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An obstruction for codimension two contact embeddings in the odd dimensional Euclidean spaces

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Abstract. We prove that the first Chern class of a codimension two closed contact submanifold of the odd dimensional Euclidean space is trivial. For any closed co-oriented contact 3-manifold with trivial Chern class, we prove that there is a contact structure on the 5-dimensional Euclidean space which admits a contact embedding of it.

1. Introduction.

A contact structure is a maximally non-integrable hyperplane field ξ on an odd dimensional manifold M^{2m+1}. If the normal bundle of ξ is orientable, we say that the contact structure ξ is co-oriented. This is equivalent to that there is a global defining 1-form α of ξ. Then, for a global contact form α, the 2-form $d\alpha$ induces a symplectic vector bundle structure on ξ. The conformal class of the symplectic bundle structure does not depend on the choice of α. A complex structure J on ξ is compatible if J_p is $(d\alpha)_p$-compatible on ξ_p for each $p \in M^{2m+1}$, i.e., if $(u, v) \mapsto (d\alpha)_p(u, J_p v)$ is a positive definite symmetric bilinear form.

Definition 1.1 (The Chern classes of a contact structure). Let $(M^{2m+1}, \xi = \ker \alpha)$ be a co-oriented contact structure. Since the conformal class of the symplectic bundle structure $(\xi, d\alpha|_\xi)$ does not depend on the choice of α and the complex bundle structure on ξ compatible with the symplectic bundle structure is homotopically unique, we define the Chern classes of ξ to be the Chern classes of this complex vector bundle.

Definition 1.2 (Contact submanifolds). Let (M, ξ) and (N, η) be co-oriented contact structures. An embedding $f : M \to N$ is said to be a contact embedding if $f_*(TM) \cap \eta|_{f(M)} = f_*\xi$. The embedded contact manifold $(f(M), f_*\xi)$ or (M, ξ) itself is called a contact submanifold of (N, η). Similarly, an immersed contact submanifold (M, ξ) and a contact immersion $M \to N$ are defined.

In this paper, we study codimension two contact embeddings in the odd dimensional Euclidean spaces. Especially, we are interested in closed contact 3-manifolds in $\mathbb{R}^5$. As a manifold, any 3-manifold embeds in $\mathbb{R}^5$ by Wall's theorem ([**9**]). On the other hand,

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we show in Section 2 that the first Chern class of a contact 3-manifold is an obstruction for contact embeddings.

THEOREM 1.3. *If a closed contact manifold* (M^{2n-1}, ξ) *is a contact submanifold of a co-oriented contact manifold* (N^{2n+1}, η) *such that* $H^2(N^{2n+1}; \mathbb{Z}) = 0$*, then the first Chern class* $c_1(\xi)$ *is trivial.*

In particular, there are infinitely many contact 3-manifolds which cannot be embedded in $\mathbb{R}^5$ as contact submanifolds for any contact structure on $\mathbb{R}^5$. On the other hand, the following theorem is known for the existence of contact immersions and contact embeddings in the standard contact structure on $\mathbb{R}^{2n+1}$. Let η_0 be the standard contact structure on $\mathbb{R}^{2n+1}$, i.e., η_0 is defined by the 1-form

$$\alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j,$$

where $(x_1, y_1, \ldots, x_n, y_n, z)$ are the coordinates on $\mathbb{R}^{2n+1}$.

THEOREM 1.4 (Mori [**7**], Martínez-Torres [**5**]). *Any closed co-orientable contact* $(2m+1)$*-manifold admits a contact immersion into* $(\mathbb{R}^{4m+1}, \eta_0)$*, and it admits a contact embedding into* $(\mathbb{R}^{4m+3}, \eta_0)$*.*

Therefore, the first Chern class is the obstruction peculiar to the cases of codimension two contact embeddings. We do not know whether every closed co-oriented contact 3-manifold with $c_1(\xi) = 0$ can be embedded as a contact submanifold in $(\mathbb{R}^5, \eta_0)$. By Gromov's h-principle, however, we show the following result in Section 3.

THEOREM 1.5. *Let* (M^3, ξ) *be a closed co-oriented contact* 3*-manifold with* $c_1(\xi) = 0$*. Then there is a contact structure* η *on* $\mathbb{R}^5$ *such that we can embed* (M^3, ξ) *in* $(\mathbb{R}^5, \eta)$ *as a contact submanifold.*

2. Proof of Theorem 1.3.

We need the following definition for the proof of Theorem 1.3.

DEFINITION 2.1 (Conformal symplectic normal bundles). Let (M, η_M) be a contact submanifold of $(N, \eta = \ker \beta)$. The vector bundle η splits along M into the Whitney sum of the two subbundles

$$\eta|_M = \eta_M \oplus (\eta_M)^{\perp},$$

where η_M is the contact plane bundle on M given by $\eta_M = TM \cap \eta|_M$ and $(\eta_M)^{\perp}$ is the symplectic orthogonal of η_M in $\eta|_M$ with respect to the form $d\beta$. We can identify $(\eta_M)^{\perp}$ with the normal bundle νM. Moreover, $d\beta$ induces a symplectic structure on $(\eta_M)^{\perp}$. The conformal class of the symplectic structure does not depend on the choice of β. We call $(\eta_M)^{\perp}$ the conformal symplectic normal bundle of M in N.

Proof of Theorem 1.3. Let $f : M^{2n-1} \to N^{2n+1}$ be an embedding such that

$$f_*(TM) \cap \eta|_{f(M)} = f_*\xi.$$

Then, the Euler class of the normal bundle of f is zero. The reason is as follows. The Thom class of the normal bundle is a relative cohomology class of the tubular neighborhood of $f(M)$. We obtain the Euler class of the normal bundle by pulling it back into the absolute cohomology group $H^2(M^{2n-1};\mathbb{Z})$ through the composition of the two restriction homomorphisms

$$H^2(N^{2n+1}, N^{2n+1}\backslash M^{2n-1};\mathbb{Z}) \to H^2(N^{2n+1};\mathbb{Z}) \to H^2(M^{2n-1};\mathbb{Z}).$$

Hence, by the condition $H^2(N^{2n+1};\mathbb{Z}) = 0$, the Euler class of the normal bundle is zero (cf. Theorem 11.3 in [**6**]). The normal bundle of f is 2-dimensional, hence it is topologically trivial. Since the conformal symplectic structure on 2-dimensional trivial vector bundle is unique, the normal bundle of the contact submanifold $f(M)$ is also trivial as a conformal symplectic vector bundle. Hence, the vector bundle η splits along $f(M)$ as

$$\eta|_{f(M)} = \eta_{f(M)} \oplus (\eta_{f(M)})^{\perp},$$

where $\eta_{f(M)} = f_*\xi$ and $(\eta_{f(M)})^{\perp}$ is a trivial symplectic bundle. By the naturality of the first Chern class and the condition that $H^2(N^{2n+1};\mathbb{Z}) = 0$, it follows that $c_1(\eta|_{f(M)}) = f^*c_1(\eta) = 0$. On the other hand, taking the Whitney sum with a trivial symplectic bundle does not change the first Chern class. Thus, it follows that $c_1(\xi) = c_1(\eta|_{f(M)}) = 0$. □

Remark 2.2. If the manifold N^{2n+1} satisfies the condition $H^{2j}(N^{2n+1};\mathbb{Z}) = 0$ in addition in Theorem 1.3, then the j-th Chern class $c_j(\xi)$ is also trivial.

3. Proof of Theorem 1.5.

In Section 3.1, we review Gromov's h-principle and prove Propositon 3.4 which we use in the proof of Theorem 1.5. In Section 3.2, we review the Wu invariant of an embedding of a closed orientable 3-manifold in $\mathbb{R}^5$. In Section 3.3, we prove Theorem 1.5.

3.1. The *h*-principle.

Definition 3.1 (Almost contact structures). Let N^{2n+1} be an odd dimensional oriented manifold. An almost contact structure on N^{2n+1} is a pair (β_1, β_2) consisting of a global 1-form β_1 and a global 2-form β_2 satisfying the condition $\beta_1 \wedge \beta_2^n \neq 0$.

Remark 3.2. We can define an almost contact structure on N^{2n+1} as a reduction of the structure group of TN^{2n+1} from $\mathrm{SO}(2n+1)$ to $\mathrm{U}(n)$. The two definitions are equivalent.

Gromov proved that an almost contact structure on an odd dimensional open manifold is homotopic to a contact structure on it.

Theorem 3.3 (Gromov [**3**], see also [**1**]). *Let N^{2n+1} be an odd dimensional open manifold. If there is an almost contact structure on N^{2n+1}, then there is a contact structure on N^{2n+1} in the same homotopy class of almost contact structures. Moreover if the almost contact structure is already a contact structure on a neighborhood of a compact submanifold M^m of N^{2n+1} with $m < 2n$, then we can choose a contact structure on N^{2n+1} which coincides with the original one on a small neighborhood of M^m.*

Let $(M^{2n-1}, \xi = \ker \alpha)$ be a closed co-oriented contact manifold and M^{2n-1} be embedded in $\mathbb{R}^{2n+1}$. Since the normal bundle is trivial as is explained in the proof of Theorem 1.3, there exists an embedding

$$F\colon M^{2n-1} \times D^2 \to \mathbb{R}^{2n+1}.$$

The form $\alpha + r^2 d\theta$ induces a contact form β on $U = F(M^{2n-1} \times D^2)$, where (r, θ) is the polar coordinate in the 2-disk D^2. By Theorem 3.3, in order to extend the given contact structure, it is enough to extend it as an almost contact structure. Almost contact structures on N^{2n+1} correspond to sections of the $\mathrm{SO}(2n+1)/\mathrm{U}(n)$ bundle associated with the tangent bundle TN^{2n+1}. Since the tangent bundle of the manifold U in $\mathbb{R}^{2n+1}$ is trivialized, we can identify the almost contact structure on U with a map

$$\tilde{g} : M^{2n-1} \times D^2 \to \mathrm{SO}(2n+1)/\mathrm{U}(n).$$

Since the extendability of the map $\tilde{g}$ over $\mathbb{R}^{2n+1}$ is equivalent to the homotopical triviality of $\tilde{g}$, we obtain the following proposition.

Proposition 3.4. *We can embed (M^{2n-1}, ξ) in $\mathbb{R}^{2n+1}$ as a contact submanifold for some contact structure, if and only if there exists an embedding*

$$F\colon M^{2n-1} \times D^2 \to \mathbb{R}^{2n+1}$$

such that the map $g : M^{2n-1} \to \mathrm{SO}(2n+1)/\mathrm{U}(n)$ induced by the underlying almost contact structure of $(M^{2n-1} \times D^2, \alpha + r^2 d\theta)$ and the standard trivialization of $\mathbb{R}^{2n+1}$ is null-homotopic.

3.2. Wu invariant.

Let M^3 be a closed oriented 3-manifold and $\mathrm{Imm}[M^3, \mathbb{R}^5]$ be the set of regular homotopy classes of immersions of M^3 into $\mathbb{R}^5$.

Theorem 3.5 (Wu [**10**], see also [**4**], [**8**]). *The normal Euler class χ_f for an immersion $f : M^3 \to \mathbb{R}^5$ is of the form $2C$ for some $C \in H^2(M^3; \mathbb{Z})$, and for any $C \in H^2(M^3; \mathbb{Z})$, there is an immersion f such that $\chi_f = 2C$. Furthermore, there is a bijection*

$$\mathrm{Imm}[M^3, \mathbb{R}^5]_\chi \approx \coprod_{C \in H^2(M^3;\mathbb{Z}) \text{ with } 2C=\chi} H^3(M^3; \mathbb{Z})/(4C \smile H^1(M^3; \mathbb{Z})),$$

where $\mathrm{Imm}[M^3, \mathbb{R}^5]_\chi$ *is the set of regular homotopy classes of immersions with normal Euler class* $\chi \in H^2(M^3;\mathbb{Z})$ *and* $\smile$ *denotes the cup product.*

Saeki, Szűcs and Takase in [**8**] examined the set $\mathrm{Emb}[M^3, \mathbb{R}^5]$ of the regular homotopy classes of immersions which contain embeddings. Since the normal bundle of an embedding of M^3 in $\mathbb{R}^5$ is trivial as is explained in the proof of Theorem 1.3,

$$\mathrm{Emb}[M^3, \mathbb{R}^5] \subset \mathrm{Imm}[M^3, \mathbb{R}^5]_0.$$

Let $\Gamma_2(M^3)$ be the finite set $\{C \in H^2(M^3;\mathbb{Z}) \mid 2C = 0\}$. By Theorem 3.5, the set $\mathrm{Imm}[M^3, \mathbb{R}^5]_0$ can be identified with $\Gamma_2(M^3) \times \mathbb{Z}$.

Definition 3.6 (Wu invariant). The projection $c : \mathrm{Imm}[M^3, \mathbb{R}^5]_0 \to \Gamma_2(M^3)$ is called the Wu invariant of the immersion of the parallelized 3-manifold with trivial normal bundle.

The following explanation due to [**8**] gives a geometrical description of the Wu invariant. A normal trivialization ν of an element $f \in \mathrm{Imm}[M^3, \mathbb{R}^5]_0$ and the trivialization of TM^3 define a map $M^3 \to \mathrm{SO}(5)$ and it induces a homomorphism

$$\pi_1(M^3) \to \pi_1(\mathrm{SO}(5)),$$

namely, an element $\tilde{c}_\nu$ in $H^1(M^3;\mathbb{Z}/2)$. If we change ν by an element

$$z \in [M^3, \mathrm{SO}(2)] = H^1(M^3;\mathbb{Z}),$$

then the class $\tilde{c}_\nu$ changes by $\rho(z)$, where ρ is the mod 2 reduction map in the Bockstein exact sequence:

$$H^1(M^3;\mathbb{Z}) \xrightarrow{\rho} H^1(M^3;\mathbb{Z}/2) \longrightarrow H^2(M^3;\mathbb{Z}) \xrightarrow{\times 2} H^2(M^3;\mathbb{Z}).$$

Hence the coset of $\tilde{c}_\nu$ in

$$H^1(M^3;\mathbb{Z}/2)/\rho(H^1(M^3;\mathbb{Z})) \cong \Gamma_2(M^3) = \ker\{\times 2 : H^2(M^3;\mathbb{Z}) \to H^2(M^3;\mathbb{Z})\}$$

does not depend on ν, and it corresponds to the Wu invariant $c(f) \in \Gamma_2(M^3)$.

Theorem 3.7 (Theorem 4 in [**8**]). *For every element* $C \in \Gamma_2(M^3)$*, there exists an embedding* $f : M^3 \to \mathbb{R}^5$ *with the Wu invariant* $c(f) = C$.

3.3. Proof of Theorem 1.5.

Proof of Theorem 1.5. By Proposition 3.4, it is enough to show the existence of an embedding $F\colon M^3 \times D^2 \to \mathbb{R}^5$ such that the map $g\colon M^3 \to \mathrm{SO}(5)/\mathrm{U}(2)$ induced by F is null-homotopic.

The condition $c_1(\xi) = 0$ is equivalent to that ξ is a trivial plane bundle over M^3. A trivialization τ of ξ and the Reeb vector field R of α give a trivialization of TM^3. Let us fix this trivialization. By Theorem 3.7, there exists an embedding $f : M^3 \to \mathbb{R}^5$ such that $c(f) = 0$, i.e., for a normal trivialization ν of f,

$$\tilde{c}_\nu \in \rho(H^1(M^3;\mathbb{Z})) \subset H^1(M^3;\mathbb{Z}/2).$$

By changing ν by an element in $\rho^{-1}(\tilde{c}_\nu)$, we obtain a normal trivialization ν such that $\tilde{c}_\nu = 0 \in H^1(M^3;\mathbb{Z}/2)$. This means the trivialization ν and the trivialization τ of TM^3 define a map $h\colon M^3 \to \mathrm{SO}(5)$ which induces the trivial map in π_1. Now, let us take a triangulation of M^3 and $M^{(l)}$ be its l dimensional skeleton, i.e.,

$$M^{(0)} \subset M^{(1)} \subset M^{(2)} \subset M^{(3)} = M^3.$$

Then, $h|_{M^{(1)}}$ is null-homotopic. Since $\pi_2(\mathrm{SO}(5)) = 0$, $h|_{M^{(2)}}$ is null-homotopic. Then for the projection $\pi\colon \mathrm{SO}(5) \to \mathrm{SO}(5)/\mathrm{U}(2)$, $\pi \circ h|_{M^{(2)}}$ is null-homotopic. Since $\pi_3(\mathrm{SO}(5)/\mathrm{U}(2)) = 0$ by the diffeomorphism $\mathrm{SO}(5)/\mathrm{U}(2) \cong \mathbb{C}P^3$ (cf. [**2**]), $\pi\circ h$ is null-homotopic. As a tubular neighborhood of $f(M^3)$ in $\mathbb{R}^5$, we can take an embedding $F : M^3 \times D^2 \to \mathbb{R}^5$ satisfying the desired condition. □

Remark 3.8. The normal Euler class for a contact immersion of (M^3, ξ) into $(\mathbb{R}^5, \eta_0)$ is equal to $-c_1(\xi)$. Furthermore, there is a bijection

$$\mathrm{CI}[(M^3,\xi),(\mathbb{R}^5,\eta_0)] \approx H^3(M^3;\mathbb{Z})/(-2c_1(\xi) \smile H^1(M^3;\mathbb{Z})),$$

where $\mathrm{CI}[(M^3,\xi),(\mathbb{R}^5,\eta_0)]$ is the set of regular homotopy classes of immersions which contain contact immersions of (M^3,ξ) into $(\mathbb{R}^5,\eta_0)$. Note that the right-hand side of the bijection is written in the terminology of Theorem 3.5.

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