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Musielak–Orlicz Hardy spaces associated to operators satisfying Davies–Gaffney estimates and bounded holomorphic functional calculus

By Xuan Thinh Duong and Tri Dung Tran

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Abstract. Let X be a metric space with doubling measure and L be an operator which satisfies Davies–Gaffney heat kernel estimates and has a bounded H_∞ functional calculus on $L^2(X)$. In this paper, we develop a theory of Musielak–Orlicz Hardy spaces associated to L, including a molecular decomposition, square function characterization and duality of Musielak–Orlicz Hardy spaces $H_{L,\omega}(X)$. Finally, we show that L has a bounded holomorphic functional calculus on $H_{L,\omega}(X)$ and the Riesz transform is bounded from $H_{L,\omega}(X)$ to $L^1(\omega)$.

1. Introduction.

The introduction and study of classical real-variable Hardy and BMO spaces on the Euclidean space $\mathbb{R}^n$ began in the 1960s with the initial paper of Stein and Weiss [**34**]. This theory was developed further by Fefferman and Stein [**19**] and studied extensively in [**10**], [**33**] as well as many others. Since then these function spaces have played an important role in modern harmonic analysis and partial differential equations, especially in the study of boundedness of singular integrals. It is well known that there are various equivalent characterizations of functions in the classical Hardy space. For instance, the Hardy space $H^1(\mathbb{R}^n)$ can be viewed as the set of functions $f \in L^1(\mathbb{R}^n)$ such that the Riesz transform $\nabla\Delta^{-1/2}f$ belongs to $L^1(\mathbb{R}^n)$. We also have alternative characterizations of $H^1(\mathbb{R}^n)$ via the atomic decomposition or by the square function and the non-tangential maximal function associated to the Poisson semigroup generated by the Laplacian. The standard theory of Hardy spaces is intimately connected with the Laplacian and harmonic functions. However, in the study of boundedness of singular integrals, there are cases in which the classical Hardy spaces are not the most appropriate spaces. For example, one considers a general elliptic operator in divergence form with complex bounded measurable coefficients. Let A be an $n \times n$ matrix with entries

$$a_{jk} : L^\infty(\mathbb{R}^n) \to \mathbb{C}, \quad j = 1, \ldots, n,\ k = 1, \ldots, n,$$

satisfying the elliptic condition

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$$\lambda|\xi|^2 \leq \Re A\xi.\bar{\xi} \text{ and } |A\xi.\zeta| \leq \Lambda|\xi||\zeta|,\ \forall\xi,\ \zeta \in \mathbb{C}^n,$$

for some constants $0 < \lambda \leq \Lambda < \infty$. Then the second order divergence form operator is given by

$$Lf := -\mathrm{div}(A\nabla f), \tag{1.1}$$

interpreted in the weak sense via a sesquilinear form. It is well known that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^2(\mathbb{R}^n)$ but could be unbounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ (see for example [**21**]). The need for investigating new Hardy spaces other than the Hardy space $H^1(\mathbb{R}^n)$ thus naturally arises for the study of singular integrals.

In recent years, function spaces, especially Hardy spaces and BMO spaces, associated with operators have been studied extensively; see, for example, [**4**], [**6**], [**7**], [**15**], [**17**], [**18**], [**20**], [**21**], [**37**] and references therein. Here we shall only recall a number of works in this topic.

Auscher, Duong and McIntosh [**4**] first introduced the Hardy space $H^1_L(\mathbb{R}^n)$ associated with an operator L whose heat kernel satisfies a pointwise Poisson type upper bound by using the area integrals, and established its molecular characterization.

Duong and Yan [**17**], [**18**] introduced its dual space $\mathrm{BMO}_L(\mathbb{R}^n)$ and established the dual relation between $H^1_L(\mathbb{R}^n)$ and $\mathrm{BMO}_L(\mathbb{R}^n)$. Yan [**37**] further generalized these results to the Hardy spaces $H^p_L(\mathbb{R}^n)$ with certain $p \leq 1$ and their dual spaces.

Auscher and Russ [**7**] studied the Hardy space H^1_L on strongly Lipschitz domains associated with a divergence form elliptic operator L with an appropriate heat kernel bound. Recently, Auscher, McIntosh and Russ [**6**] treated the Hardy space H^p with $p \in [1,\infty]$ associated to Hodge Laplacian on a Riemannian manifold with doubling measure.

Hofmann and Mayboroda [**21**] further studied the Hardy space $H^1_L(\mathbb{R}^n)$ and its dual space adapted to a second order divergence form elliptic operator L on $\mathbb{R}^n$ with bounded complex coefficients (such an operator L may not have the pointwise heat kernel bounds). Hofmann et al. [**20**] introduced the new Hardy spaces H^p_L, $1 \leq p < \infty$, on a metric space X associated to a non-negative self-adjoint operator L satisfying Davies–Gaffney estimates.

As a generalization of Hardy spaces $H^p(\mathbb{R}^n)$, the class of Orlicz–Hardy spaces on $\mathbb{R}^n$ and their dual spaces have received considerable attention as well. In particular, Strömberg [**35**] and Janson [**24**] introduced generalized Hardy spaces $H_\omega(\mathbb{R}^n)$, via replacing the norm $\|\cdot\|_{L^p(\mathbb{R}^n)}$ by the Orlicz-norm $\|\cdot\|_{L(\omega)}$ in the definition of $H^p(\mathbb{R}^n)$, where ω is an Orlicz function on $[0,\infty)$ satisfying some control conditions. Viviani [**36**] further characterized these spaces H_ω on spaces of homogeneous type via atoms. The dual spaces of these spaces were also investigated in [**35**], [**24**], [**36**], [**23**]. More recently, Jiang and Yang [**25**], [**26**] introduced the new Orlicz–Hardy spaces associated to divergence form elliptic operators and to non-negative self-adjoint operators satisfying Davies–Gaffney estimates.

In this article, we study a generalized form of Orlicz–Hardy spaces, the so-called Musielak–Orlicz Hardy spaces $H_{L,\omega}(X)$ associated to operators and their dual spaces, under the assumptions that the operators have bounded H_∞ functional calculi and satisfy

Davies–Gaffney estimates. We remark that our assumption that the operator having a bounded H_∞ functional calculus is much weaker than the usual assumption that L being non-negative self adjoint which played an important role in a number of the previous works, see for example [**20**], [**25**], [**26**]. For example, a non-negative self-adjoint operator L with Davies–Gaffney estimates would satisfy the finite speed propagation property for solutions of the corresponding wave equation, see [**32**] (see also [**20**]), and allows us to construct the Hardy space via (ω, M)-atoms, see [**20**], [**25**], [**26**].

There are two key generalizations in this article.

First, we replace the assumption that L is a non-negative self adjoint operator by the weaker assumption that L has a bounded H_∞ functional calculus on $L^2(X)$. This would allow a much larger class of applicable operators L. For example, it is well known that in general the second order divergence form operator L defined by (1.1) is not a self-adjoint operator, but L has a bounded H_∞ functional calculus on $L^2(X)$, see for example [**8**]. For another example, one considers the operator $L = b(x)\Delta$, a special case of a second order elliptic operator in non-divergence form with bounded measurable complex coefficients, where Δ denotes the Laplacian in $\mathbb{R}^n$ and b denotes an ω-accretive function on $\mathbb{R}^n$, $\omega \in [0, \pi/2)$, with bounded reciprocal, meaning that b and $\frac{1}{b}$ belong to $L^\infty(\mathbb{R}^n, \mathbb{C})$ and $|\arg b(x)| \le \omega$ for almost all $x \in \mathbb{R}^n$. The operator $L = b(x)\Delta$ is clearly not self-adjoint in general and it has a bounded H_∞ functional calculus on $L^2(X)$, see [**29**, Proposition 1.1]. Furthermore, if $\Re b(x) \ge \delta > 0$ for almost all $x \in \mathbb{R}^n$, then the semigroup $\{e^{-tL}\}_{t>0}$ satisfies the Davies–Gaffney estimate (2.5), see [**14**].

Second, the Orlicz functions $\varphi(t)$ appearing in many of previous works are replaced by more general functions $\omega(x,t)$, the so-called Musielak–Orlicz functions (cf. [**30**], [**12**]), that may vary in the spatial variables and possess some control conditions. In the particular case when $\omega = t^p, p \in (0,1]$, our results are in line with those in [**13**]. In another special case, if ω is an Orlicz function on $\mathbb{R}_+$ with $p_\omega \in (0,1]$, which is continuous, strictly increasing and concave then by Jensen's inequality it can be verified that Assumption (C) on the function ω holds (see Section 2). In this sense, this paper is an extension to [**1**].

Recently, the authors in [**38**] investigated the Musielak–Orlicz Hardy spaces associated with operators. However, our approach, which is strongly motivated by [**21**] and [**25**], differs from the approach in [**38**]. More precisely, we use the weaker assumption on the operator L that the operator L has a bounded H_∞ functional calculus on $L^2(X)$, instead of L being non-negative self adjoint in [**38**]. In addition, Musielak–Orlicz functions ω considered in [**38**] are assumed to be *growth functions* and satisfy the *uniformly reverse Hölder condition*. We do not make such assumptions but assume different conditions on ω. (See Subsection 2.4).

The paper is organized as follows. In Section 2, we shall give some preliminaries on a metric space X with a doubling measure and give some assumptions on the operator L and the Musielak–Orlicz function ω. In Sections 3, we shall introduce Musielak–Orlicz Hardy spaces $H_{L,\omega}(X)$. We show that each function in $H_{L,\omega}(X)$ can be represented as a decomposition of (ω, ϵ, M)-molecules and more importantly, the space of all finite linear combinations of (ω, ϵ, M)-molecule is dense in $H_{L,\omega}(X)$. Then the dual spaces of $H_{L,\omega}(X)$ are investigated. In the last section, we consider applications of the holomorphic functional calculus of the operator L and certain Riesz transforms associated to L. By using the molecular decomposition associated to the operator L and the Musielak–Orlicz function, we shall show that L has a bounded holomorphic functional calculus on the Musielak–Orlicz Hardy spaces $H_{L,\omega}(X)$ and the Riesz transforms are bounded from $H_{L,\omega}(X)$ to $L(\omega)$.

Throughout the paper, the letters C, c will denote (possibly different) constants that are independent of the essential variables. The symbol $X \lesssim Y$ means that there exists a positive constant C such that $X \le CY$.

2. Preliminaries.

In this section, we first recall some notions and notations on metric spaces and then describe some basic assumptions on the operator L studied in this paper; finally we present some basic properties on Musielak–Orlicz functions.

2.1. Doubling measures on metric spaces.

Let X be a metric space, with a distance d and μ is a nonnegative, Borel, doubling measure on X. Throughout this paper, we assume that $\mu(X) = \infty$.

Denote by $B(x, r)$ the open ball of radius $r > 0$ and center $x \in X$, and by $V(x, r)$ its measure $\mu(B(x, r))$. The doubling property of μ means that there exists a constant $C > 0$ so that

$$V(x, 2r) \le CV(x, r) \tag{2.1}$$

for all $x \in X$ and $r > 0$.

Notice that the doubling property (2.1) implies the following property that

$$V(x, \lambda r) \le C\lambda^n V(x, r), \tag{2.2}$$

for some positive constant n uniformly for all $\lambda \ge 1$, $x \in X$ and $r > 0$. There also exists a constant $0 \le N \le n$ such that

$$V(x, r) \le C\left(1 + \frac{d(x, y)}{r}\right)^N V(y, r), \tag{2.3}$$

uniformly for all $x, y \in X$ and $r > 0$.

To simplify notation, we will often use B for $B(x_B, r_B)$. Also given $\lambda > 0$, we will write λB for the λ-dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. For each ball $B \subset X$ we set

$$S_0(B) := B \quad \text{and} \quad S_j(B) := 2^j B \backslash 2^{j-1} B \quad \text{for} \quad j \in \mathbb{N}.$$

2.2. Holomorphic functional calculus.

We now recall some notions on holomorphic functional calculi as introduced by McIntosh [**28**].

Let $0 \le \theta < \nu < \pi$. We define the closed sector in the complex plane $\mathbb{C}$

$$S_\theta := \{z \in \mathbb{C} : |\arg z| \le \theta\}$$

and denote the interior of S_θ by S_θ^0.

We present the following subspaces of the space $H(S_\nu^0)$ of all holomorphic functions on S_ν^0:

$$H_\infty(S_\nu^0) := \{b \in H(S_\nu^0) : \|b\|_\infty < \infty\},$$

where $\|b\|_\infty := \sup\{|b(z)| : z \in S_\nu^0\}$, and

$$\Psi(S_\nu^0) := \{\psi \in H(S_\nu^0) : \exists s > 0,\ |\psi(z)| \le c|z|^s(1 + |z|^{2s})^{-1}\}.$$

Recall that a closed operator L in $L^2(X)$ is said to be of type θ if $\sigma(L) \subset S_\theta$, and for each $\nu > \theta$ there exists a constant c_ν such that

$$\|(L - \lambda I)^{-1}\| \le c_\nu |\lambda|^{-1}, \quad \lambda \notin S_\nu.$$

If L is of type θ and $\psi \in \Psi(S_\nu^0)$, for $f \in L^2(X)$, we define $\psi(L) \in \mathcal{L}(L^2(X), L^2(X))$ by putting

$$\psi(L)f = \frac{1}{2\pi i}\int_\Gamma (L - \lambda I)^{-1} f\psi(\lambda)d\lambda,$$

where Γ is the contour $\{z = re^{\pm i\xi} : r > 0\}$ parametrized clockwise around S_θ, and $\theta < \xi < \nu$. Since

$$\begin{aligned}\left\|\int_\Gamma (L - \lambda I)^{-1} f\psi(\lambda)d\lambda\right\|_{L^2(X)} &\le \int_0^\infty \|(L - \lambda I)^{-1} f\|_{L^2(X)} |\psi(\lambda)| d|\lambda| \\ &\le \|f\|_{L^2(X)} \int_0^\infty \frac{c_1 c_2 |\lambda|^s}{|\lambda|(1 + |\lambda|^{2s})} d|\lambda| < \infty,\end{aligned}$$

the integral above is absolutely convergent and defines $\psi(L)$ as a bounded operator from $L^2(X)$ into $L^2(X)$. It is straightforward to show, using Cauchy's theorem, that the definition is independent of the choice of $\xi \in (\theta, \nu)$. If, in addition, L is one-one and has dense range and if $b \in H_\infty(S_\nu^0)$, then $b(L)$ can be defined by

$$b(L) = [\psi(L)]^{-1}(b\psi)(L),$$

where $\psi(z) = z(1 + z)^{-2}$. It can be shown that $b(L)$ is a well-defined linear operator in $L^2(X)$, see [**28**]. We say that L has a bounded H_∞ calculus in $L^2(X)$ if there exists $c_{\nu,2} > 0$ such that $b(L) \in \mathcal{L}(L^2(X), L^2(X))$, and for $b \in H_\infty(S_\nu^0)$,

$$\|b(L)\| \le c_{\nu,2}\|b\|_\infty.$$

In [**28**] it was proved that L has a bounded H_∞-calculus in $L^2(X)$ if and only if for any non-zero function $\psi \in \Psi(S_\nu^0)$, L satisfies the square function estimate and its reverse

$$c_1\|g\|_2 \le \left(\int_0^\infty \|\psi_t(L)g\|_2^2 \frac{dt}{t}\right)^{1/2} \le c_2\|g\|_2 \tag{2.4}$$

for some $0 < c_1 \le c_2 < \infty$, where $\psi_t(x) = \psi(tx)$. As noted in [**28**], positive self-adjoint operators satisfy the quadratic estimate (2.4). So do normal operators with spectra in a sector, and maximal accretive operators. We refer the reader to [**39**] for precise definitions of these classes of operators. For detailed study on operators which have holomorphic functional calculi, see the work of [**28**].

2.3. Assumptions on operators L.

Let L be a linear operator of type θ on $L^2(X)$ with $\theta < \pi/2$, hence L generates a holomorphic semigroup e^{zL}, $|\arg(z)| < \pi/2 - \theta$. Throughout the whole paper, we always suppose that the operator L satisfies the following assumptions.

(i) The operator L has a bounded H_∞-calculus on $L^2(X)$.

(ii) The operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ which satisfies the Davies–Gaffney estimates, i.e., there exist positive constants C_2 and C_3 such that for all closed sets E and F in $X, t \in (0,\infty)$ and $f \in L^2(X)$ supported in E,

$$\|e^{-tL}f\|_{L^2(F)} \le C_2 \exp\left\{-\frac{d(E,F)^2}{C_3 t}\right\}\|f\|_{L^2(E)}, \tag{2.5}$$

where $d(E,F)$ is the distance between E and F in X.

REMARK 2.1. We now give a list of examples of differential operators which satisfy assumptions (i) and (ii):

(α) Second order elliptic divergence form operators defined by (1.1) in the introduction, acting on the Euclidean space $\mathbb{R}^n$. Note that these operators in general are neither self adjoint nor having Gaussian heat kernel bounds. See [**8**] and Section 2 of [**5**].

(β) The operators $L = b(x)\Delta$ as described in Section 1, page 3 of this article.

(γ) Schrödinger operators with non-negative potentials and magnetic Schrödinger operators. These operators are self adjoint and possess Gaussian upper bounds on heat kernels. See for example Section 1 and 3 of [**16**].

(δ) Laplace–Beltrami operators on all complete Riemannian manifolds. These operators are self adjoint and satisfy the Davies–Gaffney estimates (but not Gaussian heat kernel bounds) in general setting. See [**3**, Section 3.1].

LEMMA 2.1 ([**22**]). *Assume that the families of operators $\{S_t\}_{t>0}$ and $\{T_t\}_{t>0}$ satisfy Davies–Gaffney estimates* (2.5). *Then there exist two constants $C \ge 0$ and $c > 0$ such that, for every $t > 0$, every closed subsets E and F of X and every function f supported in E, one has*

$$\|S_sT_tf\|_{L^2(F)} \le C\exp\left\{-\frac{d(E,F)^2}{c\max\{s,t\}}\right\}\|f\|_{L^2(E)}.$$

Lemma 2.2. *Let L satisfy assumptions* (i) *and* (ii). *Then for any fixed $k \in \mathbb{N}$, the following family of operators $\{(tL)^k e^{-tL}\}_{t>0}$ satisfies Davies–Gaffney estimates* (2.5).

Proof. The proof of this lemma is similar to one in [**20**] and hence we omit the details here. □

2.4. Musielak–Orlicz type functions.

Let us first present here some notions on Musielak–Orlicz type functions.

A function $\omega : [0,\infty) \to [0,\infty)$ is called *an Orlicz function* if it is nondecreasing, $\omega(0) = 0$, $\omega(t) > 0$ for $t > 0$ and $\lim_{t\to\infty}\omega(t) = \infty$.

A function $\omega : X \times [0,\infty) \to [0,\infty)$ is called a *Musielak–Orlicz function* if the function $\omega(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for each $x \in X$ and the function $\omega(\cdot,t)$ is a measurable function for each $t \in [0,\infty)$.

Let ω be a Musielak–Orlicz function. The function ω is said to be of uniformly upper type p (resp. uniformly lower type p) for certain $p \in [0, \infty)$, if there exists a positive constant C such that for all $x \in X$, $t \ge 1$ (resp. $t \in (0,1]$) and $s \in (0,\infty)$,

$$\omega(x,st) \le Ct^p\omega(x,s). \tag{2.6}$$

If ω is of both uniformly upper type p_1 and lower type p_0, then ω is said to be of type (p_0, p_1). A typical example of such ω is

$$\omega(x,t) := f(x)g(t)$$

for $x \in X$ and $t \in [0,\infty)$, where f is a positive measurable function on X and g is an Orlicz function on $[0,\infty)$ of upper type p_1 and lower type p_0. Another example of Musielak–Orlicz function ω of uniformly upper type $p \in (0,1]$ is, for instance,

$$\omega(x,t) = \frac{t^p}{f(x) + [\log(e+t)]^\alpha},$$

where $\alpha \in [0,1]$ and f is a positive measurable function on X. It is also interesting to observe that if

$$\omega(x,t) = \frac{t^p}{f(x) + g(t)},$$

where f is a positive measurable function on X and g is a decreasing positive function on $[0,\infty)$ then ω is a Musielak–Orlicz function of uniformly lower type p.

Let

$$p_\omega^+ \equiv \inf\{p > 0 :\ \exists C > 0 \text{ such that (2.6) holds for all } x \in X,\ t \in [1,\infty),\ s \in (0,\infty)\},$$

and

$$p_\omega^- \equiv \sup\{p>0:\ \exists C>0 \text{ such that (2.6) holds for all } x\in X,\ t\in(0,1],\ s\in(0,\infty)\}.$$

The function ω is said to be of strictly uniformly lower type p if for all $x \in X$, $t \in (0,1)$ and $s \in (0,\infty)$, $\omega(x,st) \le t^p\omega(x,s)$. One then defines

$$p_\omega \equiv \sup\{p>0: \omega(x,st)\le t^p\omega(x,s) \text{ holds for all } x\in X,\ s\in(0,\infty) \text{ and } t\in(0,1)\}.$$

It is easy to see that $p_\omega \le p_\omega^- \le p_\omega^+$ for all ω. In what follows, p_ω, p_ω^- and p_ω^+ are called the strictly critical lower type index, the critical lower type index and the critical upper type index of ω, respectively.

In the sequel, we assume that ω satisfies the following assumptions.

Assumption (A). Suppose that ω is a Musielak–Orlicz function which is of uniformly upper type 1 and with $p_\omega \in (0,1]$. In addition, for every $x \in X$, $\omega(x,\cdot)$ is continuous, strictly increasing on $\mathbb{R}_+$.

Note that if ω satisfies Assumption (A) then it has the following properties; see [**27**, Lemma 4.1] for its proof.

Lemma 2.3. (i) *ω is uniformly σ-quasi-subadditive on $X\times[0,\infty)$, namely, there exists a positive constant C such that for all $(x,t_j)\in X\times[0,\infty)$ with $j\in\mathbb{Z}_+$, $\omega(x,\sum_{j=1}^\infty t_j)\le C\sum_{j=1}^\infty \omega(x,t_j)$.*

(ii) *Let $\widetilde{\omega}(x,t) := \int_0^t(\omega(x,s)/s)ds$ for all $(x,t)\in X\times[0,\infty)$. Then $\widetilde{\omega}$ is equivalent to ω; moreover, $\widetilde{\omega}$ also satisfies Assumption* (A).

Convention (B). From Assumption (A), it follows that $0 < p_\omega \le p_\omega^- \le p_\omega^+ \le 1$. In what follows, if (2.6) holds for p_ω^+ with $t \in [1,\infty)$, then we choose $\widetilde{p}_\omega \equiv p_\omega^+$; otherwise $p_\omega^+ < 1$ and we choose $\widetilde{p}_\omega \in (p_\omega^+, 1)$ to be close enough to p_ω^+.

Let ω satisfy Assumption (A). A measurable function f on X is said to be in the Lebesgue type space $L(\omega)$ if

$$\int_X \omega(x,|f(x)|)\,d\mu(x) < \infty.$$

Moreover, for any $f \in L(\omega)$, define

$$\|f\|_{L(\omega)} = \inf\left\{\lambda>0: \int_X \omega\left(x,\frac{|f(x)|}{\lambda}\right)d\mu(x)\le 1\right\}.$$

The function ρ defined below plays an important role in this paper.

Definition 2.1. For each $x \in X$, we define the function $\omega^{-1}(x,\cdot)$ and $\rho(x,\cdot)$ on $\mathbb{R}_+$ as follows

$$\omega^{-1}(x,t)\equiv\sup\{s\ge 0:\omega(x,s)\le t\} \tag{2.7}$$

and

$$\rho(x,t) \equiv \frac{t^{-1}}{\omega^{-1}(x,t^{-1})}. \tag{2.8}$$

Then it is easy to see that $\omega^{-1}(x,\cdot)$ is continuous, strictly increasing and for every $x \in X$,

$$\omega^{-1}(x,\omega(x,t)) = t$$

and

$$\omega(x,\omega^{-1}(x,t)) = t.$$

Moreover, the types of ω and ω^{-1} have the following relation.

LEMMA 2.4. *Let $0 < p \le q \le 1$. If ω is of type $(p,\, q)$ then ω^{-1} is of type $(q^{-1},\, p^{-1})$.*

PROOF. By the symmetry, it suffices to show that if ω is of uniformly lower type p then ω^{-1} is of uniformly upper type p^{-1}. Suppose that there exists a constant $C \ge 1$ such that for all $x \in X$, $s \ge 0$, $t \le 1$,

$$\omega(x,st) \le Ct^p\omega(x,s). \tag{2.9}$$

Then for any $x \in X$, $s \ge 0$, $t \ge 1$ and $u \ge 0$ such that $\omega(x,u) \le st$, it follows from (2.9) that

$$\omega\left(x, \frac{u}{t^{1/p}}\right) \le \frac{C\omega(x,u)}{t} \le Cs.$$

This implies that

$$u \le t^{1/p}\omega^{-1}(x,Cs)$$

and hence

$$\omega^{-1}(x,st) \le t^{1/p}\omega^{-1}(x,Cs). \tag{2.10}$$

On the other hand, observe that

$$\omega^{-1}(x,Cs) = \sup\left\{\lambda \ge 0 : \frac{\omega(x,\lambda)}{C} \le s\right\}$$

and, by (2.9), for any $\lambda \ge 0$,

$$\omega\left(x, \frac{\lambda}{C^{2/p}}\right) \le \frac{\omega(x,\lambda)}{C},$$

then we deduce that

$$\omega^{-1}(x, Cs) \leq C^{2/p}\omega^{-1}(x, s), \tag{2.11}$$

which together with (2.10) completes the proof of Lemma 2.4. □

Hereafter, we shall assume the following condition on the function ω.

Assumption (C). Let ω satisfy Assumption (A) and the following conditions:

(i) there exist positive constants C_1, C_2 such that for any $x \in X$, $C_1 \leq \omega(x, 1) \leq C_2$;
(ii) there exists a positive constant C such that for any locally integrable positive function f on X, for any ball B in X,

$$\frac{1}{|B|}\int_B \omega(x, f(x))d\mu(x) \leq C \inf_{x\in X} \omega\bigg(x, \frac{1}{|B|}\int_B f(y)d\mu(y)\bigg).$$

Remark 2.2. A typical example of Musielak–Orlicz function ω that satisfies Assumption (C) is $\omega(x,t) = h(x)\varphi(t)$ for all $x \in X$ and $t \in [0,\infty)$, where h is a measurable function on X so that there exist positive constants C_1, C_2 such that for any $x \in X$, $C_1 \leq h(x) \leq C_2$ and φ is an increasing, continuous and concave Orlicz function on $[0,\infty)$ with $p_\varphi \in (0,1]$.

3. Musielak–Orlicz Hardy spaces associated to operators.

3.1. Tent spaces on spaces of homogeneous type.

Given $x \in X$ and $\alpha > 0$, the cone of aperture α and vertex x is the set

$$\Gamma^\alpha(x) := \{(y,t) \in X \times (0,\infty) : d(y,x) < \alpha t\}.$$

For any closed subset $F \subset X$, define a saw-tooth region $R^\alpha(F) = \bigcup_{x\in F}\Gamma^\alpha(x)$. For simplicity, we will often write $R(F)$ instead of $R^1(F)$. If O is an open subset of X, and we denote by E^c the complement of a set E, then the tent over O, denoted by $\widehat{O}$, is defined as

$$\widehat{O} := [R(O^c)]^c := \{(x,t) \in X \times (0,\infty) : d(x, O^c) \geq t\}.$$

For each measurable function g on $X \times (0,\infty)$ and $x \in X$, define

$$\mathcal{A}(g)(x) := \bigg(\int_{\Gamma(x)} |g(y,t)|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\bigg)^{1/2}.$$

When $X = \mathbb{R}^n$ Coifman, Meyer and Stein [**10**] introduced the tent spaces $T_2^p(\mathbb{R}^{n+1}_+)$ for $p \in (0,\infty)$. The tent spaces $T_2^p(X)$ on spaces of homogeneous type were studied by Russ [**31**]. The function g is said to belong to the space $T_2^p(X)$ with $p \in (0,\infty)$ if $\|g\|_{T_2^p(X)} = \|\mathcal{A}(g)\|_{L^p} < \infty$. Then, Harbourne, Salinas and Viviani [**23**] introduced the tent spaces $T_\omega(\mathbb{R}^{n+1}_+)$ associated to ω. Now let ω satisfy Assumption (A). Then we define $T_\omega(X)$ as the space of all measurable functions g on $X \times (0,\infty)$ such that $\mathcal{A}(g) \in L(\omega)$, and for any $g \in T_\omega(X)$, one defines

$$\|g\|_{T_\omega(X)} = \|\mathcal{A}(g)\|_{L(\omega)}.$$

Definition 3.1. Let ω satisfy Assumption (C) and ρ be the function defined by (2.8) in Definition (2.1). A function a on $X \times (0,\infty)$ is called a $T_\omega(X)$-atom if

(i) there exists a ball $B \subset X$ such that supp $a \subset \widehat{B}$;
(ii) $\|a\|_{T_2^2(X)} \le [V(B)]^{-1/2} \inf\limits_{x\in B}[\rho(x, V(B))]^{-1}$.

Remark 3.1. (i) It is not difficult to verify that for a function ω satisfying Assumption (C), there exist positive constants K_1, K_2 such that for any $x \in X$, $K_1 \le \omega^{-1}(x,1) \le K_2$ and hence $\inf\limits_{x\in B}[\rho(x, V(B))]^{-1}$ is strictly positive.
(ii) In addition, for all $T_\omega(X)$-atoms a, we have $\|a\|_{T_\omega(X)} \lesssim 1$.

For the functions in the space $T_\omega(X)$, we have the following atomic decomposition.

Proposition 3.1. *Let ω satisfy Assumption* (C). *Then for any $f \in T_\omega(X)$, there exist $T_\omega(X)$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that for almost every $(x,t) \in X \times (0,\infty)$*

$$f(x,t) = \sum_{j=1}^\infty \lambda_j a_j(x,t). \tag{3.1}$$

Moreover, there exists a positive constant C such that for all $f \in T_\omega(X)$,

$$\begin{aligned}\Lambda(\{\lambda_j\}) &= \inf\left\{\lambda > 0 : \sum_{j=1}^\infty V(B_j) \inf_{x\in X} \omega\left(x, \frac{|\lambda_j|}{\lambda V(B_j) \sup\limits_{y\in B_j} \rho(y, V(B_j))} \le 1\right)\right\} \\ &\le C\|f\|_{T_\omega(X)},\end{aligned} \tag{3.2}$$

where $\widehat{B_j}$ appears as the support of a_j.

Proof. The proof of Proposition 3.1 is similar to those of [**10**, Theorem 1], [**31**, Theorem 1.1], [**25**, Theorem 3.1] and [**26**, Theorem 3.1] with minor modifications, thus we omit the details. ☐

The following proposition on the convergence of (3.1) plays a significant role in the remaining part of this paper. The proof of it is analogous to that of [**25**, Proposition 3.1] and we omit the details.

Proposition 3.2. *Let ω satisfy Assumption* (C). *If $f \in T_\omega(X) \cap T_2^2(X)$, then the decomposition* (3.1) *holds in both $T_\omega(X)$ and $T_2^2(X)$.*

3.2. Musielak–Orlicz Hardy spaces associated to L.

For all functions $f \in L^2(X)$, the Lusin-area function $S_L(f)$ is defined by setting,

$$S_L f(x) := \left(\int \int_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}, \quad x \in X.$$

The Musielak–Orlicz Hardy space $H_{L,\omega}(X)$ is defined as the completion of

$$\{f \in L^2(X) : \|S_L f\|_{L(\omega)} < \infty\}$$

with the norm

$$\|f\|_{H_{L,\omega}(X)} = \|S_L f\|_{L(\omega)}.$$

Noting that if $\omega(t) = t, t \in (0,\infty)$ then the space $H_{L,\omega}(X)$ turns to be the space $H^1_L(X)$ in [**20**]. Furthermore, if $\omega(t) = t^p$, $t \in (0,\infty)$ and $p \in (0,1]$, the space $H_{L,\omega}(X)$ is just the space $H^p_L(X)$ considered in [**13**]. We now introduce the notions of (ω, M, ϵ)-molecule as follows.

Let us denote by $D(T)$ the domain of an unbounded operator T and by $T^k = T \cdots T$ the k-fold composition of T with itself.

Definition 3.2. A function $m \in L^2(X)$ is called an (ω, M, ϵ)-molecule associated to the operator L if there exist a function $b \in D(L^M)$ and a ball B such that

(i) $m = L^M b$;
(ii) for every $k = 0, 1, \dots, M$, and $j \in \mathbb{Z}_+$

$$\|(r_B^2 L)^k b\|_{L^2(S_j(B))} \lesssim r_B^{2M} 2^{-j\epsilon} [V(2^j B)]^{-1/2} \inf_{x \in B} [\rho(x, V(2^j B))]^{-1}.$$

Theorem 3.1. *Let L satisfy assumptions* (i) *and* (ii), *ω satisfy Assumption* (C), *$M > (n/2)(1/p_\omega - 1/2)$ and $0 < \epsilon < 2M - n(1/p_\omega - 1/2)$. Then for all $f \in H_{L,\omega}(X) \cap L^2(X)$, there exist (ω, ϵ, M)-molecules $\{\alpha_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that*

$$f = \sum_{j=1}^\infty \lambda_j \alpha_j$$

in both $H_{L,\omega}(X)$ and $L^2(X)$. Moreover, there exists a positive constant C such that for all $f \in H_{L,\omega}(X) \cap L^2(X)$,

$$\begin{aligned}\Lambda(\{\lambda_j a_j\}_j) &= \inf \left\{ \lambda > 0 : \sum_{j=1}^\infty V(B_j) \inf_{x \in X} \omega\left(x, \frac{|\lambda_j|}{\lambda V(B_j) \sup\limits_{y \in B_j} \rho(y, V(B_j))} \le 1 \right) \right\} \\ &\le C \|f\|_{H_{L,\omega}(X)},\end{aligned}$$

where B_j is the ball associated with (ω, ϵ, M)-molecule α_j.

Before giving a proof of Theorem 3.1, we consider the following operator

$$\pi_{L,M}(F)(x) = \int_0^\infty (t^2 L e^{-t^2 L})^M (F(\cdot, t))(x) \frac{dt}{t},$$

for all $F \in L^2(X \times (0, \infty))$ with bounded support. The bound

$$\|\pi_{L,M}(F)\|_{L^2(X)} \le C \|F\|_{T_2^2(X)}, \quad \forall M \ge 1 \tag{3.3}$$

follows readily by duality and the L^2 quadratic estimate (2.4). Moreover, we have the following proposition.

PROPOSITION 3.3. *Let a be a $T_\omega(X)$-atom associated to a ball $B \subset X$ and $M > (n/2)(1/p_\omega - 1/2)$. Then, $\pi_{L,M} a$ is an (ω, ϵ, M)-molecule (up to a harmless constant); moreover, $\pi_{L,M} a \in H_{L,\omega}(X)$.*

PROOF. Setting

$$b = \int_0^\infty t^{2M} (e^{-t^2 L})^M a(\cdot, t) \frac{dt}{t}.$$

Since a is a $T_\omega(X)$-atom associated to a ball $B \subset X$, $\operatorname{supp} a \subset \widehat{B} = \{(x, t) \in X \times (0, \infty) : d(x, B^c) \ge t\} \subset B \times [0, r_B]$. Thus, the integral $b = \int_0^\infty t^{2M} (e^{-t^2 L})^M a(\cdot, t)(dt/t)$ is well defined and $\pi_{L,M} a = L^M b$.

For any $h \in L^2(S_j(B))$ with norm 1 and $k \in \{0, 1, \ldots, M\}$, one has

$$\begin{aligned}
&\left| \int_X (r_B^2 L)^k b(x) \overline{h(x)} d\mu(x) \right| \\
&\quad = \left| \int_X \int_0^\infty t^{2M} (r_B^2 L)^k (e^{-t^2 L})^M a(x, t) \overline{h(x)} \frac{dt}{t} d\mu(x) \right| \\
&\quad \le \left| \int \int_{\widehat{B}} t^{2(M-k)} (r_B^2)^k a(x, t) (e^{-t^2 L^*})^{M-k} \overline{(t^2 L^* e^{-t^2 L^*})^k h(x)} \frac{dt}{t} d\mu(x) \right| \\
&\quad \le r_B^{2M} \|a\|_{T_2^2(X)} \left(\int \int_{\widehat{B}} \left| (e^{-t^2 L^*})^{M-k} \overline{(t^2 L^* e^{-t^2 L^*})^k h(x)} \right|^2 \frac{dt}{t} d\mu(x) \right)^{1/2} \\
&\quad \le r_B^{2M} [V(B)]^{-1/2} \inf_{x \in B} [\rho(x, V(B))]^{-1} \\
&\qquad \times \left(\int \int_{\widehat{B}} \left| (e^{-t^2 L^*})^{M-k} \overline{(t^2 L^* e^{-t^2 L^*})^k h(x)} \right|^2 \frac{dt}{t} d\mu(x) \right)^{1/2}.
\end{aligned}$$

If $j \ge 3$ then we have

$$\Big(\int\int_{\widehat{B}} \big|(e^{-t^2L^*})^{M-k}\overline{(t^2L^*e^{-t^2L^*})^k h(x)}\big|^2\frac{dt}{t}d\mu(x)\Big)^{1/2}$$
$$\leq C\Big(\int_0^{r_B} e^{-d(B,S_j(B))^2/ct^2}\frac{dt}{t}\Big)^{1/2}$$
$$\leq C\Big(\int_0^{r_B}\Big(\frac{t}{2^jr_B}\Big)^{4M}\frac{dt}{t}\Big)^{1/2}$$
$$\leq C2^{-2Mj}.$$

If $j=0,1,2$, it is simple to see that

$$\Big(\int\int_{\widehat{B}} \big|(e^{-t^2L^*})^{M-k}\overline{(t^2L^*e^{-t^2L^*})^k h(x)}\big|^2\frac{dt}{t}d\mu(x)\Big)^{1/2}\leq C2^{-2Mj}.$$

All in all, one has

$$\Big|\int_X (r_B^2L)^k b(x)\overline{h(x)}d\mu(x)\Big|\leq Cr_B^{2M}2^{-2Mj}[V(B)]^{-1/2}\inf_{x\in B}[\rho(x,V(B))]^{-1}$$

which implies

$$\|(r_B^2L)^k b\|_{L^2(S_j(B))}\leq Cr_B^{2M}2^{-2Mj}[V(B)]^{-1/2}\inf_{x\in B}[\rho(x,V(B))]^{-1}.$$

Since ω is of uniformly lower type p_ω, ρ is of uniformly upper type $1/p_\omega-1$ by Lemma 2.4. Then we have

$$\|(r_B^2L)^k b\|_{L^2(S_j(B))}$$
$$\leq Cr_B^{2M}2^{-2Mj}[V(B)]^{-1/2}\inf_{x\in B}[\rho(x,V(B))]^{-1}$$
$$\leq Cr_B^{2M}2^{-2Mj}2^{(n/2)j}\Big(\frac{V(2^jB)}{V(B)}\Big)^{1/p_\omega-1}[V(2^jB)]^{-1/2}\inf_{x\in B}[\rho(x,V(2^jB))]^{-1}$$
$$\leq Cr_B^{2M}2^{(-2M+n/2+n/p_\omega-n+\epsilon)j}2^{-j\epsilon}[V(2^jB)]^{-1/2}\inf_{x\in B}[\rho(x,V(2^jB))]^{-1}$$
$$\leq Cr_B^{2M}2^{(-2M-n/2+n/p_\omega+\epsilon)j}2^{-j\epsilon}[V(2^jB)]^{-1/2}\inf_{x\in B}[\rho(x,V(2^jB))]^{-1}.$$

Due to $(-2M-n/2+n/p_\omega+\epsilon)<0$, we obtain that

$$\|(r_B^2L)^k b\|_{L^2(S_j(B))}\leq Cr_B^{2M}2^{-j\epsilon}[V(2^jB)]^{-1/2}\inf_{x\in B}[\rho(x,V(2^jB))]^{-1}.$$

Therefore, $\pi_{L,M}a$ is an (ω,ϵ,M)-molecule.

It remains to show that $\alpha=\pi_{L,M}a\in H_{L,\omega}(X)$. Write

$$\int_X \omega(x, S_L(\lambda\alpha)(x))d\mu(x) = \sum_{j=0}^{\infty}\int_X \omega(x, S_L(\lambda\alpha\chi_{S_j(B)})(x))d\mu(x) = \sum_{j=0}^{\infty} A_j$$

for all $j \in \mathbb{N}$.

By Assumption (C) and the Hölder inequality, for each $j \in \mathbb{N}$, one obtains

$$\begin{aligned}
A_j &\le \sum_{k=0}^{\infty}\int_{S_k(2^jB)} \omega(x, S_L(\lambda\alpha\chi_{S_j(B)})(x))d\mu(x)\\
&\le \sum_{k=0}^{\infty} V(2^{k+j}B)\inf_{x\in X}\omega\left(x, \frac{|\lambda|\int_{S_k(2^jB)}|S_L(\alpha\chi_{S_j(B)})(y)|d\mu(y)}{V(2^{k+j}B)}\right)\\
&\le \sum_{k=0}^{\infty} V(2^{k+j}B)\inf_{x\in X}\omega\left(x, \frac{|\lambda|\|S_L(\alpha\chi_{S_j(B)})\|_{L^2(S_k(2^jB))}}{V(2^{k+j}B)^{1/2}}\right).
\end{aligned}$$

For $k = 0, 1, 2$, one has

$$\begin{aligned}
\|S_L(\alpha\chi_{S_j(B)})\|_{L^2(S_k(2^jB))} &\le C\|\alpha\|_{L^2(S_j(B))}\\
&\le C2^{-j\epsilon}[V(2^jB)]^{-1/2}\inf_{x\in B}[\rho(x, V(2^jB))]^{-1}.
\end{aligned}$$

For $k \ge 3$, write

$$\begin{aligned}
&\|S_L(\alpha\chi_{S_j(B)})\|^2_{L^2(S_k(2^jB))}\\
&\quad = \int_{S_k(2^jB)}\left(\int_0^{d(x,x_B)/4} + \int_{d(x,x_B)/4}^{\infty}\right)\int_{d(x,y)<t}|t^2Le^{-t^2L}\alpha|^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}d\mu(x)\\
&\quad = I_j + II_j.
\end{aligned}$$

To estimate I_j, we set $U_{kj}(B) := \{y \in X : d(x,y) \le d(x,x_B)/4 \text{ for certain } x \in S_k(2^jB)\}$. Then, for each $z \in S_j(B)$ and $y \in U_{kj}(B)$, we have $d(y,z) \ge 2^{k+j-2}r_B$. It follows from the fact that

$$\int_{d(x,y)<t} V(x,t)^{-1}d\mu(x) \le C$$

and $\alpha = L^M b$ that

$$\begin{aligned}
I_j &\le C\int_0^{2^{k+j+1}r_B}\int_{S_j(B)}|(t^2L)^{M+1}e^{-t^2L}b\chi_{S_j(B)}(y)|^2d\mu(y)\frac{dt}{t^{4M+1}}\\
&\le C\|b\|^2_{L^2(S_j(B))}\int_0^{2^{k+j+1}r_B}\exp\left(-\frac{d(U_{kj}(B), S_j(B))^2}{ct^2}\right)\frac{dt}{t^{4M+1}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|b\|^2_{L^2(S_j(B))}\int_0^{2^{k+j+1}r_B}\exp\Big(-\frac{(2^{k+j-2}r_B)^2}{ct^2}\Big)\frac{dt}{t^{4M+1}}\\
&\le C\|b\|^2_{L^2(S_j(B))}\int_0^{2^{k+j+1}r_B}\Big(\frac{t}{2^{k+j}r_B}\Big)^{4M+4}\frac{dt}{t^{4M+1}}\\
&\le C\|b\|^2_{L^2(S_j(B))}\frac{2^{-4(k+j)M}}{(r_B)^{4M}}\le C2^{-4(k+j)M}2^{-2\epsilon j}[V(2^jB)]^{-1}\inf_{x\in B}[\rho(x,V(2^jB))]^{-2}.
\end{aligned}
$$

Finally, for the term II_j we obtain

$$
\begin{aligned}
II_j &\le C\int_{2^{k+j-1}r_B}^{\infty}\int_{S_j(B)}|(t^2L)^{M+1}e^{-t^2L}b\chi_{S_j(B)}(y)|^2d\mu(y)\frac{dt}{t^{4M+1}}\\
&\le C\|b\|^2_{L^2(S_j(B))}\int_{2^{k+j-1}r_B}^{\infty}\frac{dt}{t^{4M+1}}\\
&\le C2^{-4(k+j)M}2^{-2\epsilon j}[V(2^jB)]^{-1}\inf_{x\in B}[\rho(x,V(2^jB))]^{-2}.
\end{aligned}
$$

It therefore, from the estimates for I_j, II_j above, the uniformly lower type p_ω of ω together with the fact that $-2Mp_\omega+n(1-p_\omega/2)<0$, implies that

$$
\begin{aligned}
&\int_{S_j(B)}\omega(x,S_L(\alpha)(x))d\mu(x)\\
&\quad\le C\sum_{k=0}^{\infty}2^{(-2(k+j)M-j\epsilon)p_\omega}V(2^{k+j}B)\\
&\qquad\times\inf_{x\in X}\omega\Big(x,\frac{|\lambda|}{[V(2^{k+j}B)]^{1/2}[V(2^jB)]^{1/2}\sup\limits_{y\in B}\rho(y,V(2^jB))}\Big)\\
&\quad\le C\sum_{k=0}^{\infty}2^{(-2(k+j)M-j\epsilon)}2^{kn(1-p_\omega/2)p_\omega}V(2^jB)\inf_{x\in X}\omega\Big(x,\frac{|\lambda|}{[V(2^jB)]\sup\limits_{y\in B}\rho(y,V(2^jB))}\Big).
\end{aligned}
$$

Note that we can choose $\widetilde{p}_\omega$ as in Convention (B) such that $n(1/p_\omega-1/\widetilde{p}_\omega)<\epsilon$. It therefore, together with the uniformly lower type $1/\widetilde{p}_\omega-1$ of ρ by Lemma 2.4, yields

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\int_{S_j(B)}\omega(x,S_L(\lambda\alpha)(x))d\mu(x)\\
&\quad\le C\sum_{j=0}^{\infty}2^{-\epsilon p_\omega j}V(2^jB)\Big(\frac{V(B)}{V(2^jB)}\Big)^{p_\omega/\widetilde{p}_\omega}\inf_{x\in X}\omega\Big(x,\frac{|\lambda|}{V(B)\sup\limits_{y\in B}\rho(y,V(B))}\Big)\\
&\quad\le C\sum_{j=0}^{\infty}2^{-\epsilon p_\omega j}2^{(1-p_\omega/\widetilde{p}_\omega)nj}V(B)\inf_{x\in X}\omega\Big(x,\frac{|\lambda|}{V(B)\sup\limits_{y\in B}\rho(y,V(B))}\Big)
\end{aligned}
$$

$$\leq CV(B)\inf_{x\in X}\omega\bigg(x,\frac{|\lambda|}{V(B)\sup\limits_{y\in B}\rho(y,V(B))}\bigg),$$

which completes the proof of Proposition 3.3. □

Proof of Theorem 3.1. Since $f\in L^2(X)$ and T has a bounded holomorphic functional calculus on $L^2(X)$, there exists a constant $C_{M,L}$ such that

$$f=C_{M,L}\int_0^\infty (t^2Le^{-t^2L})^{M+1}f\frac{dt}{t}.$$

By definition of $H_{L,\omega}(X)$ and the quadratic estimate (2.4), $t^2Le^{-t^2L}f\in T_\omega(X)\cap T_2^2(X)$. Thanks to Proposition 3.2 and Proposition 3.3, we easily deduce

$$f=C_{M,L}\pi_{L,M}(t^2Le^{-t^2L}f)=C_{M,L}\sum_{j=0}^\infty\lambda_j\pi_{L,M}(a_j)$$

in both $L^2(X)$ and $H_{L,\omega}(X)$ and $\Lambda(\{\lambda_ja_j\}_j)\leq C\|f\|_{H_{L,\omega}(X)}$, which completes the proof of Theorem 3.1. □

By density of $H_{L,\omega}(X)\cap L^2(X)$ in $H_{L,\omega}(X)$, we conclude the following corollaries.

Corollary 3.1. *Let the operator L satisfy Assumptions* (i) *and* (ii), *ω satisfy Assumption* (C) *and $M>(n/2)(1/p_\omega-1/2)$. Then for all $f\in H_{L,\omega}(X)$ there exist a sequence of (ω,ϵ,M)-molecules $\{\alpha_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty\in\mathbb{C}$ such that $f=\sum_{j=1}^\infty\lambda_j\alpha_j$ in $H_{L,\omega}(X)$. Moreover, there exists a positive constant C independent of f such that*

$$\Lambda(\{\lambda_ja_j\}_j)\leq C\|f\|_{H_{L,\omega}(X)}.$$

Corollary 3.2. *Let the operator L satisfy Assumptions* (i) *and* (ii), *ω satisfy Assumption* (C) *and $M>(n/2)(1/p_\omega-1/2)$. Then for all $0<\epsilon<2M-n(1/p_\omega-1/2)$, the spaces $H^{mol,\epsilon,M}_{\omega,fin}$ are dense in $H_{L,\omega}(X)$ where $H^{mol,\epsilon,M}_{\omega,fin}$ denote the spaces of finite linear combinations of (ω,ϵ,M).*

3.3. Dual spaces of Musielak–Orlicz Hardy spaces.

In this subsection, we study the dual space of the Musielak–Orlicz Hardy spaces $H_{L,\omega}(X)$. Let $\phi=L^Mv$ be a function in $L^2(X)$, where $v\in D(L^M)$. Following [**21**], [**20**] for $\epsilon>0$, $M\in\mathbb{N}$ and fixed $x_0\in X$ we introduce the norm

$$\|\phi\|_{\mathcal{M}^{M,\epsilon}_\omega(L)}=\sup_{j\in\mathbb{Z}_+}\bigg\{2^{j\epsilon}[V(x_0,2^j)]^{1/2}\sup_{x\in B(x_0,2^j)}\rho(x,V(x_0,2^j))\sum_{k=0}^M\|L^kv\|_{L^2(S_j(B(x_0,1)))}\bigg\}$$

where

$$\mathcal{M}^{M,\epsilon}_\omega(L):=\big\{\phi=L^Mv\in L^2(X):\|\phi\|_{\mathcal{M}^{M,\epsilon}_\omega(L)}<\infty\big\}.$$

Let $(\mathcal{M}_\omega^{M,\epsilon}(L))^*$ be the dual of $\mathcal{M}_\omega^{M,\epsilon}(L)$ and denote either $(I+t^2L)^{-1}$ or e^{-t^2L} by A_t. Then for any $f \in (\mathcal{M}_\omega^{M,\epsilon}(L))^*$, $(I-A_t^*)^M f$ belongs to $L^2_{loc}(X)$ in the sense of distributions, see [**21**], [**20**].

For any $M \in \mathbb{N}$, one defines

$$\mathcal{M}^M_{\omega,L^*}(X) := \bigcap_{\epsilon > n(1/p_\omega - 1/p_\omega^+)} (\mathcal{M}_\omega^{M,\epsilon}(L))^*.$$

Definition 3.3. Let the operator L satisfy Assumptions (i) and (ii), ω satisfy Assumption (C) and $M > (n/2)(1/p_\omega - 1/2)$. A functional $f \in \mathcal{M}^M_{\omega,L^*}(X)$ is said to be in $BMO^M_{\rho,L}(X)$ if

$$\|f\|_{BMO^M_{\rho,L}(X)} = \sup_{B\subset X} \frac{1}{\sup\limits_{x\in B}\rho(x,V(B))}\left[\frac{1}{V(B)}\int_B |(I-e^{-r_B^2L})^M f(x)|^2 d\mu(x)\right]^{1/2} < \infty,$$

where the supremum is taken over all balls B in X.

We have the following characterizations of the spaces $BMO^M_{\rho,L}(X)$.

Proposition 3.4. *Let the operator L satisfy Assumptions* (i) *and* (ii), *ω satisfy Assumption* (C) *and $M > (n/2)(1/p_\omega - 1/2)$. Then $f \in BMO^M_{\rho,L}(X)$ if and only if $f \in \mathcal{M}^M_{\omega,L^*}(X)$ and*

$$\sup_{B\subset X} \frac{1}{\sup\limits_{x\in B}\rho(x,V(B))}\left[\frac{1}{V(B)}\int_B |(I-(I+r_B^2L)^{-1})^M f(x)|^2 d\mu(x)\right]^{1/2} < \infty.$$

Moreover,

$$\|f\|_{BMO^M_{\rho,L}(X)} \approx \sup_{B\subset X} \frac{1}{\sup\limits_{x\in B}\rho(x,V(B))}\left[\frac{1}{V(B)}\int_B |(I-(I+r_B^2L)^{-1})^M f(x)|^2 d\mu(x)\right]^{1/2}.$$

Proposition 3.5. *Let the operator L satisfy Assumptions* (i) *and* (ii), *ω satisfy Assumption* (C) *and $\epsilon > n(1/p_\omega - 1/p_\omega^+)$. Then there exists a positive constant C such that for all $f \in BMO^M_{\rho,L}(X)$,*

$$\sup_{B\subset X} \frac{1}{\sup\limits_{x\in B}\rho(x,V(B))}\left[\frac{1}{V(B)}\int_{\widehat{B}} |(t^2L)^M e^{-t^2L} f(x)|^2 \frac{d\mu(x)dt}{t}\right]^{1/2} \le C\|f\|_{BMO^M_{\rho,L}(X)}.$$

The proofs of two above propositions are similar to Lemmas 8.1 and 8.3 in [**21**] and hence we omit the details.

We are now in position to obtain the main result in this subsection.

THEOREM 3.2. *Let the operator L satisfy Assumptions* (i) *and* (ii), *ω satisfy Assumption* (C). *Then $(H_{L,\omega}(X))^*$, the dual space of $H_{L,\omega}(X)$, coincides with* $\mathrm{BMO}^M_{\rho,L^*}(X)$ *in the following sense*:

(i) *For any functional $f \in BMO^M_{\rho,L^*}(X)$ and $M > \max\{(n/2)(1/p_\omega - 1) + 1, n/4\}$, the linear functional given by*

$$\ell(g) := \langle f, g\rangle,$$

which is initially defined on $H^{mol,\epsilon,2\widetilde{M}}_{\omega,fin}$ with $\widetilde{M} > M + (N/2)(1/p_\omega - 1/2)$ and $\widetilde{M} - (n/2)(1/p_\omega - 1) > \epsilon > (N/2)(1/p_\omega - 1/2)$ (N is a constant appearing in (2.3)), *has a unique extension to $H_{L,\omega}(X)$ with*

$$\|\ell\|_{(H_{L,\omega}(X))^*} \le C\|f\|_{BMO^M_{\rho,L^*}(X)},$$

where C is a positive constant independent of f.

(ii) *Conversely, for any $\ell \in (H_{L,\omega}(X))^*$ and $M > (n/2)(1/p_\omega - 1/2)$ there exists a function $f \in BMO^M_{\rho,L^*}(X)$ such that*

$$\ell(g) = \langle f, g\rangle$$

for all $g \in H^{mol,M,\epsilon}_{\omega,fin}$ and $\|f\|_{BMO^M_{\rho,L^*(X)}} \le C\|\ell\|_{(H_{L,\omega}(X))^*}$, where C is a positive constant independent of ℓ.

Before coming to the proof of Theorem 3.2, we need the following results.

LEMMA 3.1. *There exists a collection of open sets $\{Q^k_\alpha \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$, where I_k denotes certain (possibly finite) index set depending on k, and constants $\delta \in (0,1)$, $a_0 \in (0,1)$ and $C_1 \in (0,\infty)$ such that*

(i) *$\mu(X \setminus \cup_\alpha Q^k_\alpha) = 0$ for all $k \in \mathbb{Z}$;*
(ii) *if $i \ge k$, then either $Q^i_\alpha \subset Q^k_\beta$ or $Q^i_\alpha \cap Q^k_\beta = \emptyset$;*
(iii) *for (k,α) and each $i < k$, there exists a unique β such $Q^k_\alpha \subset Q^i_\beta$;*
(iv) *the diameter of $Q^k_\alpha \le C_1\delta^k$;*
(v) *each Q^k_α contains certain ball $B(z^k_\alpha, a_0\delta^k)$.*

PROOF. The proof of this lemma can be found in [**9**]. □

THEOREM 3.3. *Let $M > \max\{(n/2)(1/p_\omega - 1/2) + 1, n/4\}$ and $0 < \epsilon < 2M - n(1/p_\omega - 1/2)$. Suppose that $f = \sum_{i=1}^l \lambda_i a_i$ where $\{a_i\}_{i=1}^l$ is a family of $(\omega,\epsilon,2M)$-molecules and $\sum_{i=1}^l |\lambda_i| < \infty$. Then there exists a representation $f = \sum_{i=1}^K \mu_i m_i$, where the $m_i's$ are (ω,ϵ,M)-molecules and*

$$\sum_{i=1}^K |\mu_i| \le C\|f\|_{H_{L,\omega}(X)},$$

with $C = C(\epsilon, M)$.

PROOF. Indeed, we can adapt the ideas in the proof of Theorem 5.3 in [**20**] with minor modifications to obtain the proof of Theorem 3.3. Instead of dealing with atoms as in [**20**], we work on molecules by decomposing the underline space X into annuli according to the balls associated with the molecules. We therefore omit the details. □

PROOF OF THEOREM 3.2. Let m be an $(\omega, \epsilon, \widetilde{M})$-molecule associated with a ball $B \subset X$. Then there exists a function b such that the conditions (i) and (ii) in Definition 3.2 hold. Then we have,

$$\begin{aligned} r_B^{2\widetilde{M}} m &= (r_B^2 L)^{\widetilde{M}} b = (I - (I + r_B^2 L)^{-1})^M (I + r_B^2 L)^M (r_B^2 L)^{\widetilde{M}-M} b \\ &= \sum_{k=0}^{M} C_M^k (I - (I + r_B^2 L)^{-1})^M (r_B^2 L)^{\widetilde{M}-k} b. \end{aligned}$$

Therefore,

$$\begin{aligned} |\langle f, m\rangle| &= r_B^{-2\widetilde{M}} |\langle f, (r_B^2 L)^{\widetilde{M}} b\rangle| \\ &\le C r_B^{-2\widetilde{M}} \sum_{k=0}^{M} \left| \int_X (I - (I + r_B^2 L^*)^{-1})^M f(x) \overline{(r_B^2 L)^{\widetilde{M}-k} b(x)} d\mu(x) \right| \\ &\le C r_B^{-2\widetilde{M}} \sum_{k=0}^{M} \sum_{j=0}^{\infty} \left(\int_{S_j(B)} \left|(I - (I + r_B^2 L^*)^{-1})^M f(x)\right|^2 d\mu(x) \right)^{1/2} \\ &\quad \times \|(r_B^2 L)^{\widetilde{M}-k} b\|_{L^2(S_j(B))} \\ &\le C \sum_{j=0}^{\infty} 2^{-\epsilon j} [V(2^j B)]^{-1/2} \sup_{x\in B} [\rho(x, V(2^j B))]^{-1} \\ &\quad \times \left(\int_{S_j(B)} \left|(I - (I + r_B^2 L^*)^{-1})^M f(x)\right|^2 d\mu(x) \right)^{1/2}. \end{aligned} \tag{3.4}$$

With notations as in Lemma 3.1 we choose an integer k_j, for each $j \in \mathbb{Z}$, such that $C_1 \delta^{k_j} \le 2^j r_B < C_1 \delta^{k_j - 1}$. Set

$$M_j = \{\beta \in I_{k_0} : Q_\beta^{k_0} \cap B(x_B, C_1 \delta^{k_j - 1}) \neq \emptyset\}.$$

Then, for each $j \in \mathbb{Z}$,

$$S_j(B) \subset B(x_B, C_1 \delta^{k_j - 1}) \subset \bigcup_{\beta \in M_j} Q_\beta^{k_0} \subset B(x_B, 2C_1 \delta^{k_j - 1}).$$

From (ii) in Lemma 3.1 we can assume that the sets $Q_\beta^{k_0}$ for all $\beta \in M_j$ are pairwise disjoint. Furthermore, it follows from (iv) and (v) that there exists $z_\beta^{k_0} \in Q_\beta^{k_0}$ such that

$$B(z_\beta^{k_0}, a_0\delta^{k_0}) \subset Q_\beta^{k_0} \subset B(z_\beta^{k_0}, C_1\delta^{k_0}) \subset B(z_\beta^{k_0}, r_B) \subset B(z_\beta^{k_0}, C_1\delta^{k_0-1}). \tag{3.5}$$

Therefore, from (2.3), Proposition 3.4 together with the fact that ρ is of type $(1/p_\omega^+ - 1, 1/p_\omega^- - 1)$ and $p_\omega \le p_\omega^- \le p_\omega^+$, we have

$$\begin{aligned}
&\Big(\int_{S_j(B)} \big|(I-(I+r_B^2L^*)^{-1})^M f(x)\big|^2 d\mu(x)\Big)^{1/2} \\
&\le \Big(\sum_{\beta\in M_j} \int_{B(z_\beta^{k_0}, r_B)} \big|(I-(I+r_B^2L^*)^{-1})^M f(x)\big|^2 d\mu(x)\Big)^{1/2} \\
&\le C\|f\|_{BMO^M_{\rho,L^*}(X)} \Big(\sum_{\beta\in M_j} V(B(z_\beta^{k_0}, r_B)) \sup_{x\in B(z_\beta^{k_0}, r_B)} \rho(x, V(B(z_\beta^{k_0}, r_B)))^2\Big)^{1/2} \\
&\le C2^{jN(1/p_\omega-1)}\|f\|_{BMO^M_{\rho,L^*}(X)} V(B(x_B, 2C_1\delta^{k_j-1}))^{1/2} \sup_{x\in B}\rho(x, V(B(x_B, r_B))) \\
&\le C2^{jN(1/p_\omega-1)}\|f\|_{BMO^M_{\rho,L^*}(X)} V(2^jB)^{1/2} \sup_{x\in B}\rho(x, V(B)).
\end{aligned} \tag{3.6}$$

Combination of two estimates (3.4) and (3.6) gives

$$\begin{aligned}
|\langle f, m\rangle| &\le C\sum_{j=0}^{\infty} 2^{-\epsilon j} 2^{jN(1/p_\omega-1)} \Big(\frac{V(B)}{V(2^jB)}\Big)^{1/p_\omega-1} \|f\|_{BMO^M_{\rho,L^*}(X)} \\
&\le C\sum_{j=0}^{\infty} 2^{j(-\epsilon+N(1/p_\omega-1))}\|f\|_{BMO^M_{\rho,L^*}(X)} \\
&\le C\|f\|_{BMO^M_{\rho,L^*}(X)}.
\end{aligned} \tag{3.7}$$

Now for any $g \in H^{mol,\epsilon,2\widetilde{M}}_{\omega,fin}$ then by Theorem 3.3, there exists a representation $g = \sum_{i=1}^K \mu_i m_i$, where the $m_i's$ are $(\omega, \epsilon, \widetilde{M})$-molecules and

$$\sum_{i=1}^K |\mu_i| \le C\|g\|_{H_{L,\omega}(X)}. \tag{3.8}$$

As a result, it follows from (3.7) and (3.8) that

$$\begin{aligned}
\langle f, g\rangle &= \sum_{i=1}^K |\mu_i|\langle f, m_i\rangle \\
&\le C\sum_{i=1}^K |\mu_i| \|f\|_{BMO^M_{\rho,L^*}(X)} \\
&\le C\|g\|_{H_{L,\omega}(X)}\|f\|_{BMO^M_{\rho,L^*}(X)}.
\end{aligned}$$

The proof of part (i) of Theorem 3.2 is complete.

Conversely, we will adapt the ideas in [**21**] to give the proof of part (ii) of Theorem 3.2. Observe that for each (ω, ϵ, M)-molecule m,

$$|\ell(m)| \leq C\|\ell\|_{(H_{L,\omega}(X))^*}.$$

Since each element in $\mathcal{M}_\omega^{M,\epsilon}(L)$ is also an (ω, ϵ, M)-molecule associated to the ball $B(x_0, 1)$ which implies that ℓ defines a linear function on $\mathcal{M}_\omega^{M,\epsilon}(L)$ for every $\epsilon > 0$, $M > (n/2)(1/p_\omega - 1/2)$. Therefore, $(I - (I + t^2L^*))^M\ell$ is well defined and belongs to L^2_{loc} for all $t > 0$. Fix a ball B and let $\phi \in L^2(B)$ such that $\|\phi\|_{L^2(B)} \leq 1$. Then one can check that

$$\widetilde{m} = (I - (I + r_B^2L)^{-1})^M\phi$$

is an (ω, ϵ, M)-molecule for every $\epsilon > 0$ and hence $\|\widetilde{m}\|_{H_{L,\omega}(X)} \leq C$. Consequently, we have

$$\begin{aligned}|\langle(I - (I + r_B^2L^*)^{-1})^M\ell, \phi\rangle| &= |\langle\ell, (I - (I + r_B^2L)^{-1})^M\phi\rangle| \\ &= |\langle\ell, \widetilde{m}\rangle| \leq C\|\ell\|_{(H_{L,\omega}(X))^*},\end{aligned}$$

which further implies that

$$\frac{1}{\sup\limits_{x\in B}\rho(x, V(B))}\left(\frac{1}{V(B)}\int_B |(I - (I + r_B^2L^*)^{-1})^M\ell(x)|^2d\mu(x)\right)^{1/2} \leq C\|\ell\|_{(H_{L,\omega}(X))^*},$$

for all balls B. Thus, $\ell \in BMO^M_{\rho,L^*}(X)$ and $\|\ell\|_{BMO^M_{\rho,L^*}(X)} \leq C\|\ell\|_{(H_{L,\omega}(X))^*}$, which completes the proof of part (ii). $\square$

4. Riesz transform and holomorphic functional calculus.

4.1. Holomorphic functional calculus.

LEMMA 4.1. *Let the operator L satisfy Assumptions* (i) *and* (ii)*, ω satisfy Assumption* (C) *and $M > (n/2)(1/p_\omega - 1/2)$. Suppose that T is linear* (*resp. nonnegative sublinear*) *operator which maps $L^2(X)$ continuous into weak-$L^2(X)$. If there exists a positive constant C such that for any (ω, ϵ, M)-molecule α*

$$\int_X \omega(x, T(\lambda\alpha)(x))d\mu(x) \leq CV(B)\inf_{x\in X}\omega\left(x, \frac{|\lambda|}{V(B)\sup\limits_{y\in B}\rho(y, V(B))}\right), \tag{4.1}$$

then T extends to a bounded linear (*resp. sublinear*) *operator from $H_{L,\omega(X)}$ to $L(\omega)$; moreover, there exists a positive constant C' such that*

$$\|Tf\|_{L(\omega)} \leq C'\|f\|_{H_{L,\omega}(X)},$$

for all $f \in H_{L,\omega}(X)$.

The proof is similar to one of Lemma 5.2 in [**26**] with minor modifications, thus we omit it here.

THEOREM 4.1. *Let L be of type θ on $L^2(X)$ with $0 \le \theta < \pi/2$ and satisfy* (i) *and* (ii)*, ω satisfy* (C) *and $\theta < \nu < \pi$. Then, for any $f \in H_\infty(S^0_\nu)$, $f(L)$ is bounded on $H_{L,\omega}(X)$, that is, for any $g \in H_{L,\omega}(X)$*

$$\|f(L)g\|_{H_{L,\omega}(X)} \le C\|f\|_\infty \|g\|_{H_{L,\omega}(X)}. \tag{4.2}$$

PROOF. Choose $M > (n/2)(1/p_\omega - 1/2)$ and $\widetilde{p}_\omega > p_\omega$ close enough to p_ω (as in Convention (B)) so that there exists ϵ satisfying

$$n\left(\frac{1}{p_\omega} - \frac{1}{\widetilde{p}_\omega}\right) < \epsilon < 2M + \frac{n}{2} - \frac{n}{p_\omega}.$$

With any (ω, ϵ, M)-molecule m associated to a ball $B \subset X$, we will claim that

$$\int_X \omega(x, S_L(\lambda f(L)m)(x))d\mu(x) \le C\|f\|_\infty V(B) \inf_{x\in X} \omega\left(x, \frac{|\lambda|}{V(B)\sup\limits_{y\in B}\rho(y, V(B))}\right). \tag{4.3}$$

Once (4.3) is proved, (4.2) follows by Lemma 4.1.

Let us prove (4.3). Write

$$\int_X \omega(x, S_L(\lambda f(L)m)(x))d\mu(x) \le \sum_{j=0}^\infty \int_X \omega(x, S_L(\lambda f(L)m \cdot \chi_{S_j(B)})(x))d\mu(x) = \sum_{j=0}^\infty A_j$$

for all $j \in \mathbb{N}$.

Since ω satisfies Assumption (C), by the Hölder inequality, for each $j \in \mathbb{N}$, one obtains

$$\begin{aligned}
A_j &\le \sum_{k=0}^\infty \int_{S_k(2^jB)} \omega(x, S_L(\lambda f(L)m \cdot \chi_{S_j(B)})(x))d\mu(x) \\
&\le \sum_{k=0}^\infty V(2^{k+j}B) \inf_{x\in X} \omega\left(x, \frac{|\lambda| \int_{S_k(2^jB)} |S_L(f(L)m \cdot \chi_{S_j(B)})(y)|d\mu(y)}{V(2^{k+j}B)}\right) \\
&\le \sum_{k=0}^\infty V(2^{k+j}B) \inf_{x\in X} \omega\left(x, \frac{|\lambda| \|S_L(f(L)m \cdot \chi_{S_j(B)})\|_{L^2(S_k(2^jB))}}{V(2^{k+j}B)^{1/2}}\right).
\end{aligned}$$

For $k = 0, 1, 2,$

$$\begin{aligned}&\|S_L(f(L)m\cdot\chi_{S_j(B)})\|_{L^2(S_k(2^jB))}\\&\quad\le C\|f(L)m\cdot\chi_{S_j(B)}\|_{L^2(X)}\\&\quad\le C\|f\|_\infty\|m\|_{S_j(B)}2^{-j\epsilon}[V(2^jB)]^{-1/2}\inf_{x\in B}[\rho(x,V(2^jB))]^{-1}.\end{aligned}$$

For $k\ge 3$, write

$$\begin{aligned}&\|S_L(f(L)m\cdot\chi_{S_j(B)})\|^2_{L^2(S_k(2^jB))}\\&\quad=\int_{S_k(2^jB)}\left(\int_0^{d(x,x_B)/4}+\int_{d(x,x_B)/4}^\infty\right)\\&\qquad\times\int_{d(x,y)<t}|t^2Le^{-t^2L}f(L)m\cdot\chi_{S_j(B)}|^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}d\mu(x)\\&\quad=I_j+II_j.\end{aligned}$$

Let us estimate I_j. It can be verified that there exists a positive constant C such that for all closed sets E and F in $X, t\in(0,\infty)$ and $g\in L^2(X)$ supported in E,

$$\|(tL)^{M+1}e^{-tL}f(L)g\|_{L^2(F)}\le C\left(\frac{t}{d(E,F)^2}\right)^{M+1}\|g\|_{L^2(E)}. \tag{4.4}$$

Setting $U_{kj}(B):=\{y\in X: d(x,y)\le d(x,x_B)/4$ for certain $x\in S_k(2^jB)\}$, then for each $z\in S_j(B)$ and $y\in U_{kj}(B)$, we have $d(y,z)\ge 2^{k+j-2}r_B$. Combining $\int_{d(x,y)<t}V(x,t)^{-1}d\mu(x)<c$, $m=L^Mb$ and (4.4), one gets

$$\begin{aligned}I_j&\le C\int_0^{2^{k+j+1}r_B}\int_{S_j(B)}|(t^2L)^{M+1}e^{-t^2L}f(L)b\cdot\chi_{S_j(B)}(y)|^2d\mu(y)\frac{dt}{t^{4M+1}}\\&\le C\|f\|^2_\infty\|b\|^2_{L^2(S_j(B))}\int_0^{2^{k+j+1}r_B}\left(\frac{ct^2}{d(U_{kj}(B),S_j(B))^2}\right)^{2M+2}\frac{dt}{t^{4M+1}}\\&\le C\|f\|^2_\infty 2^{-4(j+k)M}2^{-2\epsilon j}[V(2^jB)]^{-1}\inf_{x\in B}[\rho(x,V(2^jB))]^{-2}.\end{aligned}$$

For the term II_j, we have

$$\begin{aligned}II_j&\le C\int_{2^{k+j-1}r_B}^\infty\int_{S_j(B)}|(t^2L)^{M+1}e^{-t^2L}f(L)b\cdot\chi_{S_j(B)}(y)|^2d\mu(y)\frac{dt}{t^{4M+1}}\\&\le C\|f\|^2_\infty\|b\|^2_{L^2(S_j(B))}\int_{2^{k+j-1}r_B}^\infty\frac{dt}{t^{4M+1}}\\&\le C\|f\|^2_\infty 2^{-4(k+j)M}2^{-2\epsilon j}[V(2^jB)]^{-1}\inf_{x\in B}[\rho(x,V(2^jB))]^{-2}.\end{aligned}$$

Further going, from the estimates for I_j, II_j, the strictly lower type p_ω of ω together with the fact that $-2Mp_\omega + n(1-p_\omega/2) < 0$, we obtain

$$\begin{aligned}
&\int_{S_j(B)} \omega(x, S_L(f(L)m)(x))d\mu(x) \\
&\leq C\|f\|_\infty \sum_{k=0}^{\infty} 2^{(-2(k+j)M-j\epsilon)p_\omega} V(2^{k+j}B) \\
&\quad \times \inf_{x\in X} \omega\left(x, \frac{|\lambda|}{[V(2^{k+j}B)]^{1/2}[V(2^jB)]^{1/2}\sup\limits_{y\in B}\rho(y, V(2^jB))}\right) \\
&\leq C\|f\|_\infty \sum_{k=0}^{\infty} 2^{(-2(k+j)M-j\epsilon)} 2^{kn(1-p_\omega/2)p_\omega} V(2^jB) \inf_{x\in X} \omega\left(x, \frac{|\lambda|}{[V(2^jB)]\sup\limits_{y\in B}\rho(y, V(2^jB))}\right).
\end{aligned}$$

Since ρ is of uniformly lower type $1/\widetilde{p}_\omega - 1$, we further have

$$\begin{aligned}
&\sum_{j=0}^{\infty} \int_{S_j(B)} \omega(x, S_L(\lambda f(L)m)(x))d\mu(x) \\
&\quad\leq C\|f\|_\infty \sum_{j=0}^{\infty} 2^{-\epsilon p_\omega j} V(2^jB) \left(\frac{V(B)}{V(2^jB)}\right)^{p_\omega/\widetilde{p}_\omega} \inf_{x\in X} \omega\left(x, \frac{|\lambda|}{V(B)\sup\limits_{y\in B}\rho(y, V(B))}\right) \\
&\quad\leq C\|f\|_\infty \sum_{j=0}^{\infty} 2^{-\epsilon p_\omega j} 2^{1-(p_\omega/\widetilde{p}_\omega)nj} V(B) \inf_{x\in X} \omega\left(x, \frac{|\lambda|}{V(B)\sup\limits_{y\in B}\rho(y, V(B))}\right).
\end{aligned}$$

Noting that since $n(1/p_\omega - 1/\widetilde{p}_\omega) < \epsilon$ and $M > (n/2)(1/p_\omega - 1/2)$, we learn that

$$\sum_{j=0}^{\infty} \int_{S_j(B)} \omega(x, S_L(\lambda f(L)m)(x))d\mu(x) \leq C\|f\|_\infty V(B) \inf_{x\in X} \omega\left(x, \frac{|\lambda|}{V(B)\sup\limits_{y\in B}\rho(y, V(B))}\right).$$

□

4.2. Riesz transforms.

Assume that D is a densely defined linear operator on $L^2(X)$ which possesses the following properties:

(i) $DL^{-1/2}$ is bounded on $L^2(X)$.
(ii) The family of operators $\{\sqrt{t}De^{-tL}\}_{t>0}$ satisfies the Davies–Gaffney estimate (2.5).

Noting that operators D satisfying assumptions (i) and (ii) above include gradient operators in divergence form and Riemannian gradients on all complete Riemannian manifolds, see for example [**2**], [**3**], [**11**].

Theorem 4.2. *For any $f \in H_{L,\omega}(X)$,*

$$\|DL^{-1/2}(f)\|_{L(\omega)} \leq C\|f\|_{H_{L,\omega}(X)}.$$

Before giving the proof of Theorem 4.2, we state the following lemma.

LEMMA 4.2. *For every $M \in \mathbb{N}$, all closed sets E, F in X with $d(E,F) > 0$ and every $f \in L^2(X)$ supported in E, one has*

$$\|DL^{-1/2}(I - e^{-tL})^M f\|_{L^2(F)} \leq C\Big(\frac{t}{d(E,F)^2}\Big)^M \|f\|_{L^2(E)}, \ \forall t > 0, \tag{4.5}$$

and

$$\|DL^{-1/2}(tLe^{-tL})^M f\|_{L^2(F)} \leq C\Big(\frac{t}{d(E,F)^2}\Big)^M \|f\|_{L^2(E)}, \ \forall t > 0. \tag{4.6}$$

PROOF. The proof of Lemma 4.2 is completely analogous to one of Lemma 2.2 in [**22**] and we omit it here. □

PROOF OF THEOREM 4.2. Choose $M > (n/2)(1/p_\omega - 1/2)$. Let m is an (ω, ϵ, M)-molecule associated to a ball B, $\epsilon < 2M + n/2 - n/p_\omega$. Then there exists a function b such that $m = L^M b$. Setting $T = DL^{-1/2}$ and write

$$\begin{aligned}
\int_X \omega(x, T(\lambda m)(x))d\mu(x) &\leq \int_X \omega\big(x, |\lambda| T((I - e^{r_B^2 L})^M m(x))\big)d\mu(x)\\
&\quad + \int_X \omega\big(x, |\lambda| T([I - (I - e^{r_B^2 L})^M] m(x))\big)d\mu(x)\\
&\leq \sum_{j=0}^{\infty} \int_X \omega\big(x, |\lambda| T((I - e^{r_B^2 L})^M (m \cdot \chi_{S_j(B)})(x))\big)d\mu(x)\\
&\quad + \sum_{j=0}^{\infty} \int_X \omega\big(x, |\lambda| T([I - (I - e^{r_B^2 L})^M] (m \cdot \chi_{S_j(B)})(x))\big)d\mu(x)\\
&\leq \sum_{j=0}^{\infty} I_j + \sum_{j=0}^{\infty} II_j.
\end{aligned}$$

We estimate the term I_j first. By the Hölder inequality, we obtain

$$\begin{aligned}
I_j &\leq \sum_{k=0}^{\infty} \int_{S_k(2^j B)} \omega(x, |\lambda| T((I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})(x))d\mu(x)\\
&\leq \sum_{k=0}^{\infty} V(2^{k+j}B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{V(2^{k+j}B)} \int_{S_k(2^j B)} T((I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})(y)d\mu(y)\Big)\\
&\leq \sum_{k=0}^{\infty} V(2^{k+j}B) \inf_{x\in X} \omega\Big(x, \frac{|\lambda|}{[V(2^{k+j}B)]^{1/2}} \|T((I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})\|_{L^2(S_k(2^j B))}\Big).
\end{aligned}$$

For $k = 0, 1, 2$, it follows from Lemma 4.2 that

$$\begin{aligned}&\|T((I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})\|_{L^2(S_k(2^j B))}\\&\quad \leq C\|m\|_{L^2(S_j(B))} \leq C2^{-\epsilon j}[V(2^j B)]^{-1/2} \inf_{x\in B}[\rho(x, V(2^j B))]^{-1},\end{aligned}$$

and for $k \geq 3$ that

$$\begin{aligned}&\|T((I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})\|_{L^2(S_k(2^j B))}\\&\quad \leq C2^{-2M(k+j)}\|m\|_{L^2(S_j(B))}\\&\quad \leq C2^{-2M(k+j)}2^{-\epsilon j}[V(2^j B)]^{-1/2} \inf_{x\in B}[\rho(x, V(2^j B))]^{-1}.\end{aligned}$$

At this stage, by the same argument used in the proof of Theorem 4.1, we obtain

$$\sum_{j=0}^{\infty} I_j \leq CV(B) \inf_{x\in X} \omega\left(x, \frac{|\lambda|}{V(B)\sup_{y\in B}\rho(y, V(B))}\right).$$

We now proceed with terms $II_j, j = 0, 1, \ldots$ Also, by the Hölder inequality, we obtain

$$\begin{aligned}II_j &\leq \sum_{k=0}^{\infty} \int_{S_k(2^j B)} \omega(x, |\lambda| T(I - (I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})(x)) d\mu(x)\\&\leq \sum_{k=0}^{\infty} V(2^{k+j} B)\\&\quad \times \inf_{x\in X} \omega\left(x, \frac{|\lambda|}{V(2^{k+j} B)} \int_{S_k(2^j B)} T(I - (I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})(y) d\mu(y)\right)\\&\leq \sum_{k=0}^{\infty} V(2^{k+j} B)\\&\quad \times \inf_{x\in X} \omega\left(x, \frac{|\lambda|}{[V(2^{k+j} B)]^{1/2}} \|T(I - (I - e^{r_B^2 L})^M)(m \cdot \chi_{S_j(B)})\|_{L^2(S_k(2^j B))}\right)\\&\leq \sum_{k=0}^{\infty} II_j^k.\end{aligned}$$

Next we have

$$I - (I - e^{r_B^2 L})^M = \sum_{k=1}^{M} c_k e^{-kr_B^2 L},$$

where $c_k := (-1)^{k+1}(M!/(M - k)!k!)$. Therefore,

$$
\begin{aligned}
II_j^k &\le C \sup_{1\le k\le M} \|Te^{-kr_B^2 L} m \cdot \chi_{S_j(B)}\|_{L^2(S_k(2^jB))} \\
&\le C \sup_{1\le k\le M} \left\| T\left(\frac{k}{M} r_B^2 L e^{-(k/M)r_B^2 L}\right)^M (r_B^{-2}L^{-1})^M m\cdot \chi_{S_j(B)}\right\|_{L^2(S_k(2^jB))}.
\end{aligned}
$$

At this point, repeating the argument used to estimate I_j, we also obtain that

$$
\sum_{j=0}^{\infty} II_j \le CV(B) \inf_{x\in X} \omega\left(x, \frac{|\lambda|}{V(B)\sup\limits_{y\in B}\rho(y,V(B))}\right).
$$

Combining obtained estimates gives

$$
\int_X \omega(x, T(\lambda m(x)))d\mu(x) \le CV(B) \inf_{x\in X} \omega\left(x, \frac{|\lambda|}{V(B)\sup\limits_{y\in B}\rho(y,V(B))}\right).
$$

This, together with Lemma 4.1, therefore completes our proof. □

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Xuan Thinh DUONG
Department of Mathematics
Macquarie University
NSW, 2109, Australia
E-mail: xuan.duong@mq.edu.au

Tri Dung TRAN
Department of Mathematics
Macquarie University
NSW, 2109, Australia
Department of Mathematics
University of Pedagogy
Ho Chi Minh City, Vietnam
E-mail: dungtt@math.hcmup.edu.vn