

Simple ribbon fusions and genera of links

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Dedicated to Professor J. H. Przytycki on the occasion of his 60th birthday

(Received Oct. 2, 2014)

Abstract. Let K be the result of a 1-fusion (band sum) of a knot k and a distant trivial knot in S^3 . From results of D. Gabai and of M. G. Scharlemann, we know that the genus of K is at least that of k and that equality holds if and only if the band sum is, in fact, a connected sum (in which case K is ambient isotopic to k). In this paper, we consider a generalization of this result to an m -fusion of a link and a distant trivial link with m -components.

1. Introduction.

All links are assumed to be ordered and oriented, and they are considered up to ambient isotopy in the oriented 3-sphere S^3 .

A (m -)ribbon fusion on a link ℓ is an m -fusion of ℓ and an m -component trivial link \mathcal{O} which is split from ℓ and each of whose components is attached by a unique band to a component of ℓ (see the left side of Figure 1). Note that any ribbon knot can be obtained from the trivial knot by a ribbon fusion.

The m -ribbon fusion is called a (m -)simple ribbon fusion (or an SR -fusion) if \mathcal{O} bounds m mutually disjoint disks \mathcal{D} which are split from ℓ such that each disk of \mathcal{D} intersects with one of the bands \mathcal{B} for the ribbon fusion exactly once and each band of \mathcal{B} intersects with one disk of \mathcal{D} exactly once (see the center of Figure 1).

The following is the precise definition of the simple ribbon fusion.

Let ℓ be a link and $\mathcal{O} = O_1 \cup \cdots \cup O_m$ the m -component trivial link which is split from ℓ . Let $\mathcal{D} = D_1 \cup \cdots \cup D_m$ be a disjoint union of non-singular disks with $\partial D_i = O_i$ and $D_i \cap \ell = \emptyset$ ($i = 1, \dots, m$), and let $\mathcal{B} = B_1 \cup \cdots \cup B_m$ be a disjoint union of disks, called *bands*, for an m -fusion of ℓ and \mathcal{O} satisfying the following:

- (i) $B_i \cap \ell = \partial B_i \cap \ell = \{\text{a single arc}\}$,
- (ii) $B_i \cap \mathcal{O} = \partial B_i \cap O_i = \{\text{a single arc}\}$ and
- (iii) $B_i \cap \text{int } \mathcal{D} = B_i \cap \text{int } D_{\pi(i)} = \{\text{a single arc of ribbon type}\}$ (see the right side of Figure 1), where π is a certain permutation on $\{1, 2, \dots, m\}$.

Let L be a link obtained from ℓ and \mathcal{O} by the m -fusion along \mathcal{B} , i.e., $L = (\ell \cup \mathcal{O} \cup \partial \mathcal{B}) - \text{int}(\mathcal{B} \cap \ell) - \text{int}(\mathcal{B} \cap \mathcal{O})$. Then we say that L is obtained from ℓ by a *simple ribbon fusion* or an *SR-fusion* (*with respect to* $\mathcal{D} \cup \mathcal{B}$).

2010 *Mathematics Subject Classification.* Primary 57M25.

Key Words and Phrases. link, fusion, genus.

The second author is partially supported by JSPS KAKENHI Grant Number 22540104.

The third author is partially supported by JSPS KAKENHI Grant Number 22740050.

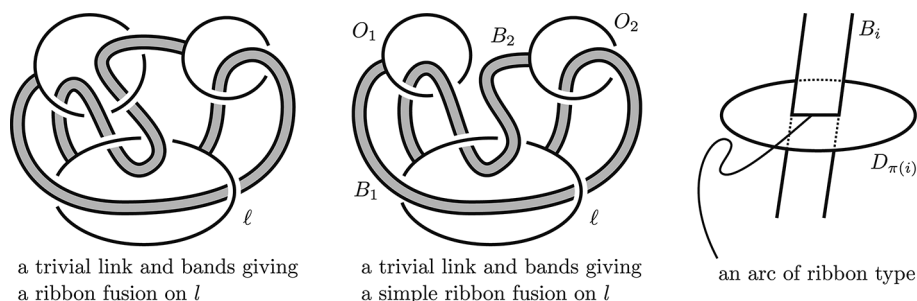


Figure 1.

The *genus* of an oriented surface is the sum of genera of its connected components. A *Seifert surface* E for a link ℓ is a compact non-singular oriented surface in S^3 with no closed components such that $\partial E = \ell$. The *genus* $g(\ell)$ of a link ℓ is the minimal number of genera of all the Seifert surfaces for ℓ . From results of D. Gabai [1] and of M.G. Scharlemann [5], we know that if a knot K is obtained from a knot k by a 1-ribbon fusion, then the genus of K is at least that of k , and that equality holds if and only if the band sum is a connected sum, i.e., K is ambient isotopic to k . However, we cannot directly generalize the result to an m -ribbon fusion on a link, since we have shown in [3] that there exists a 2-(simple) ribbon fusion on a link ℓ yielding a link L which is not ambient isotopic to ℓ but satisfies that $g(L) = g(\ell)$ (Figure 2).

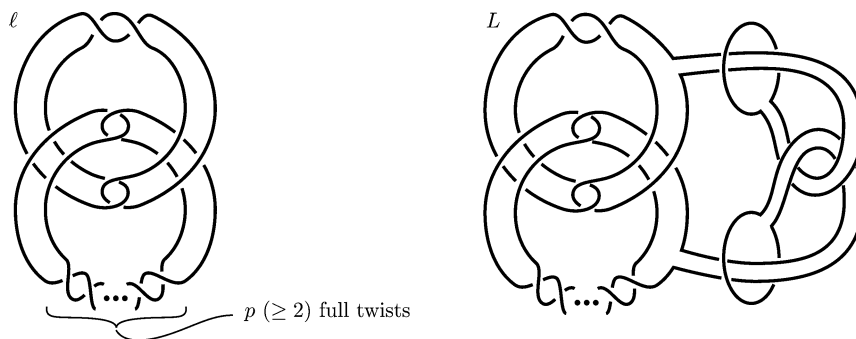


Figure 2.

Here we have the following refinement of the genus by C. Goldberg.

DEFINITION (Goldberg [2]). The *disconnectivity number* of ℓ , denoted by $\nu(\ell)$, is the maximal number of connected components of all the Seifert surfaces for ℓ . For each integer r ($1 \leq r \leq \nu(\ell)$), the r -th *genus* of ℓ , denoted by $g_r(\ell)$, is the minimal number of genera of all the Seifert surfaces for ℓ with r connected components.

Note that there exists a Seifert surface E for ℓ with $\sharp(E) = r$ for each integer r ($1 \leq r \leq \nu(\ell)$), where $\sharp(E)$ is the number of the connected components of E . From the definition, we see that $g_1(\ell)$ is the genus of ℓ , that $1 \leq \nu(\ell) \leq \sharp(\ell)$, and that $0 \leq g(\ell) = g_1(\ell) \leq g_2(\ell) \leq \dots \leq g_{\nu(\ell)}(\ell)$, where $\sharp(\ell)$ is the number of components of ℓ . For the m -component trivial link \mathcal{O} , we have that $\nu(\mathcal{O}) = m$ and that $g_r(\mathcal{O}) = 0$ for each integer r ($1 \leq r \leq m$).

An SR -fusion is *trivial* if \mathcal{O} bounds a mutually disjoint disks $\bigcup \Delta_i$ such that $\partial \Delta_i = O_i$ and $\text{int } \Delta_i$ does not intersect with $L \cup \mathcal{B}$ (note that $\bigcup \Delta_i$ may intersect with \mathcal{D} and see Figure 3 for example). Since L is ambient isotopic to ℓ through $\bigcup \Delta_i \cup \mathcal{B}$, we know that a trivial SR -fusion does not change the link type.

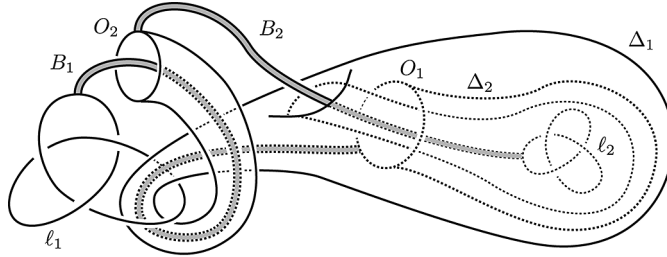


Figure 3. A trivial link and bands giving a trivial SR -fusion on a link $\ell = \ell_1 \cup \ell_2$.

THEOREM 1.1. *Let L be a link obtained from a link ℓ by an SR -fusion. Then we have that $\nu(L) \leq \nu(\ell)$ and that $g_r(L) \geq g_r(\ell)$ for each integer r ($1 \leq r \leq \nu(L)$). Moreover, the following three conditions are equivalent:*

- (1) *the SR -fusion is trivial,*
- (2) *L is ambient isotopic to ℓ and*
- (3) *$\nu(L) = \nu(\ell)$ and $g_{\nu(L)}(L) = g_{\nu(\ell)}(\ell)$.*

As for the disconnectivity number, there is an SR -fusion which realizes arbitrarily high degeneration.

THEOREM 1.2. *For any pair of positive integers v and w , there exist two links ℓ and L such that $\nu(\ell) = v + w$, $\nu(L) = v$, and L can be obtained from ℓ by an SR -fusion.*

Proofs are done by standard cut-and-paste arguments. To prove Theorem 1.1, we rename the indices of the components of \mathcal{O} , \mathcal{D} , and \mathcal{B} . From the definition of an SR -fusion, there is a permutation π such that a disk D_i of \mathcal{D} intersects with a band $B_{\pi(i)}$ of \mathcal{B} exactly once and a band B_i of \mathcal{B} intersects with a disk $D_{\pi^{-1}(i)}$ of \mathcal{D} exactly once. Since every permutation is a product of cyclic permutations, we can rename the indices of the components of \mathcal{O} , \mathcal{D} , and \mathcal{B} as

$$\mathcal{O} = \bigcup_{k=1}^n \mathcal{O}^k = \bigcup_{k=1}^n \left(\bigcup_{i=1}^{m_k} O_i^k \right), \quad \mathcal{D} = \bigcup_{k=1}^n \mathcal{D}^k = \bigcup_{k=1}^n \left(\bigcup_{i=1}^{m_k} D_i^k \right) \text{ and}$$

$$\mathcal{B} = \bigcup_{k=1}^n \mathcal{B}^k = \bigcup_{k=1}^n \left(\bigcup_{i=1}^{m_k} B_i^k \right), \text{ where}$$

- (1) $\partial D_i^k = O_i^k$,
- (2) $B_i^k \cap \mathcal{O} = \partial B_i^k \cap O_i^k$ and
- (3) $B_i^k \cap \text{int } \mathcal{D} = B_i^k \cap \text{int } D_{i+1}^k$.

We consider the lower index modulo m_k . We call each $\mathcal{D}^k \cup \mathcal{B}^k$ the $(k\text{-th})$ *elementary*

component of the SR -fusion. Let \dot{D}_i^k and \dot{B}_i^k be disks and $f : \cup_{i,k}(\dot{D}_i^k \cup \dot{B}_i^k) \rightarrow S^3$ an immersion such that $f(\dot{D}_i^k) = D_i^k$ and $f(\dot{B}_i^k) = B_i^k$. In the following of this paper, we omit the upper index k of anything if it also has a lower index and it is clear in which elementary component we are discussing.

Take an elementary component $\mathcal{D}^k \cup \mathcal{B}^k$. Denote the arc of $\text{int} \dot{D}_i \cap B_{i-1}$ by α_i , and the pre-image of α_i on \dot{D}_i (resp. \dot{B}_{i-1}) by $\dot{\alpha}_i$ (resp. $\ddot{\alpha}_i$). Let $B_{i,1}$ and $B_{i,2}$ be the subdisks of B_i such that $B_{i,1} \cup B_{i,2} = B_i$, $B_{i,1} \cap B_{i,2} = \alpha_{i+1}$, and $B_{i,1} \cap \partial D_i \neq \emptyset$ as illustrated in Figure 4 ($i = 1, \dots, m_k$).

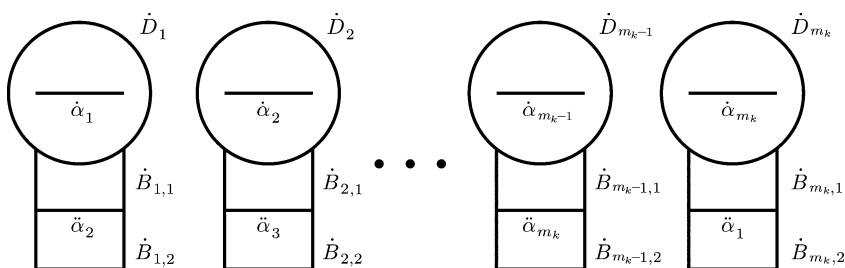


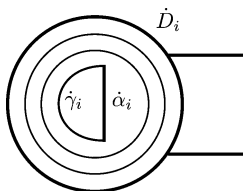
Figure 4.

2. Standard position for a Seifert surface of a link obtained by an SR -fusion.

Let L be a link obtained from a link ℓ by an SR -fusion with respect to $\mathcal{D} \cup \mathcal{B}$ and E a Seifert surface for L . We may assume that $\text{int} E$ and $\text{int}(\mathcal{D} \cup \mathcal{B})$ intersects transversely. Then the set \mathcal{S}_i of the pre-images on $\dot{D}_i \cup \dot{B}_i$ of the intersections of E and $D_i \cup B_i$ consists of arcs and loops which are mutually disjoint and simple. Let $\mathcal{S} = \cup_k \mathcal{S}^k = \cup_k (\cup_i \mathcal{S}_i)$. Define the *complexity* of E as the lexicographically ordered set (s_1, s_2, s_3) , where s_1 (resp. s_2) is the number of arcs (resp. loops) of \mathcal{S} and s_3 is the number of triple points of $E \cup (\mathcal{D} \cup \mathcal{B})$.

We say that E is in *standard position* (with respect to $\mathcal{D} \cup \mathcal{B}$) if \mathcal{S}_i consists of a single arc $\dot{\gamma}_i$ on $\text{int} \dot{D}_i$ and (may be no) loops on $\text{int} \dot{D}_i$ for any pair of i and k such that:

- (St1) $\dot{\gamma}_i \cap \dot{\alpha}_i = \partial \dot{\gamma}_i = \partial \dot{\alpha}_i$ and
- (St2) each loop bounds a disk on \dot{D}_i containing $\dot{\gamma}_i \cup \dot{\alpha}_i$.

Figure 5. $\dot{\gamma}_i$ (resp. loops) on \dot{D}_i satisfying condition (St1) (resp. (St2)).

In this section, we show that if E is incompressible in the exterior of L and has minimal complexity, then E is in standard position (Proposition 2.1). Before proving the proposition, we introduce the following two kinds of operations.

Assume that there is a simple loop ρ on E which bounds a disk δ in S^3 such that $\delta \cap E = \partial\delta = \rho$. Then we may *surger* E along δ by replacing a neighborhood of ρ in E with two parallel copies of δ (the result surface may have a closed component). If ρ is an essential simple loop, i.e., does not bound a disk on E , then this operation is a *compression* for E .

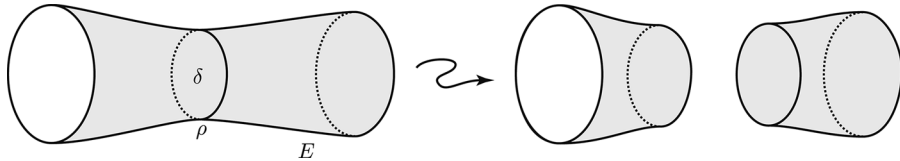


Figure 6. Surgery of E along a disk.

Extend $f : \dot{\mathcal{D}} \cup \dot{\mathcal{B}} \rightarrow S^3$ to $f : \dot{E} \cup (\dot{\mathcal{D}} \cup \dot{\mathcal{B}}) \rightarrow S^3$ such that $f(\dot{E}) = E$. We call an end \dot{p} of an arc $\dot{\gamma}$ of \mathcal{S} on $\partial(\dot{D}_i \cup \dot{B}_i) \cap \partial\dot{E}$ a *branch point*. Define the *orientation* of a branch point \dot{p} as the orientation of $\dot{\gamma}$ around \dot{p} induced by the orientation of E . We say that the orientations of two branch points which are adjacent on $\partial(\dot{D}_i \cup \dot{B}_i) \cap \partial\dot{E}$ *match* if the same (positive or negative) sides face each other. If the orientations match, then we can deform E to eliminate the branch points by isotopy with reducing the complexity as illustrated in Figure 7, where \circ indicates a branch point.

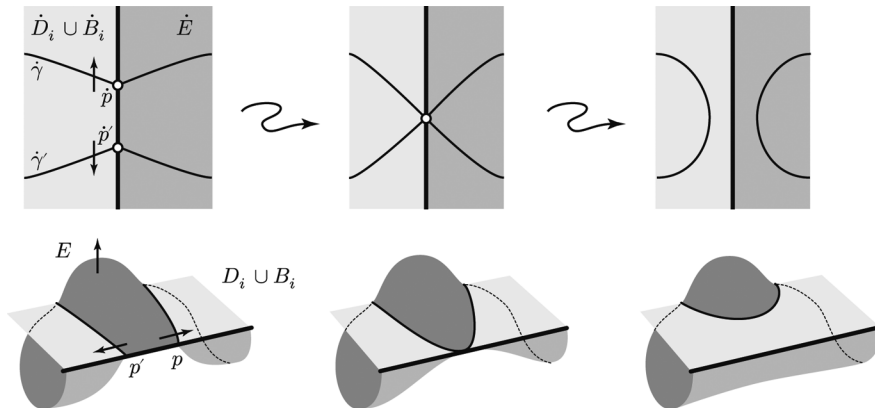


Figure 7. Elimination of branch points.

PROPOSITION 2.1. *Let L be a link obtained from a link ℓ by an SR-fusion with respect to $\mathcal{D} \cup \mathcal{B}$ and E a Seifert surface for L . If E is incompressible in the exterior of L and has minimal complexity, then E is in standard position with respect to $\mathcal{D} \cup \mathcal{B}$.*

PROOF. To prove the proposition, we take an arbitrary elementary component $\mathcal{D}^k \cup \mathcal{B}^k$ and show that \mathcal{S}^k satisfies the conditions for E to be in standard position with respect to $\mathcal{D} \cup \mathcal{B}$. First consider the loops of \mathcal{S}_i .

CLAIM 2.2. *\mathcal{S}_i does not have a loop which bounds a disk on $\dot{D}_i \cup \dot{B}_i$ intersecting with neither $\dot{\alpha}_i$ nor $\dot{\alpha}_{i+1}$.*

PROOF. Assume otherwise. Take an innermost one $\dot{\rho}$ from such loops on $\dot{D}_i \cup \dot{B}_i$ and let δ be the disk bounded by $\rho = f(\dot{\rho})$ on $D_i \cup B_i$. Since E is incompressible in the exterior of L , ρ also bounds a disk on E . Therefore by surgering E along δ , we obtain a sphere and another incompressible Seifert surface for L whose complexity is less than that of E , which contradicts that E has minimal complexity. \square

CLAIM 2.3. S_i does not have a loop which bounds a disk on $\dot{D}_i \cup \dot{B}_{i,1}$ containing exactly one boundary point of $\dot{\alpha}_i$.

PROOF. Assume that there is such a loop $\dot{\rho}$ in S_i . Then $\rho = f(\dot{\rho})$ is a simple closed curve and bounds a disk on $D_i \cup B_{i,1}$ which intersects with L in one point, and thus $\text{lk}(\rho, L) = \pm 1$. On the other hand, since ρ is also on $\text{int } E$, ρ^+ does not intersect with E , where ρ^+ is ρ pushed into the positive normal direction of E . Thus $\text{lk}(\rho, L) = \text{lk}(\rho^+, L) = 0$, since E is a Seifert surface for L . This is a contradiction. \square

CLAIM 2.4. None of the elements of S_i has a subarc which bounds a disk on $\dot{D}_i \cup \dot{B}_i$ with a proper subarc of $\dot{\alpha}_i$ or $\dot{\alpha}_{i+1}$ whose interior intersects with neither $\dot{\alpha}_i$ nor $\dot{\alpha}_{i+1}$.

PROOF. Assume otherwise. Then take an innermost one from such subarcs, that is, it bounds a disk $\dot{\delta}$ on $\dot{D}_i \cup \dot{B}_i$ with a proper subarc of $\dot{\alpha}_i$ (resp. $\dot{\alpha}_{i+1}$) whose interior does not contain any other such subarcs. Since $\dot{\delta}$ does not contain any loops from Claim 2.2, we can deform E along δ by isotopy so to reduce the one or two triple points (see Figure 8 for example), which contradicts that E has minimal complexity. \square

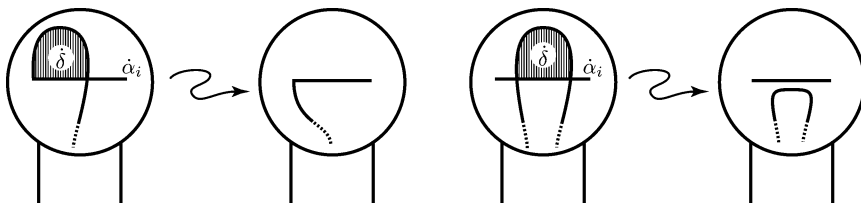


Figure 8.

Let $\dot{\gamma}_{i,1}$ and $\dot{\gamma}_{i,2}$ be the arcs on $\dot{D}_i \cup \dot{B}_i$ which have a boundary point on $\partial\dot{\alpha}_i$ (we may have that $\dot{\gamma}_{i,1} = \dot{\gamma}_{i,2}$). We need to show that $\dot{\gamma}_{i,1} = \dot{\gamma}_{i,2}$ and it is the only arc of S_i . Let $\tilde{S}_i \subset S_i$ be the set of arcs which have their boundary points on $\partial(\dot{D}_i \cup \dot{B}_i)$, i.e., \tilde{S}_i is the set obtained from S_i by removing $\dot{\gamma}_{i,1}$, $\dot{\gamma}_{i,2}$ and loops. Thus we will show that $\tilde{S}^k = \cup_i \tilde{S}_i = \emptyset$. Let $\hat{\alpha}_i = \dot{\gamma}_{i,1} \cup \dot{\alpha}_i \cup \dot{\gamma}_{i,2}$, which is an arc or a loop. We know that $\hat{\alpha}_i$ is simple from Claim 2.4 (therefore if $\dot{\gamma}_{i,1} = \dot{\gamma}_{i,2}$, then it satisfies condition (St1)).

Consider $\hat{\alpha}_i$ and the arcs of \tilde{S}_i . Transferring branch points on $\partial\dot{B}_{i,1}$ (resp. $\partial\dot{B}_{i,2}$) to $\partial\dot{D}_i - \partial\dot{B}_i$ (resp. $\dot{B}_i \cap \dot{\ell}$), we may assume that there are no branch points on ∂B_i , where $\dot{B}_i \cap \dot{\ell}$ is the subarc of $\partial\dot{B}_i$ such that $f(\dot{B}_i \cap \dot{\ell}) = B_i \cap \ell$. Note that $\partial(\dot{D}_i \cup \dot{B}_i) \cap \partial\dot{E} = \partial(\dot{D}_i \cup \dot{B}_i) - (\dot{B}_i \cap \dot{\ell})$, and thus a boundary point of an arc on $\dot{B}_i \cap \dot{\ell}$ is not a branch point. Now each of $\hat{\alpha}_i$ and the arcs of \tilde{S}_i has its boundary points on $(\partial\dot{D}_i - \partial\dot{B}_i) \cup (\dot{B}_i \cap \dot{\ell})$.

CLAIM 2.5. None of $\hat{\alpha}_i$ and the arcs of \tilde{S}_i has its boundary points on $\partial\dot{D}_i - \partial\dot{B}_i$.

PROOF. Assume otherwise and take an innermost one $\dot{\gamma}$ on $\dot{D}_i \cup \dot{B}_i$ among such arcs, i.e., $\dot{\gamma}$ bounds with a subarc of $\partial\dot{D}_i - \partial\dot{B}_i$ a disk $\dot{\delta}$ on $\dot{D}_i \cup \dot{B}_i$ whose interior contains none of $\hat{\alpha}_i$ and arcs of $\tilde{\mathcal{S}}_i$.

If $\dot{\gamma} = \hat{\alpha}_i$, then of course $\hat{\alpha}_i$ is not a loop but an arc, and each arc which has a boundary point on $\text{int}(\partial\dot{\delta} \cap \partial\dot{D}_i)$ intersects with $\hat{\alpha}_i$, since $\text{int}\dot{\delta}$ contains no arcs of $\tilde{\mathcal{S}}_i$. If there are no such arcs, then the two boundary points of $\hat{\alpha}_i$ are adjacent on $\partial\dot{D}_i - \partial\dot{B}_i$ and their orientations match (the left side of Figure 9). Hence we can eliminate the pair of branch points as illustrated in Figure 7 to reduce the complexity of E , which contradicts that E has minimal complexity. If there exist such arcs, then there is at least one adjacent pair of branch points on $\partial\dot{\delta} \cap \partial\dot{D}_i$ whose orientations match (the second to the left of Figure 9). Hence we can eliminate the pair of branch points to reduce the complexity of E , which again contradicts that E has minimal complexity.

If $\dot{\gamma}$ is an arc of $\tilde{\mathcal{S}}_i$, then $\dot{\gamma}$ intersects with $\hat{\alpha}_i$ or not. If $\dot{\gamma}$ intersects with $\hat{\alpha}_i$, then the boundary point \dot{p} of $\hat{\alpha}_i$ on $\partial\dot{\delta} \cap \partial\dot{D}_i$ is adjacent to two boundary points of $\dot{\gamma}$ on $\partial\dot{D}_i - \partial\dot{B}_i$, since $\text{int}\dot{\delta}$ does not contain arcs of $\tilde{\mathcal{S}}_i$ (the second to the right of Figure 9). Moreover, the orientation of \dot{p} matches the orientation of one boundary point of $\dot{\gamma}$. Hence we can eliminate the pair of branch points whose orientations match to reduce the complexity of E , which contradicts that E has minimal complexity. If $\dot{\gamma}$ does not intersect with $\hat{\alpha}_i$, then the two boundary points of $\dot{\gamma}$ are adjacent on $\partial\dot{D}_i - \partial\dot{B}_i$ and the orientations of the two branch points match, since $\text{int}\dot{\delta}$ does not contain arcs of $\tilde{\mathcal{S}}_i$ (the right side of Figure 9). Hence we can eliminate the pair of branch points to reduce the complexity of E , which again contradicts that E has minimal complexity. \square

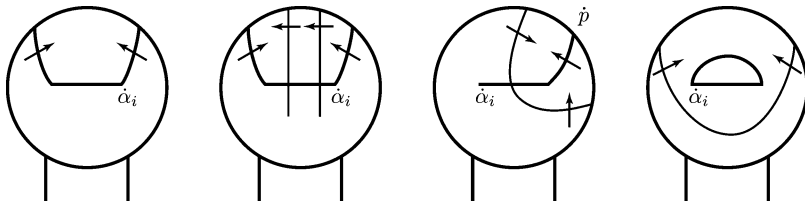


Figure 9.

Now consider the number of intersections of \mathcal{S}_i and $\hat{\alpha}_i$ (resp. $\hat{\alpha}_{i+1}$). Note that only arcs among \mathcal{S}_i can intersect with $\hat{\alpha}_i$ or $\hat{\alpha}_{i+1}$, since each loop of \mathcal{S}_i intersect with neither $\hat{\alpha}_i$ nor $\hat{\alpha}_{i+1}$ from Claims 2.3 and 2.4. Let $\dot{\gamma}_i = \dot{\gamma}_{i,1} \cup \dot{\gamma}_{i,2}$ and $\sharp(\dot{x} \cap \dot{y})$ the number of intersections of $\text{int}\dot{x}$ and $\text{int}\dot{y}$, where \dot{x} (resp. \dot{y}) is an arc or a loop on $\dot{D}_i \cup \dot{B}_i$. Then we have the following.

CLAIM 2.6. *We have that $\sharp(\dot{x} \cap \hat{\alpha}_{i+1}) \geq \sharp(\dot{x} \cap \hat{\alpha}_i)$, where \dot{x} is $\dot{\gamma}_i$ or an element $\dot{\gamma}$ of $\tilde{\mathcal{S}}_i$.*

PROOF. First we know that $\hat{\alpha}_i$ is one of the following three types from Claim 2.5; it has no boundary points, i.e., $\dot{\gamma}_{i,1} = \dot{\gamma}_{i,2}$, or either it has one boundary point on $\partial\dot{D}_i - \partial\dot{B}_i$ and one on $\dot{B}_i \cap \ell$ or it has both boundary points on $\dot{B}_i \cap \ell$ (see the left three figures of Figure 10). Then $\sharp(\dot{\gamma}_i \cap \hat{\alpha}_{i+1}) = 0, 1$, and 2 for the first, second, and third types, respectively from Claim 2.4. Since $\sharp(\dot{\gamma}_i \cap \hat{\alpha}_i) = 0$ for any of the three types also from Claim 2.4, we have that $\sharp(\dot{\gamma}_i \cap \hat{\alpha}_{i+1}) \geq \sharp(\dot{\gamma}_i \cap \hat{\alpha}_i)$.

Second $\dot{\gamma}$ is one of the following two types also from Claim 2.5; it has one boundary point on $\partial\dot{D}_i - \partial\dot{B}_i$ and one on $\dot{B}_i \cap \dot{\ell}$ or it has both boundary points on $\dot{B}_i \cap \dot{\ell}$ (see the right four figures of Figure 10). For the former type, $\sharp(\dot{\gamma} \cap \dot{\alpha}_i) = 0$ or 1, and $\sharp(\dot{\gamma} \cap \ddot{\alpha}_{i+1}) = 1$ and for the latter type, $\sharp(\dot{\gamma} \cap \dot{\alpha}_i) = 0$ or 1 and $\sharp(\dot{\gamma} \cap \ddot{\alpha}_{i+1}) = 2$ from Claim 2.4. In either case, we have that $\sharp(\dot{\gamma} \cap \ddot{\alpha}_{i+1}) \geq \sharp(\dot{\gamma} \cap \dot{\alpha}_i)$. \square

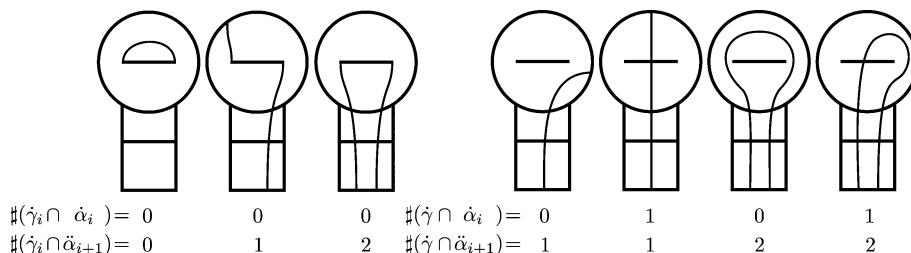


Figure 10.

Continuation of the proof of Proposition 2.1. Take a look at the number $\sharp(\mathcal{S}^k \cap \dot{\alpha}_i)$ of intersections of \mathcal{S}^k and $\dot{\alpha}_i$ ($1 \leq i \leq m_k$). From Claim 2.6 and that $f(\ddot{\alpha}_i) = f(\dot{\alpha}_i)$, we have that

$$\sharp(\mathcal{S}^k \cap \dot{\alpha}_{i+1}) = \sharp(\mathcal{S}^k \cap \ddot{\alpha}_{i+1}) \geq \sharp(\mathcal{S}^k \cap \dot{\alpha}_i).$$

Hence we have that

$$\sharp(\mathcal{S}^k \cap \dot{\alpha}_{m_k+1}) \geq \sharp(\mathcal{S}^k \cap \dot{\alpha}_{m_k}) \geq \cdots \geq \sharp(\mathcal{S}^k \cap \dot{\alpha}_1).$$

Since we consider the lower index modulo m_k , we also have that $\sharp(\mathcal{S}^k \cap \dot{\alpha}_{m_k+1}) = \sharp(\mathcal{S}^k \cap \dot{\alpha}_1)$. Therefore we have that

$$\sharp(\mathcal{S}^k \cap \dot{\alpha}_{m_k}) = \sharp(\mathcal{S}^k \cap \dot{\alpha}_{m_k-1}) = \cdots = \sharp(\mathcal{S}^k \cap \dot{\alpha}_1),$$

and thus that $\sharp(\mathcal{S}_i \cap \ddot{\alpha}_{i+1}) = \sharp(\mathcal{S}_i \cap \dot{\alpha}_i)$ ($1 \leq i \leq m_k$). Then we have that $\sharp(\dot{x} \cap \ddot{\alpha}_{i+1}) = \sharp(\dot{x} \cap \dot{\alpha}_i)$ from Claim 2.6, where \dot{x} is $\dot{\gamma}_i$ or an element $\dot{\gamma}$ of \mathcal{S}_i . Thus especially for $\dot{\gamma}_i$, we have that $\sharp(\dot{\gamma}_i \cap \ddot{\alpha}_{i+1}) = \sharp(\dot{\gamma}_i \cap \dot{\alpha}_i) = 0$ (the first figure of Figure 10). On the other hand, also for $\dot{\gamma}$, we have that $\sharp(\dot{\gamma} \cap \ddot{\alpha}_{i+1}) = \sharp(\dot{\gamma} \cap \dot{\alpha}_i) = 1$ (the fifth figure of Figure 10). Then $\dot{\gamma}$ cannot coexist with $\dot{\gamma}_{i,1} = \dot{\gamma}_{i,2}$, and thus we can conclude that $\tilde{\mathcal{S}}^k = \cup_i \mathcal{S}_i = \emptyset$. Moreover if \mathcal{S}_i has a loop, then the loop bounds a disk on \dot{D}_i containing $\dot{\gamma}_i \cup \dot{\alpha}_i$ from Claims 2.2, 2.3, and 2.4 and deforming E by isotopy if necessary. Hence \mathcal{S}^k satisfies the conditions for E to be in standard position with respect to $\mathcal{D} \cup \mathcal{B}$ for any k , and thus we complete the proof. \square

3. Simple ribbon fusions and genera of links.

Let L be a link obtained from a link ℓ by an SR -fusion with respect to $\mathcal{D} \cup \mathcal{B}$, and E a Seifert surface for L . In this section, we prove Theorem 1.1. To do this, we actually construct a Seifert surface for ℓ from E .

Assume that E is incompressible in the exterior of L and has minimal complexity with respect to $\mathcal{D} \cup \mathcal{B}$. Let $F_0 = E \cup (\mathcal{D} \cup \mathcal{B})$. Then F_0 is a singular, oriented surface such that $\partial F_0 = \ell$ and, by Proposition 2.1, the set of its singularities consists of $p = \sum_k m_k$ loops $\cup_{i,k} f(\hat{\alpha}_i) = \cup_{i,k} (\gamma_i \cup \alpha_i)$ each of which bounds a disk δ_i on D_i and q loops $\cup_j \rho_{p+j}$ each of which bounds a disk δ_{p+j} on a disk of \mathcal{D} . Extend $f : \dot{\mathcal{D}} \cup \dot{\mathcal{B}} \rightarrow S^3$ to $f : \dot{E} \cup (\dot{\mathcal{D}} \cup \dot{\mathcal{B}}) \rightarrow S^3$ such that $f(\dot{E}) = E$ and let $\dot{F}_0 = \dot{E} \cup (\dot{\mathcal{D}} \cup \dot{\mathcal{B}})$. We have that $f(\dot{F}_0) = F_0$.

Surger¹ F_0 along all the disks bounded by the loops from an innermost one in turn. Namely, we construct a sequence F_0, F_1, \dots, F_{p+q} of (singular) surfaces such that F_{t+1} is obtained by surgering F_t along $\delta_t = \delta_i$ or δ_{p+j} and give up the closed component if we obtain it by the surgery ($t = 0, \dots, p+q-1$). Thus each \dot{F}_{t+1} has no closed components and F_{p+q} is a Seifert surface for ℓ . Let $F(E)$ be F_{p+q} .

To prove Theorem 1.1, we calculate the difference of the number of components and of genera between \dot{F}_{t+1} and \dot{F}_t . Note that the genus $g(\dot{F}_t)$ is given by:

$$g(\dot{F}_t) = \sharp(\dot{F}_t) - \frac{\chi(\dot{F}_t) + \sharp(\partial \dot{F}_t)}{2} = \sharp(\dot{F}_t) - \frac{\chi(\dot{F}_t) + \sharp(\ell)}{2}.$$

Thus we have that $d_g(t) = d_\sharp(t) - d_\chi(t)/2$, where $d_g(t) = g(\dot{F}_{t+1}) - g(\dot{F}_t)$, $d_\sharp(t) = \sharp(\dot{F}_{t+1}) - \sharp(\dot{F}_t)$, and $d_\chi(t) = \chi(\dot{F}_{t+1}) - \chi(\dot{F}_t)$, respectively.

LEMMA 3.1. *We have that $\sharp(F(E)) \geq \sharp(E)$ and that $g(F(E)) \leq g(E)$.*

PROOF. Since \dot{F}_0 is homeomorphic to E , we have that $\sharp(\dot{F}_0) = \sharp(E)$ and that $g(\dot{F}_0) = g(E)$. Thus it is sufficient to show the following claim, since then we have that $d_\sharp(t) \geq 0$ and that $d_g(t) \leq 0$ for any t , which induces the conclusion. \square

CLAIM 3.2. *Let $\rho = \partial \delta_t$. Then we have one of the following :*

- (1) ρ is a non-separating loop on \dot{F}_t , $d_\sharp(t) = 0$, and $d_g(t) = -1$,
- (2) ρ is a separating loop on \dot{F}_t such that by the surgery along δ_t we obtain
 - (2-1) a sphere component, $d_\sharp(t) = 0$, and $d_g(t) = 0$,
 - (2-2) a closed surface component with genus $n(> 0)$, $d_\sharp(t) = 0$, and $d_g(t) = -n$ and
 - (2-3) no closed components, $d_\sharp(t) = 1$, and $d_g(t) = 0$.

PROOF. First note that the Euler characteristic increases by 2 by the surgery along δ_t . If ρ is a non-separating loop on \dot{F}_t , then $d_\sharp(t) = 0$, and thus $d_g(t) = 0 - 2/2 = -1$. If ρ is a separating loop on \dot{F}_t , then we either obtain a closed component or not by the surgery along δ_t . In the former case, we give up the closed component, and thus $d_\sharp(t) = 0$. If the closed component is a sphere, then $d_g(t) = 0 - (2 - 2)/2 = 0$. If the closed component is a closed surface with genus $n(> 0)$, then its Euler characteristic χ is equal to $2 - 2n$, and thus $d_g(t) = 0 - (2 - (2 - 2n))/2 = -n$. In the latter case, the number of components increase by 1, i.e., $d_\sharp(t) = 1$, and thus $d_g(t) = 1 - 2/2 = 0$. \square

¹We surger F_t along δ_t by replacing a neighborhood of $\partial \delta_t$ on $E \cup \mathcal{B} \subset F_t$ with two parallel copies of δ_t .

LEMMA 3.3. *A Seifert surface E for a link L with $\sharp(E) = \nu(L)$ and $g(E) = g_{\nu(L)}(L)$ is incompressible in the exterior of L .*

PROOF. Assume otherwise and take a compressing disk δ for E . Let E' be the surface obtained from E by surgering along δ . Then $\partial\delta$ is either a non-separating loop on E , a separating loop on E and E' has a closed component with non-zero genus, or a separating loop on E and E' has no closed components. The first and second cases contradict that $g(E) = g_{\nu(L)}(L)$ and the third case contradicts that $\sharp(E) = \nu(L)$. \square

LEMMA 3.4. *Let E be a Seifert surface for a link L with $g(E) = g_{\sharp(E)}(L)$ which is compressible in the exterior of L , and E' the surface obtained from E by a compression. Then we have that $\sharp(E') = \sharp(E) + 1$ and $g(E') = g(E)$.*

PROOF. Assume that we obtain E' by compressing E along a disk δ . If $\partial\delta$ is a non-separating loop on E or a separating loop on E such that E' has a closed component, then we have that $g_{\sharp(E)}(L) < g(E)$, which contradicts that $g(E) = g_{\sharp(E)}(L)$. Therefore $\partial\delta$ is a separating loop on E such that E' has no closed components, and thus $\sharp(E') = \sharp(E) + 1$ and $g(E') = g(E)$. \square

PROOF OF THEOREM 1.1. First we show that $\nu(\ell) \geq \nu(L)$. Take a Seifert surface E for L with $\sharp(E) = \nu(L)$, $g(E) = g_{\nu(L)}(L)$, and minimal complexity with respect to $\mathcal{D} \cup \mathcal{B}$. Since E is incompressible in the exterior of L from Lemma 3.3, we can construct $F(E)$ from E . Then we obtain that $\sharp(F(E)) \geq \sharp(E)$ from Lemma 3.1, and thus we obtain the conclusion, since $\nu(\ell) \geq \sharp(F(E))$ and $\sharp(E) = \nu(L)$.

Second we show that $g_r(\ell) \leq g_r(L)$ for any integer r ($1 \leq r \leq \nu(L)$). Take a Seifert surface \tilde{E} for L with $\sharp(\tilde{E}) = r$, $g(\tilde{E}) = g_r(L)$, and minimal complexity with respect to $\mathcal{D} \cup \mathcal{B}$. We can obtain the conclusion by showing that we can construct an incompressible Seifert surface E for L from \tilde{E} with $\sharp(E) \geq r$ and $g(E) = g_r(L)$, since then, we have that $g_r(\ell) \leq g_{\sharp(E)}(\ell) \leq g_{\sharp(F(E))}(\ell) \leq g(F(E)) \leq g(E) = g_r(L)$ from Lemma 3.1. If \tilde{E} is incompressible in the exterior of L , then it is sufficient to let E be \tilde{E} . If \tilde{E} is compressible in the exterior of L , then we can obtain an incompressible Seifert surface E for L from \tilde{E} by a finitely many times of compressions, which satisfies that $\sharp(E) > \sharp(\tilde{E}) = r$ and $g(E) = g(\tilde{E}) = g_r(L)$ from Lemma 3.4.

Third we show that the three conditions (1), (2), and (3) in the statement are equivalent. Since it is clear that (1) (resp. (2)) induces (2) (resp. (3)), it is sufficient to show that (3) induces (1), i.e., assuming that $\nu(L) = \nu(\ell)$ and $g_{\nu(L)}(L) = g_{\nu(\ell)}(\ell)$, we show that \mathcal{O} bounds a mutually disjoint disks $\bigcup_{i,k} \Delta_i$ such that $\partial\Delta_i = O_i$ and $\text{int } \Delta_i$ does not intersect with $L \cup \mathcal{B}$.

Take a Seifert surface E for L with $\sharp(E) = \nu(L)$, $g(E) = g_{\nu(L)}(L)$, and minimal complexity with respect to $\mathcal{D} \cup \mathcal{B}$. Since E is incompressible in the exterior of L from Lemma 3.3, E is in standard position from Proposition 2.1, and we can construct $F(E)$ from E . Since $\nu(L) = \nu(\ell)$ and $g_{\nu(L)}(L) = g_{\nu(\ell)}(\ell)$, we have that $\sharp(F(E)) = \sharp(E)$ and $g(F(E)) = g(E)$, and thus each $\dot{\rho}_t$ is a separating loop on \dot{F}_t and we obtain a sphere component by surgering F_t along δ_t for any t by Lemma 3.1.

Let S_i be the sphere which is to be given up after the surgery along δ_{i+1} bounded by $f(\hat{\alpha}_{i+1})$ (see the left side of Figure 11 for a case with $m_k \geq 2$). Then consider $\bigcup_{i,k} S_i$.

Note that $\cup_{i,k} S_i$ consists of mutually disjoint spheres, and that each S_i contains D_i . Now let Δ_i be the disk obtained from $S_i - \text{int } D_i$ by pushing so that $\partial \Delta_i = \partial D_i$ and $\Delta_i \cap B_{i,1} = \partial B_{i,1} \cap \partial D_i$ (see the right side of Figure 11). Then we know that Δ_i is a disk bounded by ∂D_i whose interior does not intersect with $L \cup \mathcal{B}$ and that $\cup_{i,k} \Delta_i$ are mutually disjoint disks from the construction. Therefore the SR -fusion is trivial. \square

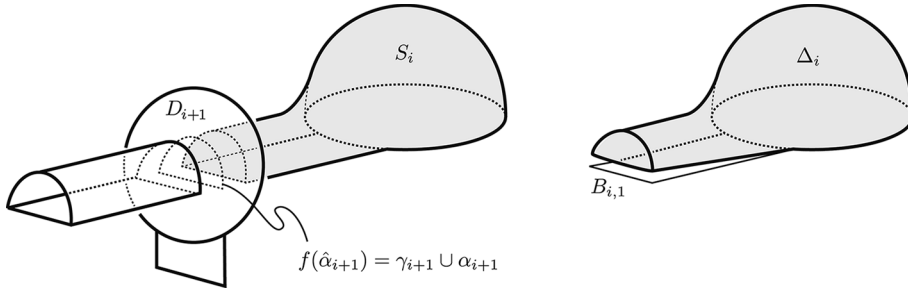


Figure 11.

4. Disconnectivity numbers of links which are related by SR -fusions.

In this section, we prove Theorem 1.2. Denote the link $\ell_1 \cup \ell_2$ by $\ell_1 \circ \ell_2$ if ℓ_1 is split from ℓ_2 .

LEMMA 4.1. *We have that $\nu(\ell_1 \circ \ell_2) = \nu(\ell_1) + \nu(\ell_2)$ for two links ℓ_1 and ℓ_2 .*

PROOF. Since ℓ_1 is split from ℓ_2 , there is a Seifert surface $\mathcal{F} (= F_1 \cup F_2)$ for $\ell_1 \circ \ell_2$ such that $\partial F_i = \ell_i$ and $\sharp(F_i) = \nu(\ell_i)$ for $i = 1, 2$. Hence $\nu(\ell_1 \circ \ell_2) \geq \sharp(\mathcal{F}) = \sharp(F_1) + \sharp(F_2) = \nu(\ell_1) + \nu(\ell_2)$.

Conversely suppose that F is a Seifert surface for $L = \ell_1 \circ \ell_2$ with $\sharp(F) = \nu(\ell_1 \circ \ell_2)$. Let Σ be a 2-sphere which separates ℓ_1 from ℓ_2 . If $\Sigma \cap F \neq \emptyset$, then it consists of mutually disjoint simple loops. Here we may assume that $g(F) = g_{\nu(L)}(L)$ and $\sharp(\Sigma \cap F)$ is minimal among such Seifert surfaces. Take an innermost one ρ on Σ among the loops of $\Sigma \cap F$. Since ρ bounds a disk δ on Σ which does not contain any other loops, we can surger F along δ to obtain F' . Since F is incompressible in the exterior of L from Lemma 3.3, ρ is a separating loop on F and F' has a sphere component. However this contradicts the minimality of $\sharp(\Sigma \cap F)$. Therefore $\Sigma \cap F = \emptyset$. Then let X be a 3-ball in S^3 with $\partial X = \Sigma$ and $F_1 = F \cap X$ and $F_2 = F \cap (S^3 - X)$. Then $\partial F_i = \ell_i$ for $i = 1, 2$ and $\nu(\ell_1 \circ \ell_2) = \sharp(F) = \sharp(F_1) + \sharp(F_2) \leq \nu(\ell_1) + \nu(\ell_2)$. Therefore we complete the proof. \square

PROOF OF THEOREM 1.2. It is sufficient to show the case where $v = 1$, since then the links $\ell \circ \mathcal{O}^{v-1}$ and $L \circ \mathcal{O}^{v-1}$ are the required pair by Lemma 4.1, where \mathcal{O}^{v-1} is the $(v-1)$ -component trivial link. Now let ℓ be the $(w+1)$ -component trivial link $o_0 \cup o_1 \cup \cdots \cup o_w$ and $L = L^w$ the $(w+1)$ -component link obtained from ℓ by an SR -fusion as illustrated in Figure 12.

We show that $\nu(L^w) = 1$ by induction on w . Let l be the link as illustrated in the right side of Figure 13. Since l is not a boundary link ([4, p. 140 Example]), we have that $\nu(l) = 1$. Thus we have that $\nu(L^w) = 1$ in the case where $w = 1$, since L^1 is l .

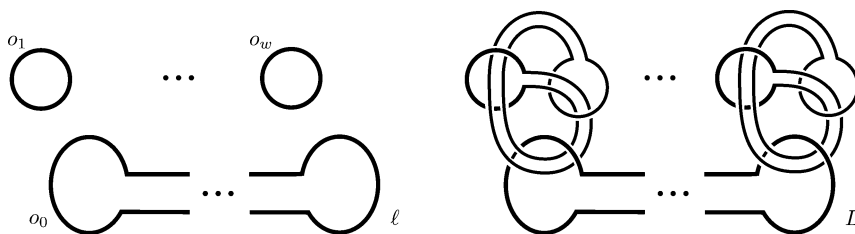


Figure 12.

Now consider the case where $w = k + 1$ under the assumption that $\nu(L^w) = 1$ in the cases where $w \leq k$. Note that L^{k+1} is obtained by a 1-fusion of L^k and l along a band Y as illustrated in the left side of Figure 13. It is sufficient to show that $\nu(L^{k+1}) \leq 1$. Let Σ be a 2-sphere which intersects with L^{k+1} only in two points on $L^{k+1} \cap Y$ and separates L^k from l . Let F be a Seifert surface for L^{k+1} with $\sharp(F) = \nu(L^{k+1})$ and $g(F) = g_{\nu(L^{k+1})}(L^{k+1})$. Assume that $\sharp(\Sigma \cap F)$ is minimal among all such Seifert surfaces. If $\Sigma \cap F \neq \emptyset$, then it consists of an arc, say α , and loops which are mutually disjoint and simple. Take an innermost one ρ on $\Sigma - \alpha$ among all the loops of $\Sigma \cap F$. Since ρ bounds a disk δ on Σ which contains neither α nor the other loops, we can surger F along δ to obtain F' . Since F is incompressible in the exterior of L from Lemma 3.3, ρ is a separating loop on F and F' has a sphere component. However this contradicts the minimality of $\sharp(\Sigma \cap F)$. Therefore $\Sigma \cap F = \{\alpha\}$. Thus $F_0 (= cl(F - N(\alpha : F)))$ is a Seifert surface for $L^k \circ l$ and $F_0 \cap \Sigma = \emptyset$. Hence we obtain that $\nu(L^{k+1}) = \sharp(F) = \sharp(F_0) - 1 \leq \nu(L^k \circ l) - 1 = \nu(L^k) + \nu(l) - 1 = 1 + 1 - 1 = 1$ by Lemma 4.1. Therefore we complete the proof. \square

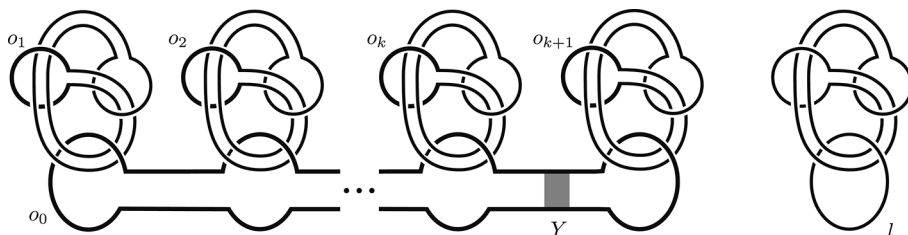


Figure 13.

ACKNOWLEDGEMENT. The authors would like to thank Professor Yasutaka Nakanishi and Professor Shin Satoh for reminding us of the example on page 140 of [4], which helps us to simplify the proof of Theorem 1.2.

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