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Log canonical algebras and modules

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Abstract. Let $(X/Z, B)$ be a lc pair with $K_X + B$ pseudo-effective/Z and Z affine. We show that $(X/Z, B)$ has a good log minimal model if and only if its log canonical algebra and modules are finitely generated.

1. Introduction.

Let $X \to Z$ be a projective morphism of varieties over $\mathbb{C}$ with $Z = \operatorname{Spec} A$ being affine. For any Cartier divisor L on X we have the graded ring

$$R(L) := \bigoplus_{m \geq 0} H^0(X, mL)$$

which is a graded A-algebra. On the other hand, for each $\mathscr{O}_X$-module $\mathscr{F}$ on X and each integer p, we have the graded $R(L)$-module $M^p_{\mathscr{F}}(L) = \bigoplus_{m \in \mathbb{Z}} M_m$ where $M_m = 0$ if $m < p$ but

$$M_m = H^0(X, \mathscr{F}(mL))$$

if $m \geq p$. Here $\mathscr{F}(mL)$ stands for $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(mL)$ and the module structure is given via the pairing

$$H^0(X, mL) \otimes H^0(X, \mathscr{F}(nL)) \to H^0(X, \mathscr{F}((m+n)L)).$$

If $\mathscr{F} = \mathscr{O}_X(D)$ for some divisor D, we usually write $M^p_D(L)$ instead of $M^p_{\mathscr{O}_X(D)}(L)$.

When $L = I(K_X + B)$ for a log canonical pair (X, B) and integer $I > 0$, we refer to $R(L)$ as a *log canonical algebra* and refer to the module $M^p_{\mathscr{F}}(L)$ as a *log canonical module*. The following theorem is the main result of this short note.

Theorem 1.1. *Assume that $(X/Z, B)$ is lc where $Z = \operatorname{Spec} A$, and let I be a positive integer so that $L := I(K_X + B)$ is Cartier. If $K_X + B$ is pseudo-effective/Z, then the following are equivalent*:

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(1) *$(X/Z, B)$ has a good log minimal model*;
(2) *$R(L)$ is a finitely generated A-algebra, and for any very ample/Z divisor G and integer p the module $M^p_G(L)$ is finitely generated over $R(L)$.*

The klt case of the theorem is a result of Demailly-Hacon-Păun [**3**]. Our proof below is somewhat different and more algebraic in nature, and it also works in the lc case. Note that we have assumed Z to be affine for simplicity of notation; the general case can be formulated and proved in a similar way.

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2. Preliminaries.

Varieties are assumed to be over $\mathbb{C}$ unless stated otherwise. We use the notion and notation of pairs and log minimal models as in [**1**]. Singularities such as lc, klt, and dlt are as in [**5**]. We use the numerical Kodaira dimension κ_σ introduced by Nakayama [**6**].

Rings are assumed to be commutative with identity. A graded ring is of the form $R = \bigoplus_{m\geq 0} R_m$, that is, graded by non-negative integers, and a graded module is of the form $M = \bigoplus_{m\in\mathbb{Z}} M_m$, that is, it is graded by the integers. For an element $(\dots, 0, \alpha, 0, \dots)$ of degree m we often abuse notation and just write α but keep in mind that α has degree m.

Remark 2.1 (Truncation principle).

(1) Let $R = \bigoplus_{m\geq 0} R_m$ be a graded ring and I a positive integer. Define the truncated ring $R^{[I]} = \bigoplus_{m\geq 0} R'_m$ by putting $R'_m = R_m$ if $I|m$ and $R'_m = 0$ otherwise. Note that the degree structure is different from the usual definition of truncation. However, it is more convenient for us to define it in this way.
(2) With R and I as in (1), assume that R_0 is a Noetherian ring and that R is an integral domain. It is well-known that: R is a finitely generated R_0-algebra if and only if $R^{[I]}$ is a finitely generated R_0-algebra.
(3) Again R and I are as in (1). Let $M = \bigoplus_{m\in\mathbb{Z}} M_m$ be a graded R-module. Let $N_i = \bigoplus_{m\in\mathbb{Z}} N_{m,i}$ where $N_{m,i} = M_m$ if $m \equiv i (\mathrm{mod}\ I)$ but $N_{m,i} = 0$ otherwise. Then, each N_i is a graded module over $R^{[I]}$ and we have the decomposition

$$M \simeq N_0 \oplus N_1 \oplus \cdots \oplus N_{I-1}$$

as graded $R^{[I]}$-modules. If the modules $N_0, \dots, N_{I-1}$ are finitely generated over $R^{[I]}$, then M is also a finitely generated $R^{[I]}$-module hence a finitely generated R-module too.

THEOREM 2.2. *Let $X \to Z$ be a projective morphism of normal varieties with $Z = \operatorname{Spec} A$, and let L be a Cartier divisor on X such that $R(L)$ is a finitely generated A-algebra. Fix an integer p. Then we have*:

1. *Assume that $M^p_G(L)$ is a finitely generated $R(L)$-module for any very ample/Z divisor G. Then $M^p_{\mathscr{F}}(L)$ is a finitely generated $R(L)$-module for every torsion-free coherent sheaf $\mathscr{F}$.*
2. *If L is big/Z, then $M^p_{\mathscr{F}}(L)$ is a finitely generated $R(L)$-module for every torsion-free coherent sheaf $\mathscr{F}$.*
3. *Let $\mathscr{F}$ be a coherent sheaf and $I > 0$ an integer. For each $0 \le i < I$, assume that $M^{q_i}_{\mathscr{F}(iL)}(IL)$ is a finitely generated $R(IL)$-module where $q_i \in \mathbb{Z}$ is the smallest number satisfying $q_i I + i \ge p$. Then $M^p_{\mathscr{F}}(L)$ is a finitely generated $R(L)$-module.*

PROOF. (1) Let G be a very ample/Z divisor and pick a reflexive coherent sheaf $\mathscr{F}$. There is a surjective morphism $\bigoplus_{j=1}^r \mathscr{O}_X(-l_jG) \to \mathscr{F}^\vee$ for some $l_j > 0$ where $\vee$ stands for dual. Taking the dual of this morphism gives an injective morphism

$$\mathscr{F} \simeq \mathscr{F}^{\vee\vee} \to \mathscr{E} = \bigoplus_{j=1}^r \mathscr{O}_X(l_jG)$$

which in turn gives an injective map $M^p_{\mathscr{F}}(L) \to M^p_{\mathscr{E}}(L)$. By assumptions, $M^p_{\mathscr{E}}(L)$ is finitely generated over $R(L)$ which in particular means that $M^p_{\mathscr{E}}(L)$ is Noetherian as $R(L)$ is Noetherian. Therefore, each submodule of $M^p_{\mathscr{E}}(L)$ is also finitely generated over $R(L)$, in particular, $M^p_{\mathscr{F}}(L)$.

Now assume that $\mathscr{F}$ is just a torsion-free coherent sheaf. The natural morphism $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is injective (cf. [**4**]). So, we get an injective map $M^p_{\mathscr{F}}(L) \to M^p_{\mathscr{F}^{\vee\vee}}(L)$ and the claim follows since $\mathscr{F}^{\vee\vee}$ is a reflexive sheaf.

(2) By (1), it is enough to verify the finite generation of $M^p_G(L)$ for very ample/Z divisors G. Since L is big/Z, there is $n > 0$ such that $nL \sim E + G$ for some effective Cartier divisor E. Thus, there is an injective map

$$M^p_G(L) \to M^p_{E+G}(L) \simeq M^p_{nL}(L).$$

So, it is enough to show that $M^p_{nL}(L)$ is a finitely generated $R(L)$-module. This in turn follows from finite generation of $M^{-n}_{nL}(L)$. Now the elements of degree $-n$ are $H^0(X, -nL + nL) = H^0(X, \mathscr{O}_X)$ which contains $1 \in \mathscr{O}_X(X)$. If $\alpha \in M^{-n}_{nL}(L)$ is a homogeneous element of degree $m \geq -n$, that is, an element of $H^0(X, mL + nL)$, then $\alpha = \alpha \cdot 1$ where we consider the second α as an element of $R(L)$ of degree $m + n$ and we consider 1 as an element of $M^{-n}_{nL}(L)$ of degree $-n$. So, $M^{-n}_{nL}(L)$ is generated over $R(L)$ by the element 1 of degree $-n$.

(3) We can write $M^p_{\mathscr{F}}(L) \simeq N_0 \oplus N_1 \oplus \cdots \oplus N_{I-1}$ as in Remark 2.1 (3). Let N'_i be the module over $R(IL)$ whose m-th degree summand is just $N_{mI+i,i} = M_{mI+i}$. In fact, $N'_i = M^{q_i}_{\mathscr{F}(iL)}(IL)$ where $q_i \in \mathbb{Z}$ is the smallest number satisfying $q_i I + i \geq p$. Note that the degree n elements of $R(IL)$ are the same as the degree nI elements of $R(L)^{[I]}$, and the degree m elements of N'_i are the same as the degree $mI + i$ elements of N_i. By assumptions, N'_i is a finitely generated $R(IL)$-module. Therefore, N_i is a finitely generated $R(L)^{[I]}$-module, and so by Remark 2.1 (3) we are done. □

3. Proof of Theorem 1.1.

Throughout this section we let $X \to Z$ be a projective morphism of normal varieties over $\mathbb{C}$ with $Z = \operatorname{Spec} A$.

Lemma 3.1. *Assume that $Z = \mathrm{pt}$ and L is a Cartier divisor on X. Further assume that for any very ample divisor G the module $M^0_G(L)$ is finitely generated over $R(L)$. Then, $\kappa(L) = \kappa_\sigma(L)$.*

Proof. The inequality $\kappa(L) \leq \kappa_\sigma(L)$ follows from the fact that $\kappa(L) = \kappa(JL)$ and $\kappa_\sigma(L) = \kappa_\sigma(JL)$ for any positive integer J and the fact that for some J and certain constants c_1, $c_2 > 0$ we have

$$c_1 m^{\kappa(L)} \leq h^0(X, mJL) \leq c_2 m^{\kappa(L)}$$

for any $m \gg 0$.

For the converse $\kappa(L) \geq \kappa_\sigma(L)$, we may assume that $\kappa_\sigma(L) \geq 0$ and we can choose a very ample divisor G so that $\kappa_\sigma(L)$ satisfies

$$\limsup_{m \to +\infty} \frac{h^0(X, mL + G)}{m^{\kappa_\sigma(L)}} > 0.$$

By assumptions, $M^0_G(L)$ is a finitely generated $R(L)$-module. Let $\{\alpha_1, \dots, \alpha_r\}$ be a set of generators of homogeneous elements with $n_i := \deg \alpha_i$. For any $\alpha \in M^0_G(L)$ of degree m, there are homogeneous elements $a_i \in R(L)$ such that $\alpha = \sum_i a_i \alpha_i$. It is clear that $\deg a_i = m - n_i$. Thus,

$$h^0(X, (m - n_1)L) + \cdots + h^0(X, (m - n_r)L) \geq h^0(X, mL + G)$$

which implies that

$$\limsup_{m\to+\infty} \frac{h^0(X,(m-n_1)L)+\cdots+h^0(X,(m-n_r)L)}{m^{\kappa_\sigma(L)}} > 0$$

hence $\kappa(L) \geq \kappa_\sigma(L)$. □

The next result is well-known but we include its proof for convenience.

LEMMA 3.2. *Let L be a Cartier divisor on X with $h^0(X, nL) \neq 0$ for some $n > 0$. Then, the following are equivalent*:

(1) *$R(L)$ is a finitely generated A-algebra*;
(2) *there exist a projective birational morphism $f\colon W \to X$ from a smooth variety, a positive integer J, and Cartier divisors E and F such that $|F|$ is base point free, and*

$$\operatorname{Mov} f^*mJL = mF \quad \textit{and} \quad \operatorname{Fix} f^*mJL = mE$$

for every positive integer m.

PROOF. Assume that $R(L)$ is a finitely generated A-algebra. Perhaps after replacing L with JL for some positive integer J, we may assume that the algebra $R(L)$ is generated by elements $\alpha_1, \dots, \alpha_r$ of degree 1, and that there is a resolution $f\colon W \to X$ on which $f^*L = F + E$ where F is free, $\operatorname{Mov} f^*L = F$, and $\operatorname{Fix} f^*L = E$. We could in addition assume that $F \geq 0$ with no common components with E. Obviously, $\operatorname{Fix} mf^*L \leq mE$ for any $m > 0$. Suppose that equality does not hold for some $m > 0$. Take $m > 0$ minimal with this property. Since $E = \operatorname{Fix} f^*L$, $m > 1$. There is $\alpha \in H^0(W, mf^*L)$ and a component S of E such that $\mu_S(\alpha) < 0$ where μ stands for multiplicity, that is, the coefficient and (α) is the divisor associated to the rational function α. By assumptions, $\alpha = \sum a_i\alpha_i$ where a_i are elements of $H^0(W, (m-1)f^*L)$. Thus,

$$\mu_S(\alpha) \geq \min\{\mu_S(a_i) + \mu_S(\alpha_i)\} = \mu_S(a_j) + \mu_S(\alpha_j)$$

for some j. The choice of m ensures that $\mu_S(\alpha_j) \geq 0$ and $\mu_S(a_j) \geq 0$. This contradicts $\mu_S(\alpha) < 0$.

Conversely, assume that there exist $f\colon W \to X$, J, E, and F as in the theorem. Then, $R(JL) \simeq R(f^*JL) \simeq R(F)$ is a finitely generated A-algebra as $|F|$ is base point free. This implies that $R(L)$ is a finitely generated A-algebra by Remark 2.1. □

LEMMA 3.3. *Let L be a Cartier divisor on X with $h^0(X, nL) \neq 0$ for some $n > 0$ and with $R(L)$ a finitely generated A-algebra. Assume further that $M^0_{G'}(L)$ is a finitely generated $R(L)$-module for any very ample/Z divisor G'. Let f, W, F, E, J be as in Lemma 3.2. Fix a nonnegative integer r and a very ample/Z divisor G on W. Then,*

$$\operatorname{Supp}\operatorname{Fix}(m(f^*JL + rF) + G) = \operatorname{Supp} E$$

for every integer $m \gg 0$.

PROOF. Let G'' be a very ample/Z divisor on W and let G' be a very ample/Z divisor on X such that $G'' \leq f^*G'$. By assumptions, $R(L)$ is a Noetherian ring and $M = M^0_{G'}(L)$ is a Noetherian $R(L)$-module. Moreover, $R(L)$ is integral over the ring $R(L)^{[J]}$ which implies that M is a finitely generated $R(L)^{[J]}$-module. Put $N_0 = \bigoplus_{m\geq 0} N_{m,0}$ where $N_{m,0} = M_m$ if $J|m$ but $N_{m,0} = 0$ otherwise, as in Remark 2.1 (3). Since N_0 is an $R(L)^{[J]}$-submodule of M, it is finitely generated over $R(L)^{[J]}$. This corresponds to saying that $M^0_{G'}(JL)$ is a finitely generated $R(JL)$-module. On the other hand, $M^0_{G''}(f^*JL)$ is a submodule of $M^0_{f^*G'}(f^*JL)$ hence a finitely generated $R(f^*JL)$-module. Thus, after replacing L with f^*JL and X with W we can assume that $J = 1$ and $W = X$. We may also assume that $F, G \geq 0$ and that $F + G$ has no common component with E.

Obviously,

$$\operatorname{Supp}\operatorname{Fix}(m(L + rF) + G) \subseteq \operatorname{Supp} E$$

for every integer $m > 0$. Assume that there is a component S of E which does not belong to $\operatorname{Supp}\operatorname{Fix}(m(L+rF)+G)$ for some $m > 0$. Let $\alpha \in H^0(X, m(L+rF)+G)$ so that

$$S \nsubseteq \operatorname{Supp}((\alpha) + m(L + rF) + G)$$

which in particular means that $\mu_S(\alpha) = -m\mu_S E$. Since

$$(m + mr)L + G = m(L + rF) + G + mrE$$

and $mrE \geq 0$, there is α' in $M^0_G(L)$ of degree $m + mr$ such that $\alpha' = \alpha$ as rational functions on X.

Assume that $\{\alpha_1, \ldots, \alpha_r\}$ is a set of homogeneous generators of $M^0_G(L)$ with $n_i := \deg \alpha_i$. We can write $\alpha' = \sum a_i\alpha_i$ where $a_i \in R(L)$ is homogenous of degree $m + mr - n_i$. Therefore,

$$\mu_S(\alpha') \geq \min\{\mu_S(a_i) + \mu_S(\alpha_i)\}.$$

Since

$$\mathrm{Fix}(m + mr - n_i)L = (m + mr - n_i)E$$

we have $\mu_S(a_i) \geq 0$ hence if the above minimum is attained at index j, then

$$-m\mu_S E = \mu_S(\alpha) = \mu_S(\alpha') \geq \mu_S(\alpha_j)$$

from which we get $m\mu_S E \leq -\mu_S(\alpha_j)$. This means that such m cannot be too large so the theorem holds for $m \gg 0$. □

Proof of Theorem 1.1. (1) $\implies$ (2): Assume that $(X/Z, B)$ has a good log minimal model $(Y/Z, B_Y)$. By Theorem 2.2, we can replace I with a multiple so that we can assume that $|I(K_Y + B_Y)|$ is base point free. Let $f\colon W \to X$ and $g\colon W \to Y$ be a common resolution. Then, we can write

$$f^*I(K_X + B) = g^*I(K_Y + B_Y) + E$$

where $E \geq 0$ and exceptional/Y [**1**, Remark 2.4]. Then, by letting $L_Y := I(K_Y + B_Y)$ we have $R(L) \simeq R(L_Y)$ as A-algebras and this is a finitely generated A-algebra as $|L_Y|$ is base point free by assumptions. Let $\mathscr{G}$ be any torsion-free coherent sheaf on Y and let $\pi\colon Y \to T/Z$ be the contraction defined by $|L_Y|$. There is an ample/Z divisor N on T such that $L_Y \sim \pi^*N$. Then, by the projection formula

$$\pi_*(\mathscr{G}(mL_Y)) \simeq (\pi_*\mathscr{G})(mN)$$

hence

$$H^0(Y, \mathscr{G}(mL_Y)) \simeq H^0(T, (\pi_*\mathscr{G})(mN)).$$

So $R(L) \simeq R(L_Y) \simeq R(N)$ as A-algebras and $M^p_{\mathscr{G}}(L_Y) \simeq M^p_{\pi_*\mathscr{G}}(N)$ as modules. By Theorem 2.2 (2), $M^p_{\pi_*\mathscr{G}}(N)$ is a finitely generated $R(N)$-module hence $M^p_{\mathscr{G}}(L_Y)$ is a finitely generated $R(L_Y)$-module.

Next we prove the finite generation of $M^p_{\mathscr{F}}(L)$ for any coherent torsion-free sheaf $\mathscr{F}$ on X. By Theorem 2.2 (1), we may assume that $\mathscr{F} = \mathscr{O}_X(G)$ where G is some very ample/Z divisor. For each m we have an isomorphism

$$H^0(X, mL + G) \simeq H^0(W, f^*mL + f^*G)$$

and this is isomorphic to a subspace of $H^0(Y, mL_Y + g_*f^*G)$. So, $M^p_G(L)$ is isomorphic to a submodule of $M^p_{g_*f^*G}(L_Y)$. Therefore, $M^p_G(L)$ is a finitely generated $R(L)$-module because $M^p_{g_*f_*G}(L_Y)$ is a finitely generated $R(L_Y)$-module and $R(L_Y)$ is Noetherian.

(2) $\implies$ (1): We may assume that $X \to Z$ is a contraction. Let V be the generic fibre of $X \to Z$, and let K be the function field of Z. As Z is affine, by base change theorems, $R(L|_V) \simeq R(L) \otimes_A K$ is a finitely generated K-algebra, and for any very ample/Z divisor G on X the module $M^0_{G|_V}(L|_V) \simeq M^0_G(L) \otimes_A K$ is finitely generated over $R(L|_V)$. By Theorem 2.2 and Lemma 3.1, $\kappa(L|_V) \geq 0$ which in particular implies that $h^0(X, nL) \neq 0$ for some $n > 0$.

Let f, W, E, F, J be as in Theorem 3.2. We may assume that f gives a log resolution of $(X/Z, B)$. Let B_W be $B^\sim$ plus the reduced exceptional divisor of f where $B^\sim$ is the birational transform of B. We can write

$$JI(K_W + B_W) = JIf^*(K_X + B) + E'$$

where $E' \geq 0$ is exceptional/X. It is enough to construct a good log minimal model for $(W/Z, B_W)$ [**1**, Remark 2.4]. We will show that $(W/Z, B_W)$ also satisfies the finite generation assumptions. Pick any $\alpha \in H^0(W, mJI(K_W + B_W))$ and let

$$P := (\alpha) + mJI(K_W + B_W) = (\alpha) + mF + mE + mE'.$$

Since $P - mE' \equiv 0/X$ and $f_*(P - mE') \geq 0$, we have $P - mE' \geq 0$ by the negativity lemma. Moreover, $\operatorname{Fix}(P - mE') = mE$ hence $P - mE' \geq mE$. This implies that

$$\operatorname{Fix} mJI(K_W + B_W) = mE + mE' \quad \text{and} \quad \operatorname{Mov} mJI(K_W + B_W) = mF.$$

Therefore, $R(L_W) \simeq R(L)$ where $L_W = I(K_W + B_W)$. On the other hand, if G is a very ample/Z divisor on W, then there is a very ample/Z divisor G' on X such that $G \leq f^*G'$ hence $M^p_G(L_W)$ is a finitely generated $R(L_W)$-module as it is a submodule of $M^p_{f^*G'}(L_W) \simeq M^p_{G'}(L)$. Therefore, by replacing $(X/Z, B)$ with $(W/Z, B_W)$ from now on we can assume that $W = X$ and that f is the identity.

Let $g\colon X \to T$ be the contraction/Z defined by $|F|$. Let F' be a general element of $|rF|$ for some $r \in \mathbb{N}$. We can choose a very ample/Z divisor $G \geq 0$ so that $K_X + B + F' + G$ is nef/Z and that $(X/Z, B + F' + G)$ is dlt. Run the LMMP/Z on $K_X + B + F'$ with scaling of G. By boundedness of the length of extremal rays due to Kawamata, if r is sufficiently large, then the LMMP is over T, i.e. only extremal rays over T are contracted. Suppose that, perhaps after some log flips and divisorial contractions, we get an infinite sequence of log flips $X_i \dashrightarrow X_{i+1}/Z_i$. Let λ_i be the numbers appearing in the LMMP with scaling in the above sequence of log flips, that is, $K_{X_i} + B_i + F'_i + \lambda_i G_i$ is nef/Z and numerically trivial over Z_i where B_i, F'_i, G_i are birational transforms on X_i. By [**2**], $\lambda := \lim \lambda_i = 0$. Moreover, by the base point free theorem, each $K_{X_i} + B_i + F'_i + \lambda_i G_i$ is semi-ample/Z (of course $K_{X_i} + B_i + F'_i + \lambda_i G_i$ may not be klt but we can use the ampleness of G to reduce the claim to the klt case). Thus, if S is a component of E not contracted by the LMMP, then there exist

$$0 \leq N_i \sim_{\mathbb{Q}} K_X + B + F' + \lambda_i G$$

not containing S. This contradicts Lemma 3.3 in view of Theorem 3.4 below. Therefore, E is contracted by the LMMP and $K_{X_i} + B_i + F'_i$ is $\mathbb{Q}$-linearly a multiple of F'_i. But $|F'_i|$ is base point free as the LMMP we ran is over T. Thus, the LMMP terminates with a good log minimal model. □

The following theorem was proved by Nakayama [**6**, Theorem 6.1.3]. He treated the case $Z = \mathrm{pt}$ but his proof works for general Z. For convenience of the reader we present his proof.

THEOREM 3.4. *Assume that $W \to Z = \operatorname{Spec} A$ is a projective morphism from a smooth variety, $w \in W$ a closed point, and D a Cartier divisor on W. Assume further that for some effective divisor C there exist an infinite sequence of positive rational numbers $t_1 > t_2 > \cdots$ with $\lim t_i = 0$, and effective $\mathbb{Q}$-divisors $N_i \sim_{\mathbb{Q}} D + t_i C$ with $w \notin \operatorname{Supp} N_i$. Then, there is a very ample/Z divisor G on W such that $w \notin \operatorname{Bs} |mD + G|$ for any $m > 0$.*

PROOF. Let $f \colon W' \to W$ be the blow up at w with E the exceptional divisor, $D' = f^*D$, $C' = f^*C$, and $N'_i = f^*N_i$. Let G be a very ample/Z divisor on W such that $H := f^*G - K_{W'} - E$ is ample/Z and $H - \epsilon C'$ is also ample/Z for some $\epsilon > 0$. Put $G' = f^*G$. For each $m > 0$, we can write

$$\begin{aligned} mD' + G' = K_{W'} + E + H + mD' &= K_{W'} + E + H - mt_i C' + m(t_i C' + D') \\ &\sim_{\mathbb{Q}} K_{W'} + E + H - mt_i C' + mN'_i \end{aligned}$$

where we choose t_i so that $mt_i < \epsilon$. By assumptions, E does not intersect N'_i. Thus, the multiplier ideal sheaf $\mathscr{I}_i$ of mN'_i is isomorphic to $\mathscr{O}_{W'}$ near E. In particular, we have the natural exact sequence

$$0 \to \mathscr{I}_i(mD' + G' - E) \to \mathscr{I}_i(mD' + G') \to \mathscr{O}_E(mD' + G') \to 0$$

from which we derive the exact sequence

$$\begin{aligned} H^0(W', \mathscr{I}_i(mD'+G')) &\to H^0(E, (mD'+G')|_E) \\ &\to H^1(W', \mathscr{I}_i(mD'+G'-E)) = 0 \end{aligned}$$

where the last vanishing follows from Nadel vanishing. On the other hand, $(mD'+G')|_E \sim 0$ hence some section of $\mathscr{I}_i(mD'+G')$ does not vanish on E. But

$$\mathscr{I}_i(mD'+G') \subseteq \mathscr{O}_{W'}(mD'+G')$$

so some section of $\mathscr{O}_{W'}(mD'+G')$ does not vanish on E, which simply means that w is not in $\operatorname{Bs}|mD+G|$. □

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