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Harmonic functions on asymptotic cones with Euclidean volume growth

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Abstract. We study harmonic functions with polynomial growth on asymptotic cones of a nonnegatively Ricci curved manifold with Euclidean volume growth. Especially, we will give the classification of such harmonic functions.

1. Introduction.

Let M be a complete n-dimensional nonnegatively Ricci curved manifold and $V_M := \lim_{R\to\infty} \mathrm{vol}\, B_R(m)/R^n$, where $m \in M$. Note that by Bishop-Gromov volume comparison theorem, the limit exists and does not depend on the choice of m. Assume that the following *Euclidean volume growth condition* holds:

$$V_M > 0.$$

Let (M_∞, m_∞) be an asymptotic cone of M, to which the rescaled Riemannian manifolds $(M, R_i^{-1} d_M, m)$ for a divergent sequence of positive numbers R_i, converges to (M_∞, m_∞) with respect to the Gromov-Hausdorff topology, where d_M is the distance function of M. In this paper, we will study harmonic functions with polynomial growth on M_∞. See Section 2 for the definition of harmonic functions on M_∞. For $d \geq 0$, let $H^d(M_\infty)$ be the space of harmonic functions f on M_∞ satisfying that there exists $C > 1$ such that $|f(x)| \leq C(1 + \overline{m_\infty, x}^d)$ for every $x \in M_\infty$, where $\overline{m_\infty, x} = d_{M_\infty}(m_\infty, x)$. By Cheeger-Colding's cerebrated work [**5**], we see that there exists a compact geodesic space X with $\mathrm{diam}\, X \leq \pi$ such that (M_∞, m_∞) is isometric to the metric cone $(C(X), p)$ of X, where $C(X) := \boldsymbol{R}_{\geq 0} \times X/(\{0\} \times X)$, the distance is defined by $\overline{(t_1, x_1), (t_2, x_2)} := \sqrt{t_1^2 + t_2^2 - 2t_1t_2 \cos \overline{x_1, x_2}}$, and $p = [\{0\} \times X]$. Note that X is H^{n-1}-rectifiable and that (X, H^{n-1}) satisfies a weak Poincaré inequality of type $(1, 2)$, where H^{n-1} is the $(n-1)$-dimensional Hausdorff measure. See [**20**, Lemma 4.3] and [**38**, Corollary 3.2]. Thus, by [**8**, Theorem 6.25], we see that there exists the canonical self-adjoint operator (called *Laplacian*) Δ_X on $L^2(X)$. Let $E_\lambda(X)$ be the space of functions on X spanned by eigenfunctions of Δ_X on X associated with the eigenvalues $\leq \lambda$.

A main result in this paper is the following:

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Theorem 1.1 (Harmonic functions with polynomial growth on asymptotic cones). *Let* $d \geq 0$. *Then we have*

$$\dim H^d(C(X)) = \dim E_{d(d+n-2)}(X).$$

Especially, we have $\dim H^d(C(X)) < \infty$.

Note that we can regard the last statement $\dim H^d(C(X)) < \infty$ as a solution of *an asymptotic cone's version of Yau's conjecture* (see [**14**, Conjecture 0.1], [**74**] and [**75**]).

We will also show the following asymptotic estimates. This is *an asymptotic cone's version of Weyl type asymptotic bounds on manifolds* given by Colding-Minicozzi in [**18**]:

Theorem 1.2 (Weyl type asymptotic bounds). *For every* $V > 0$, *there exists* $d(n,V) \geq 1$ *such that*

$$C(n)^{-1} V_M d^{n-1} \leq \dim H^d(M_\infty) \leq C(n) V_M d^{n-1}$$

holds for every n*-dimensional complete nonnegatively Ricci curved manifold* M *with* $V_M \geq V$, *every* $d \geq d(n,V)$ *and every asymptotic cone* (M_∞, m_∞) *of* M, *where* $C(n)$ *is a positive constant depending only on* n.

It is important that we can get *two sided bounds* on asymptotic cones as above. Compare with [**18**, Theorem 0.26] and [**18**, Proposition 6.1].

We will also give a relationship between harmonic functions with polynomial growth on M and that of asymptotic cones:

Theorem 1.3 (Liouville type theorem). *There exists a unique* $d_1 \geq 1$ *such that the following hold*:

1. $H^d(M) = \{$*constant functions*$\}$ *for every* $0 < d < d_1$.
2. $H^d(M_\infty) = \{$*constant functions*$\}$ *for every* $0 < d < d_1$ *and every asymptotic cone* (M_∞, m_∞) *of* M.
3. $H^{d_1}(\hat{M}_\infty) \neq \{$*constant functions*$\}$ *for some asymptotic cone* $(\hat{M}_\infty, \hat{m}_\infty)$ *of* M.

As a corollary of Theorem 1.3, we have the following: Assume that there exists $0 < r \leq 1$ such that every asymptotic cone of M is isometric to the cone $C(\boldsymbol{S}^{n-1}(r))$ of $\boldsymbol{S}^{n-1}(r) = \{x \in \boldsymbol{R}^n; |x| = r\}$ (note that in general, the asymptotic cones of M are not unique, however if M has nonnegative sectional curvature, then the asymptotic cone of M is unique. See for instance [**6**], [**59**]. Moreover, recently Colding-Minicozzi showed that if $\mathrm{Ric}_M \equiv 0$, $V_M > 0$ and an asymptotic cone has a smooth cross section, then the asymptotic cone is unique. See [**19**].) Let

$$d_1 := \frac{-(n-1) + \sqrt{(n-2)^2 + 4n/r^2}}{2}.$$

Then we have $H^d(M) = \{\text{constant functions}\}$ for every $d < d_1$. Note that $d_1 \to \infty$ as $r \to 0$.

Essential tools to show Theorems 1.1, 1.2 and 1.3 are important results about asymptotic behavior of harmonic functions on manifolds given by Colding-Minicozzi in [**15**], [**18**] and a new notion about a convergence of Lipschitz functions with respect to the Gromov-Hausdorff topology given by the author in [**38**].

Organization of this paper is as follows:

In Section 2, we will introduce several fundamental notions for metric measure spaces, the structure theory of limit spaces of Riemannian manifolds developed by Cheeger-Colding, several results given in [**38**] which will be used in this paper and two important (gradient) estimates by Cheng-Yau and Li-Schoen.

In Section 3, we will discuss about *frequency functions* for harmonic functions introduced by Colding-Minicozzi in [**15**]. Roughly speaking, we will show that C^0-convergence of harmonic functions with respect to the Gromov-Hausdorff topology yields convergence of frequency functions of them. See Proposition 3.4 for the precise statement. By using Proposition 3.4, several important results about asymptotic behavior of harmonic functions given in [**15**], and several properties for convergence of harmonic functions with respect to the Gromov-Hausdorff topology given in [**38**], we will give a proof of Theorem 1.1.

In Section 4, we will give a proof of Theorem 1.2 by using results given in Section 3 and [**18**].

In Section 5, we will study the topology of the moduli space of asymptotic cones of M. Roughly speaking, we will show that the Gromov-Hausdorff topology on the moduli space and the spectral topology given by Kasue-Kumura in [**42**], [**43**] coincide. This is a solution of *an asymptotic cone's version of Fukaya's conjecture* [**26**, (0.5) Conjecture]. The main result in Section 5 is Theorem 5.4.

In Section 6, we will show *a comparison theorem* between $H^d(M)$ and $H^{\hat{d}}(M_\infty)$. See Theorem 6.1. As corollaries, we will give an alternative proof of Weyl type asymptotic bounds for harmonic functions on manifolds by Colding-Minicozzi, and a proof of Theorem 1.3 via Theorem 5.4.

Section 7 is an appendix. We will show a co-area formula on a non-collapsing metric cone. This performs a crucial role in this paper.

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2. Preliminaries.

For a positive number $\epsilon > 0$ and real numbers a, b, we use the following notation:

$$a = b \pm \epsilon \Longleftrightarrow |a - b| < \epsilon.$$

We denote by $\Psi(\epsilon_1, \epsilon_2, \ldots, \epsilon_k; c_1, c_2, \ldots, c_l)$ (or by Ψ for short) some positive function on $\boldsymbol{R}^k_{>0} \times \boldsymbol{R}^l$ satisfying

$$\lim_{\epsilon_1, \epsilon_2, \ldots, \epsilon_k \to 0} \Psi(\epsilon_1, \epsilon_2, \ldots, \epsilon_k; c_1, c_2, \ldots, c_l) = 0$$

for fixed real numbers $c_1, c_2, \ldots, c_l$. We often denote by $C(c_1, c_2, \ldots, c_l)$ some positive constant depending only on fixed real numbers $c_1, c_2, \ldots, c_l$.

2.1. Metric measure spaces.

For a metric space Z, a point $z \in Z$ and positive numbers r, R with $r < R$, let $B_r(z) := \{x \in Z; \overline{z, x} < r\}$, $\overline{B}_r(z) := \{x \in Z; \overline{z, x} \le r\}$, $\partial B_r(z) := \{x \in Z; \overline{z, x} = r\}$ and $A_z(r, R) := B_R(z) \setminus \overline{B}_r(z)$, where $\overline{x, y}$ is the distance between x and y, we often denote the distance by $d_Z(y, x)$. For $z \in Z$, we denote the distance function from z by r_z, i.e., $r_z(w) := \overline{z, w}$. For a Lipschitz function f on Z and a point $z \in Z$ which is not isolated in Z, we put

$$\mathrm{Lip}\, f(z) := \limsup_{r \to 0} \left(\sup_{x \in B_r(z) \setminus \{z\}} \frac{|f(x) - f(z)|}{\overline{x, z}} \right).$$

If z is isolated in Z, then we put $\mathrm{Lip}\, f(z) := 0$. We also denote the Lipschitz constant of f by $\mathbf{Lip}\ f := \sup_{x \neq y}(|f(x) - f(y)|/\overline{x, y})$. For an open subset $U \subset Z$, we denote the set of Lipschtiz functions on U with compact support by $\mathcal{K}(U)$. We say that Z is *proper* if every bounded subset of Z is relatively compact. We also say that Z is a *geodesic space* if for every $x_1, x_2 \in Z$, there exists an isometric embedding γ from $[0, \overline{x_1, x_2}]$ to Z such that $\gamma(0) = x_1$, $\gamma(\overline{x_1, x_2}) = x_2$ (γ is called a *minimal geodesic from* x_1 *to* x_2). In this paper, for a proper metric space Z and a Radon measure υ of Z, we say that a pair (Z, υ) is *a metric measure space* if the following hold:

1. (Positivity). $\upsilon(B_1(z)) > 0$ for every $z \in Z$.
2. (Doubling condition). For every $R > 0$, there exists $\kappa = \kappa(R) \ge 0$ such that $\upsilon(B_{2r}(z)) \le 2^{\kappa} \upsilon(B_r(z))$ for every $0 < r < R$.

We now recall the notion of *rectifiability* for metric measure spaces given by Cheeger-Colding in [**8**]:

Definition 2.1. Let (Z, υ) be a metric measure space. We say that Z *is* υ-*rectifiable* if there exist a positive integer m, a collection of Borel subsets $\{C_{k,i}\}_{1 \le k \le m, i \in \boldsymbol{N}}$ of Z, and a collection of bi-Lipschitz embedding maps $\{\phi_{k,i} : C_{k,i} \to \boldsymbol{R}^k\}_{k,i}$ such that the following hold:

1. $\upsilon(Z \setminus \bigcup_{k,i} C_{k,i}) = 0$.

2. υ is Ahlfors k-regular at each $x \in C_{k,i}$, i.e., there exists $A_{k,i} > 1$ such that $(A_{k,i})^{-1} r^k \leq \upsilon(B_r(x)) \leq A_{k,i} r^k$ for every $0 < r < (A_{k,i})^{-1}$.
3. For every k, every $x \in \bigcup_{i \in \boldsymbol{N}} C_{k,i}$ and every $0 < \delta < 1$, there exists i such that $x \in C_{k,i}$ and that the map $\phi_{k,i}$ is $(1 \pm \delta)$-bi-Lipschitz to the image $\phi_{k,i}(C_{k,i})$.

It is important that the cotangent bundle on a rectifiable metric measure space exists in some sense. We do not explain the construction, however we now give several fundamental properties of the cotangent bundle only:

THEOREM 2.2 (Cheeger, Cheeger-Colding, [**3**], [**8**]). *Let (Z, υ) be a rectifiable metric measure space. Then, there exist a topological space T^*Z and a Borel map $\pi : T^*Z \to Z$ such that the following hold*:

1. $\upsilon(Z \setminus \pi(T^*Z)) = 0$.
2. $\pi^{-1}(w)$ *is a finite dimensional real Hilbert space with the inner product $\langle \cdot, \cdot \rangle(w)$ for every $w \in \pi(T^*Z)$.*
3. *For every open subset U of Z and every Lipschitz function f on U, there exist a Borel subset V of U, and a Borel map df (called the differential section of f or the differential of f) from V to T^*Z such that $\upsilon(U \setminus V) = 0$ and that $\pi \circ df(w) = w$, $|df|(w) = \mathrm{Lip}\, f(w) = lipf(w)$ for every $w \in V$, where $|v|(w) := \sqrt{\langle v, v\rangle(w)}$.*
4. *Assume that (Z, υ) satisfies a weak Poincaré inequality of type $(1, 2)$ and that Z is compact. Then a bilinear form*

$$\int_Z \langle df_1, df_2 \rangle d\upsilon$$

is closable. Especially, the canonical self-adjoint operotor Δ_Z (called the Laplacian on Z) on $L^2(Z)$ is well-defined. Moreover, $(1 + \Delta_Z)^{-1}$ is a compact operator.

See Section 4 in [**3**] and Section 7 in [**8**] for the definition of a weak Poincaré inequality on metric measure spaces and the details of Theorem 2.2.

Let $1 < p < \infty$ and let (Z, υ) be a metric measure space satisfying a weak Poincaré inequality of type $(1, p)$. Then for every open subset U of Z, it is known that the Sobolev space $H_{1,p}(U)$ on U is well-defined and that the differential df as above of $f \in H_{1,p}(U)$ is also well-defined. See also Section 4 in [**3**] for the detail.

Finally we end this subsection by giving the definition of *harmonic functions* on metric measure spaces by Cheeger. We say that $f \in H_{1,2}(U)$ *is harmonic on* U if

$$\int_U |d(f + k)|^2 d\upsilon \geq \int_U |df|^2 d\upsilon$$

holds for every $k \in \mathcal{K}(U)$. See Section 7 in [**3**] for several fundamental properties of harmonic functions.

2.2. Gromov-Hausdorff convergence and the structure theory of limit spaces of Riemannian manifolds.

Let $\{(Z_i, z_i)\}_{1 \leq i \leq \infty}$ be a sequence of pointed proper geodesic spaces. We say that

(Z_i, z_i) *converges to* (Z_∞, z_∞) *with respect to the pointed Gromov-Hausdorff topology* if there exist sequences $\{\epsilon_i\}_i$, $\{R_i\}_i$ of positive numbers, and $\{\phi_i\}_i$ of Borel maps ϕ_i from $(B_{R_i}(z_i), z_i)$ to $(B_{R_i}(z_\infty), z_\infty)$ such that $\epsilon_i \to 0$, $R_i \to \infty$ as $i \to \infty$, $B_{R_i}(z_\infty) \subset B_{\epsilon_i}(\mathrm{Image}\,\phi_i)$ and $|\overline{\alpha,\beta} - \overline{\phi_i(\alpha),\phi_i(\beta)}| \le \epsilon_i$ for every $\alpha, \beta \in B_{R_i}(x_i)$. We denote it by $(Z_i, z_i) \xrightarrow{(\phi_i, R_i, \epsilon_i)} (Z_\infty, z_\infty)$ or by $(Z_i, z_i) \to (Z_\infty, z_\infty)$ for short. Assume $(Z_i, z_i) \to (Z_\infty, z_\infty)$. Let $\{x_i\}_{1 \le i \le \infty}$ be a sequence of points $x_i \in Z_i$. We say that x_i *converges to* x_∞ if $x_i \in B_{R_i}(z_i)$ and $\overline{\phi_i(x_i), x_\infty} \to 0$. Then, we denote it by $x_i \to x_\infty$. Let $\{\upsilon_i\}_{1 \le i \le \infty}$ be a sequence of Radon measures υ_i on Z_i. We say that (Z_i, z_i, υ_i) *converges to* $(Z_\infty, z_\infty, \upsilon_\infty)$ *with respect to the measured Gromov-Hausdorff topology* if $\lim_{i\to\infty} \upsilon_i(B_r(x_i)) = \upsilon_\infty(B_r(x_\infty))$ for every $r > 0$ and every $x_i \to x_\infty$. Then we denote it by $(Z_i, z_i, \upsilon_i) \to (Z_\infty, z_\infty, \upsilon_\infty)$. See also Section 1 in [**6**] or [**26**, (0.2) Definition]. Let $(X, x), (W, w)$ be proper geodesic spaces. We say that (X, x) *is a tangent cone of* W *at* w if there exists a sequence $\{\epsilon\}_i$ of positive numbers such that $\epsilon_i \to 0$ and $(X, x, \epsilon_i^{-1} d_X) \to (W, w)$. For a metric space Y and a positive integer n, let $\mathcal{R}_n(Y) := \{y \in Y;$ every tangent cone of Y at y is isometric to $(\boldsymbol{R}^n, 0_n).\}$.

We end this subsection by introducing several important properties of the *non-collapsing* limit space of a sequence of Riemannian manifolds with a lower Ricci curvature bound by Cheeger-Colding [**5**], [**6**], [**7**], [**8**], Colding [**12**]. See [**5**], [**6**], [**7**], [**8**] for collapsing case. Let $\{(M_i, m_i)\}_{i<\infty}$ be a sequence of pointed n-dimensional complete Riemannian manifolds and (M_∞, m_∞) the Gromov-Hausdorff limit space. Assume that there exist $\nu > 0$ and $K < 0$ such that $\mathrm{Ric}_{M_i} \ge K(n-1)$ and $\mathrm{vol}\, B_1(m_i) \ge \nu$ for every $i < \infty$. Let $\mathcal{R}_n = \mathcal{R}_n(M_\infty)$. Then, we have the following:

1. (GH-convergence implies measured GH-convergence [**6**, Theorem 5.9]). $(M_i, m_i, \mathrm{vol}) \to (M_\infty, m_\infty, H^n)$ where H^n is the n-dimensional spherical Hausdorff measure.
2. (Regular sets have full measure [**6**, Theorem 2.1]). $H^n(M_\infty \setminus \mathcal{R}_n) = 0$.
3. (Limit spaces are rectifiable [**8**, Theorems 5.5 and 5.7]). M_∞ is H^n-recitifiable.

2.3. Convergence of the differentials of Lipschitz functions.

In this subsection, we recall the definition of a convergence of the differential of Lipschitz functions with respect to the measured Gromov-Hausdorff topology given in [**38**]. We consider the same setting as in the previous subsection: Let $(M_i, m_i) \to (M_\infty, m_\infty)$ with $\mathrm{Ric}_{M_i} \ge K(n-1)$ and $\mathrm{vol}\, B_1(m_i) \ge \nu$. Fix $R > 0$, $L \ge 1$ and an L-Lipschitz function f_i on $B_R(m_i)$ for every $i \le \infty$. We say that f_i *converges to* f_∞ *at* x_∞ if $f_i(x_i) \to f_\infty(x_\infty)$ for every $x_i \to x_\infty$. Then we denote it by $f_i \to f_\infty$ at x_∞. The following notion performs a crucial role in this paper:

Definition 2.3 ([**38**, Definition 1.1, Definition 4.4]). We say that df_i *converges to* df_∞ *at* x_∞ if for every $\epsilon > 0$ and every $z_i \to z_\infty$, there exists $r > 0$ such that

$$\limsup_{i\to\infty} \left| \frac{1}{\mathrm{vol}\, B_t(x_i)} \int_{B_t(x_i)} \langle dr_{z_i}, df_i \rangle d\,\mathrm{vol} - \frac{1}{H^n(B_t(x_\infty))} \int_{B_t(x_\infty)} \langle dr_{z_\infty}, df_\infty \rangle dH^n \right| < \epsilon$$

and

$$\limsup_{i\to\infty} \frac{1}{\operatorname{vol} B_t(x_i)} \int_{B_t(x_i)} |df_i|^2 d\operatorname{vol} \le \frac{1}{H^n(B_t(x_\infty))} \int_{B_t(x_\infty)} |df_\infty|^2 dH^n + \epsilon$$

for every $0 < t < r$ and every $x_i \to x_\infty$. Then we denote it by $df_i \to df_\infty$ at x_∞.

We use the following notation: $(f_i, df_i) \to (f_\infty, df_\infty)$ at x_∞ if $f_i \to f_\infty$ and $df_i \to df_\infty$ at x_∞. We end this subsection by giving two fundamental properties of this convergence:

1. [**38**, Corollary 4.7]. Assume that f_i is harmonic for every $i < \infty$ and that $f_i \to f_\infty$ on $B_R(m_\infty)$ (i.e., $f_i \to f_\infty$ at every $x_\infty \in B_R(m_\infty)$). Then we see that $df_i \to df_\infty$ on $B_R(m_\infty)$ and that f_∞ is harmonic on $B_R(m_\infty)$.
2. [**38**, Corollary 4.4]. Let k be a positive integer, $r > 0$ with $r < R$, $\{w_i\}_i$ a sequence of points $w_i \in M_i$ with $w_i \to w_\infty$, $\{f_i^l, g_i^l\}_{1\le i\le\infty, 1\le l\le k}$ a collection of Lipschitz functions f_i^l, g_i^l on $B_R(m_i)$ with $\sup_{i,l}(\mathbf{Lip}\ f_i^l + \mathbf{Lip}\ g_i^l) < \infty$, and $\{F_i\}_{1\le i\le\infty} \subset C^0(\boldsymbol{R}^k)$. Assume that the following hold:
   (a) F_i converges to F_∞ with respect to the compact uniformly topology.
   (b) $df_i^l \to df_\infty^l$ and $dg_i^l \to dg_\infty^l$ at a.e. $\alpha \in B_R(m_\infty) \setminus B_r(w_\infty)$ for every $1 \le l \le k$.
   Then we have
$$\lim_{i\to\infty} \int_{B_R(m_i)\setminus B_r(w_i)} F_i\big(\langle df_i^1, dg_i^1\rangle, \dots, \langle df_i^k, dg_i^k\rangle\big) d\operatorname{vol}$$
$$= \int_{B_R(m_\infty)\setminus B_r(w_\infty)} F_\infty\big(\langle df_\infty^1, dg_\infty^1\rangle, \dots, \langle df_\infty^k, dg_\infty^k\rangle\big) dH^n.$$

See [**38**] for more fundamental properties of this convergence: $df_i \to df_\infty$.

2.4. Gradient estimates.

In this subsection, we recall the following two very important estimates. These estimates will be used many times in this paper:

1. (Cheng-Yau's gradient estimate [**11**]). Let $K \ge 0$, $R > 0$ and let (M, m) be a pointed complete n-dimensional Riemannian manifold with $\mathrm{Ric}_M \ge K(n-1)$. Then for every positive valued harmonic function f on $B_R(m)$, we have
$$\frac{|\nabla f|^2}{f^2} \le C(n, K, r, R)$$
   on $B_r(m)$ for every $r < R$.
2. (Li-Schoen's mean value inequality [**51**]). Let (M, m) be a pointed complete n-dimensional nonnegatively Ricci curved manifold, $R > 0$ and f a nonnegative valued subharmonic function $B_{3R/2}(m)$. Then we have
$$\sup_{B_R(m)} f \le \frac{C(n)}{\operatorname{vol} B_{3R/2}(m)} \int_{B_{3R/2}(m)} f\, d\operatorname{vol}.$$

Note that if $\mathrm{Ric}_M \ge 0$, then $|\nabla h|^2$ is a subharmonic function for every harmonic function h. This is a direct consequence of Bochner's formula.

3. Convergence of frequency functions.

Our goal in this section is to give a proof of Theorem 1.1. Throughout this section, we will always assume that the dimensions of all manifolds under consideration are greater than 2. See [**22**], [**39**], [**53**] for important works about two dimensional case.

Throughout this section, we fix an n-dimensional complete nonnegatively Ricci curved Riemannian manifold M with $V_M^{g_M} := V_M > 0$, where g_M is the Riemannian metric of M. It is easy to check that $V_M^{r^{-2}g_M} = V_M^{g_M}$ holds for every $r > 0$. Fix $m \in M$. Note that the entire Green's function $G^{g_M}(x, y)$ on M exists. See for instance [**63**].

First we recall an important result about asymptotic behavior of G^{g_M} by Colding-Minicozzi:

Theorem 3.1 (Colding-Minicozzi [**16**]). *We have*

$$\lim_{\overline{m,x} \to \infty} \frac{G^{g_M}(m, x)}{\overline{m,x}^{2-n}} = \frac{\operatorname{vol} B_1(0_n)}{V_M}.$$

Note that for every $r > 0$, we have

$$G^{r^{-2}g_M}(m, x) = \frac{G^{g_M}(m, x)}{r^{2-n}}.$$

It is known that there exists $C_1 > 1$ such that $\overline{m,x}^{2-n} \le G^{g_M}(m, x) \le C_1 \overline{m,x}^{2-n}$ for every $m \neq x$. Define a smooth function $b_m^{g_M}$ on $M \setminus \{m\}$ by

$$b_m^{g_M}(x) := \left(\frac{V_M}{\operatorname{vol} B_1(0_n)} G^{g_M}(m, x) \right)^{1/(2-n)}.$$

Note $b_m^{r^{-2}g_M} = b_m^{g_m}/r$. We use the notation $b^{g_M} = b_m^{g_M}$ for short. Thus we have

$$\left(\frac{V_M}{\operatorname{vol} B_1(0_n)} \right)^{2-n} \overline{m,y}^{r^{-2}g_M} \le b^{r^{-2}g_M}(y) \le \left(\frac{C_1 V_M}{\operatorname{vol} B_1(0_n)} \right)^{2-n} \overline{m,y}^{r^{-2}g_M}$$

for every $r > 0$. Let $b^{g_M}(m) := 0$. It is easy to check

$$\nabla^{g_M} b^{g_M} = \frac{V_M}{(2-n) \operatorname{vol} B_1(0_n)} (b^{g_M})^{n-1} \nabla^{g_M} G^{g_M}(m, \cdot).$$

On the other hand, for every $\epsilon > 0$, there exists $R(\epsilon) > 0$ such that

$$\frac{1}{\operatorname{vol}\{b^{g_M} \le R\}} \int_{b^{g_M} \le R} \left| |\nabla b^{g_M}|^2 - 1 \right|^2 + |\operatorname{Hess}_{(b^{g_M})^2} - 2g_M|^2 d \operatorname{vol} \le \epsilon$$

for every $R > R(\epsilon)$ and that

$$\left| \frac{b^{g_M}(x)}{\overline{m, x}^{g_M}} - 1 \right| < \epsilon$$

for every $x \in M \setminus B_{R(\epsilon)}(m)$. See (2.23), (2.24) and (2.25) in [**15**] or Section 4 in [**16**] for the proofs of these results.

Lemma 3.2. *We have*

$$\lim_{R \to \infty} \frac{\operatorname{vol}\{b^{g_M} \leq R\}}{\operatorname{vol} B_R^{g_M}(m)} = 1.$$

Proof. Fix $0 < \epsilon < 1$. Let $R(\epsilon) > 0$ as above and

$$\hat{R}(\epsilon) := \left(\frac{C_1 V_M}{\operatorname{vol} B_1(0_n)} \right)^{2-n} R(\epsilon) + R(\epsilon).$$

Fix $R > \hat{R}(\epsilon)$.

First we will show $B_R(m) \subset \{b^{g_M} \leq (1+\epsilon)R\}$. Let $y \in B_R(m)$. If $\overline{m, y} \leq R(\epsilon)$, then

$$b^{g_M}(y) \leq \left(\frac{C_1 V_M}{\operatorname{vol} B_1(0_n)} \right)^{2-n} \overline{m, y} \leq \left(\frac{C_1 V_M}{\operatorname{vol} B_1(0_n)} \right)^{2-n} R(\epsilon) \leq \hat{R}(\epsilon) \leq R.$$

If $\overline{m, y} > R(\epsilon)$, then $|b^{g_M}(y) - \overline{m, y}| < \epsilon \overline{m, y}$. Especially, we have $b^{g_M}(y) \leq (1+\epsilon)\overline{m, y} < (1+\epsilon)R$. Thus, we have $B_R(m) \subset \{b^{g_M} \leq (1+\epsilon)R\}$.

On the other hand, for every $x \in \{b^{g_M} \leq (1+\epsilon)R\}$ with $\overline{m, x} \geq R(\epsilon)$, we have $(1-\epsilon)\overline{m, x} \leq b^{g_M}(x) \leq (1+\epsilon)R$. Thus, we have $\{b^{g_M} \leq (1+\epsilon)R\} \subset B_{(1+\epsilon)R/(1-\epsilon)}(m)$. Since

$$\lim_{R \to \infty} \frac{\operatorname{vol} B_{aR}(m)}{\operatorname{vol} B_{bR}(m)} = \left(\frac{a}{b} \right)^n$$

for every $a, b > 0$, we have the assertion. $\square$

We now recall the definition of the frequency function for a harmonic function on M by Colding-Minicozzi. For $R > 0$, $0 < r < R$ and a harmonic function u on $\{b^{g_M} < R\}$, let

$$I_u^{g_M}(r) := r^{1-n} \int_{b^{g_M} = r} u^2 |\nabla^{g_M} b^{g_M}| d \operatorname{vol}_{n-1}^{g_M}, \quad D_u^{g_M}(r) = r^{2-n} \int_{b^{g_M} \leq r} |\nabla^{g_M} u|^2 d \operatorname{vol}^{g_M}$$

and

$$F_u^{g_M}(r) := r^{3-n} \int_{b^{g_M} = r} \left| \frac{\partial u}{\partial n} \right|^2 |\nabla b^{g_M}| d \operatorname{vol}_{n-1}^{g_M},$$

where n is the outer unit normal vector of $\{b^{g_M} = r\}$, $\operatorname{vol}_{n-1}^{g_M}$ is the $(n-1)$-dimensional

Hausdorff measure with respect to the Riemannian metric g_M. Moreover, let

$$U_u^{g_M}(r) := \frac{D_u^{g_M}(r)}{I_u^{g_M}(r)} \text{ if } I_u^{g_M}(r) \neq 0, \quad U_u^{g_M}(r) := 0 \text{ if } I_u^{g_M}(r) = 0.$$

We call the function $U_u^{g_M}$ on $(0, R)$ *the frequency function for* u. Note that the critical set of b^{g_M} has codimension two at least. See [**10**], [**34**]. By the maximum principle, $U_u^{g_M}(r) = 0$ for some $0 < r < R$ if and only if u is a constant function. Note that

$$D_u^{g_M}(r) \le \left(\frac{r}{s}\right)^{2-n} D_u^{g_M}(s), \qquad \frac{dI_u^{g_M}}{dr} = 2\frac{D_u^{g_M}(r)}{r}$$

and

$$I_u^{g_M}(s) = \exp\left(2\int_r^s \frac{U_u^{g_M}(t)}{t}dt\right) I_u^{g_M}(r)$$

for $r < s$. See (2.10), (2.12), (2.13) and (2.14) in [**15**] for the proofs. For $\tau, r > 0$, $R > r\tau$ and a harmonic function u on $\{b^{g_M} < R\}$, we put $u_\tau := u/\tau$. Then for a rescaled metric $\tau^{-2}g_M$, it is easy to check that $D_{u_\tau}^{\tau^{-2}g_M}(r) = \tau^{-2}D_u^{g_M}(r\tau)$, $I_{u_\tau}^{\tau^{-2}g_M}(r) = \tau^{-2}I_u^{g_M}(r\tau)$, $F_{u_\tau}^{\tau^{-2}g_M}(r) = \tau^{-2}F_u^{g_M}(r\tau)$ and $U_{u_\tau}^{\tau^{-2}g_M}(r) = U_u^{g_M}(r\tau)$.

Fix an asymptotic cone (M_∞, m_∞) of M for a divergent sequence of positive numbers R_i: $(M, m, R_i^{-1}d_M) \to (M_\infty, m_\infty)$. Note that by the assumption $V_M > 0$, we have $(M, m, R_i^{-1}d_M, \mathrm{vol}^{R_i^{-2}g_M}) \to (M_\infty, m_\infty, H^n)$. On the other hand, by [**5**, Theorem 7.6], we see that there exists a compact geodesic space X with $\operatorname{diam} X \le \pi$ such that (M_∞, m_∞) is isometric to $(C(X), p)$.

Let $R > 0$, $0 < r < R$ and let u be a Lipschitz function on $\overline{B}_R(p)$. Assume that u is harmonic on $B_R(p)$. Put, for $r \in (0, R)$,

$$I_u(r) := r^{1-n}\int_{\partial B_r(p)} u^2 dH^{n-1}, \quad D_u(r) := r^{2-n}\int_{B_r(p)} |du|^2 dH^n$$

and

$$U_u(r) := \frac{D_u(r)}{I_u(r)} \text{ if } I_u(r) \neq 0, \quad U_u(r) := 0 \text{ if } I_u(r) = 0.$$

By Proposition 7.6, we see that the function

$$F_u(r) := r^{3-n}\int_{\partial B_r(p)} \langle dr_p, du\rangle^2 dH^{n-1}$$

is well defined for a.e. $r \in (0, R)$.

Remark 3.3. Let $R > 0$ and let $\{u_i\}_{i<\infty}$ be a sequence of harmonic functions u_i on $B_{RR_i}^{g_M}(m)$. Assume that $\sup_i |(u_i)_{R_i}|_{L^\infty(B_r^{R_i^{-2}g_M}(m))} < \infty$ for every $0 < r < R$. Then we have $\sup_i \mathbf{Lip}\left((u_i)_{R_i}|_{B_r^{R_i^{-2}g_M}(m)}\right) < \infty$ for every $0 < r < R$. The proof is as follows. Fix $\hat{r}$ with $r < \hat{r} < R$. Since $\overline{B}_r(p)$ is convex, it is not difficult to see that there exists i_0 such that $\mathrm{Image}(\gamma_i) \subset \overline{B}_{\hat{r}}^{R_i^{-2}g_M}(m)$ for every $i \geq i_0$, every $x_1(i), x_2(i) \in \overline{B}_r^{R_i^{-2}g_M}(m)$ and every minimal geodesic γ_i from $x_1(i)$ to $x_2(i)$. Therefore, by Cheng-Yau's gradient estimate, we have $\limsup_{i\to\infty} \mathbf{Lip}\left((u_i)_{R_i}|_{B_r^{R_i^{-2}g_M}(m)}\right) < \infty$ for every $0 < r < R$.

PROPOSITION 3.4. *Let $R > 0$ and let $\{u_i\}_{i<\infty}$ be a sequence of harmonic functions u_i on $B^{g_M}_{RR_i}(m)$ and u_∞ a Lipschitz function on $B_R(p)$. Assume that $\sup_i |(u_i)_{R_i}|_{L^\infty(B_t^{R_i^{-2}g_M}(m))} < \infty$ and $(u_i)_{R_i} \to u_\infty$ on $B_t(p)$ for every $0 < t < R$. Then, we have*

$$\lim_{i\to\infty} \sup_{t\in[r,s]} \left| D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t) - D_{u_\infty}(t)\right| = 0 \quad \textit{and} \quad \lim_{i\to\infty} \sup_{t\in[r,s]} \left| I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t) - I_{u_\infty}(t)\right| = 0$$

for every $0 < r < s < R$.

PROOF. Let r, s be positive numbers with $r < s < R$. Fix positive numbers $\hat{r}, \hat{s}$ with $\hat{r} < r < s < \hat{s} < R$. Let $L \geq 1$ with $\|u_\infty\|_{L^\infty(B_{\hat{s}}(x_\infty))} + \mathbf{Lip}\, u_\infty \leq L$. Fix $\epsilon > 0$ with $\epsilon \ll \min\{\hat{r}, R - \hat{s}\}$. Then, by the proof of Lemma 3.2, there exists $R_1(\epsilon) > 1$ such that $B^{g_M}_{(1-\epsilon^2)R}(m) \subset \{b^{g_M} \leq R\} \subset B^{g_M}_{(1+\epsilon^2)R}(m)$ and

$$\frac{1}{\mathrm{vol}\{b^{g_M} \leq R\}} \int_{b^{g_M}\leq R} \left||\nabla^{g_M} b^{g_M}|^2 - 1\right|^2 \leq \epsilon^8$$

for every $R > R_1(\epsilon)$. The Cauchy-Schwartz inequality yields

$$\frac{1}{\mathrm{vol}\{b^{g_M} \leq R\}} \int_{b^{g_M}\leq R} \left||\nabla^{g_M} b^{g_M}|^2 - 1\right| \leq \epsilon^4$$

and

$$\frac{1}{\mathrm{vol}\{b^{g_M} \leq R\}} \int_{b^{g_M}\leq R} \left||\nabla^{g_M} b^{g_M}| - 1\right| \leq \epsilon^2.$$

For every $0 < t < R$, let

$$F_i(t) := \int_{b^{R_i^{-2}g_M}\leq t} (u_i)^2_{R_i} \left|\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}\right|^2 d\,\mathrm{vol}^{R_i^{-2}g_M}.$$

Then, we have

$$\frac{dF_i}{dt}(t) = \int_{b^{R_i^{-2}g_M}=t} (u_i)^2_{R_i} \left|\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}\right| d\,\mathrm{vol}^{R_i^{-2}g_M}_{n-1} = I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t) t^{n-1}.$$

Thus, we have

$$\begin{aligned}\frac{d^2F_i}{dt^2}(t) &= 2t^{n-1}\frac{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)}{t} + (n-1)I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)t^{n-2} \\ &= 2\int_{b^{R_i^{-2}g_M}\le t}\left|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\right|^2 d\,\mathrm{vol}^{R_i^{-2}g_M} \\ &\quad + \frac{n-1}{t}\int_{b^{R_i^{-2}g_M}=t}(u_i)^2_{R_i}\left|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}\right|^2 d\,\mathrm{vol}_{n-1}^{R_i^{-2}g_M}.\end{aligned}$$

Recall that for every $a, s, t \in \boldsymbol{R}$, and every C^2-function f on $\boldsymbol{R}$, we have

$$f(t) = f(a) + (t-a)f'(a) - \int_a^t (s-t)f''(s)ds.$$

Therefore, for every $0 < t < R$, we have

$$\begin{aligned}&\left|\frac{F_i(t+\epsilon)-F_i(t)}{\epsilon} - \int_{b^{R_i^{-2}g_M}=t}(u_i)^2_{R_i}\left|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}\right|d\,\mathrm{vol}^{R_i^{-2}g_M}\right| \\ &\le \int_t^{t+\epsilon}2\int_{b^{R_i^{-2}g_M}\le a}\left|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\right|^2 d\,\mathrm{vol}^{R_i^{-2}g_M}\,da \\ &\quad + (n-1)\int_t^{t+\epsilon}a^{-1}\int_{b^{R_i^{-2}g_M}=a}(u_i)^2_{R_i}\left|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}\right|d\,\mathrm{vol}^{R_i^{-2}g_M}\,da \\ &\le 2\epsilon\int_{b^{R_i^{-2}g_M}\le t+\epsilon}\left|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\right|^2 d\,\mathrm{vol}^{R_i^{-2}g_M} \\ &\quad + \frac{n-1}{t}\int_{t\le b^{g_M}\le t+\epsilon}(u_i)^2_{R_i}\left|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}\right|^2 d\,\mathrm{vol}^{R_i^{-2}g_M}.\end{aligned}$$

By [**38**, Proposition 2.4], there exists $i_0 \in \boldsymbol{N}$ such that $R_i\hat{r} \ge 10R_1(\epsilon)$, $\|(u_i)_{R_i}\|_{L^\infty(B_{\hat{s}}^{R_i^{-2}g_M}(m))} \le 10L$ and

$$\sup_{a\in[0,R]}\left|\mathrm{vol}^{R_i^{-2}g_M}\ B_a^{R_i^{-2}g_M}(m) - H^n(B_a(p))\right| < \epsilon^2$$

for every $i \ge i_0$. Then, since $H^n(B_R(p)) = R^nH^n(B_1(p)) \le R^nC(n)$, Cheng-Yau's gradient estimate yields

$$\begin{aligned}\int_{b^{R_i^{-2}g_M}\le t+\epsilon}\left|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\right|^2 d\,\mathrm{vol}^{R_i^{-2}g_M} &\le \int_{B^{R_i^{-2}g_M}_{(1+\epsilon)(t+\epsilon)}(m)}\left|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\right|^2 d\,\mathrm{vol}^{R_i^{-2}g_M} \\ &\le C(n,L,R)\end{aligned}$$

for every $i \ge i_0$ and every $r < t < s$. Moreover, we have

$$
\begin{aligned}
&\int_{t\le b^{R_i^{-2}g_M}\le t+\epsilon}(u_i)^2_{R_i}\big|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}\big|^2 d\,\mathrm{vol}^{R_i^{-2}g_M}\\
&\quad\le \int_{t\le b^{R_i^{-2}g_M}\le t+\epsilon}(u_i)^2_{R_i}d\,\mathrm{vol}^{R_i^{-2}g_M}\\
&\qquad+\int_{t\le b^{R_i^{-2}g_M}\le t+\epsilon}(u_i)^2_{R_i}\big||\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}|^2-1\big|d\,\mathrm{vol}^{R_i^{-2}g_M}\\
&\quad\le \int_{t\le b^{R_i^{-2}g_M}\le t+\epsilon}(u_i)^2_{R_i}d\,\mathrm{vol}^{R_i^{-2}g_M}+100L^2\,\mathrm{vol}^{R_i^{-2}g_M}\left\{t\le b^{R_i^{-2}g_M}\le t+\epsilon\right\}\\
&\quad\le 200L^2\,\mathrm{vol}^{R_i^{-2}g_M}\left\{t\le b^{R_i^{-2}g_M}\le t+\epsilon\right\}\\
&\quad\le 200L^2\,\mathrm{vol}^{R_i^{-2}g_M}\ A_m^{R_i^{-2}g_M}\big((1-\epsilon^2)t,(1+\epsilon^2)(t+\epsilon)\big)\\
&\quad\le 200L^2H^n\big(A_p((1-\epsilon^2)t,(1+\epsilon^2)(t+\epsilon))\big)+300L^2\epsilon^2.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\frac{F_i(t+\epsilon)-F_i(t)}{\epsilon}&=\frac{1}{\epsilon}\int_{t\le b^{R_i^{-2}g_M}\le t+\epsilon}(u_i)^2_{R_i}d\,\mathrm{vol}^{R_i^{-2}g_M}\\
&\quad\pm\frac{1}{\epsilon}\int_{t\le b^{R_i^{-2}g_M}\le t+\epsilon}(u_i)^2_{R_i}\big||\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}|^2-1\big|d\,\mathrm{vol}^{R_i^{-2}g_M},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\epsilon}\int_{t\le b^{R_i^{-2}g_M}\le t+\epsilon}(u_i)^2_{R_i}\big|\big|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}\big|^2-1\big|d\,\mathrm{vol}^{R_i^{-2}g_M}\\
&\quad\le\frac{100L^2}{\epsilon}\int_{b^{R_i^{-2}g_M}\le t+\epsilon}\big|\big|\nabla^{R_i^{-2}g_M}b^{R_i^{-2}g_M}\big|^2-1\big|d\,\mathrm{vol}^{R_i^{-2}g_M}\\
&\quad\le\frac{100L^2}{\epsilon}\epsilon^2\,\mathrm{vol}^{R_i^{-2}g_M}\left\{b^{R_i^{-2}g_M}\le t+\epsilon\right\}\\
&\quad\le 100L^2\epsilon\frac{\mathrm{vol}^{g_M}\ B^{g_M}_{(1+\epsilon^2)(t+\epsilon)R_i}(m)}{R_i^n}\\
&\quad\le\epsilon C(n,L,R).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\left|\int_{t\le b^{R_i^{-2}g_M}\le t+\epsilon}(u_i)^2_{R_i}d\,\mathrm{vol}^{R_i^{-2}g_M}-\int_{A_m^{R_i^{-2}g_M}(t,t+\epsilon)}(u_i)^2_{R_i}d\,\mathrm{vol}^{R_i^{-2}g_M}\right|\\
&\quad\le 100L^2\,\mathrm{vol}^{R_i^{-2}g_M}\left(\left\{t\le b^{R_i^{-2}g_M}\le t+\epsilon\right\}\triangle A_m^{R_i^{-2}g_M}(t,t+\epsilon)\right),
\end{aligned}
$$

where $A\triangle B=(A\setminus B)\cup(B\setminus A)$.

CLAIM 3.5. *We have*

$$\big\{t \le b^{R_i^{-2}g_M} \le t+\epsilon\big\} \triangle A_m^{R_i^{-2}g_M}(t,t+\epsilon)$$
$$\subset A_m^{R_i^{-2}g_M}\big((1-\epsilon^2)(t+\epsilon),(1+\epsilon^2)(t+\epsilon)\big) \cup A_m^{R_i^{-2}g_M}\big((1-\epsilon^2)t,(1+\epsilon^2)t\big)$$

for every $i \ge i_0$ and every $r < t < s$.

The proof is as follows. Put $A_i^\epsilon(t) := \{t \le b^{R_i^{-2}g_M} \le t+\epsilon\} \triangle A_m^{R_i^{-2}g_M}(t,t+\epsilon)$. Let $y \in \{t \le b^{R_i^{-2}g_M} \le t+\epsilon/2\} \cap A_i^\epsilon(t)$. Then we have $y \in B_{(1+\epsilon^2)(t+\epsilon/2)}^{R_i^{-2}g_M}(m)$. In particular, we have

$$\overline{m,y}^{R_i^{-2}g_M} \le (1+\epsilon^2)\bigg(t+\frac{\epsilon}{2}\bigg) < t+\epsilon.$$

Since $y \in M \setminus A_m^{R_i^{-2}g_M}(t,t+\epsilon)$, we have $y \in B_t^{R_i^{-2}g_M}(m)$. Thus, we have $\{t \le b^{R_i^{-2}g_M} \le t+\epsilon/2\} \cap A_i^\epsilon(t) \subset B_t^{R_i^{-2}g_M}(m) \setminus B_{(1-\epsilon^2)t}^{R_i^{-2}g_M}(m)$. Similarly, we have $\{t+\epsilon/2 \le b^{R_i^{-2}g_M} \le t+\epsilon\} \cap A_i^\epsilon(t) \subset B_{(1+\epsilon^2)(t+\epsilon)}^{R_i^{-2}g_M}(m) \setminus B_{t+\epsilon}^{R_i^{-2}g_M}(m)$. Therefore, we have

$$\big\{t \le b^{R_i^{-2}g_M} \le t+\epsilon\big\} \cap A_i^\epsilon(t) \subset A_m^{R_i^{-2}g_M}((1-\epsilon^2)t,t) \cup A_m^{R_i^{-2}g_M}(t+\epsilon,(1+\epsilon^2)(t+\epsilon)).$$

Let $x \in A_i^\epsilon(t) \cap A_m^{R_i^{-2}g_M}(t,t+\epsilon/2)$. Then we have

$$b^{R_i^{-2}g_M}(x) \le (1+\epsilon^2)\overline{m,x}^{R_i^{-2}g_M} \le (1+\epsilon^2)(t+\epsilon/2) < t+\epsilon.$$

Since $x \in M \setminus \{t \le b^{R_i^{-2}g_M} \le t+\epsilon\}$, we have $b^{R_i^{-2}g_M}(x) < t$. Therefore, we have $x \in B_{(1+\epsilon^2)t}^{R_i^{-2}g_M}(m)$. Thus, we have $A_m^{R_i^{-2}g_M}(t,t+\epsilon/2) \cap A_i^\epsilon(t) \subset A_m^{R_i^{-2}g_M}(t,(1+\epsilon^2)t)$. Similarly, we have $A_m^{R_i^{-2}g_M}(t+\epsilon/2,t+\epsilon) \cap A_i^\epsilon(t) \subset A_m^{R_i^{-2}g_M}(t+\epsilon,(1+\epsilon^2)(t+\epsilon))$. Therefore we have Claim 3.5.

By Claim 3.5 and Bishop-Gromov volume comparison theorem, we have

$$\begin{aligned}
&\epsilon^{-1}\operatorname{vol}^{R_i^{-2}g_M}\Big(\big\{t \le b^{R_i^{-2}g_M} \le t+\epsilon\big\} \triangle A_m^{R_i^{-2}g_M}(t,t+\epsilon)\Big)\\
&\quad \le \epsilon^{-1}\operatorname{vol}^{R_i^{-2}g_M}\Big(A_m^{R_i^{-2}g_M}\big((1-\epsilon^2)(t+\epsilon),(1+\epsilon^2)(t+\epsilon)\big)\Big)\\
&\qquad + \epsilon^{-1}\operatorname{vol}^{R_i^{-2}g_M}\Big(A_m^{R_i^{-2}g_M}\big((1-\epsilon^2)t,(1+\epsilon^2)t\big)\Big)\\
&\quad \le 3\epsilon^{-1}\epsilon^2\operatorname{vol}_{n-1}^{R_i^{-2}g_M}\Big(\partial B_{(1-\epsilon^2)(t+\epsilon)}^{R_i^{-2}g_M}(m) \setminus C_m\Big) + 3\epsilon^{-1}\epsilon^2\operatorname{vol}_{n-1}^{R_i^{-2}g_M}\Big(\partial B_{(1-\epsilon^2)t}^{R_i^{-2}g_M}(m) \setminus C_m\Big)\\
&\quad \le 6\epsilon\operatorname{vol}\partial B_R(0_n).
\end{aligned}$$

Therefore we have

$$\left| \int_{t \le b^{R_i^{-2} g_M} \le t+\epsilon} (u_i)^2_{R_i} d\,\mathrm{vol}^{R_i^{-2} g_M} - \int_{A_m^{R_i^{-2} g_M}(t, t+\epsilon)} (u_i)^2_{R_i} d\,\mathrm{vol}^{R_i^{-2} g_M} \right|$$
$$\le 600 L^2 \epsilon^2 \,\mathrm{vol}\, \partial B_R(0_n)$$

for every $i \ge i_0$ and every $r < t < s$.

Define an 1-Lipschitz map π_t from $C(X)$ to $\overline{B}_t(p)$ by $\pi_t(s, x) := (\hat{\pi}_t(s), x)$, where $\hat{\pi}_t(s) = s$ if $s \le t$, and $\hat{\pi}_t(s) = t$ if $s > t$. Put $u^t_\infty := (u_\infty)^2 \circ \pi_t$. Then Proposition 7.6 yields

$$\left| \int_t^{t+\epsilon} \int_{\partial B_a(p)} (u_\infty)^2 dH^{n-1} da - \int_t^{t+\epsilon} \int_{\partial B_a(p)} u^t_\infty dH^{n-1} da \right|$$
$$\le \int_{A_p(t, t+\epsilon)} \left| (u_\infty)^2 - u^t_\infty \right| dH^n$$
$$\le \mathbf{Lip}(u_\infty)^2 \epsilon H^n(A_p(t, t+\epsilon))$$

for every $r < t < s$. On the other hand, we have

$$\begin{aligned} \int_t^{t+\epsilon} \int_{\partial B_a(p)} u^t_\infty dH^{n-1} da &= \int_t^{t+\epsilon} \left(\frac{a}{t}\right)^{n-1} \int_{\partial B_t(p)} (u_\infty)^2 dH^{n-1} da \\ &= \int_{\partial B_t(p)} (u_\infty)^2 dH^{n-1} \int_t^{t+\epsilon} \left(\frac{a}{t}\right)^{n-1} da \\ &= I_{u_\infty}(t) \frac{(t+\epsilon)^n - t^n}{n} \\ &= I_{u_\infty}(t)(\epsilon t^{n-1} \pm \Psi(\epsilon; n, R)\epsilon). \end{aligned}$$

Therefore we have

$$\lim_{i \to \infty} \sup_{t \in [r,s]} \left| I^{R_i^{-2} g_M}_{(u_i)_{R_i}}(t) - I_{u_\infty}(t) \right| = 0.$$

Finally, we will show

$$\lim_{i \to \infty} \sup_{t \in [r,s]} \left| D^{R_i^{-2} g_M}_{(u_i)_{R_i}}(t) - D_{u_\infty}(t) \right| = 0.$$

We use the same notations as above. It is clear that

$$t^{2-n} \int_{B^{R_i^{-2} g_M}_{(1-\epsilon^2)t}(m)} \left| \nabla^{R_i^{-2} g_M} (u_i)_{R_i} \right|^2 d\,\mathrm{vol}^{R_i^{-2} g_M}$$
$$\le D^{R_i^{-2} g_M}_{(u_i)_{R_i}}(t) \le t^{2-n} \int_{B^{R_i^{-2} g_M}_{(1+\epsilon^2)t}(m)} \left| \nabla^{R_i^{-2} g_M} (u_i)_{R_i} \right|^2 d\,\mathrm{vol}^{R_i^{-2} g_M}$$

holds for every $i \geq i_1$ and every $r < t < s$. On the other hand, we have

$$\begin{aligned}
&\int_{A_m^{R_i^{-2}g_M}((1-\epsilon^2)t,(1+\epsilon^2)t)} \left|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\right|^2 d\,\mathrm{vol}^{R_i^{-2}g_M}(m) \\
&\quad \leq C(n,L,R)\,\mathrm{vol}^{R_i^{-2}g_M}\ A_m^{R_i^{-2}g_M}((1-\epsilon^2)t,(1+\epsilon^2)t) \\
&\quad \leq C(n,L,R)\bigl(H^n(A_p((1-\epsilon^2)t,(1+\epsilon^2)t))+\epsilon\bigr).
\end{aligned}$$

Therefore, by [**38**, Corollary 4.7], we have the assertion. $\square$

Let $0 < r < R$, and let u be a harmonic function on $\{b^{g_M} < R\}$. Put

$$E_u^{g_M}(r) := r^{2-n}\int_{b^{g_M}\leq r} |\nabla^{g_M}u|^2|\nabla^{g_M}b^{g_M}|^2 d\,\mathrm{vol}^{g_M}.$$

It is easy to check that $E_{u_\tau}^{\tau^{-2}g_M}(r) = \tau^{-2}E_u^{g_M}(\tau r)$ for every $r > 0$ with $R > r\tau$. By [**16**, Proposition 3.3] and the proof of Proposition 3.4, we have the following:

PROPOSITION 3.6. *With the same assumption as in Proposition* 3.4*, we have*

$$\lim_{i\to\infty}\sup_{t\in[r,s]}\left|E_{(u_i)_{R_i}}^{R_i^{-2}g_M}(t) - D_{u_\infty}(t)\right| = 0$$

for every $0 < r < s < R$.

We now introduce an important result [**21**, Theorem 2.1] for harmonic functions on asymptotic cones by Ding:

THEOREM 3.7 (Ding, [**21**]). *Let $R > 0$ and let u_∞ be a harmonic function on $B_R(p)$. Then u_∞ is Lipschitz on $B_r(p)$ for every $r < R$. Moreover, for every $0 < r < s < R$, there exist a subsequence $\{i(j)\}_j$ of $\boldsymbol{N}$ and a sequence of harmonic functions $u_{i(j)}$ on $B_s^{R_{i(j)}^{-2}g_M}(m)$ such that $u_{i(j)} \to u_\infty$ on $B_r(x_\infty)$.*

PROOF. We give an outline of the proof only. Let r, s be positive numbers with $r < s < R$. First, we will show that u_∞ is Lipschitz on $B_s(p)$. By [**45**, Proposition 5.1], for every $u \in H_{1,2}(M)$ and every $\hat{R} > 0$, we have

$$\begin{aligned}
&\int_M u(y)^2 H^{\hat{R}^{-2}g_M}(t,y,x)d\,\mathrm{vol}_y^{\hat{R}^{-2}g_M} \\
&\quad \leq 2t\int_M \left|d^{\hat{R}^{-2}g_M}u\right|^2 d\,\mathrm{vol}_y^{\hat{R}^{-2}g_M} + \left(\int_M u(y)H^{\hat{R}^{-2}g_M}(t,y,x)d\,\mathrm{vol}_y^{\hat{R}^{-2}g_M}\right)^2
\end{aligned}$$

for a.e. $x \in M$, where $H^{\hat{R}^{-2}g_M}(t,y,x)$ is the heat kernel of a rescaled manifold $(M, \hat{R}^{-2}g_M)$. By [**3**, Lemma 10.3] and [**20**, Theorem 5.54], for every $u \in \mathcal{K}(C(X))$, we have

$$\int_{C(X)} u(y)^2 H_\infty(t,y,x)dH^n(y)$$
$$\leq 2t\int_{C(X)} |du|^2 dH^n(y) + \left(\int_{C(X)} u(y)H_\infty(t,y,x)dH^n(y)\right)^2$$

for a.e. $x \in C(X)$, where H_∞ is as in [**20**, Theorem 5.54]. Since $\mathcal{K}(C(X))$ is a dense subspace of $H_{1,2}(C(X))$, the inequality above holds for every $u \in H_{1,2}(C(X))$. Fix $x \in X$ and $0 < t < R$. It is easy to check that $H^n(B_t((1,x))) \geq C(n,V_M)t^n$. For every $R > 0$, define a map ϕ_R from $A_p(R-t,R+t)$ to $A_p(1-(t/R),1+(t/R))$ by $\phi_R((\hat{t},x)) = (\hat{t}/R,x)$. Since $H^n(\phi_R(A)) = R^n H^n(A)$ for every Borel subset A of $A_p(R-t,R+t)$, we have $H^n(B_t(R,x)) = R^n H^n(B_{t/R}(1,x)) \geq C(n,V_M)t^n$. Therefore, $(C(X),H^n)$ is an Ahlfors n-regular metric measure space. By [**20**, Theorem 6.1], [**20**, Theorem 6.20] and [**45**, Theorem 1.1], we see that u_∞ is a locally Lipschitz function on $B_R(p)$. By the convexity of $B_s(p)$ and the proof of [**45**, Theorem 1.1], we see that u_∞ is Lipschitz on $B_s(p)$.

Let $L \geq 1$ with $\mathbf{Lip}(u_\infty|_{B_s(p)}) + \|u_\infty\|_{L^\infty(B_s(p))} \leq L$. Without loss of generality, we can assume that there exists a sequence of Lipschitz functions f_i on $\overline{B}_s^{R_i^{-2}g_M}(m)$ such that $\mathbf{Lip}\, f_i + |f_i|_{L^\infty(B_s(p))} \leq 10L$ and $f_i \to u_\infty$ on $B_s(p)$. Let u_i be a harmonic function on $B_s^{R_i^{-2}g_M}(m)$ satisfying that $u_i|_{\partial B_s^{R_i^{-2}g_M}(m)} = f_i|_{\partial B_s^{R_i^{-2}g_M}(m)}$ in the sense of *Perron's method* for f_i.

We now give a short review of Perron's method for subharmonic functions in this setting. See for instance Section 2.8 in [**28**] for the detail. For $f \in C^0(B_s^{R_i^{-2}g_M}(m))$, we say that f is subharmonic (superharmonic) in $B_s^{R_i^{-2}g_M}(m)$ if for every $w \in B_s^{R_i^{-2}g_M}(m)$, every $r_1 > 0$ with $\overline{B}_{r_1}^{R_i^{-2}g_M}(w) \subset B_s^{R_i^{-2}g_M}(m)$, and every $h \in C^0(\overline{B}_{r_1}^{R_i^{-2}g_M}(w))$ satisfying that $h|_{B_{r_1}^{R_i^{-2}g_M}(w)}$ is harmonic and that $h|_{\partial B_{r_1}^{R_i^{-2}g_M}(w)} \leq (\geq) f|_{\partial B_{r_1}^{R_i^{-2}g_M}(w)}$, we also have $h \leq (\geq) f$ on $B_{r_1}^{R_i^{-2}g_M}(w)$. For $g \in C^0(\overline{B}_s^{R_i^{-2}g_M}(m))$, we say that g is a subfunction (superfunction) relative to $f_i|_{B_s^{R_i^{-2}g_M}(m)}$ if $g|_{B_s^{R_i^{-2}g_M}(m)}$ is a subharmonic function (superharmonic function) and $g|_{\partial B_s^{R_i^{-2}g_M}(m)} \leq (\geq) f_i|_{\partial B_s^{R_i^{-2}g_M}(m)}$. Let S_{f_i} denote the set of subfunctions relative to $f_i|_{B_s^{R_i^{-2}g_M}(m)}$. Define a function u_i on $B_s^{R_i^{-2}g_M}(m)$ by $u_i(w) = \sup_{v \in S_{f_i}} v(w)$. By an argument similar to that of the proof of [**28**, Theorem 2.12], it is easy to check that u_i is harmonic on $B_s^{R_i^{-2}g_M}(m)$.

Fix $\tau > 0$, $x \in \partial B_s(p)$ and $z \in \partial B_{2s}(p)$ with $\tau < 2R$ and $\overline{p,x} + \overline{x,z} = \overline{p,z}$. Let $\{x(i)\}_i$, $\{z(i)\}_i$ be sequences of points $x(i) \in \partial B_s^{R_i^{-2}g_M}(m)$, $z(i) \in \partial B_{2s}^{R_i^{-2}g_M}(m)$ with $x(i) \to x$ and $z(i) \to z$. Then it is easy to check that $C_1(n,R)\overline{x,\alpha}^2 \leq \overline{z,\alpha} - \overline{z,x} \leq \overline{x,\alpha}$ for every $\alpha \in B_s(p)$. Fix $\alpha \in B_r(p)$. Let $\{\alpha(i)\}_i$ be a sequence of points $\alpha_i \in B_s^{R_i^{-2}g_M}(m)$ with $\alpha(i) \to \alpha$. Define a function b^i on $B_s^{R_i^{-2}g_M}(m)$ by $b^i = (r_{z(i)}^{R_i^{-2}g_M})^{2-n} - (r_{z(i)}^{R_i^{-2}g_M})^{2-n}(x(i))$. By Laplacian comparison theorems on manifolds, for every sufficiently large i, we see that b^i is a superharmonic, $f_i(x(i)) + 100L\tau + C(n,L,R)b^i/\tau^2$ is a superfunction relative to $f_i|_{\partial B_s^{R_i^{-2}g_M}(m)}$, and that $f_i(x(i)) - 100L\tau - C(n,L,R)b^i/\tau^2$ is a subfunction relative to $f_i|_{\partial B_s^{R_i^{-2}g_M}(m)}$. By an argument similar to that of the proof of [**28**, Lemma 2.13], we have

$$|f_i(x(i)) - u_i(\alpha(i))| \le C(n,R,L)\tau + \frac{C(n,R,L)}{\tau^2}\overline{x(i),\alpha(i)}^{R_i^{-2}g_M}$$

for every sufficiently large i. On the other hand, by Cheng-Yau's gradient estimate and [**38**, Corollary 4.7], without loss of generality we can assume that there exists a harmonic function $\hat{u}_\infty$ on $B_s(p)$ such that $\hat{u}_\infty|_{B_{\hat{s}}(p)}$ is a Lipschitz function and that $u_i \to u_\infty$ on $B_{\hat{s}}(p)$ for every $0 < \hat{s} < s$. Thus we have

$$|u_\infty(x) - \hat{u}_\infty(\alpha)| \le C(n,R,L)\tau + \frac{C(n,R,L)}{\tau^2}\overline{x,\alpha}$$

for every $\alpha \in B_s(p)$. For every $x \in \partial B_s(p)$ and every $\alpha \in B_s(p)$, by letting $\tau = \overline{x,\alpha}^{1/3}$, we have

$$|u_\infty(x) - \hat{u}_\infty(\alpha)| \le C(n,R,L)\overline{x,\alpha}^{1/3}.$$

Since $\hat{u}_\infty \in H_{1,2}(B_{\hat{s}}(p))$ for every $0 < \hat{s} < s$, and that u_∞ is Lipschitz on $\overline{B}_s(p)$, by [**64**, Cororally 6.6], we have $\sup_{B_s(p)} |u_\infty - \hat{u}_\infty| = \lim_{\hat{s}\to s}(\sup_{\partial B_{\hat{s}}(p)} |u_\infty - \hat{u}_\infty|) = 0$. Therefore, we have the assertion. $\square$

From now on, we will replace the most of many important statements about harmonic functions on manifolds given in [**15**] with corresponding statements on asymptotic cones:

PROPOSITION 3.8. *For every $0 < r < s < R$ and every harmonic function u_∞ on $B_R(p)$, we have*

$$D_{u_\infty}(r) \le \left(\frac{r}{s}\right)^{2-n} D_{u_\infty}(s)$$

and

$$I_{u_\infty}(s) - I_{u_\infty}(r) = 2\int_r^s \frac{D_{u_\infty}(t)}{t}dt.$$

Moreover, if $I_{u_\infty}(r) > 0$, then we have

$$I_{u_\infty}(s) = \exp\left(2\int_r^s \frac{U_{u_\infty}(t)}{t}dt\right) I_{u_\infty}(r).$$

PROOF. By Theorem 3.7, without loss of generality, we can assume that the assumption of Proposition 3.4 holds. Since

$$D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r) \le \left(\frac{r}{s}\right)^{2-n} D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(s),$$

by letting $i \to \infty$, Proposition 3.4 yields the first assertion. Similarly, since

$$I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(s) - I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r) = 2\int_r^s \frac{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)}{t} dt,$$

by letting $i \to \infty$ the second assertion follows from Proposition 3.4 and the dominated convergence theorem. Especially, we see that I_{u_∞} is a continuous function and that a monotonicity $I_{u_\infty}(r) \le I_{u_\infty}(s)$ holds.

Finally, we now check the third assertion. By Proposition 3.4 and the monotonicity of I_{u_∞}, we have $\liminf_{i\to\infty}\left(\inf_{\alpha\in[r,s]} I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(\alpha)\right) > 0$. Therefore, Cheng-Yau's gradient estimate and Remark 3.3 yield

$$\limsup_{i\to\infty}\left(\sup_{\alpha\in[r,s]} U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(\alpha)\right) < \infty.$$

On the other hand, since

$$I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(s) = \exp\left(2\int_r^s \frac{U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)}{t} dt\right) I^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r),$$

by letting $i \to 0$, the dominated convergence theorem and Proposition 3.4, we have the third assertion. □

Corollary 3.9. *Let $0 < r < R$ and let u_∞ be a harmonic function on $B_R(p)$. If $U_{u_\infty}(r) = 0$, then u_∞ is a constant function on $B_r(p)$.*

Proof. First, assume $I_{u_\infty}(r) = 0$. Then, by Proposition 3.8, we have $D_{u_\infty}(t) = 0$ for a.e. $0 < t < r$. Since D_{u_∞} is continuous, we have $D_{u_\infty}(r) = 0$. Thus, by the weak Poincaré inequality of type $(1, 2)$ on $C(X)$, we have

$$\begin{aligned}\frac{1}{\upsilon(B_r(p))}\int_{B_r(p)}\left|f - \frac{1}{\upsilon(B_r(p))}\int_{B_r(p)} f d\upsilon\right| d\upsilon \\ \le C(n,R) r\sqrt{\frac{1}{\upsilon(B_r(p))}\int_{B_r(p)} (\operatorname{Lip} f)^2 d\upsilon} = 0.\end{aligned}$$

Since f is Lipschitz on $B_r(p)$, we see that f is a constant function on $B_r(p)$. Next, assume $U_{u_\infty}(r) = 0$ and $I_{u_\infty}(r) > 0$. Then, by the definition, we have $D_{u_\infty}(r) = 0$. Therefore, by the argument above, we have the assertion in this case. □

The following corollary follows directly from Proposition 3.8 and the continuity of a function: $t \mapsto H^n(B_t(p))$.

COROLLARY 3.10. *Let $R > 0$ and let u_∞ be a harmonic function on $B_R(p)$. Then we see that I_{u_∞} is a C^1-function on $(0, R)$. Moreover we have*

$$\frac{dI_{u_\infty}}{dt}(t) = \frac{2D_{u_\infty}(t)}{t}.$$

Let $0 < r < R$ and let u be a harmonic function on $B_R^{g_M}(m)$ with $u \not\equiv 0$ on $B_R^{g_M}(m)$. Put

$$W_u^{g_M}(r) := \frac{E_u^{g_M}(r)}{I_u^{g_M}(r)}.$$

Note that with the same assumption as in Proposition 3.4, if u_∞ is not a constant function on $B_r(p)$, then Proposition 3.4 and Proposition 3.6 yield

$$\lim_{i\to\infty} W_{(u_i)_{R_i}}^{R_i^{-2}g_M}(r) = U_{u_\infty}(r).$$

PROPOSITION 3.11. *Let $0 < r < s < R$ and let u_∞ be a harmonic function on $B_{7R}(p)$. Then we have*

$$U_{u_\infty}(r) \le U_{u_\infty}(s).$$

PROOF. By Theorem 3.7, without loss of generality, we can assume that $U_{u_\infty}(r) > 0$ and that there exists a sequence of harmonic functions u_i on $B_{6RR_i}^{g_M}(m)$ such that $\sup_i \mathbf{Lip}(u_i)_{R_i} < \infty$ and $(u_i)_{R_i} \to u_\infty$ on $B_{6R}(p)$. Fix $\epsilon > 0$. We now use the same notation as in [**15**, Proposition 4.11]. Put $\Omega_0 := s/r$, $\gamma := D_{u_\infty}(2s)/D_{u_\infty}(r) + 1$. Let $\hat{R} := R(m, \gamma, \epsilon, \Omega_0)$ as in [**15**, Proposition 4.11]. By Proposition 3.4, there exists i_0 such that $R_i r > \hat{R}$ and

$$\frac{D_{u_i}^{g_M}(2\Omega_0 R_i r)}{D_{u_i}^{g_M}(R_i r)} = \frac{D_{u_i}^{g_M}(2R_i s)}{D_{u_i}^{g_M}(R_i r)} = \frac{D_{(u_i)_{R_i}}^{R_i^{-2}g_M}(2s)}{D_{(u_i)_{R_i}}^{R_i^{-2}g_M}(r)} \le \gamma$$

for every $i \ge i_0$. Therefore [**15**, Proposition 4.11] yields

$$\int_{R_i r}^{R_i s} \frac{d \log W_{u_i}^{g_M}}{dt} dt \ge -\epsilon,$$

i.e.,

$$\log W_{u_i}^{g_M}(R_i s) - \log W_{u_i}^{g_M}(R_i t) \ge -\epsilon.$$

Since $W_{u_i}^{g_M}(R_i s) = W_{(u_i)_{R_i}}^{R_i^{-2}g_M}(s)$, by letting $i \to \infty$, we have

$$\log U_{u_\infty}(s) - \log U_{u_\infty}(r) \ge -\epsilon.$$

Since ϵ is arbitrary, we have the assertion. $\square$

REMARK 3.12. The most of important results given in [**15**] are about global harmonic functions on M. However, by the proofs, they also hold for harmonic functions on a *big* domain as in the proof of Proposition 3.11. We will often use these facts without an attention.

We recall that $\mathcal{H}^d(M_\infty)$ is the space of harmonic functions u_∞ on M_∞ satisfying that there exists $C > 1$ such that $|u_\infty(x)| \le C(1 + \overline{m_\infty, x}^d)$ for every $x \in M_\infty$. The next proposition follows directly from the proof of [**16**, Lemma 1.29] via Propositions 3.8 and 3.11:

PROPOSITION 3.13. *We have $U_{u_\infty}(t) \le d$ for every $t > 0$ and every $u_\infty \in \mathcal{H}^d(M_\infty)$.*

PROPOSITION 3.14. *Let $0 < s < t < \alpha < R$ and let u_∞ be a harmonic function on $B_{7R}(p)$. Then we have*

$$I_{u_\infty}(t) \le \left(\frac{t}{s}\right)^{2U_{u_\infty}(\alpha)} I_{u_\infty}(s).$$

PROOF. First, assume that u_∞ is not a constant function on $B_s(p)$. By Theorem 3.7, without loss of generality, we can assume that there exists a sequence of harmonic functions u_i on $B^{g_M}_{6RR_i}(m)$ such that $\sup_i \mathbf{Lip}(u_i)_{R_i} < \infty$ and $(u_i)_{R_i} \to u_\infty$ on $B_{6R}(p)$. Fix $\epsilon > 0$. By the assumption and Corollary 3.9, there exists $0 < r < s$ such that $U_{u_\infty}(r) > 0$. We now apply [**15**, Corollary 4.37]. Put $\Omega_0 := 2\alpha/r$, $\Omega = \alpha/r$ and $\gamma := D_{u_\infty}(2\Omega r)/D_{u_\infty}(r) + 1$. Let $\hat{R} := R(m, \gamma, \epsilon, \Omega_0)$ as in [**15**, Corollary 4.37]. There exists i_0 such that $R_i r > \hat{R}$ and

$$\frac{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(2\Omega r)}{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r)} < \gamma$$

for every $i \ge i_0$. Thus [**15**, Corollary 4.37] yields

$$I^{g_M}_{u_i}(R_i t) \le \left(\frac{R_i t}{R_i s}\right)^{2(1+\epsilon)W^{g_M}_{u_i}(\Omega R_i r)} I^{g_M}_{u_i}(R_i s).$$

Thus by letting $i \to \infty$ and $\epsilon \to 0$, we have the assertion.

Next assume that u_∞ is a constant function on $B_s(p)$. Put $\hat{s} := \sup\{\beta \in [0, R]; u_\infty$ is a constant function on $B_\beta(p)\}$. If $\hat{s} \ge t$, then, since $I_{u_\infty}(t) = I_{u_\infty}(s)$, we have the assertion. Thus assume $\hat{s} < t$. Let $\tilde{s} > 0$ with $\hat{s} < \tilde{s} < t$. Then, by the argument above, we have

$$I_{u_\infty}(t) \le \left(\frac{t}{\tilde{s}}\right)^{2U_{u_\infty}(\alpha)} I_{u_\infty}(\tilde{s}).$$

Since $s \le \hat{s}$ and $I_{u_\infty}(s) = I_{u_\infty}(\hat{s})$, by letting $\tilde{s} \to \hat{s}$, we have the assertion. $\square$

Corollary 3.15. *Let $0 < s < R$ and let u_∞ be a harmonic function on $B_{7R}(p)$. Assume $U_{u_\infty}(s) = 0$. Then u_∞ is a constant function on $B_R(p)$.*

Proof. First, assume that $I_{u_\infty}(s) = 0$. Then, by Proposition 3.14, we have $I_{u_\infty}(t) = 0$ for every $s < t < R$. Therefore, the assertion follows from Proposition 3.9.

Next, assume $I_{u_\infty}(s) > 0$ and $U_{u_\infty}(s) = 0$. If we put $\hat{u}_\infty := u_\infty - u_\infty(p)$, then $\hat{u}_\infty \equiv 0$ on $B_s(p)$. Since $I_{\hat{u}_\infty}(s) = 0$, by the argument above, we have the assertion. $\square$

Proposition 3.16. *Let $R > 0$ and let u_∞ be a harmonic function on $B_{7R}(p)$ with $u_\infty(p) = 0$. Assume $u \not\equiv 0$ on $B_R(p)$. Then, we have*

$$U_{u_\infty}(s) \geq 1$$

for every $0 < s < R$.

Proof. By Theorem 3.7, without loss of generality, we can assume that there exists a sequence of harmonic functions u_i on $B^{g_M}_{6RR_i}(m)$ such that $\sup_i \mathbf{Lip}(u_i)_{R_i} < \infty$, $(u_i)_{R_i} \to u_\infty$ on $B_{6R}(p)$, and $(u_i)_{R_i}(m) = 0$. Note that by Corollary 3.15, we have $U_{u_\infty}(r) > 0$ for every $0 < r < R$. Fix a sufficiently small $\epsilon > 0$. We now apply [**15**, Corollary 4.40]. Let $\Omega_L := \Omega_L(n, \epsilon) \geq 2$ as in [**15**, Corollary 4.40] (or [**15**, Corollary 3.29]). Put $\Omega_0 := 5\Omega_L$, $r := s/2(2\Omega_L)^2 < s$ and $\gamma := D_{u_\infty}(s)/D_{u_\infty}(r) + 1$. Let $\hat{R} := R(m, \gamma, \epsilon, \Omega_0)$ as in [**15**, Corollary 4.40]. Then there exists i_0 such that $R_i r > \hat{R}$ and

$$\frac{D^{g_M}_{u_i}(2(2\Omega_L)^2 R_i r)}{D^{g_M}_{u_i}(R_i r)} = \frac{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(2(2\Omega_L)^2 r)}{D^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r)} \leq \gamma$$

for every $i \geq i_0$. Then [**15**, Corollary 4.40] yields

$$1 - 3\epsilon \leq U^{g_M}_{u_i}(2\Omega_L R_i r) = U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(2\Omega_L r).$$

By letting $i \to \infty$, Proposition 3.4 and Proposition 3.11, we have $1 - 3\epsilon \leq U_{u_\infty}(2\Omega_L r) \leq U_{u_\infty}(s)$. Since ϵ is arbitrary, we have the assertion. $\square$

Proposition 3.17. *Let $0 < r < s < R$, $\delta > 0, d_0 > 0$ and let u_∞ be a harmonic function on $B_{7R}(p)$. Assume that $U_{u_\infty}(s) \leq d_0$,*

$$\left| \log \frac{U_{u_\infty}(s)}{U_{u_\infty}(r)} \right| < \delta$$

and that u_∞ is not a constant function on $B_R(p)$. Then, we have

$$\int_{A_p(r,s)} r_p^{-n} \left| r_p \langle dr_p, du_\infty \rangle - U_{u_\infty}(r_p) u_\infty \right|^2 dH^n \leq \Psi(\delta; n, d_0) I_{u_\infty}(s).$$

PROOF. By Theorem 3.7, without loss of generality, we can assume that there exists a sequence of harmonic functions u_i on $B^{g_M}_{6RR_i}(m)$ such that $\sup_i \mathbf{Lip}(u_i)_{R_i} < \infty$ and $(u_i)_{R_i} \to u_\infty$ on $B_{6R}(p)$. We now apply [**15**, Proposition 4.50]. Put $\Omega_0 := 2s/r$, $\Omega := s/r$ and $\gamma := D_{u_\infty}(2\Omega r)/D_{u_\infty}(r) + 1$ as in [**15**, Proposition 4.50]. Then, by Proposition 3.4, there exists i_0 such that

$$\frac{D^{R_i^{-2}g_M}_{u_\infty}(2\Omega r)}{D^{R_i^{-2}g_M}_{u_\infty}(r)} \le \gamma, \quad \max_{r \le t \le \Omega r} U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t) \le 2d_0$$

and

$$\left| \log \frac{U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(\Omega r)}{U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r)} \right| \le \delta$$

for every $i \ge i_0$. Thus, by [**15**, Proposition 4.50], we have

$$\int_{rR_i \le b^{g_M} \le sR_i} (b^{g_M})^{-n} \left(b^{g_M} \frac{\partial u_i}{\partial n} - U^{g_M}_{u_i}(b^{g_M}) |\nabla^{g_M} b^{g_M}| \right)^2 d\,\mathrm{vol}^{g_M} \le \Psi(\delta; n, d_0) I^{g_M}_{u_i}(R_i s)$$

for every sufficiently large i. On the other hand, Cheng-Yau's gradient estimate yields

$$\begin{aligned}
|\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}| &= \frac{V_M}{(n-2)\,\mathrm{vol}\, B_1(0_n)} |b^{R_i^{-2}g_M}|^{n-1} |\nabla^{R_i^{-2}g_M} G^{R_i^{-2}g_M}(m,\cdot)| \\
&\le \frac{V_M}{(n-2)\,\mathrm{vol}\, B_1(0_n)} 2\big(r_m^{R_i^{-2}g_M}\big)^{n-1} C(n) \big(r_m^{R_i^{-2}g_M}\big)^{-1} |G^{R_i^{-2}g_M}(m,\cdot)| \\
&\le C(n) \big(r_m^{R_i^{-2}g_M}\big)^{-1} \big(r_m^{R_i^{-2}g_M}\big)^{n-1} \big(r_m^{R_i^{-2}g_M}\big)^{2-n} \\
&\le C(n)
\end{aligned}$$

on $A^{R_i^{-2}g_M}_m(r,s)$ for every sufficiently large i. Thus by [**38**, Corollary 4.7] and Theorem 3.1, we have $(b^{R_i^{-2}g_M}, db^{R_i^{-2}g_M}) \to (r_p, dr_p)$ on $A_p(r,s)$. On the other hand, note that

$$\begin{aligned}
&\int_{r \le b^{R_i^{-2}g_M} \le s} \big(b^{R_i^{-2}g_M}\big)^{-n} \Big(b^{R_i^{-2}g_M}(R_i^{-2}g_M) \big(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}\big) \\
&\qquad\qquad - U^{R_i^{-2}g_M}_{(u_i)_{R_i}}(b^{R_i^{-2}g_M}) |\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|^2 \Big)^2 d\,\mathrm{vol}^{R_i^{-2}g_M} \\
&= \int_{rR_i \le b^{g_M} \le sR_i} (b^{g_M})^{-n} |\nabla b^{g_M}|^2 \left(b^{g_M} \frac{\partial u_i}{\partial n} - U^{g_M}_{u_i}(b^{g_M}) |\nabla^{g_M} b^{g_M}| \right)^2 d\,\mathrm{vol}^{g_M} \\
&\le C(n) \int_{rR_i \le b^{g_M} \le sR_i} (b^{g_M})^{-n} \left(b^{g_M} \frac{\partial u_i}{\partial n} - U^{g_M}_{u_i}(b^{g_M}) |\nabla^{g_M} b^{g_M}| \right)^2 d\,\mathrm{vol}^{g_M}
\end{aligned}$$

$$\leq \Psi(\delta; n, d_0) I_{u_i}^{g_M}(R_i s) = \Psi(\delta; n, d_0) I_{(u_i)_{R_i}}^{R_i^{-2} g_M}(s)$$

for every sufficiently large i. Therefore, by letting $i \to \infty$, [**38**, Corollary 4.4] and Proposition 3.4 yield the assertion. $\square$

The following corollary follows directly from Proposition 3.17.

Corollary 3.18. *Let r, s, R be positive numbers and let u_∞ be a harmonic function on $B_{7R}(p)$ with $r < s < R$ and $u_\infty(p) = 0$. Assume $U_{u_\infty}(r) = U_{u_\infty}(s)$. Then we have*

$$r_p(w)\langle du_\infty, dr_p\rangle(w) = U_{u_\infty}(s) u_\infty(w)$$

for a.e $w \in A_p(r, s)$.

Proposition 3.19. *With the same assumption as in Corollary 3.18, we have*

$$u_\infty(\hat{t}, x) = \frac{u_\infty(t, x)}{t^C} \hat{t}^C$$

for every $r \leq t \leq \hat{t} \leq s$ and every $x \in X$, where $C = U_{u_\infty}(r)$.

Proof. Define a Borel function a on $A_p(r, s)$ by

$$a(t, x) := \limsup_{h \to 0} \frac{u_\infty(t + h, x) - u_\infty(t, x)}{h}.$$

By [**38**, Theorem 3.3] and Corollary 3.18, there exists a Borel subset A of $A_p(r, s)$ such that $H^n(A_p(r, s) \setminus A) = 0$ and $\langle dr_p, du_\infty\rangle(z) = a(z) = Cu_\infty(z)/r_p(z)$ for every $z \in A$. Fix r_0, s_0 with $0 < s \leq r_0 \leq s_0 \leq s$ and define a bi-Lipschitz map ϕ from $A_p(r_0, s_0)$ to $[r_0, s_0] \times X$ by $\phi(t, x) = (t, x)$. Then we have $H^n([r_0, s_0] \times X \setminus \phi(A)) = 0$. Therefore by Fubini's theorem, there exists a Borel subset $\hat{X}$ of X such that $H^{n-1}(X \setminus \hat{X}) = 0$ and $H^1([r_0, s_0] \times \{x\} \setminus \phi(A)) = 0$ for every $x \in X$. Thus we have $H^1(\phi^{-1}([r_0, s_0] \times \{x\} \setminus \phi(A))) = 0$ for $x \in \hat{X}$. For every $x \in \hat{X}$, Rademacher's theorem yields

$$\begin{aligned} u_\infty(s_0, x) - u_\infty(r_0, x) &= \int_{r_0}^{s_0} a(t, x) dt \\ &= \int_{r_p(\phi^{-1}([r_0, s_0] \times \{x\} \cap \phi(A)))} a(t, x) dt \\ &= \int_{r_p(\phi^{-1}([r_0, s_0] \times \{x\} \cap \phi(A)))} \frac{Cu_\infty(t, x)}{t} dt \\ &= \int_{r_0}^{s_0} \frac{Cu_\infty(t, x)}{t} dt. \end{aligned}$$

Note that for every $x \in X$, there exists a sequence of points x_i in $\hat{X}$ such that $x_i \to x$.

Therefore by the dominated convergence theorem, the equality above holds for every $x \in X$. Thus, for every $x \in X$, we see that a function $f_x(\tilde{t}) = u_\infty(\tilde{t}, x)$ on $[r, s]$ is a C^1-function. Moreover we have

$$\frac{df_x}{d\tilde{t}}(\tilde{t}) = \frac{Cf_x(\tilde{t})}{\tilde{t}}.$$

Therefore, we have the assertion. □

PROPOSITION 3.20. *Let r, s, δ, R, d_0 be positive numbers with $0 < r < s < R$, and u_∞, v_∞ harmonic functions on $B_{7R}(p)$. Assume that $\max_{r \le t \le s} U_{v_\infty}(t) \le d_0$,*

$$\left| \log \frac{U_{v_\infty}(s)}{U_{v_\infty}(r)} \right| < \delta$$

and that v_∞ is not a constant function on $B_R(p)$. Then, we have

$$\left| s_0^{1-n} \int_{\partial B_{s_0}(p)} u_\infty v_\infty dH^{n-1} - \exp\left(2 \int_{r_0}^{s_0} \frac{U_{v_\infty}(\hat{s})}{\hat{s}} d\hat{s}\right) r_0^{1-n} \int_{\partial B_{r_0}(p)} u_\infty v_\infty dH^{n-1} \right|^2$$
$$\le \Psi(\delta; n, d_0) \left(\frac{s_0}{r_0}\right)^{6d_0+3} I_{u_\infty}(s_0) I_{v_\infty}(s_0)$$

for every $r \le r_0 \le s_0 \le s$.

PROOF. By Theorem 3.7, without loss of generality, we can assume that there exist sequences of harmonic functions u_i, v_i on $B^{g_M}_{6RR_i}(m)$ such that $\sup_i(\mathbf{Lip}(u_i)_{R_i} + \mathbf{Lip}(v_i)_{R_i}) < \infty$, $(u_i)_{R_i} \to u_\infty$, $(v_i)_{R_i} \to v_\infty$ on $B_{6R}(p)$. By the proof of Proposition 3.17, there exists i_0 such that

$$\int_{rR_i \le b^{g_M} \le sR_i} (b^{g_M})^{-n} \left(b^{g_M} \frac{\partial v_i}{\partial n} - U^{g_M}_{v_i}(b^{g_M}) |\nabla^{g_M} b^{g_M}| \right)^2 d\,\mathrm{vol}^{g_M} \le \Psi(\delta; n, d_0) I^{g_M}_{v_i}(R_i s)$$

for every $i \ge i_0$. Thus, [**15**, Corollary 5.24] yields

$$\Bigg| (R_i s_0)^{1-n} \int_{b^{g_M} = R_i s_0} u_i v_i d\,\mathrm{vol}^{g_M}_{n-1}$$
$$- \exp\left(2 \int_{r_0 R_i}^{s_0 R_i} \frac{U^{g_M}_{v_i}(\hat{s})}{\hat{s}} d\hat{s}\right) (R_i r_0)^{1-n} \int_{b^{g_M} = R_i r_0} u_i v_i d\,\mathrm{vol}^{g_M}_{n-1} \Bigg|^2$$
$$\le \Psi(\delta; n, d_0) \left(\frac{s_0}{r_0}\right)^{6d_0+3} I^{g_M}_{u_i}(R_i s_0) I^{g_M}_{v_i}(R_i s_0)$$

for every $i \ge i_0$. Thus, we have

$$\Bigg| s_0^{1-n} \int_{b^{R_i^{-2}g_M}=s_0} (u_i)_{R_i}(v_i)_{R_i}\, d\,\mathrm{vol}_{n-1}^{R_i^{-2}g_M}$$

$$- \exp\left(2\int_{r_0}^{s_0} \frac{U_{(v_i)_{R_i}}^{R_i^{-2}g_M}(\hat{s})}{\hat{s}} d\hat{s}\right) r_0^{1-n} \int_{b^{R_i^{-2}g_M}=r_0} (u_i)_{R_i}(v_i)_{R_i}\, d\,\mathrm{vol}_{n-1}^{R_i^{-2}g_M} \Bigg|^2$$

$$\le \Psi(\delta; n, d_0)\left(\frac{s_0}{r_0}\right)^{6d_0+3} I_{(u_i)_{R_i}}^{R_i^{-2}g_M}(s_0) I_{(v_i)_{R_i}}^{R_i^{-2}g_M}(s_0).$$

On the other hand, Proposition 3.4 yields

$$\int_{b^{R_i^{-2}g_M}=s_0} (u_i)_{R_i}(v_i)_{R_i}\, d\,\mathrm{vol}_{n-1}^{R_i^{-2}g_M}$$

$$= \frac{1}{2}\int_{b^{R_i^{-2}g_M}=s_0} ((u_i)_{R_i} + (v_i)_{R_i})^2\, d\,\mathrm{vol}_{n-1}^{R_i^{-2}g_M} - \frac{1}{2}\int_{b^{R_i^{-2}g_M}=s_0} (u_i)_{R_i}^2\, d\,\mathrm{vol}_{n-1}^{R_i^{-2}g_M}$$

$$- \frac{1}{2}\int_{b^{R_i^{-2}g_M}=s_0} (v_i)_{R_i}^2\, d\,\mathrm{vol}_{n-1}^{R_i^{-2}g_M}$$

$$\xrightarrow{i\to\infty} \frac{1}{2}\int_{\partial B_{s_0}(p)} (u_\infty + v_\infty)^2 dH^{n-1} - \frac{1}{2}\int_{\partial B_{s_0}(p)} u_\infty^2 dH^{n-1} - \frac{1}{2}\int_{\partial B_{s_0}(p)} v_\infty^2 dH^{n-1}$$

$$= \int_{\partial B_{s_0}(p)} u_\infty v_\infty dH^{n-1}.$$

Therefore we have the assertion. □

The following is a direct consequence of Proposition 3.20:

COROLLARY 3.21. *Let r, s, R be positive numbers with $0 < r < s < R$, and u_∞, v_∞ harmonic functions on $B_{7R}(p)$. Assume that $U_{v_\infty}(r) = U_{v_\infty}(s)$ and that v_∞ is not a constant function on $B_R(p)$. Then, we have*

$$s_0^{1-n}\int_{\partial B_{s_0}(p)} u_\infty v_\infty dH^{n-1} = \left(\frac{s_0}{r_0}\right)^{2C} r_0^{1-n}\int_{\partial B_{r_0}(p)} u_\infty v_\infty dH^{n-1}$$

for every $r \le r_0 \le s_0 \le s$, where $C = U_{v_\infty}(r)$.

We now consider a convergence of F:

PROPOSITION 3.22. *With the same assumption as in Proposition 3.4, we have*

$$\lim_{i\to\infty}\int_r^s F_{(u_i)_{R_i}}^{R_i^{-2}g_M}(t)dt = \int_r^s F_{u_\infty}(t)dt$$

for every $0 < r < s < R$.

Proof. Since $(b^{R_i^{-2}g_M}, db^{R_i^{-2}g_M}) \to (r_p, dr_p)$ on $A_p(r,s)$, by [**38**, Corollary 4.5], we have

$$\begin{aligned}
&\int_r^s F^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)dt \\
&\quad = \int_r^s t^{3-n} \int_{b^{R_i^{-2}g_M}=t} (R^{-2}g_M)\left(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \frac{\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}}{|\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|}\right)^2 \\
&\qquad \times \left|\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}\right| d\,\mathrm{vol}^{R_i^{-2}g_M}\, dt \\
&\quad = \int_{r \le b^{R_i^{-2}g_M} \le s} (R^{-2}g_M)\left(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \frac{\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}}{|\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}|}\right)^2 \\
&\qquad \times \left|\nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}\right|^2 (b^{R_i^{-2}g_M})^{3-n} d\,\mathrm{vol}^{R_i^{-2}g_M} \\
&\quad = \int_{r \le b^{R_i^{-2}g_M} \le s} (R^{-2}g_M)\big(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \nabla^{R_i^{-2}g_M} b^{R_i^{-2}g_M}\big)^2 (b^{R_i^{-2}g_M})^{3-n} d\,\mathrm{vol}^{R_i^{-2}g_M} \\
&\quad \xrightarrow{i\to\infty} \int_{A_p(r,s)} r_p^{3-n} \langle du_\infty, dr_p\rangle^2 d\upsilon = \int_r^s F_{u_\infty}(t)dt. \qquad \square
\end{aligned}$$

Proposition 3.23. *Let $0 < r < s < R$ and let u_∞ be a harmonic function on $B_{7R}(p)$. Then we have*

$$D_{u_\infty}(s) - D_{u_\infty}(r) = \int_r^s \frac{2F_{u_\infty}(t)}{t} dt.$$

Proof. Without loss of generality, we can assume that the assumption of Proposition 3.4 holds. By (4.3) in [**15**], we have

$$\begin{aligned}
&E^{R_i^{-2}g_M}_{(u_i)_{R_i}}(s) - E^{R_i^{-2}g_M}_{(u_i)_{R_i}}(r) \\
&\quad = \int_r^s \frac{2F^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)}{t} dt + \int_r^s \frac{2E^{R_i^{-2}g_M}_{(u_i)_{R_i}}(t)}{t} dt \\
&\qquad - \int_r^s t^{1-n} \int_{b^{R_i^{-2}g_M} \le t} 2\left|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\right|^2 d\,\mathrm{vol}^{R_i^{-2}g_M}\, dt \\
&\qquad \pm \int_r^s t^{1-n} \int_{b^{R_i^{-2}g_M} \le t} \Big| \mathrm{Hess}^{R_i^{-2}g_M}_{(b^{R_i^{-2}g_M})^2} \big(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \nabla^{R_i^{-2}g_M}(u_i)_{R_i}\big) \\
&\qquad\qquad - 2(R_i^{-2}g_M)\big(\nabla^{R_i^{-2}g_M}(u_i)_{R_i}, \nabla^{R_i^{-2}g_M}(u_i)_{R_i}\big)\Big| d\,\mathrm{vol}^{R_i^{-2}g_M}\, dt.
\end{aligned}$$

By [**38**, Corollary 4.5] and Theorem 3.1, we have

$$\lim_{i\to\infty}\int_{b^{R_i^{-2}g_M}\le t}|d(u_i)_{R_i}|^2 d\,\mathrm{vol}^{R_i^{-2}g_M}=\int_{B_t(p)}|du_\infty|^2 dH^n.$$

By the dominated convergence theorem, we have

$$\begin{aligned}\lim_{i\to\infty}\int_r^s t^{1-n}\int_{b^{R_i^{-2}g_M}\le t}2\big|\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\big|^2 d\,\mathrm{vol}^{R_i^{-2}g_M}\,dt&=\int_r^s t^{1-n}\int_{B_t(p)}2|du_\infty|^2 dH^2 dt\\&=\int_r^s\frac{2E_{u_\infty}(t)}{t}dt.\end{aligned}$$

On the other hand, since

$$\lim_{R\to\infty}\frac{1}{\mathrm{vol}^{g_M}(\{b^{g_M}\le R\})}\int_{b^{g_M}\le R}|\operatorname{Hess}_{(b^{g_M})^2}-2g_M|d\,\mathrm{vol}^{g_M}=0,$$

we have

$$\begin{aligned}\lim_{i\to\infty}\int_{b^{R_i^{-2}g_M}\le t}\Big|&\operatorname{Hess}^{R_i^{-2}g_M}_{(b^{R_i^{-2}g_M})^2}\big(\nabla^{R_i^{-2}g_M}(u_i)_{R_i},\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\big)\\&-2(R_i^{-2}g_M)\big(\nabla^{R_i^{-2}g_M}(u_i)_{R_i},\nabla^{R_i^{-2}g_M}(u_i)_{R_i}\big)\Big|d\,\mathrm{vol}^{R_i^{-2}g_M}\,dt=0.\end{aligned}$$

Therefore we have the assertion. $\square$

We now give a short review of important results given by Ding in [**20**], [**21**]. We denote the differential of a Lipschitz function f on X by $d_X f$. For every $i\ge 0$, let ϕ_i be an i-th eigenfunction of the Laplacian Δ_X associated with the i-th eigenvalue $\lambda_i=\lambda_i(X)\ge 0$: $\Delta_X\phi_i=\lambda_i\phi_i, 0=\lambda_0<\lambda_1\le\lambda_2\le\cdots$. Define $\alpha_i\ge 0$ by satisfying $\lambda_i=\alpha_i(\alpha_i+n-2)$. According to [**21**], we see that $v_i(r,x)=r^{\alpha_i}\phi_i(x)$ is harmonic on $C(X)$ for every i. In fact, by [**20**, Theorem 4.15] and Proposition 7.6, for every $f\in\mathcal{K}(C(X)\setminus\{p\})$, we have

$$\begin{aligned}&\int_{C(X)}\langle df,dv_i\rangle dH^n\\&=\int_0^\infty\int_{\partial B_r(p)}\Big(-\alpha_i(\alpha_i-1)r^{\alpha_i-2}f\phi_i-\frac{n-1}{r}\alpha_i r^{\alpha_i-1}+\frac{1}{r^2}\langle d_X f,d_X\phi_i\rangle\Big)dH^{n-1}dr\\&=\int_0^\infty\int_{\partial B_r(p)}\Big(-\alpha_i(\alpha_i-1)r^{\alpha_i-2}f\phi_i-(n-1)\alpha_i r^{\alpha_i-2}f\phi_i+\lambda_i r^{\alpha_i-2}f\phi_i\Big)dH^{n-1}dr\\&=0.\end{aligned}$$

Thus, v_i is harmonic on $C(X)\setminus\{p\}$. Therefore [**20**, Corollary 4.25] yields that v_i is harmonic on $C(X)$. By Theorem 3.7, we see that v_i is locally Lipschitz. Especially, we see that ϕ_i is Lipschitz and that $\lambda_1\ge n-1$. See [**21**, Corollary 2.4] and [**21**, Corollary 2.5] for the details. On the other hand, it is easy to check $U_{v_i}(s)=\alpha_i$ for every $s>0$.

We say that v_i *is a homogeneous harmonic function with growth* α_i. Our goal in the following argument is to show that *we can apply* [**20**, Theorem 4.15] *for every* $d \geq 0$ *and every* $u_\infty \in \mathcal{H}^d(M_\infty)$. As a corollary, we will prove Theorem 1.1.

Let $\operatorname{ord}_\infty u_\infty := \lim_{r\to\infty} U_{u_\infty}(r)$ and $\operatorname{ord}_0 u_\infty := \lim_{r\to 0} U_{u_\infty}(r)$ for every harmonic function u_∞ on $C(X)$. Then the following proposition follows directly from Proposition 3.16:

Proposition 3.24. *For every non-constant harmonic function* u_∞ *on* $C(X)$ *with* $u_\infty(p) = 0$, *we have*

$$\operatorname{ord}_0 u_\infty \geq 1.$$

By an argument similar to that of the proof of [**15**, Lemma 1.36], we have the following proposition:

Proposition 3.25. *Let* u_∞, v_∞ *be harmonic functions on* $C(X)$. *Then we have*

$$\operatorname{ord}_\infty(u_\infty + v_\infty) \leq \max\{\operatorname{ord}_\infty u_\infty, \operatorname{ord}_\infty v_\infty\}.$$

Definition 3.26. Let u_∞, v_∞ be harmonic functions on $C(X)$. We say that u_∞ *and* v_∞ *are orthogonal* if

$$\int_{\partial B_1(p)} u_\infty v_\infty dv = 0.$$

The proof of the next proposition is little more delicate than that of [**15**, Lemma 1.49]:

Proposition 3.27. *Let* u_∞ *be a harmonic function on* $C(X)$. *Assume that* $\operatorname{ord}_\infty u_\infty = d < \infty$ *and that* v *and* u_∞ *are orthogonal for every homogeneous harmonic function* v *with growth* α *satisfying* $\alpha < d$. *Then, we have*

$$D_{u_\infty}(s) \geq \left(\frac{s}{r}\right)^{2d} D_{u_\infty}(r)$$

for every $0 < r < s < \infty$.

Proof. Corollary 3.21 yields

$$\int_{\partial B_t(p)} v_i u_\infty dH^{n-1} = 0$$

for every $t > 0$ and every $\alpha_i < d$. Let $\lambda := d(d + n - 2)$. Note that $\alpha_i < d$ holds if and only if $\lambda_i < \lambda$ holds. Let $i_d := \max\{i \in \boldsymbol{N} | \alpha_i < d\}$. Thus, we have $\lambda_{i_d} < \lambda \leq \lambda_{i_d+1}$. Note

$$\lambda_{i_d+1} = \inf\left\{ \frac{\int_X |d_X u|^2 dH^{n-1}}{\int_X u^2 dH^{n-1}} \middle| u \in H_{1,2}(X),\ u \neq 0, \right.$$
$$\left. \int_X u\phi_j dH^{n-1} = 0 \text{ for every } 0 \le j \le i_d \right\}.$$

Since the k-th eigenvalue λ_k^t of $\Delta_{\partial B_t(p)}$ is equal to $t^{-2}\lambda_k$, we have

$$\frac{\int_{\partial B_t(p)} |d_{\partial B_t(p)} u_\infty|^2 dH^{n-1}}{\int_{\partial B_t(p)} (u_\infty)^2 dH^{n-1}} \ge \frac{\lambda}{t^2},$$

where $d_{\partial B_t(p)} f$ is the differential of a Lipshitz function $f|_{\partial B_t(p)}$. On the other hand, by [**38**, Theorem 3.21] and Proposition 7.6, for a.e. $t > 0$, we have $|du_\infty|^2(w) = (\langle dr_p, du_\infty\rangle(w))^2 + |d_{\partial B_r(p)} u_\infty|^2(w)$ for a.e. $w \in \partial B_t(p)$. Therefore, we have

$$\int_{\partial B_t(p)} (|du_\infty|^2 - \langle dr_p, du_\infty\rangle^2) dH^{n-1} \ge \frac{\lambda}{t^2} \int_{\partial B_t(p)} u_\infty^2 dH^{n-1}$$

i.e.

$$t^{3-n} \int_{\partial B_t(p)} |du_\infty|^2 dH^{n-1} - F_{u_\infty}(t) \ge \lambda I_{u_\infty}(t)$$

for a.e. $t > 0$. We now use the notation: $f' := df/dt$ for a locally Lipschitz function f on $\boldsymbol{R}$. By Proposition 3.23, we see that D_{u_∞} is a locally Lipschitz function on $(0, \infty)$. Proposition 7.6 and Rademacher's theorem yield

$$D'_{u_\infty}(t) = (2-n)t^{1-n} \int_{B_t(p)} (\mathrm{Lip}\, u_\infty)^2 dH^n + t^{2-n} \int_{\partial B_t(p)} (\mathrm{Lip}\, u_\infty)^2 dH^{n-1}$$

for a.e. $t > 0$. Therefore, we have

$$tD'_{u_\infty}(t) - (2-n)D_{u_\infty}(t) - F_{u_\infty}(t) \ge \lambda I_{u_\infty}(t)$$

for a.e. $t > 0$. On the other hand, Proposition 3.23 yields $D'_{u_\infty}(t) = 2F_{u_\infty}(t)/t$ for a.e. $t > 0$. Therefore, we have

$$\frac{t}{2} D'_{u_\infty}(t) - (2-n)D_{u_\infty}(t) \ge \lambda I_{u_\infty}(t)$$

for a.e. $t > 0$. Thus we have

$$\frac{D'_{u_\infty}(t)}{D_{u_\infty}(t)} - \frac{2(2-n)}{t} \ge \frac{2\lambda I_{u_\infty}(t)}{tD_{u_\infty}(t)} \ge \frac{2\lambda}{dt}$$

for a.e. $t > 0$. Therefore, we have

$$\begin{aligned}\frac{D'_{u_\infty}(t)}{D_{u_\infty}(t)} &\geq \frac{1}{t}\left(\frac{2\lambda}{d}+2(2-n)\right)\\ &=\frac{1}{t}\frac{2\lambda+4d-2nd}{d}\\ &=\frac{1}{t}\frac{2d(d+n-2)+4d-2nd}{d}\\ &=\frac{2d}{t}\end{aligned}$$

for a.e. $t > 0$. Integrating the both sides of the inequality above on $[r, s]$ yields the assertion. □

PROPOSITION 3.28. *Let g be a Lipschitz function on X and f a C^2-function on $\boldsymbol{R}_{>0}$. Assume that $f(1) = 1$, $\lim_{r\to 0} f(r) = 0$, $g \not\equiv 0$ and that a function $u(r, x) = f(r)g(x)$ on $C(X) \setminus \{p\}$ is locally Lipschitz and harmonic. Then there exists $\lambda > 0$ such that $\lambda \geq n-1$, $\Delta_X g = \lambda g$ and $f(r) = r^q$ for every $r > 0$, where $q \geq 0$ satisfying $\lambda = q(q+n-2)$.*

PROOF. Let $g = \sum_{i=1}^{\infty} a_i\phi_i$ in $H_{1,2}(X)$. For every function h on X, define a function h^r on $\partial B_r(p)$ by $h^r(r, x) = h(x)$. Let ϕ_i^r be an i-th eigenfunction of $\Delta_{\partial B_r(p)}$ on $\partial B_r(p)$ associated with the eigenvalue λ_i^r. It is clear that $g^r = \sum_{i=1}^{\infty} a_i\phi_i^r$ in $H_{1,2}(\partial B_r(p))$, $\Delta_{\partial B_r(p)}\phi_i^r = \lambda_i^r\phi_i^r$ and $\lambda_i^r = r^{-2}\lambda_i$. [**20**, Theorem 4.15] and [**38**, Corollary 4.7] yield

$$\begin{aligned}0 &= \int_{C(X)} \langle du, d\phi\rangle dH^n\\ &= \int_0^\infty \int_{\partial B_r(p)} \left(\phi\left(-\frac{d^2f}{dr^2}(r)g(x) - \frac{n-1}{r}\frac{df}{dr}(r)g(x)\right)\right.\\ &\qquad\qquad\qquad \left. + \langle d_{\partial B_r(p)}\phi, d_{\partial B_r(p)}g^r\rangle f(r)\right) dH^{n-1}dr\\ &= \int_0^\infty \int_{\partial B_r(p)} \phi\left(-\frac{d^2f}{dr^2}(r)g(x) - \frac{n-1}{r}\frac{df}{dr}(r)g(x) + f(r)\sum_{i=1}^{\infty} a_i\lambda_i^r\phi_i^r\right) dH^{n-1}dr\end{aligned}$$

for every $\phi \in \mathcal{K}(C(X) \setminus \{p\})$. Therefore, we have

$$\int_0^\infty a(r)\int_{\partial B_r(p)} b(x)\left(-\frac{d^2f}{dr^2}(r)g(x) - \frac{n-1}{r}\frac{df}{dr}(r)g(x) + f(r)\sum_{i=1}^{\infty} a_i\lambda_i^r\phi_i^r\right) dH^{n-1}dr = 0$$

for every $a \in \mathcal{K}(\boldsymbol{R}_{>0})$ and every Lipschitz function b on X. Since

$$\sum_{i=1}^{\infty}(\lambda_i^r)^2 a_i^2 \int_{\partial B_r(p)} (\phi_i^r)^2 dH^{n-1} = \int_{\partial B_r(p)} |d_{\partial B_r(p)}g^r|^2 dH^{n-1} < \infty,$$

a function

$$-\frac{d^2f}{dr^2}(r)g(x)-\frac{n-1}{r}\frac{df}{dr}(r)g(x)+f(r)\sum_{i=1}^{\infty}a_i\lambda_i^r\phi_i^r$$

on $\partial B_r(p)$ is in $L^2(\partial B_r(p))$. Since the space of Lipschitz functions on $\partial B_r(p)$ is dence in $L^2(\partial B_r(p))$, we have

$$\begin{aligned} 0&=\int_0^{\infty}a(r)\int_{\partial B_r(p)}\left|-\frac{d^2f}{dr^2}(r)g(x)-\frac{n-1}{r}\frac{df}{dr}(r)g(x)+f(r)\sum_{i=1}^{\infty}a_i\lambda_i^r\phi_i^r\right|^2dH^{n-1}dr\\ &=\int_0^{\infty}a(r)\int_{\partial B_r(p)}\left|-\frac{d^2f}{dr^2}(r)g(x)-\frac{n-1}{r}\frac{df}{dr}(r)g(x)+\frac{f(r)}{r^2}\sum_{i=1}^{\infty}a_i\lambda_i\phi_i(x)\right|^2dH^{n-1}dr. \end{aligned}$$

On the other hand, it is easy to check that a function on $\boldsymbol{R}_{>0}$:

$$r\mapsto\int_{\partial B_r(p)}\left|-\frac{d^2f}{dr^2}(r)g(x)-\frac{n-1}{r}\frac{df}{dr}(r)g(x)+\frac{f(r)}{r^2}\sum_{i=1}^{\infty}a_i\lambda_i\phi_i(x)\right|^2dH^{n-1}$$

is continuous. Therefore for every $r>0$, there exists a Borel subset $A(r)$ of X such that $H^{n-1}(X\setminus A(r))=0$ and that

$$-\frac{d^2f}{dr^2}(r)g(x)-\frac{n-1}{r}\frac{df}{dr}(r)g(x)+\frac{f(r)}{r^2}\sum_{i=1}^{\infty}a_i\lambda_i\phi_i(x)=0$$

for every $x\in A(r)$. Let

$$\lambda:=\frac{d^2f}{dr^2}(1)+(n-1)\frac{df}{dr}(1).$$

Then, for every Lipschitz function ϕ on X, we have

$$\int_X\lambda g\phi dH^{n-1}=\int_X\phi\sum_{i=1}^{\infty}a_i\lambda_i\phi_idH^{n-1}=\int_X\langle d_X\phi,d_Xg\rangle dH^{n-1}.$$

Thus, g is a λ-eigenfunction of Δ_X. Therefore, we have $\lambda\geq n-1$. For every $r>0$, we have

$$\begin{aligned} 0&=-\frac{d^2f}{dr^2}(r)\int_Xg^2dH^{n-1}-\frac{n-1}{r}\frac{df}{dr}(r)\int_Xg^2dH^{n-1}+\frac{f(r)}{r^2}\int_Xg\sum_{i=1}^{\infty}a_i\lambda_i\phi_i(x)dH^{n-1}\\ &=-\frac{d^2f}{dr^2}(r)\int_Xg^2dH^{n-1}-\frac{n-1}{r}\frac{df}{dr}(r)\int_Xg^2dH^{n-1}+\frac{f(r)}{r^2}\int_X|d_Xg|^2dH^{n-1}\\ &=-\frac{d^2f}{dr^2}(r)\int_Xg^2dH^{n-1}-\frac{n-1}{r}\frac{df}{dr}(r)\int_Xg^2dH^{n-1}+\frac{f(r)}{r^2}\lambda\int_Xg^2dH^{n-1}. \end{aligned}$$

Thus, we have

$$-\frac{d^2 f}{dr^2}(r) - \frac{n-1}{r}\frac{df}{dr}(r) + \frac{f(r)}{r^2}\lambda = 0.$$

Therefore, we have the assertion. $\square$

Next corollary follows from Propositions 3.19 and 3.28 directly:

COROLLARY 3.29. *Let u_∞ be a nonconstant harmonic function on $C(X)$ with $u_\infty(p) = 0$. Assume that* $\mathrm{ord}_0\, u_\infty = \mathrm{ord}_\infty\, u_\infty = d < \infty$. *Then, we have the following*:

1. *A function $g(x) = u_\infty(1, x)$ on X is a $d(d+n-2)$-eigenfunction of Δ_X.*
2. *$u_\infty(r, x) = r^d g(x)$.*

By Corollary 3.29, the following follows directly from an argument similar to that of the proof of [**15**, Corollary 1.63].

COROLLARY 3.30. *Let u_∞ be a nonconstant harmonic function on $C(X)$. Assume that $u_\infty(p) = 0$,* $\mathrm{ord}_\infty\, u_\infty = d < \infty$ *and that v and u_∞ are orthogonal for every homogeneous harmonic function v with growth α satisfying $\alpha < d$. Then, we have the following*:

1. *A function $g(x) = u_\infty(1, x)$ on X is a $d(d+n-2)$-eigenfunction of Δ_X.*
2. *$u_\infty(r, x) = r^d g(x)$.*

We now give a proof of Theorem 1.1:

A PROOF OF THEOREM 1.1. Theorem 1.1 follows directly from the results given in this section and an argument similar to that of the proof of [**15**, Theorem 1.67]. $\square$

4. Weyl type asymptotic bounds.

Our goal in this section is to give a proof of Theorem 1.2.

PROPOSITION 4.1. *Let $d \geq 0$, and let $\hat{M}$ be an n-dimensional complete nonnegatively Ricci curved manifold with $V_{\hat{M}} > 0$, and $(\hat{M}_\infty, \hat{m}_\infty)$ an asymptotic cone of $\hat{M}$. Then we have* $\dim H^d(\hat{M}_\infty) \leq C(n) d^{n-1}$. *Moreover, for every $V > 0$, there exists $d(V, n) > 1$ such that*

$$\dim H^d(M_\infty) \leq C(n) V_M d^{n-1}$$

for every n-dimensional complete nonnegatively Ricci curved manifold M with $V_M \geq V$, every $d > d(V, n)$ and every asymptotic cone (M_∞, m_∞) of M.

PROOF. The assertion follows from Theorem 1.1 and arguments similar to that of the proofs of [**18**, Proposition 3.1] and [**18**, Proposition 6.1]. We now only introduce important ideas used in the proofs of their propositions and give an outline of a proof of our assertion. Fix $V > 0$, an n-dimensional complete nonnegatively Ricci curved manifold M with $V_M \geq V$, and $(M_\infty, m_\infty) = (C(X), p) \in \hat{\mathcal{M}}_M$. Let $d_1 := d_1(n) \geq 1$ with $d(d+n-2) \leq 2d^2$ for every $d \geq d_1$. Let ϕ_i be a $\lambda_i(X)$-eigenfunction of Δ_X with

$$\int_X \phi_i \phi_j dH^{n-1} = \delta_{ij}$$

for every i, j. Let $N_d := \max\{l \in \boldsymbol{N}; \lambda_l(X) \leq d(d+n-2)\}$. Then we have

$$\int_X |d\phi_i|^2 dH^{n-1} = \lambda_i(X) \leq d(d+n-2)$$

for every $1 \leq i \leq N_d$. On the other hand, by Proposition 7.9 and the proof of [**18**, Proposition 6.1], there exists $d_2 := d_2(n, V_M) \geq d_1$ such that for every $d \geq d_2$ and every $\{y_i\}_{1 \leq i \leq l} \subset X$ which is a maximal $1/d$-separated subset of X, we have $l \leq C(n) V_M d^{n-1}$. Fix $C > 1$ and $d \geq d_2$ (we will choose C depending only on n later). Let $\{x_j\}_{1 \leq j \leq l}$ be a maximal $1/(Cd)$-separated subset of X and $\mathcal{V} := \mathrm{span}\{\phi_i; 1 \leq i \leq N_d\}$. Define a linear map $\mathcal{M}$ from $\mathcal{V}$ to $\boldsymbol{R}^l$ by

$$\mathcal{M}(v) = \left(\int_{B_{2/Cd}(x_1)} v dH^{n-1}, \dots, \int_{B_{2/Cd}(x_l)} v dH^{n-1} \right).$$

Let $\mathcal{K} := \mathrm{Ker}\,\mathcal{M}$, and let $\{w_j\}_{1 \leq j \leq k}$ be an L^2-orthonormal basis of $\mathcal{K}$, and $\{w_j\}_{k+1 \leq j \leq N_d} \subset \mathcal{V}$ satisfying that $\{w_i\}_{1 \leq i \leq N_d}$ is a L^2-orthonormal basis of $\mathcal{V}$. By the weak Poincaré inequality of type $(1,2)$ on X, we have

$$\int_{B_{2/Cd}(x_i)} w_j^2 dH^{n-1} \leq \frac{C(n)}{(Cd)^2} \int_{B_{2/Cd}(x_i)} |dw_j|^2 dH^{n-1}$$

for every $1 \leq j \leq k$ and every $1 \leq i \leq l$. Therefore, we have

$$\begin{aligned} 1 &\leq \sum_{i=1}^l \int_{B_{2/Cd}(x_i)} w_j^2 dH^{n-1} \leq \frac{C(n)}{(Cd)^2} \sum_{i=1}^l \int_{B_{2/Cd}(x_i)} |dw_j|^2 dH^{n-1} \\ &\leq \frac{C(n)}{(Cd)^2} \int_X |dw_j|^2 dH^{n-1} \end{aligned}$$

for every $1 \leq j \leq k$. Thus we have

$$\begin{aligned} k &\leq \frac{C(n)}{(Cd)^2} \sum_{j=1}^k \int_X |dw_j|^2 dH^{n-1} \leq \frac{C(n)}{(Cd)^2} \sum_{j=1}^{N_d} \int_X |dw_j|^2 dH^{n-1} \\ &\leq \frac{C(n)}{(Cd)^2} 2d^2 N_d \leq \frac{C(n)}{C^2} N_d. \end{aligned}$$

Let $C := \sqrt{2C(n)}$, where $C(n)$ is as above. Then we have $k \leq N_d/2$. Since $N_d = k + \dim(\mathrm{Image}\,\mathcal{M})$, we have $N_d \leq 2l \leq C(n) V_M d^{n-1}$. On the other hand, by Theorem

1.1, we have $\dim H^d(M_\infty) \leq N_d$. Therefore, we have the assertion. $\square$

Note that Theorem 1.2 follows directly from Proposition 4.1 and the following:

Proposition 4.2. *For every $V > 0$, there exists $d(V, n) > 1$ such that the following holds: Let M be an n-dimensional complete nonnegatively Ricci curved Riemannian manifold with $V_M \geq V$. Then, for every $d \geq d(V, n)$ and every asymptotic cone (M_∞, m_∞) of M, we have*

$$\dim H^d(M_\infty) \geq C(n) V_M d^{n-1}.$$

Proof. Fix $V > 0$, an n-dimensional complete nonnegatively Ricci curved manifold M with $V_M \geq V$, and $X \in \mathcal{M}_M$. The next claim follows directly from Proposition 7.9.

Claim 4.3. *Let $\epsilon > 0$, $k \in \boldsymbol{N}$ and $\{x_i\}_{1 \leq i \leq k} \subset X$. Assume that $\{x_i\}_{1 \leq i \leq k}$ is an ϵ-separated subset of X. Then we have $k \leq C(n)/\epsilon^{n-1}$.*

We give an upper bound of the first eigenvalue with respect to the Dirichlet problem on $B_r(x) (\subset X)$:

Claim 4.4. *We have*

$$\inf_{k \in \mathcal{K}(B_r(x)), k \not\equiv 0} \frac{\int_{B_r(x)} |dk|^2 dH^{n-1}}{\int_{B_r(x)} k^2 dH^{n-1}} \leq \frac{C(n)}{r^2}$$

for every $x \in X$ and every $0 < r \leq \pi$.

The proof is as follows. Define a Lipschitz function k on X by $k(w) := \max\{r/2 - \overline{x, w}, 0\}$. By the definition, we have $k \in \mathcal{K}(B_r(x))$,

$$\int_{B_r(x)} |d_X k|^2 dH^{n-1} = H^{n-1}(B_{r/2}(x))$$

and

$$\int_{B_r(x)} k^2 dH^{n-1} \geq \int_{B_{r/4(x)}(x)} k^2 dH^{n-1} \geq \int_{B_{r/4}(x)} \frac{r^2}{16} dH^{n-1} \geq \frac{r^2}{16} H^{n-1}(B_{r/4}(x)).$$

Proposition 7.9 yields

$$\frac{\int_{B_r(x)} |dk|^2 dH^{n-1}}{\int_{B_r(x)} k^2 dH^{n-1}} \leq \frac{16}{r^2} \frac{H^{n-1}(B_{r/2}(x))}{H^{n-1}(B_{r/4}(x))} \leq \frac{C(n)}{r^2}.$$

Thus, we have Claim 4.4.

Claim 4.5. *We have*

$$\limsup_{r \to 0} \frac{H^{n-1}(B_r(x))}{r^{n-1}} \le C(n)$$

for every $x \in X$.

The proof is as follows. For every sufficiently small $r > 0$, let $A_r := \{(s, w) \in C(X); 1 - r \le s \le 1 + r, w \in B_r(x)\}$. Proposition 7.6 yields

$$\begin{aligned} H^n(B_{5r}(1, x)) &= \int_{1-r}^{1+r} H^{n-1}(\partial B_t(p) \cap B_{5r}(1, x))dt \\ &\ge \int_{1-r}^{1+r} H^{n-1}(\partial B_t(p) \cap A_r)dt \\ &\ge C(n) r H^{n-1}(B_r(x)). \end{aligned}$$

Since

$$\lim_{r \to 0} \frac{H^n(B_r(1, x))}{H^n(B_r(0_n))} \le 1,$$

where $0_n \in \boldsymbol{R}^n$, we have Claim 4.5.

Claim 4.6. *We have*

$$\lambda_d(X) \le C(n) \left(\frac{d}{H^{n-1}(X)} \right)^{2/(n-1)}$$

for every $d \ge 1$.

The proof is as follows. Fix $0 < C < 1$ (we will choose C depending only on n later). Let

$$\epsilon := C \left(\frac{H^{n-1}(X)}{d} \right)^{1/(n-1)}$$

and let $\{x_i\}_{1 \le i \le k}$ be a maximum ϵ-separated subset of X. By Claim 4.3, we have $k \le C(n)/\epsilon^{n-1} \le C(n) d^{n-1}/(C^{n-1} H^{n-1}(X))$. On the other hand, we have

$$\sum_{i=1}^k H^{n-1}(B_{2\epsilon}(x_i)) \ge H^{n-1}(X).$$

By Claim 4.5 and Proposition 7.9, we have $H^{n-1}(B_{5\epsilon}(x_i)) \le C(n)\epsilon^{n-1}$. Thus, we have

$$H^{n-1}(X) \le \sum_{i=1}^k H^{n-1}(B_{2\epsilon}(x_i)) \le k C(n) \epsilon^{n-1}.$$

Therefore, we have

$$k \geq \frac{C_1(n)H^{n-1}(X)}{\epsilon^{n-1}} = \frac{C_1(n)}{C^{n-1}}\frac{H^{n-1}(X)d}{H^{n-1}(X)} \geq \frac{C_1(n)}{C^{n-1}}d,$$

where $C_1(n)$ is a sufficiently small positive constant depending only on n. Let $C := C_1(n)^{1/(n-1)}/2$. Then we have $k \geq 2d\,(\geq d+1)$. By Claim 4.4, for every $1 \leq i \leq k$, there exists $\phi_i \in \mathcal{K}(B_{\epsilon/10}(x_i))$ such that $\phi_i \not\equiv 0$ and

$$\frac{\int_{B_{\epsilon/10}(x_i)} |d\phi_i|^2 dH^{n-1}}{\int_{B_{\epsilon/10}(x_i)} (\phi_i)^2 dH^{n-1}} \leq \frac{C(n)}{\epsilon^2}.$$

Since $\{B_{\epsilon/10}(x_i)\}_i$ is a pairwise disjoint collection, we see that $\{\phi_i\}_i$ are linearly independent in $L^2(X)$. For every $(a_1, \ldots, a_k) \in \boldsymbol{R}^k \setminus \{0_n\}$, we have

$$\begin{aligned}\int_X \left|d\left(\sum_{i=1}^k a_i\phi_i\right)\right|^2 dH^{n-1} &= \sum_{i=1}^k \int_X |d(a_i\phi_i)|^2 dH^{n-1} \\ &\leq \sum_{i=1}^k \frac{C(n)}{\epsilon^2}\int_X (a_i\phi_i)^2 dH^{n-1} \\ &= \frac{C(n)}{\epsilon^2}\int_X \left|\sum_{i=1}^k a_i\phi_i\right|^2 dH^{n-1}.\end{aligned}$$

Thus, we have $\lambda_{k-1}(X) \leq C(n)/\epsilon^2$. Therefore, we have

$$\lambda_d(X) \leq \lambda_{k-1}(X) \leq \frac{C(n)}{\epsilon^2} \leq C(n)\left(\frac{d}{H^{n-1}(X)}\right)^{2/(n-1)}.$$

Thus, we have Claim 4.6.

The assertion follows directly from Claim 4.6 and Theorem 1.1. □

5. Gromov-Hausdorff topology on the moduli space of asymptotic cones.

In this section, we will study the moduli space of asymptotic cones of a fixed non-negatively Ricci curved manifold M with Euclidean volume growth. In general, the asymptotic cones of M are *not* unique. See [**6**] and [**59**] for such examples. Therefore we now consider the set of compact geodesic spaces X such that $(C(X), p)$ are asymptotic cones of M, denoted by $\mathcal{M}_M$. Define a topology on $\mathcal{M}_M$ by the Gromov-Hausdorff distance d_{GH}. On the other hand, let $\hat{\mathcal{M}}_M := \{(C(X), p); X \in \mathcal{M}_M\}$ and define a topology on $\hat{\mathcal{M}}_M$ by the pointed Gromov-Hausdorff topology. Then the canonical map $\pi : \mathcal{M}_M \to \hat{\mathcal{M}}_M$ defined by $\pi(X) = (C(X), p)$ is a homeomorphism. Note that Gromov's compactness theorem yields that $\hat{\mathcal{M}}_M$ is compact. Thus $\mathcal{M}_M$ is compact.

5.1. Continuity of eigenvalues.

The main result in this subsection is the following theorem. We can regard it as a "$\mathcal{M}_M$-version" of [**26**, (0.4) Theorem] by Fukaya or of [**8**, Theorem 7.9] by Cheeger-Colding.

Theorem 5.1. *Assume that X_i converges to X_∞ in $\mathcal{M}_M$. Then (X_i, H^{n-1}) converges to (X_∞, H^{n-1}). Moreover, we have*

$$\lim_{i \to \infty} \lambda_k(X_i) = \lambda_k(X_\infty)$$

for every $k \geq 1$, where $\lambda_k(X)$ is the k-th eigenvalue of Δ_X.

Proof. Let $\{x_i\}_i$ be a sequence of points $x_i \in X_i$ with $x_i \to x_\infty$, and r, ϵ positive numbers. Let $A_\epsilon^r(x_i) := \{(t,x) \in C(X_i); x \in B_r(x_i), 1-\epsilon \leq t \leq 1+\epsilon.\}$. Then, by [**38**, Proposition 4.7], we have

$$\lim_{i \to \infty} H^n(A_\epsilon^r(x_i)) = H^n(A_\epsilon^r(x_\infty)).$$

On the other hand, by Proposition 7.6, we have

$$H^n(A_\epsilon^r(x_i)) = \int_{1-\epsilon}^{1+\epsilon} H^{n-1}(\partial B_t(p_i) \cap A_\epsilon^r(x_i))dt = C(n)\epsilon H^{n-1}(B_r^{X_i}(x_i))$$

for every $1 \leq i \leq \infty$, where p_i is the pole of $C(X_i)$. Thus, we have $(X_i, H^{n-1}) \to (X_\infty, H^{n-1})$.

We now give a proof of the second assertion by induction for k. Fix a subsequence $\{i(j)\}_j$ of $\boldsymbol{N}$. Let $f_1^{i(j)}$ be a $\lambda_1(X_{i(j)})$-eigenfunction on $X_{i(j)}$ with

$$\frac{1}{H^{n-1}(X_{i(j)})} \int_{X_{i(j)}} (f_1^{i(j)})^2 dH^{n-1} = 1.$$

Thus we have

$$\frac{1}{H^{n-1}(X_{i(j)})} \int_{X_{i(j)}} |df_1^{i(j)}|^2 dH^{n-1} = \lambda_1(X_{i(j)}).$$

Define a harmonic function $u_1^{i(j)}$ on $C(X_{i(j)})$ by $u_1^{i(j)}(r,x) := r^{\alpha_1^{i(j)}} f_1^{i(j)}(x)$, where $\alpha_1^{i(j)} \geq 0$ with $\lambda_1(X_{i(j)}) = \alpha_1^{i(j)}(\alpha_1^{i(j)} + n - 2)$. Note that $\lambda_1(X_{i(j)}) \geq n-1$ and $\alpha_1^{i(j)} \geq 1$. Then, [**38**, Proposition 3.21] and Proposition 7.6 yield

$$\begin{aligned} &\int_{B_7(p_{i(j)})} \left(\operatorname{Lip} u_1^{i(j)}\right)^2 dH^n \\ &= \int_0^7 \int_{\partial B_r(p_{i(j)})} \left(\alpha_1^{i(j)}\right)^2 \left(r^{\alpha_1^{i(j)}-1}\right)^2 \left(f_1^{i(j)}\right)^2 dH^{n-1} dr \end{aligned}$$

$$
\begin{aligned}
&+\int_0^7 \int_{\partial B_r(p_{i(j)})} r^{2\alpha_1^{i(j)}-2}\left|d_X f_1^{i(j)}\right|^2 dH^{n-1}dr \\
&= \int_0^7 \left(\alpha_1^{i(j)}\right)^2 r^{2\alpha_1^{i(j)}-2} r^{n-1} H^{n-1}(X_{i(j)})dr \\
&+\int_0^7 r^{2\alpha_1^{i(j)}+n-1-2}\lambda_1(X_{i(j)})H^{n-1}(X_{i(j)})dr \\
&= H^{n-1}(X_{i(j)})\left(\frac{7^{2\alpha_1^{i(j)}+n-2}(\alpha_1^{i(j)})^2}{2\alpha_1^{i(j)}+n-2}+\frac{7^{\alpha_1^{i(j)}+n}\lambda_1(X_{i(j)})}{2\alpha_1^{i(j)}+n-2}\right).
\end{aligned}
$$

By Li-Schoen's mean value inequality and Theorem 3.7, we have

$$
\mathbf{Lip}\left(u_1^{i(j)}|_{B_2(p_{i(j)})}\right) \leq \frac{C(n)}{H^n(B_7(p_{i(j)}))}\int_{B_7(p_{i(j)})}\left(\operatorname{Lip} u_1^{i(j)}\right)^2 dH^n.
$$

On the other hand, Claim 4.6 yields

$$
\lambda_l(X_{i(j)}) \leq C(n)\left(\frac{l}{H^{n-1}(X_{i(j)})}\right)^{2/(n-1)}
$$

for every l. Thus, we have

$$
\mathbf{Lip}\left(u_1^{i(j)}|_{B_2(p_{i(j)})}\right) \leq C(n, V_M).
$$

Then there exist a subsequence $\{j(l)\}_l$ of $\{i(j)\}_i$, a Lipschitz harmonic function u_1^∞ on $B_2(p_\infty)$, a Lipschitz function f_1^∞ on X_∞ and $\alpha_1^\infty \geq 0$ such that $u_1^{j(l)} \to u_1^\infty$ on $B_2(p_\infty)$, $f_1^{j(l)} \to f_1^\infty$ on X_∞ and $\alpha_1^{j(l)} \to \alpha_1^\infty$. Thus, we see that $u_1^\infty(r,x) = r^{\alpha_1^\infty} f_1^\infty(x)$ on $B_2(p_\infty)$, and

$$
\lim_{i\to\infty}\int_{X_{m(i)}} (f_1^{m(i)})^2 dH^{n-1} = \int_{X_\infty}(f_1^\infty)^2 dH^{n-1}.
$$

On the other hand, Proposition 3.4 and Theorem 3.7 yield

$$
\begin{aligned}
&\lim_{l\to\infty}\int_{1-\epsilon}^1 t^{3-n}\int_{\partial B_t(p_{j(l)})}\left|d_{\partial B_t(p_{j(l)})} u_1^{j(l)}\right|^2 dH^{n-1}dt \\
&= \lim_{l\to\infty}\left(\int_{1-\epsilon}^1 tD_{u_1^{j(l)}}(t)dt - \int_{1-\epsilon}^1 F_{u_1^{j(l)}}(t)dt\right) \\
&= \int_{1-\epsilon}^1 tD_{u_1^\infty}(t)dt - \int_{1-\epsilon}^1 F_{u_1^\infty}(t)dt \\
&= \int_{1-\epsilon}^1 t^{3-n}\int_{\partial B_t(p_\infty)} |d_{\partial B_t(p_\infty)} u_1^\infty|^2 dH^{n-1}dt
\end{aligned}
$$

for every $0 < \epsilon < 1$. Since $|d_{\partial B_t(p_{j(l)})} u_1^{j(l)}|^2 = t^{2\alpha_1^{j(l)}-2} |d_{X(j(l))} u_1^{j(l)}|^2$, we have

$$\begin{aligned}
&\int_{1-\epsilon}^1 t^{3-n} \int_{\partial B_t(p_{j(l)})} \left| d_{\partial B_t(p_{j(l)})} u_1^{j(l)} \right|^2 dH^{n-1} dt \\
&\quad = \int_{1-\epsilon}^1 t^{3-n} t^{2\alpha_1^{j(l)}-2} t^{n-1} \int_{X_{j(l)}} \left| d_{X_{j(l)}} u_1^{j(l)} \right|^2 dH^{n-1} dt \\
&\quad = \int_{1-\epsilon}^1 t^{2\alpha_1^{j(l)}} \lambda_1(X_{j(l)}) H^{n-1}(X_{j(l)}) dt \\
&\quad = \frac{1-(1-\epsilon)^{2\alpha_1^{j(l)}+1}}{2\alpha_1^{j(l)}+1} \lambda_1(X_{j(l)}) H^{n-1}(X_{j(l)}).
\end{aligned}$$

Similarly, we have

$$\int_{1-\epsilon}^1 t^{3-n} \int_{\partial B_t(p_\infty)} |d_{\partial B_t(p_\infty)} u_1^\infty|^2 dH^{n-1} dt = \frac{1-(1-\epsilon)^{2\alpha_1^\infty+1}}{2\alpha_1^\infty+1} \int_{X_\infty} |df_1^\infty|^2 dH^{n-1}.$$

Therefore, we have

$$\lim_{l\to\infty} \frac{1}{H^{n-1}(X_{j(l)})} \int_{X_{j(l)}} \left| df_1^{j(l)} \right|^2 dH^{n-1} = \lim_{l\to\infty} \lambda_1(X_{j(l)}) = \frac{1}{H^{n-1}(X_\infty)} \int_{X_\infty} |df_1^\infty|^2 dH^{n-1}.$$

Since $\{i(j)\}_j$ is arbitrary, we have

$$\liminf_{i\to\infty} \lambda_1(X_i) \geq \lambda_1(X_\infty).$$

On the other hand, by [**8**, Theorem 7.1], we have

$$\limsup_{i\to\infty} \lambda_1(X_i) \leq \lambda_1(X_\infty).$$

Therefore we see that

$$\lim_{i\to\infty} \lambda_1(X_i) = \lambda_1(X_\infty),$$

and that f_1^∞ is a $\lambda_1(X_\infty)$-eigenfunction.

Next, fix an integer $k \geq 2$. Assume that the following hold:

1. $\lim_{i\to\infty} \lambda_j(X_i) = \lambda_j(X_\infty)$ holds for every $1 \leq j \leq k-1$.
2. For every subsequence $\{i(j)\}_i$ of $\boldsymbol{N}$, there exist a subsequence $\{j(l)\}_l$ of $\{i(j)\}_j$, a $\lambda_m(X_{j(l)})$-eigenfunction $f_m^{j(l)}$ on $X_{j(l)}$ and a $\lambda_m(X_\infty)$-eigenfunction f_m^∞ on X_∞ for every $1 \leq m \leq k-1$ such that the following hold:
   (a) $f_m^{j(l)} \to f_m^\infty$ on X_∞.

(b) $\mathbf{Lip}(f_m^{j(l)}|_{B_2(p_{j(l)})}) \leq C(n, m, V_M)$ for every $1 \leq m \leq k-1$.

(c)
$$\frac{1}{H^{n-1}(X_{j(l)})} \int_{X_{j(l)}} f_s^{j(l)} f_t^{j(l)} dH^{n-1} = \delta_{st}$$

holds for every $1 \leq s \leq t \leq k-1$.

Fix a subsequence $\{i(j)\}_j$ of $\boldsymbol{N}$. Let $\{j(l)\}_l$, $\{f_m^{j(l)}\}_{l \in \boldsymbol{N}, m \leq k-1}$ be as above, and $\{f_k^{j(l)}\}_{l<\infty}$ a sequence of $\lambda_k(X_{j(l)})$-eigenfunctions $f_k^{j(l)}$ with

$$\frac{1}{H^{n-1}(X_{j(l)})} \int_{X_{j(l)}} (f_k^{j(l)})^2 dH^{n-1} = 1.$$

Define a harmonic function $u_k^{j(l)}$ on $C(X_{j(l)})$ by $u_k^{j(l)}(r, x) := r^{\alpha_k^{j(l)}} f_k^{j(l)}(x)$, where $\alpha_k^{j(l)} \geq 0$ with $\alpha_k^{j(l)}(\alpha_k^{j(l)} + n - 2) = \lambda_k(X_{j(l)})$.

By an argument similar to that of the case $k = 1$, without loss of generality, we can assume that there exist a Lipschitz harmonic function u_k^∞ on $B_2(p_\infty)$, a Lipschitz function f_k^∞ on X_∞ and $\alpha_k^\infty \geq 0$ such that $\mathbf{Lip}\left(u_k^{j(l)}|_{B_2(p_{j(l)})}\right) \leq C(n, k, V_M)$, $\mathbf{Lip}\, f_k^{j(l)} \leq C(n, k, V_M)$, $u_k^{j(l)} \to u_k^\infty$ on $B_2(p_\infty)$, $f_k^{j(l)} \to f_k^\infty$ on X_∞ and $\alpha_k^{j(l)} \to \alpha_k^\infty$. Thus, we have $u_k^\infty(r, x) = r^{\alpha_k^\infty} f_k^\infty(x)$. By an argument similar to that of the case $k = 1$, we have

$$\lim_{l \to \infty} \int_{X_{j(l)}} \left|df_k^{j(l)}\right|^2 dH^{n-1} = \int_{X_\infty} |df_k^\infty|^2 dH^{n-1}.$$

On the other hand, we have

$$\lim_{l \to \infty} \int_{X_{j(l)}} f_s^{j(l)} f_t^{j(l)} dH^{n-1} = \int_{X_\infty} f_s^\infty f_t^\infty dH^{n-1}$$

for every $1 \leq s \leq t \leq k$. Thus, we have $f_k^\infty \in (\mathrm{span}\{1, f_1^\infty, \ldots, f_{k-1}^\infty\})^\perp$ and $f_k^\infty \not\equiv 0$. Therefore, we have

$$\lambda_k(X_\infty) \leq \int_{X_\infty} |df_k^\infty|^2 dH^{n-1}.$$

Since $\{i(j)\}_j$ is arbitrary, we have

$$\liminf_{i \to \infty} \lambda_k(X_i) \geq \lambda_k(X_\infty).$$

On the other hand, [**8**, Theorem 7.1] yields

$$\limsup_{i \to \infty} \lambda_k(X_i) \leq \lambda_k(X_\infty).$$

Therefore, we see that

$$\lim_{i \to \infty} \lambda_k(X_i) = \lambda_k(X_\infty),$$

and that f_k^∞ is a $\lambda_k(X_\infty)$-eigenfunction. Thus we have the assertion. $\square$

Remark 5.2. By the proof of Theorem 5.1, we also have the following: With the same assumption as in Theorem 5.1, if a sequence of $\lambda_k(X_i)$-eigenfunctions f_k^i on X_i converges to a Lipschitz function f_k^∞ on X_∞, then f_k^∞ is also an $\lambda_k(X_\infty)$-eigenfunction.

5.2. Spectral convergence.

In this subsection, we now study a convergence of the heat kernel $h_X(t,x,y)$ of $X \in \mathcal{M}_M$ with respect to the Gromov-Hausdorff topology via *spectral convergence* introduced by Kasue-Kumura in [**42**], [**43**]. See for instance [**68**], [**69**], [**70**] for the heat kernels on metric measure spaces. See also [**65**] for the case of Alexandrov spaces.

Definition 5.3 (Spectral distance, [**42**], [**43**]). Let $X, \hat{X} \in \mathcal{M}_M$. A Borel map $f : X \to \hat{X}$ is called *an ϵ-spectral approximation* if it satisfies $e^{t+1/t}|h_X(t,x_1,x_2) - h_{\hat{X}}(t,f(x_1),f(x_2))| < \epsilon$ for every $t > 0$ and every $x_1, x_2 \in X$. We define the spectral distance $SD(X,\hat{X})$ between X and $\hat{X}$ by the infimum of $\epsilon > 0$ such that both ϵ-spectral approximations $f : X \to \hat{X}$ and $g : \hat{X} \to X$ exist.

See [**42**], [**43**] for fundamental properties of this spectral distance. The following theorem is the main result in this section. Note that the following implies directly Theorem 5.1.

Theorem 5.4. *Let $X_i \to X_\infty$ in $\mathcal{M}_M$. Then $SD(X_i, X_\infty) \to 0$.*

Proof. By Theorems 1.1, 1.2 and the compactness of $\mathcal{M}_M$, it is not difficult to check that

$$p_X(t,x,x) \leq C_1(n,V_M)t^{-C_2(n,V_M)}$$

for every $t > 0$, every $X \in \mathcal{M}_M$, and every $x \in X$. Then, by Theorem 5.1 and an argument similar to that in Section 2 in [**43**], we have the assertion. $\square$

6. A dimension comparison theorem and a Liouville type theorem.

In this subsection, we will give a comparison theorem (Theorem 6.1) between the dimension of a space of harmonic functions on a fixed nonnegatively Ricci curved manifold with Euclidean volume growth, and that on an asymptotic cone of the manifold. Essential tools to show it are [**14**, Lemma 3.1] (or [**15**, Lemma 7.1]) and Proposition 3.4. We apply Theorem 6.1 to give a new Liouville type theorem (Theorem 1.3) and an alternative proof of Colding-Minicozzi's result about Weyl type asymptotic bounds for harmonic functions on manifolds in [**18**]. Fix an n-dimensional complete nonnegatively Ricci curved manifold M with $V_M > 0$.

Theorem 6.1. *Let d, ϵ be positive numbers and k, l nonnegative integers with $0 \leq l \leq k \leq \dim H^d(M) - 1$. Then, there exists $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$ such that*

$$l \leq \dim H^{(k/(k-l+1))(d-1+(n/2))+1-(n/2)+\epsilon}(M_\infty) - 1.$$

Proof. Without loss of generality, we can assume $l \geq 1$. Let $\{u_j\}_{1\leq j\leq k}$ be a collection of linearly independent harmonic functions in $H^d(M)$ with $u_i(m) = 0$. Put

$$J_r(u_i, u_j) := \int_{b^{g_M}\leq r} \langle du_i, du_j\rangle d\,\mathrm{vol}^{g_M}.$$

For every $r > 0$ and every $i, j \in \{1, \ldots, k\}$, define a harmonic function $w_{i,r}$ and a real number $\lambda_{ji}(r)$ so that $u_{\hat{i}} = \sum_{\hat{j}=1}^{\hat{i}-1} \lambda_{\hat{j}\hat{i}}(r)u_{\hat{j}} + w_{\hat{i},r}$ for every $\hat{i}$, and that $J_r(w_{\tilde{i},r}, w_{\tilde{j},r}) = 0$ for every $\tilde{i} \neq \tilde{j}$. Let

$$f_i(r) := \int_{b^{g_M}\leq r} |dw_{i,r}|^2 d\,\mathrm{vol}^{g_M}.$$

Claim 6.2. *We have the following*:

1. *There exists* $K > 0$ *such that* $f_i(r) \leq K(r^{2d-2+n} + 1)$ *for every* i *and every* $r > 0$.
2. $f_i(r) > 0$ *for every* i *and every* $r > 0$.
3. $f_i(r) \leq f_i(s)$ *for every* i *and every* $r \leq s$.
4. *For every* i, f_i *is a barrier for* $t^{n-2}D^{g_M}_{w_{i,s}}(t)$ *at every* $s > 0$. *Here, for functions* g, h *on* $\boldsymbol{R}$ *and a real number* r, *we say that* f *is a barrier for* g *at* r *if* $f(r) = g(r)$ *and* $f(s) \leq g(s)$ *for* $s < r$ (*see* [**14**, Definition 4.6]).

Claim 6.2 follows from the trivial monotonicity of $t^{n-2}D^{g_M}_u(t)$ and an argument similar to that of the proof of [**15**, Proposition 8.6] (or [**14**, Proposition 4.7]).

Let $\lambda := k/(k-l+1)$. By [**14**, Lemma 3.1], for every $N \geq 2$, there exist a subsequence $\{m(N,i)\}_{i\in\boldsymbol{N}}$ of $\boldsymbol{N}$ and pairwise distinct integers $\alpha_1^N, \ldots, \alpha_l^N \in \{1, \ldots, k\}$ such that $f_j(N^{m(N,i)+1}) \leq 2N^{\lambda(2d-2+n)} f_j(N^{m(N,i)})$ for every $j \in \{\alpha_1^N, \ldots, \alpha_k^N\}$ and every $i \in \boldsymbol{N}$. Without loss of generality, we can assume that $\alpha_i^N = i$ for every $1 \leq i \leq l$. Claim 6.2 yields

$$\frac{f_j(N^{m(N,i)+1})}{f_j(N^{m(N,i)})} \geq \frac{(N^{m(N,i)+1})^{n-2} D^{g_M}_{w_{j,N^{m(N,i)+1}}}(N^{m(N,i)+1})}{(N^{m(N,i)})^{n-2} D^{g_M}_{w_{j,N^{m(N,i)+1}}}(N^{m(N,i)})}.$$

Thus, we have

$$\frac{D^{g_M}_{w_{j,N^{m(N,i)+1}}}(N^{m(N,i)+1})}{D^{g_M}_{w_{j,N^{m(N,i)+1}}}(N^{m(N,i)})} \leq 2N^{\lambda(2d-2+n)+2-n}.$$

Define a harmonic function $w_j^{N,i}$ on $B^{(N^{m(N,i)})^{-2}g_M}_{N/10}(m)$ by

$$w_j^{N,i}(w) := w_{j,N^{m(N,i)+1}} \times \left(N^{m(N,i)}\sqrt{\frac{1}{\mathrm{vol}^{g_M}\{b^{g_M} \le N^{m(N,i)}\}}\int_{b^{g_M}\le N^{m(N,i)}} \left|dw_{j,N^{m(N,i)+1}}\right|^2 d\,\mathrm{vol}^{g_M}}\right)^{-1}.$$

Assume that N is sufficiently large. Then Li-Schoen's mean value inequality yields

$$\begin{aligned}
&\left|w_j^{N,i}(x_1) - w_j^{N,i}(x_2)\right| \\
&\le \sup_{B_{N^{m(N,i)}\frac{N}{5}}(m)} |\nabla w_{j,N^{m(n,i)+1}}|\overline{x_1, x_2}^{g_M} \\
&\quad \times \left(N^{m(N,i)}\sqrt{\frac{1}{\mathrm{vol}^{g_M}\{b^{g_M} \le N^{m(N,i)}\}}\int_{b^{g_M}\le N^{m(N,i)}} \left|dw_{j,N^{m(N,i)+1}}\right|^2 d\,\mathrm{vol}^{g_M}}\right)^{-1} \\
&\le C(n)\sqrt{\frac{1}{\mathrm{vol}^{g_M}\{b^{g_M} \le N^{m(N,i)}2N/3\}}\int_{b^{g_M}\le N^{m(N,i)}2N/3} \left|dw_{j,N^{m(N,i)+1}}\right|^2 d\,\mathrm{vol}^{g_M}} \\
&\quad \times \left(\sqrt{\frac{1}{\mathrm{vol}^{g_M}\{b^{g_M} \le N^{m(N,i)}\}}\int_{b^{g_M}\le N^{m(N,i)}} \left|dw_{j,N^{m(N,i)+1}}\right|^2 d\,\mathrm{vol}^{g_M}}\right)^{-1} \\
&\quad \times \overline{x_1, x_2}^{(N^{m(N,i)})^{-2}g_M} \\
&\le C(n)N^{\lambda(d-1+n/2)+1-n/2}\overline{x_1, x_2}^{(N^{m(N,i)})^{-2}g_M}
\end{aligned}$$

for every $x_1, x_2 \in B_{N/10}^{(N^{m(N,i)})^{-2}g_M}(m)$. Without loss of generality, there exist $\{X_N\}_{N\ge 2} \subset \mathcal{M}_M$ and a collection $\{w_j^{N,\infty}\}_{1\le j\le l, N\ge 2}$ of Lipschitz functions $w_j^{N,\infty}$ on $B_{N/10}(p_N)$ such that $(M, m, (N^{m(N,i)})^{-1}d_M) \to (C(X_N), p_N)$ and that $w_j^{N,i} \to w_j^{N,\infty}$ on $B_{N/10}(p_N)$. On the other hand, we have

$$\begin{aligned}
&\frac{1}{\mathrm{vol}^{(N^{m(N,i)})^{-2}g_M}\ B_1^{(N^{m(N,i)})^{-2}g_M}(m)} \\
&\quad \times \int_{B_1^{(N^{m(N,i)})^{-2}g_M}(m)} \left|d^{(N^{m(N,i)})^{-2}g_M} w_j^{N,i}\right|^2 d\,\mathrm{vol}^{(N^{m(N,i)})^{-2}g_M} \\
&= \frac{1}{\mathrm{vol}^{g_M}\ B_{N^{m(N,i)}}(m)}\int_{B_{N^{m(N,i)}}(m)} |dw_{j,N^{m(N,i)+1}}|^2 (N^{m(N,i)})^2 d\,\mathrm{vol}^{g_M} \\
&\quad \times \left(N^{2m(N,i)}\frac{1}{\mathrm{vol}^{g_M}\{b^{g_M} \le N^{m(N,i)}\}}\int_{b^{g_M}\le N^{m(N,i)}} |dw_{j,N^{m(N,i)+1}}|^2 d\,\mathrm{vol}^{g_M}\right)^{-1} \\
&= 1 \pm \Psi(i^{-1}; n, N).
\end{aligned}$$

By [**38**, Corollary 4.7] and Theorem 3.1, we have

$$\frac{1}{H^n(B_1(p_N))}\int_{B_1(p_N)}\left|dw_j^{N,\infty}\right|^2 dH^n = 1.$$

Similarly, we have

$$\int_{B_1(p_N)}\langle dw_i^{N,\infty}, dw_j^{N,\infty}\rangle dH^n = 0$$

for every $i \neq j$. Therefore, we see that $\{w_j^{N,\infty}\}_j$ is a collection of linearly independent harmonic functions. Proposition 3.11 yields

$$\begin{aligned}\frac{I_{w_j^{N,\infty}}(N/100)}{I_{w_j^{N,\infty}}(1)} &= \frac{U_{w_j^{N,\infty}}(1)}{U_{w_j^{N,\infty}}(N/100)}\frac{D_{w_j^{N,\infty}}(N/100)}{D_{w_j^{N,\infty}}(1)} \\ &\leq \frac{D_{w_j^{N,\infty}}(N/100)}{D_{w_j^{N,\infty}}(1)} \leq 2N^{\lambda(2d-2+n)+2-n}.\end{aligned}$$

Therefore, by Proposition 3.8, we have

$$\exp\left(\int_1^{N/100} 2\frac{U_{w_j^{N,\infty}}(t)}{t}dt\right) \leq 2N^{\lambda(2d-2+n)+2-n}.$$

Thus, by Proposition 3.11, for every $1 \leq \hat{l} < N/100$, we have

$$\left(\frac{N}{100\hat{l}}\right)^{2U_{w_j^{N,\infty}}(\hat{l})} \leq 2N^{\lambda(2d-2+n)+2-n}$$

i.e.

$$2U_{w_j^{N,\infty}}(\hat{l}) \leq \frac{\log N}{\log N - \log(100\hat{l})} + \frac{\log N}{\log N - \log(100\hat{l})}(\lambda(2d-2+n)+2-n).$$

Therefore, for every $\hat{l} \geq 1$, there exists $N_{\hat{l}}$ such that $U_{w_j^{N,\infty}}(a) \leq \lambda(d-1+n/2)+1-n/2+\epsilon$ for every $N \geq N_{\hat{l}}$ and every $1 \leq a \leq \hat{l}$. Let $x_1 \in B_{\hat{l}/10}(p_N)$. Li-Schoen's mean value inequality and Theorem 3.7 yield

$$\begin{aligned}\operatorname{Lip} w_j^{N,\infty}(x_1) &\leq C(n)\sqrt{\frac{1}{H^n(B_{\hat{l}}(p_N))}\int_{B_{\hat{l}}(p_N)}\left(\operatorname{Lip} w_j^{N,\infty}\right)^2 dH^n} \\ &\leq C(n,V_M)\sqrt{\hat{l}^{-n}\int_{B_{\hat{l}}(p_N)}\left|dw_j^{N,\infty}\right|^2 dH^n}\end{aligned}$$

$$\leq C(n, V_M, \lambda, d)\sqrt{\hat{l}^{-1-n}\int_{\partial B_{\hat{l}}(p_N)} \left|w_j^{N,\infty}\right|^2 dH^n}$$

$$\leq C(n, V_M, \lambda, d)\hat{l}^{-1}\sqrt{\frac{1}{H^{n-1}(\partial B_{\hat{l}}(p_N))}\int_{\partial B_{\hat{l}}(p_N)} \left|w_j^{N,\infty}\right|^2 dH^n}.$$

On the other hand, by Proposition 3.11, we have

$$\begin{aligned} I_{w_j^{N,\infty}}(\hat{l}) &= \exp\left(\int_1^{\hat{l}} \frac{2U_{w_j^{N,\infty}}(t)}{t} dt\right) I_{w_j^{N,\infty}}(1) \\ &\leq \exp\left(\int_1^{\hat{l}} \frac{\lambda(2d-2+n)+2-n+2\epsilon}{t} dt\right) I_{w_j^{N,\infty}}(1) \\ &\leq \hat{l}^{\lambda(2d-2+n)+2-n+2\epsilon} I_{w_j^{N,\infty}}(1) \end{aligned}$$

for every $N \geq N_l$. By Proposition 3.16, we have

$$0 \leq I_{w_j^{N,\infty}}(1) \leq I_{w_j^{N,\infty}}(1) U_{w_j^{N,\infty}}(1) \leq D_{w_j^{N,\infty}}(1) = 1.$$

Thus, we have $I_{w_j^{N,\infty}}(\hat{l}) \leq \hat{l}^{\lambda(2d-2+n)+2-n+2\epsilon}$. Therefore, we have

$$\mathbf{Lip}\left(w_j^{N,\infty}|_{B_{\hat{l}/10}(p_N)}\right) \leq C(n, V_M, \lambda, d)\hat{l}^{\lambda(d-1+n/2)-n/2+\epsilon}.$$

Since $\mathcal{M}_M$ is compact, without loss of generality, we can assume that there exist $X_\infty \in \mathcal{M}_M$ and a collection of locally Lipschitz harmonic functions $\{w_j^\infty\}_j \subset H^{\lambda(d-1+n/2)+1-n/2+\epsilon}(C(X_\infty))$ such that $X_N \to X_\infty$ and $w_j^{N,\infty} \to w_j^\infty$ on $B_R(p_\infty)$ as $N \to \infty$ for every j and every $R > 0$. [**38**, Corollary 4.7] yields

$$\frac{1}{H^n(B_1(p_\infty))}\int_{B_1(p_\infty)} \langle dw_j^\infty, dw_i^\infty\rangle dH^n = \delta_{ij}.$$

In particular, we see that $\{w_j^\infty\}_j$ is a collection of linearly independent nonconstant harmonic functions. Therefore we have the assertion. □

As a corollary of Theorem 6.1, we will give an alternative proof of the following important result by Colding-Minicozzi:

COROLLARY 6.3 (Colding-Minicozzi, [**18**]). *For every $V > 0$, there exists $d(V, n) > 1$ such that*

$$\dim H^d(M) \leq C(n)V_M d^{n-1}$$

for every $d > d(V, n)$ and every n-dimensional complete nonnegatively Ricci curved manifold M with $V_M \geq V$.

PROOF. By letting $k = l = [(\dim H^d(M) - 1)/2]$, where $[a] := \inf\{l \in \boldsymbol{Z}; a \leq l\}$ for every $a \in \boldsymbol{R}$, the assertion follows from Theorems 1.2 and 6.1. □

We now give a proof of Theorem 1.3:

A PROOF OF THEOREM 1.3. Let $\lambda_1 := \inf\{\lambda_1(X); X \in \mathcal{M}_M\}$ and

$$d_1 := \frac{-(n-1) + \sqrt{(n-2)^2 + 4\lambda_1}}{2} \geq 1.$$

By Theorems 3.27, 5.1 and the compactness of $\mathcal{M}_M$, we have the following:

1. $H^d(M_\infty) = \{\text{constant functions}\}$ for every $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$ and every $0 < d < d_1$.
2. $H^{d_1}(\hat{M}_\infty) \neq \{\text{constant functions}\}$ for some $(\hat{M}_\infty, \hat{m}_\infty) \in \hat{\mathcal{M}}_M$.

Assume that there exists $d > 0$ such that $d < d_1$ and $\dim H^d(M_\infty) > 1$. Let $\epsilon > 0$ with $\epsilon < d_1 - d$. Applying Theorem 6.1 as $k = l = 1$ yields that there exists $(M_\infty, m_\infty) \in \hat{\mathcal{M}}_M$ such that $2 \leq \dim H^{d+\epsilon}(M_\infty)$. This is a contradiction. □

We end this subsection by giving the following. See also [**16**, Conjecture 0.9].

COROLLARY 6.4. *Let d be a positive number and $u \in H^d(M)$. Then we have*

$$\liminf_{t \to \infty} \left(\sup_{s \in K} U_u^{g_M}(ts) \right) \leq d$$

for every compact subset K of $(0, \infty)$.

PROOF. Without loss of generality, we can assume that u is not a constant function. By the proof of Theorem 6.1, for every $\epsilon > 0$, there exist sequences of positive numbers $\{R_i\}_i$, $\{\hat{R}_i\}_i$, an asymptotic cone (M_∞, m_∞) of M and a nonconstant harmonic function $u_\infty \in H^{d+\epsilon}(M_\infty)$ such that $R_i \to \infty$, $\hat{R}_i \to \infty$, $(M, m, R_i^{-1} d_M) \to (M_\infty, m_\infty)$, $\sup_i \mathbf{Lip}^{R_i^{-1} d_M} \left((u)_{\hat{R}_i}|_{B_R^{R_i^{-1} d_M}(m_i)} \right) < \infty$ for every $R > 0$, and that $(u)_{\hat{R}_i} \to u_\infty$ on M_∞. By the definition of $U_u^{g_M}(t)$, we have $U_{(u)_{\hat{R}_i}}^{R_i^{-2} g_M}(s) = U_u^{R_i^{-2} g_M}(s) = U_u^{g_M}(R_i s)$ for every $s > 0$ and every i. Thus, since $\lim_{i \to \infty} \left(\sup_{s \in K} \left| U_{(u)_{\hat{R}_i}}^{R_i^{-2} g_M}(s) - U_{u_\infty}(s) \right| \right) = 0$ and $U_{u_\infty} \leq d + \epsilon$, we have $\liminf_{t \to \infty} (\sup_{s \in K} U_u^{g_M}(ts)) \leq d + \epsilon$. Therefore, we have the assertion. □

7. Appendix: a co-area formula on metric cones.

In this section we will prove a co-area formula for the distance function from the pole on a noncollapsed metric cone used in the previous sections. See Proposition 7.6 for the precise statement. Throughout this subsection, we fix a pointed metric measure space (Y, y, υ) which is the Gromov-Hausdorff limit of a sequence of n-dimensional complete Riemannian manifolds $\{(M_i, m_i, \mathrm{vol} / \mathrm{vol}\, B_1(m_i))\}_{i < \infty}$ with $\mathrm{Ric}_{M_i} \geq -(n-1)$. Assume that the following hold:

1. There exists a compact geodesic space X such that $\operatorname{diam} X \le \pi$ and $(Y, y) = (C(X), p)$.
2. $\dim_H X = n-1$, where $\dim_H X$ is the Hausdorff dimension of X.

Remark 7.1. We use the renormalized measure $\underline{\mathrm{vol}} := \mathrm{vol} / \mathrm{vol}\, B_1(m)$ here in order to apply several results given in [**36**] directly.

Note that by [**6**, Theorem 5.9], there exists $C > 0$ such that $\upsilon = CH^n$. First, we recall the following definitions of lower dimensional Hausdorff measures associated to υ, and of standard (spherical) Hausdorff measures. See Section 2 in [**7**]. For convenience, we use the notaion: $r^{-\alpha}\upsilon(B_r(x)) = 0$ if $r = 0$. For every $\alpha \ge 0$, every $\delta > 0$ and every subset A of Y, let

$$(\upsilon_{-\alpha})_\delta(A) := \inf\left\{\sum_{i=1}^\infty r_i^{-\alpha}\upsilon(B_{r_i}(x_i));\ x_i \in Y,\ 0 \le r_i < \delta,\ A \subset \bigcup_{i=1}^\infty B_{r_i}(x_i)\right\},$$

$$(H^\alpha)_\delta(A) := \inf\left\{\sum_{i=1}^\infty \omega_\alpha r_i^\alpha;\ x_i \in Y,\ 0 \le r_i < \delta,\ A \subset \bigcup_{i=1}^\infty B_{r_i}(x_i)\right\}$$

and

$$\upsilon_{-\alpha}(A) := \lim_{\delta\to 0}(\upsilon_{-\alpha})_\delta(A), \quad H^\alpha(A) := \lim_{\delta\to 0}(H^\alpha)_\delta(A),$$

where $\omega_\alpha = 2\pi^{\alpha/2}/\Gamma(1+\alpha/2)$ and $\Gamma(t)$ is the gamma function. On the other hand, for every subset A of $\{1\}\times X (\subset C(X))$, let

$$(\upsilon_{-\alpha})_{X,\delta}(A) := \left\{\sum_{i=1}^\infty r_i^{-\alpha}\upsilon(B_{r_i}(x_i));\ x_i \in \{1\}\times X,\ 0 \le r_i < \delta,\ A \subset \bigcup_{i=1}^\infty B_{r_i}(x_i)\right\},$$

$$(H^\alpha)_{X,\delta}(A) := \left\{\sum_{i=1}^\infty \omega_\alpha r_i^\alpha;\ x_i \in \{1\}\times X,\ 0 \le r_i < \delta,\ A \subset \bigcup_{i=1}^\infty B_{r_i}(x_i)\right\}$$

and

$$(\upsilon_{-\alpha})_X(A) := \lim_{\delta\to 0}(\upsilon_{-\alpha})_\delta(A), \quad H^\alpha_X(A) := \lim_{\delta\to 0}(H^\alpha)_\delta(A).$$

Note that it is easy to check the following:

1. We have $\upsilon_{-\alpha}(A) \le (\upsilon_{-\alpha})_X(A)$ and $H^\alpha(A) \le H^\alpha_X(A)$ for every subset A of $\{1\}\times X$.
2. Let ϕ be a map from (X, d_X) to $(\{1\}\times X, d_{C(X)})$ defined by $\phi(x) = (1, x)$. Then we have $H^{n-1}(A) = H^{n-1}_X(\phi(A))$ for every subset A of X.

Lemma 7.2. *We have $\upsilon_{-1}(A) = (\upsilon_{-1})_X(A)$ for every Borel subset A of $\{1\}\times X$.*

Proof. Fix $\delta, \epsilon > 0$. Then, there exists an open covering $\{B_{r_i}(x_i)\}_i$ of A such that $0 \le r_i < \delta$, $x_i = (t_i, w_i) \in C(X) (= \boldsymbol{R}_{\ge 0} \times X/(\{0\}\times X))$ and $\left|(\upsilon_{-1})_\delta(A) - \sum_{i=1}^\infty r_i^{-1}\upsilon(B_{r_i}(x_i))\right| < \epsilon$. Without loss of generality, we can assume that $B_{r_i}(x_i) \cap A \neq \emptyset$

for every i. Let $y_i = (1, w_i) \in C(X)$ and $\hat{y}_i = (1, w_i) \in (\boldsymbol{R} \times X, \sqrt{d_{\boldsymbol{R}}^2 + d_X^2})$. It is easy to check that the map $\Phi_i(s, z) = (s, z)$ from $B_{5r_i}(x_i)$ to $\boldsymbol{R} \times X$ is an $(1 \pm \Psi(\delta))$-bi-Lipschitz embedding. Therefore, we have $B_{r_i}(x_i) \cap (\{1\} \times X) \subset B_{(1+\Psi(\delta))\sqrt{r_i^2 - \overline{x_i, y_i}^2}}(\hat{y}_i)$. On the other hand, since $|t_i - 1| \le \delta$, the map $\hat{\Phi}_i(t, w) = (t + t_i - 1, w)$ from $B_{(1+\Psi(\delta))r_i}(\hat{y}_i)$ to $C(X)$ is an $(1 \pm \Psi(\delta))$-bi-Lipschitz embedding. Since $\hat{\Phi}_i(\hat{y}_i) = x_i$, we have Image $\hat{\Phi} \subset B_{(1+\Psi(\delta))r_i}(x_i)$. Therefore Bishop-Gromov volume comparison theorem for H^n yields $H^n(B_{(1+\Psi(\delta))r_i}(\hat{y}_i)) \le (1 + \Psi(\delta))H^n(B_{(1+\Psi(\delta))r_i}(x_i)) \le (1 + \Psi(\delta))H^n(B_{r_i}(x_i))$. Thus, since $\upsilon = CH^n$, we have

$$\begin{aligned}(\upsilon_{-1})_{X,(1+\Psi(\delta))\delta}(A) &\le \sum_{i=1}^{\infty}((1 + \Psi(\delta))r_i)^{-1}CH^n(B_{(1+\Psi(\delta))r_i}(\hat{y}_i)) \\ &\le (1 + \Psi(\delta))\sum_{i=1}^{\infty} r_i^{-1}CH^n(B_{r_i}(x_i)) \\ &\le (1 + \Psi(\delta))((\upsilon_{-1})_{X,\delta}(A) + \epsilon).\end{aligned}$$

By letting $\epsilon \to 0$ and $\delta \to 0$, we have the assertion. $\square$

Similarly, we have the following:

Lemma 7.3. *We have $H_X^{n-1}(A) = H^{n-1}(A)$ for every Borel subset A of $\{1\} \times X$.*

Remark 7.4. It is easy to check that there exists $C_1 > 1$ such that $C_1^{-1}\upsilon_{-1}(A) \le H^{n-1}(A) \le C_1\upsilon_{-1}(A)$ for every Borel subset A of $C(X)$. The proof is as follows. By Bishop-Gromov volume comparison theorem for υ, there exists $V > 1$ such that $V^{-1} \le \lim_{r\to 0}\upsilon(B_r(x))/r^n \le V$ for every $x \in B_2(p)$. On the other hand, since $\upsilon = CH^n$, we have $\lim_{r\to 0}\upsilon(B_r((t, w)))/r^n = \lim_{r\to 0}\upsilon(B_r((s, w)))/r^n$ for every $0 < s < t < \infty$ and every $w \in X$. Then the assertion follows from these facts.

Lemma 7.5. *The product measure $H^1 \times H^{n-1}$ on $\boldsymbol{R} \times X$ is equal to H^n.*

Proof. Fix a Borel subset A of X. It suffices to check that $H^n([0, a] \times A) = aH^{n-1}(A)$ for every $a > 0$. Note that there exists a Borel subset $\hat{X}$ of X such that the following hold:

1. $H^{n-1}(X \setminus \hat{X}) = 0$.
2. For every $x \in \hat{X}$ and every $\epsilon > 0$, there exists $r_x^{\epsilon} > 0$ such that for every $0 < r < r_x^{\epsilon}$, there exist a compact subset C_r^x of $\overline{B}_r(x)$ and an $(1 \pm \epsilon)$-bi-Lipschitz embedding ϕ_r^x from C_r^x to $\boldsymbol{R}^{n-1}$ such that

$$\frac{H^{n-1}(\overline{B}_r(x) \setminus C_r^x)}{H^{n-1}(\overline{B}_r(x))} \le \epsilon.$$

For every $x \in \hat{X}$ and every $\epsilon > 0$, by Fubini's theorem, we have

$$\begin{aligned} H^n([0,a] \times C_r^x) &= (1 \pm \Psi(\epsilon; n)) H^n([0,a] \times \phi_r^x(C_r^x)) \\ &= (1 \pm \Psi(\epsilon; n)) a H^{n-1}(\phi_r^x(C_r^x)) \\ &= (1 \pm \Psi(\epsilon; n)) a H^{n-1}(C_r^x) \\ &= (1 \pm \Psi(\epsilon; n)) a H^{n-1}(\overline{B}_r(x)) \end{aligned}$$

for every sufficiently small $r > 0$. On the other hand, by the proof of [**37**, Lemma 5.2], we have $H^n([0,a] \times \hat{A}) \le C(n) a H^{n-1}(\hat{A})$ for every Borel subset $\hat{A}$ of X. Thus, we have

$$\lim_{r \to 0} \frac{H^n([0,a] \times \overline{B}_r(x))}{a H^{n-1}(\overline{B}_r(x))} = 1$$

for every $x \in \hat{X}$. Therefore, there exists a Borel subset $\hat{A}$ of A such that $H^{n-1}(A \setminus \hat{A}) = 0$,

$$\lim_{r \to 0} \frac{H^n([0,a] \times \overline{B}_r(x))}{a H^{n-1}(\overline{B}_r(x))} = 1$$

and

$$\lim_{r \to 0} \frac{H^{n-1}(A \cap \overline{B}_r(x))}{H^{n-1}(\overline{B}_r(x))} = 1$$

for every $x \in \hat{A}$. Note that $H^n([0,a] \times (A \setminus \hat{A})) \le C(n) a H^{n-1}(A \setminus \hat{A}) = 0$. Fix a sufficiently small $\epsilon > 0$. By a standard covering argument (c.f. [**38**, Proposition 2.2]), there exists a pairwise disjoint collection $\{\overline{B}_{r_i}(x_i)\}_i$ such that $x_i \in \hat{A}$, $r_i < \epsilon$, $\hat{A} \setminus \bigcup_{i=1}^N \overline{B}_{r_i}(x_i) \subset \bigcup_{i=N+1}^\infty \overline{B}_{5r_i}(x_i)$ for every $N \in \boldsymbol{N}$, and that

$$\left| \frac{H^n([0,a] \times \overline{B}_{r_i}(x_i))}{a H^{n-1}(\overline{B}_{r_i}(x_i))} - 1 \right| + \left| \frac{H^{n-1}(A \cap \overline{B}_{r_i}(x_i))}{H^{n-1}(\overline{B}_{r_i}(x_i))} - 1 \right| < \epsilon$$

for every i. Let $N_0 \in \boldsymbol{N}$ with $\sum_{i=N_0+1}^\infty H^{n-1}(\overline{B}_{r_i}(x_i)) < \epsilon$. Then, we have

$$\begin{aligned} H^n([0,a] \times \hat{A}) &\le \sum_{i=1}^{N_0} H^n([0,a] \times \overline{B}_{r_i}(x_i)) + \sum_{i=N_0+1}^{\infty} H^n([0,a] \times \overline{B}_{5r_i}(x_i)) \\ &\le \sum_{i=1}^{N_0} H^n([0,a] \times \overline{B}_{r_i}(x_i)) + aC(n) \sum_{i=N_0+1}^{\infty} H^{n-1}(\overline{B}_{5r_i}(x_i)) \\ &\le \sum_{i=1}^{N_0} H^n([0,a] \times \overline{B}_{r_i}(x_i)) + \Psi(\epsilon; n, a, C_1) \\ &\le a(1+\epsilon) \sum_{i=1}^{N_0} H^{n-1}(\overline{B}_{r_i}(x_i)) + \Psi(\epsilon; n, a, C_1) \\ &\le a(1+\epsilon)^2 (H^{n-1}(\hat{A}) + \epsilon) + \Psi(\epsilon; n, a, C_1). \end{aligned}$$

Therefore, by letting $\epsilon \to 0$, we have

$$H^n([0,a] \times A) \le aH^{n-1}(A).$$

On the other hand, we have

$$\begin{aligned} aH^{n-1}(A) &= a\left(\sum_{i=1}^{N_0} H^{n-1}(\overline{B}_{r_i}(x_i)) + \Psi(\epsilon; n, C_1)\right) \\ &\le (1+\epsilon)\sum_{i=1}^{N_0} H^n([0,a] \times \overline{B}_{r_i}(x_i)) + \Psi(\epsilon; n, a, C_1) \end{aligned}$$

and

$$\frac{H^n([0,a] \times (\overline{B}_{r_i}(x_i) \setminus A))}{H^n([0,a] \times \overline{B}_{r_i}(x_i))} \le C(n)(1+\epsilon)\frac{aH^{n-1}(\overline{B}_{r_i}(x_i) \setminus A)}{aH^{n-1}(\overline{B}_{r_i}(x_i))} \le \Psi(\epsilon; n).$$

Therefore, we have

$$\begin{aligned} aH^{n-1}(A) &\le (1+\epsilon)\sum_{i=1}^{N_0} H^n([0,a] \times \overline{B}_{r_i}(x_i)) + \Psi(\epsilon; n, a, C_1) \\ &\le (1+\Psi(\epsilon; n))\sum_{i=1}^{N_0} H^n\left([0,a] \times (\overline{B}_{r_i}(x_i) \cap A)\right) + \Psi(\epsilon; n, a, C_1) \\ &\le (1+\Psi(\epsilon; n))H^n([0,a] \times A) + \Psi(\epsilon; n, a, C_1). \end{aligned}$$

Therefore, by letting $\epsilon \to 0$, we have

$$aH^{n-1}(A) \le H^n([0,a] \times A).$$

Thus, we have the assertion. □

We now give a co-area formula on $C(X)$:

Proposition 7.6. *We have*

$$\int_{C(X)} f dH^n = \int_0^\infty \int_{\partial B_t(p)} f dH^{n-1} dt$$

for every $f \in L^1(C(X))$.

Proof. By [**36**, Theorem 5.2] and Remark 7.4, it suffices to check that

$$\lim_{r \to 0} \frac{1}{H^n(B_r(x))} \int_0^\infty H^{n-1}(\partial B_t(p) \cap \overline{B}_r(x)) dt = 1$$

for every $x \in C(X) \setminus \{p\}$. Fix $x \in C(X) \setminus \{p\}$ and a sufficiently small positive number r. Let $R := \overline{p, x} > 0$. Then, since the map $\Phi(t, w) = (t, w)$ from $B_r(x)$ to $\boldsymbol{R} \times X$ is an $(1 \pm \Psi(r))$-bi-Lipschitz embedding, we have

$$B_{(1-\Psi(r))r}(\Phi(x)) \subset \Phi(B_r(x)) \subset B_{(1+\Psi(r))r}(\Phi(x)).$$

On the other hand, Lemma 7.5 and Fubini's Theorem yield

$$H^n\big(B_{(1+\Psi(r))r}(\Phi(x))\big) = \int_{R-(1+\Psi(r))r}^{R+(1+\Psi(r))r} H^{n-1}\big((\{t\} \times X) \cap B_{(1+\Psi(r))r}(\Phi(x))\big) dt.$$

Since $\Phi(\partial B_t(p) \cap B_r(x)) \subset (\{t\} \times X) \cap B_{(1+\Psi(r))r}(\Phi(x))$, we have

$$H^n\big(B_{(1+\Psi(r))r}(\Phi(x))\big) \geq (1 - \Psi(r; n)) \int_{R-(1+\Psi(r))r}^{R+(1+\Psi(r))r} H^{n-1}(\partial B_t(p) \cap B_r(x)) dt.$$

Therefore, we have

$$1 \geq \limsup_{r \to 0} \frac{1}{H^n(B_r(x))} \int_0^\infty H^{n-1}(\partial B_t(p) \cap B_r(x)) dt.$$

Similarly, we have

$$1 \leq \liminf_{r \to 0} \frac{1}{H^n(B_r(x))} \int_0^\infty H^{n-1}(\partial B_t(p) \cap B_r(x)) dt.$$

Therefore, we have the assertion. □

Proposition 7.7. *We have $\upsilon_{-1}(A) = C(n)CH^{n-1}(A)$ for every Borel subset A of $\{1\} \times X$.*

Proof. By [**12**], we have

$$\lim_{r \to 0} \frac{H^n(B_r(z))}{\omega_n r^n} = 1$$

for every $z \in \mathcal{R}_n(Y)$. Since $\mathcal{R}_n(Y) \cap (\{1\} \times X) = \{1\} \times \mathcal{R}_{n-1}(X)$, by Proposition 7.6, we have $H^{n-1}(X \setminus \mathcal{R}_{n-1}(X)) = 0$. Fix $\epsilon, \delta, \tau > 0$. Let

$$A_\tau^\epsilon := \left\{ a \in A \cap (\{1\} \times \mathcal{R}_{n-1}(X));\ \left| \frac{H^n(B_r(a))}{\omega_n r^n} - 1 \right| < \epsilon \text{ for every } 0 < r \leq \tau \right\}.$$

By the definition of υ_{-1}, there exists an open covering $\{B_{r_i}(x_i)\}_i$ of A_τ^ϵ such that $x_i \in A_\tau^\epsilon$, $r_i < \min\{\delta, \tau\}$ and $\left|\upsilon_{-1}(A_\tau^\epsilon) - \sum_{i=1}^\infty r_i^{-1} \upsilon(B_{r_i}(x_i))\right| < \epsilon$. Thus, we have

$$\begin{aligned}(H^{n-1})_\delta(A_\tau^\epsilon) &\le \sum_{i=1}^{\infty}\omega_{n-1}r_i^{n-1}\\ &\le \sum_{i=1}^{\infty}\frac{\omega_{n-1}}{\omega_n}r_i^{-1}(1+\epsilon)H^n(B_{r_i}(x_i))\\ &= \sum_{i=1}^{\infty}\frac{\omega_{n-1}}{\omega_n}(1+\epsilon)r_i^{-1}C^{-1}\upsilon(B_{r_i}(x_i))\\ &\le \sum_{i=1}^{\infty}\frac{\omega_{n-1}}{\omega_n}(1+\epsilon)C^{-1}(\upsilon_{-1}(A_\tau)+\epsilon).\end{aligned}$$

By letting $\delta \to 0$, $\tau \to 0$ and $\epsilon \to 0$, we have

$$CH^{n-1}(A) \le \frac{\omega_{n-1}}{\omega_n}\upsilon_{-1}(A).$$

Claim 7.8. *Let $Z = \mathcal{R}_n(Y) \cap (\{1\} \times X)$. Then we have $H^{n-1}((\{1\}\times X)\setminus Z) = 0$ and*

$$\lim_{r\to 0}\frac{H^{n-1}(\overline{B}_r(z)\cap(\{1\}\times X))}{\omega_{n-1}r^{n-1}} = 1$$

for every $z \in Z$.

The proof is as follows. Let $x \in X$ and let $\{r_i\}_i$ be a sequence of positive numbers with $r_i \to 0$. Assume that there exists a tangent cone $(T_xX, 0_x)$ of X at x such that $(X, x, r_i^{-1}d_X) \to (T_xX, 0_x)$. By [**6**, Theorem 5.9] and [**37**, Claim 4.5], we have $(C(X), r_i^{-1}d_{C(X)}, (1,x), H^n) \to (\boldsymbol{R}\times T_xX, (0,0_x), H^n)$. Moreover, since T_xX is H^{n-1}-rectifiable (see [**38**, Corollary 3.53]), by an argument similar to that in the proof of Lemma 7.5, we have $H^1 \times H^{n-1} = H^n$ on $\boldsymbol{R}\times T_xX$. Since $([-r_i, r_i]\times \overline{B}^{d_X}_{r_i}(x), r_i^{-1}d_{C(X)})$ Gromov-Hausdorff converges to $[-1,1]\times\overline{B}_1(0_x)$ as $i \to \infty$, [**38**, Proposition 4.12] yields

$$\lim_{i\to\infty}H^n\big([-r_i,r_i]\times\overline{B}^{d_X}_{r_i}(x)\big) = H^n\big([-1,1]\times\overline{B}_1(0_x)\big).$$

By Proposition 7.6, we have $H^n([-r_i,r_i]\times\overline{B}^{d_X}_{r_i}(x)) = (1\pm\Psi(r_i;n))2H^{n-1}(\overline{B}_1^{r_i^{-1}d_X}(x))$. In particular, we have

$$\lim_{i\to\infty}H^{n-1}\big(\overline{B}_1^{r_i^{-1}d_X}(x)\big) = H^{n-1}(\overline{B}_1(0_x)).$$

Therefore, we have Claim 7.8.

Let $W := \mathrm{Leb}(A\cap Z)$ with respect to the $(n-1)$-dimensional Hausdorff measure H^{n-1}. By a standard covering argument, there exists a pairwise disjoint collection $\{\overline{B}_{r_i}(a_i)\}_i$ such that $a_i \in W$, $r_i < \delta/100$, $W\setminus\bigcup_{i=1}^N\overline{B}_{r_i}(a_i) \subset \bigcup_{i=N+1}^\infty\overline{B}_{5r_i}(a_i)$ for every N, and

$$\left|\frac{H^n(B_{r_i}(a_i))}{\omega_n r_i^n}-1\right|+\left|\frac{H^{n-1}(\overline{B}_{r_i}(a_i)\cap W)}{\omega_{n-1}r_i^{n-1}}-1\right|<\epsilon$$

for every i. Let $N_0 \in \boldsymbol{N}$ with $\sum_{i=N_0+1}^{\infty} H^{n-1}(\overline{B}_{r_i}(a_i) \cap W) < \epsilon$. Then, we have $\sum_{i=N_0+1}^{\infty} H^{n-1}(\overline{B}_{5r_i}(a_i) \cap W) < \Psi(\epsilon; n, C_1)$. By the assumption, we have $\sum_{i=N_0+1}^{\infty} \omega_{n-1} r_i^{n-1} \leq \Psi(\epsilon; n, C_1)$. Therefore, we have

$$\begin{aligned}
(\upsilon_{-1})_\delta(W) &\leq \sum_{i=1}^{N_0} r_i^{-1}\upsilon(\overline{B}_{r_i}(a_i)) + \sum_{i=N_0+1}^{\infty}(5r_i)^{-1}\upsilon(\overline{B}_{5r_i}(a_i)) \\
&\leq \sum_{i=1}^{N_0} r_i^{-1}CH^n(\overline{B}_{r_i}(a_i)) + \sum_{i=N_0+1}^{\infty} C(n)Cr_i^{n-1} \\
&\leq \sum_{i=1}^{N_0} r_i^{-1}CH^n(\overline{B}_{r_i}(a_i)) + \Psi(\epsilon; n, C, C_1) \\
&\leq \sum_{i=1}^{N_0} C\omega_n r_i^{n-1}(1+\epsilon) + \Psi(\epsilon; n, C, C_1) \\
&\leq \frac{C\omega_n}{\omega_{n-1}}(1+\epsilon)\sum_{i=1}^{N_0} H^{n-1}(\overline{B}_{r_i}(a_i)\cap W) + \Psi(\epsilon; n, C, C_1) \\
&\leq \frac{C\omega_n}{\omega_{n-1}}(1+\epsilon)H^{n-1}(W) + \Psi(\epsilon; n, C, C_1).
\end{aligned}$$

By letting $\delta \to 0$ and $\epsilon \to 0$, we have

$$\upsilon_{-1}(A) \leq \frac{C\omega_n}{\omega_{n-1}} H^{n-1}(A).$$

Thus, we have the assertion. $\square$

We end this section by giving a proof of the following proposition:

Proposition 7.9. *We have*

$$H^{n-1}(B_t(x)) \leq C(n)\frac{t^{n-1}}{s^{n-1}}H^{n-1}(B_s(x))$$

for every $0 < s < t \leq \pi$ *and every* $x \in X$.

Proof. Note that there exists a universal constant $C_2 > 1$ such that for every metric space $\hat{X}$, the bi-Lipschitz map from $\hat{X}$ to $\{1\} \times \hat{X} \subset C(\hat{X})$ defined by $f_{\hat{X}}(\hat{x}) = (1, \hat{x})$ satisfies $\mathbf{Lip}\, f_{\hat{X}} + \mathbf{Lip}\, f_{\hat{X}}^{-1} \leq C_2$. Therefore, by [**36**, Theorem 5.7] and Proposition 7.6, we have

$$
\begin{aligned}
H^{n-1}(B_t(x)) &\leq C(n)H^{n-1}(B_{C_2t}(1,x)\cap(\{1\}\times X))\\
&= C(n)C^{-1}\upsilon_{-1}(B_{C_2t}(1,x)\cap(\{1\}\times X))\\
&\leq \frac{C(n)\upsilon(C_p(B_{C_2t}(1,x)\cap(\{1\}\times X))\cap A_p(\max\{0,1-C_2t\},1))}{C\operatorname{vol}A_p(\max\{0,1-C_2t\},1)}\\
&\leq \frac{C(n)}{Ct}\upsilon(B_{5C_2t}(1,x))\\
&\leq \frac{C(n)}{Ct}\frac{t^n}{s^n}\upsilon\big(B_{C_2^{-1}s}(1,x)\big)\\
&\leq C(n)\frac{t^{n-1}}{s^n}\int_{\max\{0,1-C_2^{-1}s\}}^{1+C_2^{-1}s} H^{n-1}\big(\partial B_r(p)\cap B_{C_2^{-1}s}(1,x)\big)dr\\
&\leq C(n)\frac{t^{n-1}}{s^n}\int_{\max\{0,1-C_2^{-1}s\}}^{1+C_2^{-1}s} r^{n-1}H^{n-1}\big(\partial B_1(p)\cap B_{C_2^{-1}s}(1,x)\big)dr\\
&\leq C(n)\frac{t^{n-1}}{s^n}sH^{n-1}\big(\partial B_1(p)\cap B_{C_2^{-1}s}(1,x)\big)\\
&\leq C(n)\frac{t^{n-1}}{s^{n-1}}H^{n-1}\big(\partial B_1(p)\cap B_{C_2^{-1}s}(1,x)\big)\\
&\leq C(n)\frac{t^{n-1}}{s^{n-1}}H^{n-1}(B_s(x)). \qquad \square
\end{aligned}
$$

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