and Proposition 5.18 we deduce the following proposition.

PROPOSITION 6.6. *The composition in Agr is well-defined.*

We define an assignment F from the orbifold category Orb to the category of marked atlas groupoids Agr as follows. On the level of objects, F maps the orbifold (Q, \mathcal{U}) to $\Gamma(Q, \mathcal{U})$. Suppose that $[\hat{f}]$ is a morphism from the orbifold (Q, \mathcal{U}) to the orbifold (Q', \mathcal{U}') . Then F maps $[\hat{f}]$ to the morphism $[F_1(\hat{f})]$ from $\Gamma(Q, \mathcal{U})$ to $\Gamma(Q', \mathcal{U}')$.

THEOREM 6.7. *The assignment F is a covariant functor from Orb to Agr. Even more, F is an isomorphism of categories. The functor F and its inverse are constructive.*

In the following example we show that the representatives of orbifold maps from Example 4.6 define different orbifold maps. In this example we use $G(x, y)$ to denote the set of arrows g of the groupoid G with $s(g) = x$ and $t(g) = y$.

EXAMPLE 6.8. Recall the representatives of orbifold maps

$$\hat{f}_1 = (f, \tilde{f}, P, \nu_1) \quad \text{and} \quad \hat{f}_2 = (f, \tilde{f}, P, \nu_2)$$

from Examples 4.6 and 4.8. We claim that \hat{f}_1 and \hat{f}_2 are representatives of different orbifold maps. Assume for contradiction that \hat{f}_1 and \hat{f}_2 define the same orbifold map on (Q, \mathcal{U}_1) . This means that the groupoid homomorphisms φ and ψ from Example 4.8 are Morita equivalent. Hence there exist marked atlas groupoids K and H , unit weak equivalences

$$\begin{array}{ll} \alpha = (\alpha_0, \alpha_1): K \rightarrow \Gamma, & \gamma = (\gamma_0, \gamma_1): H \rightarrow \Gamma, \\ \beta = (\beta_0, \beta_1): K \rightarrow \Gamma, & \delta = (\delta_0, \delta_1): H \rightarrow \Gamma, \end{array}$$

and a homomorphism $\chi = (\chi_0, \chi_1): K \rightarrow H$ such that the diagram

$$\begin{array}{ccccc} & & \Gamma & \xrightarrow{\varphi} & \Gamma \\ & \nearrow \alpha & & & \searrow \gamma \\ K & \xrightarrow{\chi} & H & & \\ & \searrow \beta & & & \swarrow \delta \\ & & \Gamma & \xrightarrow{\psi} & \Gamma \end{array}$$

commutes. Since α is a (unit) weak equivalence, there exists $x \in K$ and $g \in \Gamma$ with $s(g) = \alpha_0(x)$ and $t(g) = 0$. Necessarily, $g \in \{\text{germ}_0(\pm \text{id})\}$, and hence $\alpha_0(x) = 0$. In turn, α_1 induces a bijection between $K(x, x)$ and $\Gamma(0, 0)$. Thus $K(x, x)$ consists of two elements, say $K(x, x) = \{k_1, k_2\}$. Let $x' := \chi_0(x)$. Then

$$0 = \varphi_0(\alpha_0(x)) = \gamma_0(x').$$

This shows that γ_1 induces a bijection between $H(x', x')$ and $\Gamma(0, 0)$. For $j = 1, 2$ we have

$$\gamma_1(\chi_1(k_j)) = \varphi_1(\alpha_1(k_j)) = \text{germ}_0 \text{id},$$

which implies that $\chi_1(k_1) = \chi_1(k_2)$. Further β_1 induces a bijection between $K(x, x)$ and $\Gamma(\beta_0(x), \beta_0(x))$. Hence $\beta_0(x) = 0$, and thus

$$\psi_1(\beta_1(k_1)) \neq \psi_1(\beta_1(k_2)).$$

But this contradicts to

$$\psi_1(\beta_1(k_1)) = \delta_1(\chi_1(k_1)) = \delta_1(\chi_1(k_2)) = \psi_1(\beta_1(k_2)).$$

In turn, φ and ψ are not Morita equivalent.

7. Extension to marked proper effective étale Lie groupoids.

Let Pgr denote the category of marked proper effective étale Lie groupoids which is defined analogously to the category Agr of marked atlas groupoids with the only difference that any marked atlas groupoid may be a marked proper effective étale Lie groupoid. Note that the objects in Pgr are not the single marked proper effective étale Lie groupoids themselves but the unit Morita equivalence classes of these.

In this final section we show that these two categories are isomorphic via the canonical embedding. As before we consider two different flavors of categories distinguished by whether the involved topological spaces are required to be just paracompact or even second-countable. These differences do not cause any distinctions in statements or proofs and therefore will be implicit.

THEOREM 7.1. *The embedding functor $\text{Agr} \rightarrow \text{Pgr}$ is an isomorphism.*

PROOF. We prove that we can lift any given homomorphism (weak equivalence) between two proper effective étale Lie groupoids to a homomorphism (weak equivalence) between weakly equivalent atlas groupoids:

$$\begin{array}{ccc} G & \xrightarrow{\quad g \quad} & G' \\ \swarrow & & \searrow \\ \Gamma(\mathcal{U}) & \dashrightarrow & \Gamma(\mathcal{U}'). \end{array}$$

Let G be a proper effective étale Lie groupoid. By [8, Corollary 5.31] there exists an orbifold atlas \mathcal{U} on the base space G_0 of G such that $\Gamma(\mathcal{U})$ is weakly equivalent to G . More precisely, let $(U_i)_{i \in I}$ be a covering of G_0 by open connected subsets such that for each $i \in I$, there exists a point $x \in U_i$ such that

$$G_i := G_x := \{g \in G_1 \mid x \xrightarrow{g} x\} = G_1|_{U_i}$$

and U_i is G_i -stable, and such that the action groupoid $G_i \ltimes U_i$ is isomorphic to $G|_{U_i}$. Then the family

$$\mathcal{U} := \{(U_i, G_i, \text{pr}) \mid i \in I\}$$

is an orbifold atlas on G_0 . We identify U_i with $\{i\} \times U_i$. Let $H := \Gamma(\mathcal{U})$. Then the weak equivalence $\varphi: H \rightarrow G$ is given by

$$\varphi_0: H_0 \rightarrow G_0, \quad (i, x) \mapsto x$$

and

$$\varphi_1: H_1 \rightarrow G_1, \quad (i, x) \xrightarrow{k} (j, y) \quad \mapsto \quad x \xrightarrow{k} y.$$

Now let G, G' be two proper effective étale Lie groupoids and $\chi: G \rightarrow G'$ be a homomorphism (weak equivalence). Choose orbifold atlases (in the way as above)

$$\mathcal{U} := \{(U_i, G_i, \text{pr}) \mid i \in I\}, \quad \mathcal{U}' := \{(U'_j, G'_j, \text{pr}) \mid j \in J\}$$

on G_0 resp. G'_0 such that there exists a map $\beta: I \rightarrow J$ such that

$$\chi_0(U_i) \subseteq U'_{\beta(i)}$$

for all $i \in I$ (this is possible). Set $H := \Gamma(\mathcal{U})$, $H' := \Gamma(\mathcal{U}')$ and define $\alpha: H \rightarrow H'$ via

$$\begin{aligned} \alpha_0: H_0 &\rightarrow H'_0, \quad (i, x) \mapsto (\beta(i), \chi_0(x)) \\ \alpha_1: H_1 &\rightarrow H'_1, \quad (i, x) \xrightarrow{k} (j, y) \quad \mapsto \quad (\beta(i), \chi_0(x)) \xrightarrow{\chi_1(k)} (\beta(j), \chi_0(y)). \end{aligned}$$

One easily checks that α is a homomorphism (weak equivalence).

With this lifting property we can bring back all equivalence relations between marked proper effective étale Lie groupoids and between maps between these groupoids to statements purely between marked atlas groupoids and maps between those. This proves the theorem. \square

References

- [1] A. Adem, J. Leida and Y. Ruan, Orbifolds and stringy topology, Cambridge Tracts in Mathematics, vol. 171, Cambridge University Press, Cambridge, 2007.
- [2] J. Borzellino and V. Brunsden, A manifold structure for the group of orbifold diffeomorphisms of a smooth orbifold, *J. Lie Theory*, **18** (2008), 979–1007.
- [3] W. Chen and Y. Ruan, Orbifold Gromov–Witten theory, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., **310**, Amer. Math. Soc., Providence, RI, 2002, pp. 25–85.
- [4] A. Haefliger, Groupoïdes d’holonomie et classifiants, Astérisque, No. 116, (1984), 70–97.
- [5] E. Lerman, Orbifolds as stacks?, *Enseign. Math. (2)*, **56** (2010), 315–363.
- [6] E. Lupercio and B. Uribe, Gerbes over orbifolds and twisted K -theory, *Comm. Math. Phys.*, **245** (2004), 449–489.
- [7] I. Moerdijk, Orbifolds as groupoids: an introduction, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., **310**, Amer. Math. Soc., Providence, RI, 2002, pp. 205–222.
- [8] I. Moerdijk and J. Mrčun, Introduction to foliations and Lie groupoids, Cambridge Studies in Advanced Mathematics, **91**, Cambridge University Press, Cambridge, 2003.
- [9] I. Moerdijk and D. Pronk, Orbifolds, sheaves and groupoids, *K-Theory*, **12** (1997), 3–21.
- [10] D. Pronk, Etendues and stacks as bicategories of fractions, *Compositio Math.*, **102** (1996), 243–303.
- [11] I. Satake, On a generalization of the notion of manifold, *Proc. Nat. Acad. Sci. U.S.A.*, **42** (1956), 359–363.
- [12] I. Satake, The Gauss–Bonnet theorem for V -manifolds, *J. Math. Soc. Japan*, **9** (1957), 464–492.
- [13] A. Schmeding, The diffeomorphism group of a non-compact orbifold., *Diss. Math.*, **507** (2015), 179.
- [14] W. Thurston, The geometry and topology of three-manifolds, available at www.msri.org/publications/books/gt3m/.

Anke D. POHL

Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstr, 3-5
37073 Göttingen, Germany
E-mail: anke.pohl@math.uni-goettingen.de