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Asymptotic analysis of oscillatory integrals via the Newton polyhedra of the phase and the amplitude

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Abstract. The asymptotic behavior at infinity of oscillatory integrals is in detail investigated by using the Newton polyhedra of the phase and the amplitude. We are especially interested in the case that the amplitude has a zero at a critical point of the phase. The properties of poles of local zeta functions, which are closely related to the behavior of oscillatory integrals, are also studied under the associated situation.

1. Introduction.

In this paper, we investigate the asymptotic behavior of oscillatory integrals, that is, integrals of the form

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau f(x)} \varphi(x)\chi(x)dx, \tag{1.1}$$

for large values of the real parameter τ, where f, φ, χ are real-valued smooth functions defined on $\mathbb{R}^n$ and χ is a cut-off function with small support which identically equals one in a neighborhood of the origin in $\mathbb{R}^n$. Here f and $\varphi\chi$ are called the *phase* and the *amplitude*, respectively.

By the principle of stationary phase, the main contribution in the behavior of the integral (1.1) as $\tau \to +\infty$ is given by the local properties of the phase around its critical points. We assume that the phase has a critical point at the origin, i.e., $\nabla f(0) = 0$. The following deep result has been obtained by using Hironaka's resolution of singularities [**16**] (cf. [**21**]). If f is real analytic on a neighborhood of the origin and the support of χ is contained in a sufficiently small neighborhood of the origin, then the integral $I(\tau)$ has an asymptotic expansion of the form

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$$I(\tau) \sim e^{i\tau f(0)} \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha k} \tau^{\alpha} (\log \tau)^{k-1} \quad \text{as } \tau \to +\infty, \tag{1.2}$$

where α runs through a finite number of arithmetic progressions not depending on φ and χ, which consist of negative rational numbers. Our interest focuses the largest α occurring in (1.2). Let $S(f, \varphi)$ be the set of pairs (α, k) such that for each neighborhood of the origin in $\mathbb{R}^n$, there exists a cut-off function χ with support in this neighborhood for which $C_{\alpha k} \neq 0$ in the asymptotic expansion (1.2). We denote by $(\beta(f, \varphi), \eta(f, \varphi))$ the maximum of the set $S(f, \varphi)$ under the lexicographic ordering, i.e. $\beta(f, \varphi)$ is the maximum of values α for which we can find k so that (α, k) belongs to $S(f, \varphi)$; $\eta(f, \varphi)$ is the maximum of integers k satisfying that $(\beta(f, \varphi), k)$ belongs to $S(f, \varphi)$. We call $\beta(f, \varphi)$ *oscillation index* of (f, φ) and $\eta(f, \varphi)$ its *multiplicity.* (This multiplicity, less one, is equal to the corresponding multiplicity in [**1**, p. 183].)

From various points of view, the following is an interesting problem: What kind of information of the phase and the amplitude determines (or estimates) the oscillation index $\beta(f, \varphi)$ and its multiplicity $\eta(f, \varphi)$? There have been many interesting studies concerning this problems ([**28**], [**6**], [**25**], [**7**], [**5**], [**13**], [**14**], [**15**], etc.). In particular, the significant work of Varchenko [**28**] shows the following by using the theory of toric varieties: By the geometry of the Newton polyhedron of f, the oscillation index can be estimated and, moreover, this index and its multiplicity can be exactly determined when $\varphi(0) \neq 0$, under a certain nondegenerate condition of the phase (see Theorem 2.1 in Section 2). Since his study, the investigation of the behavior of oscillatory integrals has been more closely linked with the theory of singularities. Refer to the excellent expositions [**19**], [**1**] for studies in this direction. Besides [**28**], recent works of Greenblatt [**11**], [**12**], [**13**], [**14**], [**15**] are also interesting. He explores a certain resolution of singularities, which is obtained from an elementary method, and investigates the asymptotic behavior of $I(\tau)$. His analysis is also available for a wide class of phases without the above nondegenerate condition.

In this paper, we generalize and improve the above results of Varchenko [**28**]. To be more precise, we are especially interested in the behavior of the integral (1.1) as $\tau \to +\infty$ when φ has a zero at a critical point of the phase. Indeed, under some assumptions, we obtain more accurate results by using the Newton polyhedra of not only the phase but also the amplitude. Closely related issues have been investigated by Arnold, Gusein-Zade and Varchenko [**1**] and Pramanik and Yang [**24**], and they obtained similar results to ours. From the point of view of our investigations, their results will be reviewed in Remark 2.8 in Section 2 and Section 7.4. In our results, delicate geometrical conditions of the Newton polyhedra of the phase and the amplitude affect the behavior of oscillatory integrals. There exist some faces of the Newton polyhedron of the amplitude, which play a crucial role in determining the oscillation index and its multiplicity. Furthermore, in order to determine the oscillation index in general, we need not only geometrical properties of their Newton polyhedra but also information about the coefficients of the terms, corresponding to the above faces, in the Taylor series of the amplitude. (See Theorem 2.7 in Section 2.2 and Example 2 in Section 7.3.)

It is known (see, for instance, [**17**], [**1**], [**19**], and Section 6.1 in this paper) that the asymptotic analysis of oscillatory integral (1.1) can be reduced to an investigation of the poles of the functions $Z_+(s)$ and $Z_-(s)$ (see (5.1) below), which are similar to the local zeta function

$$Z(s) = \int_{\mathbb{R}^n} |f(x)|^s \varphi(x)\chi(x)dx, \tag{1.3}$$

where f, φ, χ are the same as in (1.1) with $f(0) = 0$. The substantial analysis in this paper is to investigate the properties of poles of the local zeta function $Z(s)$ and the functions $Z_\pm(s)$ by using the Newton polyhedra of the functions f and φ. See Section 5 for more details.

Many problems in analysis, including partial differential equations, mathematical physics, harmonic analysis and probability theory, lead to the need to study the behavior of oscillatory integrals of the form (1.1) as $\tau \to +\infty$. We explain the original motivation for our investigation. In the function theory of several complex variables, it is an important problem to understand boundary behavior of the Bergman kernel for pseudoconvex domains. In [**18**], the special case of domains of finite type is considered and the behavior as $\tau \to +\infty$ of the Laplace integral

$$\tilde{I}(\tau) = \int_{\mathbb{R}^n} e^{-\tau f(x)} \varphi(x)dx$$

plays an important role in boundary behavior of the above kernel. Here f,φ are C^∞ functions satisfying certain conditions. The computation of asymptotic expansion of the above kernel in [**18**] requires precise analysis of $\tilde{I}(\tau)$ when φ has a zero at the critical point of f. Our analysis in this paper can be applied to the case of the above Laplace integrals. See also [**2**], [**3**].

This paper is organized as follows. In Section 2, after explaining some important notions and terminology, we state main results relating to oscillatory integrals. In Section 3, we consider an important assumption in Theorem 2.7 in Section 2, which is related to elementary convex geometry (cf. [**29**]). In Section 4, we overview the theory of toric varieties and explain a certain resolution of singularities. In Section 5, we investigate the properties of poles of the local zeta function $Z(s)$ and the functions $Z_{\pm}(s)$ by using the resolution of singularities constructed in Section 4. In Section 6, we give proofs of theorems on the behavior of oscillatory integrals stated in Section 2. In Section 7, we give some examples, which clarify the subtlety of our results. Lastly, we check a related result in [**1**] with these examples.

Notation and Symbols.

- We denote by $\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ the subsets consisting of all nonnegative numbers in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, respectively.
- We use the multi-index as follows. For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$, define

$$|x| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}, \quad \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n,$$
$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \langle \alpha \rangle = \alpha_1 + \cdots + \alpha_n.$$

- For $A, B \subset \mathbb{R}^n$ and $c \in \mathbb{R}$, we set

$$A + B = \{a + b \in \mathbb{R}^n; a \in A \text{ and } b \in B\}, \quad c \cdot A = \{ca \in \mathbb{R}^n; a \in A\}.$$

- We express by $\mathbf{1}$ the vector $(1, \ldots, 1)$ or the set $\{(1, \ldots, 1)\}$.
- For a finite set A, ${}^{\#}A$ means the cardinality of A.
- For a C^∞ function f, we denote by $\mathrm{Supp}(f)$ the support of f, i.e., $\mathrm{Supp}(f) = \overline{\{x \in \mathbb{R}^n; f(x) \neq 0\}}$.

2. Definitions and main results.

2.1. Newton polyhedra.

Let us explain some necessary notions to state our main theorems. The definitions of more fundamental terminologies (polyhedra, faces, dimensions, etc.) will be given in Section 3.1.

Let f be a real-valued C^∞ function defined on a neighborhood of the origin in $\mathbb{R}^n$, which has the Taylor series $\sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha$ at the origin. Then, the *Taylor support* of f is the set $S_f = \{\alpha \in \mathbb{Z}_+^n; c_\alpha \neq 0\}$ and the *Newton polyhedron* of f is the integral polyhedron:

$$\Gamma_+(f) = \text{the convex hull of the set } \bigcup\{\alpha + \mathbb{R}_+^n; \alpha \in S_f\} \text{ in } \mathbb{R}_+^n$$

(i.e., the intersection of all convex sets which contain $\bigcup\{\alpha + \mathbb{R}_+^n; \alpha \in S_f\}$). The union of the compact faces of the Newton polyhedron $\Gamma_+(f)$ is called the *Newton diagram* $\Gamma(f)$ of f, while the topological boundary of $\Gamma_+(f)$ is denoted by $\partial\Gamma_+(f)$.

The *principal part* of f is defined by $f_0(x) = \sum_{\alpha\in\Gamma(f)\cap\mathbb{Z}_+^n} c_\alpha x^\alpha$. For a compact subset $\gamma \subset \partial\Gamma_+(f)$, let $f_\gamma(x) = \sum_{\alpha\in\gamma\cap\mathbb{Z}_+^n} c_\alpha x^\alpha$. f is said to be *nondegenerate* over $\mathbb{R}$ with respect to the Newton polyhedron $\Gamma_+(f)$ if for every compact face $\gamma \subset \Gamma(f)$, the polynomial f_γ satisfies

$$\nabla f_\gamma = \left(\frac{\partial f_\gamma}{\partial x_1}, \ldots, \frac{\partial f_\gamma}{\partial x_n}\right) \neq (0, \ldots, 0) \quad \text{on the set } \{x \in \mathbb{R}^n; x_1 \cdots x_n \neq 0\}.$$

f is said to be *convenient* if the Newton diagram $\Gamma(f)$ intersects all the coordinate axes.

Let f, φ be real-valued C^∞ functions defined on a neighborhood of the origin in $\mathbb{R}^n$ and assume that $\Gamma(f)$ and $\Gamma(\varphi)$ are nonempty. We define the *Newton distance* of (f, φ) by

$$d(f, \varphi) = \min\{d > 0; d \cdot (\Gamma_+(\varphi) + \mathbf{1}) \subset \Gamma_+(f)\}. \tag{2.1}$$

It is easy to see $d(f, \varphi) = \max\{d > 0; \partial\Gamma_+(f) \cap d \cdot (\Gamma_+(\varphi) + \mathbf{1}) \neq \emptyset\}$. The number $d(f, \varphi)$ corresponds to what is called the *coefficient of inscription* of $\Gamma_+(\varphi)$ in $\Gamma_+(f)$ in [**1**, p. 254]. (Their definition in [**1**] must be slightly modified.) Let $\Gamma(\varphi, f)$ be the subset in $\mathbb{R}^n$ defined by

$$\Gamma(\varphi, f) + \mathbf{1} = \left(\frac{1}{d(f, \varphi)} \cdot \partial\Gamma_+(f)\right) \cap (\Gamma_+(\varphi) + \mathbf{1}).$$

In the above definition, $\partial\Gamma_+(\varphi)$ can be used instead of $\Gamma_+(\varphi)$ (see Remark 3.2). Lemma 3.1, below, implies that $\Gamma(\varphi, f)$ is some union of faces of $\Gamma_+(\varphi)$.

Let $\Gamma^{(k)}$ be the union of k-dimensional faces of $\Gamma_+(f)$. Then $\Gamma_+(f)$ is stratified as $\Gamma^{(0)} \subset \Gamma^{(1)} \subset \cdots \subset \Gamma^{(n-1)} (= \partial\Gamma_+(f)) \subset \Gamma^{(n)} (= \Gamma_+(f))$. Let $\tilde{\Gamma}^{(k)} = \Gamma^{(k)} \setminus \Gamma^{(k-1)}$ for $k = 1, \ldots, n$ and $\tilde{\Gamma}^{(0)} = \Gamma^{(0)}$. A map $\rho_f : \Gamma_+(f) \to \{0, 1, \ldots, n\}$ is defined as $\rho_f(\alpha) = k$ if $\alpha \in \tilde{\Gamma}^{(n-k)}$. In other words, $\rho_f(\alpha)$ is the codimension of the face of $\Gamma_+(f)$, whose relative interior contains the point α. We define the *Newton multiplicity* of (f, φ) by

$$m(f, \varphi) = \max\{\rho_f(d(f, \varphi)(\alpha + \mathbf{1})); \alpha \in \Gamma(\varphi, f)\}.$$

Let Γ_0 be the subset of $\Gamma(\varphi, f)$ defined by

$$\Gamma_0 = \{\alpha \in \Gamma(\varphi, f); \rho_f(d(f, \varphi)(\alpha + \mathbf{1})) = m(f, \varphi)\},$$

which is called the *essential set* on $\Gamma(\varphi, f)$. Proposition 3.3, below, shows that Γ_0

is a disjoint union of faces of $\Gamma_+(\varphi)$.

Consider the case $\varphi(0) \neq 0$. Then $\Gamma_+(\varphi) = \mathbb{R}_+^n$. In this case, $d(f,\varphi)$ and $m(f,\varphi)$ are denoted by d_f and m_f, respectively. (Note that $d(f,\varphi) \leq d_f$ for general φ.) It is easy to see that the point $q = (d_f,\ldots,d_f)$ is the intersection of the line $\alpha_1 = \cdots = \alpha_n$ in $\mathbb{R}^n$ and $\partial\Gamma_+(f)$, and that $m_f = \rho_f(q)$. $\Gamma(\varphi,f) \supset \Gamma_0 \supset \{0\}$. More generally, in the case that $\Gamma_+(\varphi) = \{p\} + \mathbb{R}_+^n$ with $p \in \mathbb{Z}_+^n$, the geometrical meanings of the quantities $d(f,\varphi)$ and $m(f,\varphi)$ will be considered in Proposition 5.4 below.

2.2. Main results.

Let us explain our results relating to the behavior of the oscillatory integral $I(\tau)$ in (1.1) as $\tau \to +\infty$.

Throughout this subsection, f, φ, χ satisfy the following conditions: Let U be an open neighborhood of the origin in $\mathbb{R}^n$.

(A) $f : U \to \mathbb{R}$ is a real analytic function satisfying that $f(0) = 0$, $|\nabla f(0)| = 0$ and $\Gamma(f) \neq \emptyset$;
(B) $\varphi : U \to \mathbb{R}$ is a C^∞ function satisfying $\Gamma(\varphi) \neq \emptyset$;
(C) $\chi : \mathbb{R}^n \to \mathbb{R}_+$ is a C^∞ function which identically equals one in some neighborhood of the origin and has a small support which is contained in U.

As mentioned in the Introduction, it is known that the oscillatory integral (1.1) has an asymptotic expansion of the form (1.2). Before stating our results, recall a part of famous results due to Varchenko in [**28**]. In our language, they are stated as follows.

Theorem 2.1 (Varchenko [**28**]). *Suppose that f is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron. Then*

(i) $\beta(f,\varphi) \leq -1/d_f$ *for any* φ;
(ii) *If* $\varphi(0) \neq 0$ *and* $d_f > 1$, *then* $\beta(f,\varphi) = -1/d_f$ *and* $\eta(f,\varphi) = m_f$;
(iii) *The progression* $\{\alpha\}$ *in* (1.2) *belongs to finitely many arithmetic progressions, which are obtained by using the theory of toric varieties based on the geometry of the Newton polyhedron* $\Gamma_+(f)$. (*See Remark* 2.6, *below.*)

Now, let us explain our results. First, we investigate more precise situation in the estimate in the part (i) of Theorem 2.1. Indeed, when φ has a zero at the origin, the oscillation index $\beta(f,\varphi)$ can be more accurately estimated by using the Newton distance $d(f,\varphi)$, which is called "the coefficient of inscription of $\Gamma_+(\varphi)$ in $\Gamma_+(f)$" in [**1**].

Theorem 2.2. *Suppose that* (i) *f is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron and* (ii) *at least one of the following conditions is satisfied*:

(a) f *is convenient*;
(b) φ *is convenient*;
(c) φ *is real analytic on* U;
(d) φ *is expressed as* $\varphi(x) = x^p \tilde{\varphi}(x)$ *on* U, *where* $p \in \mathbb{Z}_+^n$ *and* $\tilde{\varphi}$ *is a* C^∞ *function defined on* U *with* $\tilde{\varphi}(0) \neq 0$.

Then, we have $\beta(f, \varphi) \leq -1/d(f, \varphi)$.

REMARK 2.3. A more precise estimate for $I(\tau)$ is obtained as follows: If the support of χ is contained in a sufficiently small neighborhood of the origin, then there exists a positive constant C independent of τ such that

$$|I(\tau)| \leq C\tau^{-1/d(f,\varphi)}(\log \tau)^{A-1} \quad \text{for } \tau \geq 1,$$

where

$$A := \begin{cases} m(f, \varphi) & \text{if } 1/d(f, \varphi) \text{ is not an integer,} \\ \min\{m(f, \varphi) + 1, n\} & \text{otherwise.} \end{cases}$$

The details will be explained in the proof of the above theorem in Section 6.

REMARK 2.4. Let us consider the above theorem under the assumptions (i), (ii)-(d) without the condition: $\tilde{\varphi}(0) \neq 0$. Then the estimate $\beta(f, \varphi) \leq -1/d(f, \varphi)$ does not always hold. In fact, consider the two-dimensional example: $f(x_1, x_2) = x_1^2$, $\varphi(x_1, x_2) = x_1^2(x_1^2 + e^{-1/x_2^2})$. The proof of Theorem 2.2, however, implies that the estimate $\beta(f, \varphi) \leq -1/d(f, x^p)$ holds under the above assumptions. This assertion with $p = (0, \ldots, 0)$ shows the assertion (i) in Theorem 2.1.

Vassiliev [**27**] obtained a similar result to that in the case of (d).

REMARK 2.5. The condition (d) implies $\Gamma_+(\varphi) = \{p\} + \mathbb{R}_+^n$. When φ is a C^∞ function, however, the converse is not true in general. We give an example in Section 7.2, which shows that the assumption (d) cannot be replaced by the condition: $\Gamma(\varphi) = \{p\} + \mathbb{R}_+^n$ in Theorem 2.2.

REMARK 2.6. From the proof of the above theorem, we can see that under the same condition, the progression $\{\alpha\}$ in (1.2) is contained in the set

$$\left\{ \frac{1}{d(f, \varphi)} + \frac{\nu}{l(a)}; a \in \tilde{\Sigma}^{(1)}, \nu \in \mathbb{Z}_+ \right\} \left(\subset \left\{ \frac{1}{d_f} + \frac{\nu}{l(a)}; a \in \tilde{\Sigma}^{(1)}, \nu \in \mathbb{Z}_+ \right\} \right),$$

where the symbol $l(a)$ is as in (4.1) and $\tilde{\Sigma}^{(1)}$ is as in Theorem 5.1, below. This explicitly shows the assertion (iii) in Theorem 2.1.

Next, let us give an analogous result to the part (ii) in Theorem 2.1, due to Varchenko. Indeed, the following theorem deals with the case that the equation $\beta(f,\varphi) = -1/d(f,\varphi)$ holds.

Theorem 2.7. *Suppose that* (i) *f is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron,* (ii) *at least one of the following two conditions is satisfied*:

(a) *$d(f,\varphi) > 1$;*
(b) *f is nonnegative or nonpositive on U,*

and (iii) *at least one of the following two conditions is satisfied*:

(c) *φ is expressed as $\varphi(x) = x^p\tilde{\varphi}(x)$ on U, where every component of $p \in \mathbb{Z}_+^n$ is even and $\tilde{\varphi}$ is a C^∞ function defined on U with $\tilde{\varphi}(0) \neq 0$;*
(d) *f is convenient and φ_{Γ_0} is nonnegative or nonpositive on U.*

Then the equations $\beta(f,\varphi) = -1/d(f,\varphi)$ and $\eta(f,\varphi) = m(f,\varphi)$ hold.

Remark 2.8. Considering the assumptions: (i), (ii)-(a), (iii)-(c) with $p = (0,\dots,0)$ in the above theorem, we see the assertion (ii) in Theorem 2.1.

Pramanik and Yang [**24**] obtained a similar result in the case that the dimension is two and $\varphi(x) = |g(x)|^\epsilon$ where g is real analytic and ϵ is positive. Their result in Theorem 3.1 (a) does not need strong assumptions. We explain this reason roughly. They use the *weighted Newton distance*, whose definition is different from our Newton distance. The definition of their distance is more intrinsic and is based on a good choice of coordinate system, which induces a clear resolution of singularity. Moreover, the nonnegativity of φ implies the positivity of the coefficient of the expected leading term of the asymptotic expansion (1.2). On the other hand, in our case, the corresponding coefficient possibly vanishes without the assumption (c) or (d). See Sections 7.1 and 7.3.

Remark 2.9. In Section 3, we discuss the set Γ_0 and the function φ_{Γ_0} in the condition (d) in detail. If $\varphi(0) = 0$ and φ takes the local minimal (resp. the local maximal) at the origin, then φ_{Γ_0} is nonnegative (resp. nonpositive) on some neighborhood of the origin.

Remark 2.10. It is easy to show that Theorem 2.7 can be rewritten in a slightly stronger form by replacing the condition (c) by the following (c′):

(c′) φ is expressed as $\varphi(x) = \sum_{j=1}^{l} x^{p_j}\tilde{\varphi}_j(x)$ on U, where $p_j \in \mathbb{Z}_+^n$ and $\tilde{\varphi}_j \in C^\infty(U)$ for all j satisfies that if $p_j \in \Gamma_0$, then every component of p_j is even and $\tilde{\varphi}_j(0) > 0$ (or $\tilde{\varphi}_j(0) < 0$) for all j.

We will give an example in Section 7.3, which satisfies the conditions (a), (d) but does not satisfy the condition (c′). (Consider the case that the parameter t satisfies $0 < |t| < 2$ in the example.)

Remark 2.11. In the one-dimensional case, the conditions (c) and (d) are equivalent.

Lastly, let us discuss a "symmetrical" property with respect to the phase and the amplitude. Observe the one-dimensional case. Let f, φ satisfy that $f(0) = f'(0) = \cdots = f^{(q-1)}(0) = \varphi(0) = \varphi'(0) = \cdots = \varphi^{(p-1)}(0) = 0$ and $f^{(q)}(0)\varphi^{(p)}(0) \neq 0$, where $p, q \in \mathbb{N}$ are even. Applying the computation in Chapter 8 in [**26**] (see also Section 7.1 in this paper), we can see that if the support of χ is sufficiently small, then

$$\int_{-\infty}^{\infty} e^{i\tau x f(x)} \varphi(x)\chi(x)dx \sim \tau^{-(p+1)/(q+1)} \sum_{j=0}^{\infty} C_j \tau^{-j/(q+1)} \qquad \text{as } \tau \to \infty,$$

where C_0 is a nonzero constant. Note that the above expansion can be obtained for C^∞ functions f and φ. In particular, $\beta(xf, \varphi) = -(p+1)/(q+1)$ holds. Similarly, we can get $\beta(x\varphi, f) = -(q+1)/(p+1)$. From this observation, the following question seems interesting: When does the equality $\beta(x^{\mathbf{1}}f, \varphi)\beta(x^{\mathbf{1}}\varphi, f) = 1$ hold in higher dimensional case? The following theorem is concerned with this question.

Theorem 2.12. *Let f, φ be nonnegative or nonpositive real analytic functions defined on U. Suppose that both f and φ are convenient and nondegenerate over $\mathbb{R}$ with respect to their Newton polyhedra. Then we have $\beta(x^{\mathbf{1}}f, \varphi)\beta(x^{\mathbf{1}}\varphi, f) \geq 1$. Moreover, the following two conditions are equivalent*:

(i) $\beta(x^{\mathbf{1}}f, \varphi)\beta(x^{\mathbf{1}}\varphi, f) = 1$;
(ii) *There exists a positive rational number d such that $\Gamma_+(x^{\mathbf{1}}f) = d \cdot \Gamma_+(x^{\mathbf{1}}\varphi)$.*

If the condition (i) *or* (ii) *is satisfied, then we have $\eta(x^{\mathbf{1}}f, \varphi) = \eta(x^{\mathbf{1}}\varphi, f) = n$.*

3. Convex polyhedra and essential sets.

3.1. Polyhedra.

Let us give precise definitions for polyhedra, faces, dimensions and so on. Refer to [**29**], etc. for general theory of convex polyhedra.

A (*convex*) *polyhedron* is an intersection of closed halfspaces: a set $P \subset \mathbb{R}^n$ presented in the form

$$P = \bigcap_{j=1}^{m} \{x \in \mathbb{R}^n; \langle a^j, x\rangle \geq z_j\},$$

for some $a^1, \ldots, a^m \in \mathbb{R}^n$ and $z_1, \ldots, z_m \in \mathbb{R}$. It is known (cf. [**29**]) that the Newton polyhedron $\Gamma_+(f)$ in Section 2.1 is a polyhedron.

Let P be a polyhedron in $\mathbb{R}^n$. A pair $(a, z) \in \mathbb{R}^n \times \mathbb{R}$ is *valid* for P if a linear inequality $\langle a, x\rangle \geq z$ is satisfied for all points $x \in P$. For $(a, z) \in \mathbb{R}^n \times \mathbb{R}$, define

$$H(a, z) = \{x \in \mathbb{R}^n; \langle a, x\rangle = z\}. \tag{3.1}$$

A *face* of P is any set of the form $F = P \cap H(a, z)$, where (a, z) is valid for P. Since $(0, 0)$ is always valid, we consider P itself as a trivial face of P; the other faces are called *proper faces*. Conversely, it is easy to see that any face is a polyhedron. Considering the valid pair $(0, -1)$, we see that the empty set is always a face of P. The *dimension* of a face F is the dimension of its affine hull of F (i.e., the intersection of all affine flats that contain F). The faces of dimensions $0, 1$ and $\dim(P) - 1$ are called *vertices*, *edges* and *facets*, respectively. The *boundary* of a polyhedron P, denoted by ∂P, is the union of all proper faces of P. For a face F, ∂F is similarly defined.

3.2. Essential sets.

Let us consider the properties of $\Gamma(\varphi, f)$ and the essential set Γ_0 defined in Section 2.1. Moreover, we consider the condition (d) in Theorem 2.7.

Lemma 3.1. *Let P_1, P_2 be n-dimensional polyhedra in $\mathbb{R}^n$. If $P_1 \subset P_2$, then $P_1 \cap \partial P_2$ is the union of proper faces of P_1.*

Proof. There exist finite pairs $(a^1, z_1), \ldots, (a^l, z_l) \in \mathbb{R}^n \times \mathbb{R}$ such that every (a^j, z_j) is valid for P_2 and $\partial P_2 = \bigcup_{j=1}^{l} (P_2 \cap H(a^j, z_j))$.

$$\begin{aligned} P_1 \cap \partial P_2 &= P_1 \cap \left[\bigcup_{j=1}^{l} (P_2 \cap H(a^j, z_j)) \right] \\ &= \bigcup_{j=1}^{l} \big(P_1 \cap P_2 \cap H(a^j, z_j)\big) = \bigcup_{j=1}^{l} \big(P_1 \cap H(a^j, z_j)\big). \end{aligned} \tag{3.2}$$

Since every (a^j, z_j) is also valid for P_1 and $P_1 \cap H(a^j, z_j)$ is a proper face of P_1, then we get the lemma. $\square$

Hereafter in this section, we assume that f, φ are C^∞ functions defined on a neighborhood of the origin and their Newton polyhedra are nonempty.

By applying the above lemma to the case:

$$P_1 = \Gamma_+(\varphi), \quad P_2 = \frac{1}{d(f,\varphi)} \cdot \Gamma_+(f) - \mathbf{1}, \tag{3.3}$$

we see that $\Gamma(\varphi, f) = P_1 \cap \partial P_2 (\neq \emptyset)$ is the union of faces of $\Gamma_+(\varphi)$.

REMARK 3.2. From (3.2), we see $P_1 \cap \partial P_2 \subset \partial P_1$. Thus, $\partial P_1 \cap \partial P_2 = P_1 \cap \partial P_2$ holds.

PROPOSITION 3.3. *There exist the faces $\gamma_1, \ldots, \gamma_l$ of $\Gamma_+(\varphi)$ such that*

$$\Gamma_0 = \biguplus_{j=1}^{l} \gamma_j \qquad (\textit{disjoint union}).$$

Moreover, the dimension of γ_j is not greater than $n - m(f,\varphi)$ for any j.

PROOF. Set P_1 and P_2 as in (3.3) and let $k_0 = n - m(f,\varphi)$.

In the case $k_0 = 0$, Γ_0 is the set of vertices of P_1, which implies the proposition. Consider the case $1 \leq k_0 \leq n$. Let $F_1, \ldots, F_l$ be the k_0-dimensional faces of P_2 such that $F_j \cap P_1 \neq \emptyset$. Now, let us show the sets $\gamma_j := F_j \cap P_1$ satisfy the condition in the proposition. Of course, the union of all γ_j is Γ_0 and the dimensions of γ_j are not greater than $k_0 (= n - m(f,\varphi))$ for any j. It suffices to show that each γ_j is a face of P_1 and that $\gamma_j \cap \gamma_k = \emptyset$ if $j \neq k$. For each j, there is a pair $(a^j, z_j) \in \mathbb{R}^n \times \mathbb{R}$ such that it is valid for P_2 and $F_j = P_2 \cap H(a^j, z_j)$. Thus, $\gamma_j = P_1 \cap F_j = P_1 \cap P_2 \cap H(a^j, z_j) = P_1 \cap H(a^j, z_j)$, which implies γ_j is a face of P_1. Next, by the minimality of k_0, γ_j is contained in the relative interior of F_j (i.e., $\gamma_j \subset F_j \setminus \partial F_j$). Since all relative interiors of F_j are disjoint, we have $\gamma_j \cap \gamma_k = \emptyset$ if $j \neq k$. □

LEMMA 3.4. *Let γ be a compact face of $\Gamma_+(\varphi)$. If φ is nonnegative (or nonpositive) in a neighborhood of the origin, so is φ_γ.*

PROOF. A pair $(a, z) = ((a_1, \ldots, a_n), z) \in \mathbb{R}^n \times \mathbb{R}$ corresponds to γ, i.e., $H(a, z) \cap \Gamma_+(\varphi) = \gamma$. Taylor's formula implies that for any $N \in \mathbb{N}$, φ can be expressed as

$$\varphi(x) = \sum_{\alpha \in S_\varphi \cap U_N} c_\alpha x^\alpha + \sum_{p \in \mathbb{Z}_+^n, \langle p \rangle = N} x^p \varphi_p(x), \tag{3.4}$$

where $U_N := \{\alpha \in \mathbb{R}_+^n; \langle \alpha \rangle < N\}$, c_α are constants and φ_p are C^∞ functions defined on a neighborhood of the origin. Here, take a sufficiently large N such that γ is contained in the set U_N. For $\xi = (\xi_1, \dots, \xi_n) \in (-\epsilon, \epsilon)^n$, $t \in (-\epsilon, \epsilon)$, where $\epsilon > 0$ is small, a simple computation gives

$$\begin{aligned}\varphi(\xi_1 t^{a_1}, \dots, \xi_n t^{a_n}) &= \sum_{\alpha \in S_\varphi \cap U_N} c_\alpha \xi^\alpha t^{\langle a, \alpha \rangle} + \sum_{p \in \mathbb{Z}_+^n, \langle p \rangle = N} \xi^\alpha t^{\langle a, p \rangle} \varphi_p(\xi_1 t^{a_1}, \dots, \xi_n t^{a_n}) \\ &= t^z (\varphi_\gamma(\xi) + a(\xi, t) t),\end{aligned}$$

where $a(\xi, t)$ is a C^∞ function defined on a neighborhood of $(0,0) \in \mathbb{R}^n \times \mathbb{R}$. From the above, it is easy to show the lemma. $\square$

Remark 3.5. Even if φ_γ is nonnegative (resp. nonpositive) near the origin for every faces γ of $\Gamma_+(\varphi)$, φ is not always nonnegative (resp. nonpositive) near the origin: Consider the example $\varphi(x_1, x_2) = (x_1 - x_2)^2 - x_2^4$.

Proposition 3.6. *Let $\gamma_1, \dots, \gamma_l$ be the faces of $\Gamma_+(\varphi)$ as in Proposition* 3.3 *and suppose Γ_0 is compact. Then the following two conditions are equivalent*:

(i) φ_{Γ_0} *is nonnegative* (*resp. nonpositive*) *near the origin*;
(ii) φ_{γ_j} *is nonnegative* (*resp. nonpositive*) *near the origin for all j.*

Proof. From Lemma 3.4, we can see that (i) implies (ii). Since Γ_0 is the disjoint union of the faces γ_j, we have $\varphi_{\Gamma_0}(x) = \sum_{j=1}^l \varphi_{\gamma_j}(x)$. This shows that (ii) implies (i). $\square$

Corollary 3.7. *If f is convenient and φ is nonnegative or nonpositive near the origin, then the condition* (d) *in Theorem* 2.7 *is satisfied.*

Proof. The convenience of f implies the compactness of Γ_0. By Lemma 3.4, the assertion (ii) in Proposition 3.6 is satisfied. $\square$

4. Toric resolution.

The purpose of this section is to give the resolution of the singularities of the critical points of some functions from the theory of toric varieties. Refer to [**20**], [**22**], [**9**], [**23**], etc. for general theory of toric varieties.

4.1. Cones and fans.

In order to construct a toric resolution obtained from the Newton polyhedron, we recall the definitions of important terminology: *cone* and *fan*.

A *rational polyhedral cone* $\sigma \subset \mathbb{R}^n$ is a cone generated by finitely many elements of $\mathbb{Z}^n$. In other words, there are $u_1, \dots, u_k \in \mathbb{Z}^n$ such that

$$\sigma = \{\lambda_1 u_1 + \cdots + \lambda_k u_k \in \mathbb{R}^n; \lambda_1, \ldots, \lambda_k \geq 0\}.$$

We say that σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$.

By regarding a cone as a polyhedron in $\mathbb{R}^n$, the definitions of *dimension*, *face*, *edge*, *facet* for the cone are given in the same way as in Section 3.

The *fan* is defined to be a finite collection Σ of cones in $\mathbb{R}^n$ with the following properties:

- Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone;
- If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$;
- If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each.

For a fan Σ, the union $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$ is called the *support* of Σ. For $k = 0, 1, \ldots, n$, we denote by $\Sigma^{(k)}$ the set of k-dimensional cones in Σ. The *skeleton* of a cone $\sigma \in \Sigma$ is the set of all of its primitive integer vectors (i.e., with components relatively prime in $\mathbb{Z}_+$) in the edges of σ. It is clear that the skeleton of σ generates σ itself. Thus, the set of skeletons of the cones belonging to $\Sigma^{(k)}$ is also expressed by the same symbol $\Sigma^{(k)}$.

4.2. Simplicial subdivision.

We denote by $(\mathbb{R}^n)^*$ the dual space of $\mathbb{R}^n$ with respect to the standard inner product. For $a = (a_1, \ldots, a_n) \in (\mathbb{R}^n)^*$, define

$$l(a) = \min\{\langle a, \alpha\rangle; \alpha \in \Gamma_+(f)\} \tag{4.1}$$

and $\gamma(a) = \{\alpha \in \Gamma_+(f); \langle a, \alpha\rangle = l(a)\}$ $(= \Gamma_+(f) \cap H(a, l(a)))$. We introduce an equivalence relation $\sim$ in $(\mathbb{R}^n)^*$ by $a \sim a'$ if and only if $\gamma(a) = \gamma(a')$. For any k-dimensional face γ of $\Gamma_+(f)$, there is an equivalence class γ^* which is defined by

$$\gamma^* = \{a \in (\mathbb{R}^n)^*; \gamma(a) = \gamma, \text{ and } a_j \geq 0 \text{ for } j = 1, \ldots, n\}.$$

It is easy to see that the closure of γ^* is an $(n-k)$-dimensional strongly convex rational polyhedral cone in $(\mathbb{R}^n)^*$. Moreover, the collection of the closures of γ^* gives a fan Σ_0. Note that $|\Sigma_0| = \mathbb{R}^n_+$.

It is known that there exists a *simplicial subdivision* Σ of Σ_0, that is, Σ is a fan satisfying the following properties:

- The fans Σ_0 and Σ have the same support;
- Each cone of Σ lies in some cone of Σ_0;
- The skeleton of any cone belonging to Σ can be completed to a base of the lattice dual to $\mathbb{Z}^n$.

4.3. Construction of toric varieties.

Fix a simplicial subdivision Σ of Σ_0. For n-dimensional cone $\sigma \in \Sigma$, let $a^1(\sigma), \dots, a^n(\sigma)$ be the skeleton of σ, ordered once and for all. Here, we set the coordinates of the vector $a^j(\sigma)$ as

$$a^j(\sigma) = (a_1^j(\sigma), \dots, a_n^j(\sigma)).$$

With every such cone σ, we associate a copy of $\mathbb{C}^n$ which is denoted by $\mathbb{C}^n(\sigma)$. We denote by $\pi(\sigma) : \mathbb{C}^n(\sigma) \to \mathbb{C}^n$ the map defined by $\pi(\sigma)(y_1, \dots, y_n) = (x_1, \dots, x_n)$ with

$$x_j = y_1^{a_j^1(\sigma)} \cdots y_n^{a_j^n(\sigma)}, \qquad j = 1, \dots, n. \tag{4.2}$$

Let X_Σ be the union of $\mathbb{C}^n(\sigma)$ for σ which are glued along the images of $\pi(\sigma)$. Indeed, for any n-dimensional cones $\sigma, \sigma' \in \Sigma$, two copies $\mathbb{C}^n(\sigma)$ and $\mathbb{C}^n(\sigma')$ can be identified with respect to a rational mapping: $\pi^{-1}(\sigma') \circ \pi(\sigma) : \mathbb{C}^n(\sigma) \to \mathbb{C}^n(\sigma')$ (i.e. $x \in \mathbb{C}^n(\sigma)$ and $x' \in \mathbb{C}^n(\sigma')$ will coalesce if $\pi^{-1}(\sigma') \circ \pi(\sigma) : x \mapsto x'$). Then it is known that

- X_Σ is an n-dimensional complex algebraic manifold;
- The map $\pi : X_\Sigma \to \mathbb{C}^n$ defined on each $\mathbb{C}^n(\sigma)$ as $\pi(\sigma) : \mathbb{C}^n(\sigma) \to \mathbb{C}^n$ is proper.

The manifold X_Σ is called the *toric variety* associated with Σ. The transition functions between local maps of the manifold X_Σ are real on the real part of the manifold X_Σ which will be denoted by Y_Σ. The restriction of the projection π to Y_Σ is also denoted by π. Then we have

- Y_Σ is an n-dimensional real algebraic manifold;
- The map $\pi : Y_\Sigma \to \mathbb{R}^n$ defined on each $\mathbb{R}^n(\sigma)$ as $\pi(\sigma) : \mathbb{R}^n(\sigma) \to \mathbb{R}^n$ is proper.

Note 4.1. The map $\pi(\sigma)$ plays an important role in our analysis and the following kind of computation often appears: Let $p = (p_1, \dots, p_n) \in \mathbb{R}^n$, then (4.2) implies

$$\begin{aligned} x^p &= (\pi(\sigma)(y))^p = \left(y_1^{a_1^1(\sigma)} \cdots y_n^{a_1^n(\sigma)}\right)^{p_1} \cdots \left(y_1^{a_n^1(\sigma)} \cdots y_n^{a_n^n(\sigma)}\right)^{p_n} \\ &= y_1^{\langle a^1(\sigma), p \rangle} \cdots y_n^{\langle a^n(\sigma), p \rangle}. \end{aligned}$$

4.4. Resolution of singularities.

For $I \subset \{1, \dots, n\}$, define the set T_I in $\mathbb{R}^n$ by

$$T_I = \{y \in \mathbb{R}^n; y_j = 0 \text{ for } j \in I,\ y_j \neq 0 \text{ for } j \notin I\}. \tag{4.3}$$

The following proposition shows that $\pi : Y_\Sigma \to \mathbb{R}^n$ is a real resolution of the singularity of the critical point of a real analytic function satisfying the nondegenerate property.

Proposition 4.2 ([**28**, Lemma 2.13, Lemma 2.15]). *Suppose that f is a real analytic function in a neighborhood U of the origin. Then we have the following.*

(i) *There exists a real analytic function f_σ defined on the set $\pi(\sigma)^{-1}(U)$ such that $f_\sigma(0) \neq 0$ and*

$$(f \circ \pi(\sigma))(y_1, \dots, y_n) = y_1^{l(a^1(\sigma))} \cdots y_n^{l(a^n(\sigma))} f_\sigma(y_1, \dots, y_n). \tag{4.4}$$

(ii) *The Jacobian of the mapping $\pi(\sigma)$ is equal to*

$$J_{\pi(\sigma)}(y) = \pm y_1^{\langle a^1(\sigma)\rangle - 1} \cdots y_n^{\langle a^n(\sigma)\rangle - 1}. \tag{4.5}$$

(iii) *The set of the points in $\mathbb{R}^n$ in which $\pi(\sigma)$ is not an isomorphism is a union of coordinate planes.*

Moreover, if f is nondegenerate over $\mathbb{R}$ with respect to $\Gamma_+(f)$ and $\pi(\sigma)(T_I) = 0$, then the set $\{y \in T_I; f_\sigma(y) = 0\}$ is nonsingular, that is, the gradient of the restriction of the function f_σ to T_I does not vanish at the points of the set $\{y \in T_I; f_\sigma(y) = 0\}$.

5. Poles of local zeta functions.

Throughout this section, the functions f, φ, χ always satisfy the conditions (A), (B), (C) in the beginning of Section 2.2.

The purpose of this section is to investigate the properties of poles of the functions:

$$Z_+(s) = \int_{\mathbb{R}^n} f(x)_+^s \varphi(x)\chi(x)dx, \quad Z_-(s) = \int_{\mathbb{R}^n} f(x)_-^s \varphi(x)\chi(x)dx, \tag{5.1}$$

where $f(x)_+ = \max\{f(x), 0\}$ and $f(x)_- = \max\{-f(x), 0\}$ and the *local zeta function*:

$$Z(s) = \int_{\mathbb{R}^n} |f(x)|^s \varphi(x)\chi(x)dx. \tag{5.2}$$

From the properties of $Z_+(s)$ and $Z_-(s)$, we can easily obtain analogous properties of $Z(s)$ by using the relationship: $Z(s) = Z_+(s) + Z_-(s)$.

It is easy to see that the above functions are holomorphic functions in the region $\mathrm{Re}(s) > 0$. Moreover, it is known (see [**1**], [**19**], etc.) that if the support of χ is sufficiently small, then these functions can be analytically continued to the complex plane as meromorphic functions and their poles belong to finitely many arithmetic progressions constructed from negative rational numbers. More precisely, Varchenko [**28**] describes the positions of the candidate poles and their orders by using the toric resolution constructed in Section 4. In this section, we give more accurate results in the case that φ has a zero at the origin.

5.1. The monomial case.

First, let us consider the case that the function φ is a monomial, i.e., $\varphi(x) = x^p = x_1^{p_1} \cdots x_n^{p_n}$ with $p = (p_1, \ldots, p_n) \in \mathbb{Z}_+^n$. Fedorjuk [**8**] was the first to consider this kind of issue in two-dimensional case. Moreover, there have been closely related studies to ours in [**6**], [**7**], [**4**], [**5**], which contain other interesting results.

Theorem 5.1. *Suppose that* (i) *f is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron and* (ii) *$\varphi(x) = x^p$ with $p \in \mathbb{Z}_+^n$. If the support of χ is contained in a sufficiently small neighborhood of the origin, then the poles of the functions $Z_+(s)$, $Z_-(s)$ and $Z(s)$ are contained in the set*

$$\left\{ -\frac{\langle a, p + \mathbf{1} \rangle + \nu}{l(a)} ; \nu \in \mathbb{Z}_+,\ a \in \tilde{\Sigma}^{(1)} \right\} \cup (-\mathbb{N}),$$

where $l(a)$ is as in (4.1) *and $\tilde{\Sigma}^{(1)} = \{a \in \Sigma^{(1)}; l(a) > 0\}$.*

In Remark 5.3 after the proof, we will explain in more detail the reason why the set $(-\mathbb{N})$ is necessary to express the poles.

Proof. Let Σ_0 be the fan constructed from the Newton polyhedron of f. Fix a simplicial subdivision Σ of Σ_0 and let (Y_Σ, π) be the real resolution associated with Σ as in Section 4.

By using the mapping $x = \pi(y)$, $Z_+(s)$ and $Z_-(s)$ are expressed as

$$\begin{aligned} Z_\pm(s) &= \int_{\mathbb{R}^n} f(x)_\pm^s x^p \chi(x) dx \\ &= \int_{Y_\Sigma} ((f \circ \pi)(y))_\pm^s (\pi(y))^p (\chi \circ \pi)(y) |J_\pi(y)| dy, \end{aligned}$$

where dy is a volume element in Y_Σ, $J_\pi(y)$ is the Jacobian of the mapping π. It is easy to see that there exists a set of C_0^∞ functions $\{\chi_\sigma : Y_\Sigma \to \mathbb{R}_+; \sigma \in \Sigma^{(n)}\}$ satisfying the following properties:

- For each $\sigma \in \Sigma^{(n)}$, the support of the function χ_σ is contained in $\mathbb{R}^n(\sigma)$ and χ_σ identically equals one in some neighborhood of the origin.
- $\sum_{\sigma\in\Sigma^{(n)}} \chi_\sigma \equiv 1$ on the support of $\chi \circ \pi$.

Applying Proposition 4.2, we have

$$Z_\pm(s) = \sum_{\sigma\in\Sigma^{(n)}} Z_\pm^{(\sigma)}(s)$$

with

$$\begin{aligned} Z_\pm^{(\sigma)}(s) &= \int_{\mathbb{R}^n} ((f\circ\pi(\sigma))(y))_\pm^s (\pi(\sigma)(y))^p (\chi\circ\pi(\sigma))(y)\chi_\sigma(y)|J_{\pi(\sigma)}(y)|dy \\ &= \int_{\mathbb{R}^n} \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} f_\sigma(y)\right)_\pm^s \left(\prod_{j=1}^n y_j^{\langle a^j(\sigma),p\rangle}\right) \left|\prod_{j=1}^n y_j^{\langle a^j(\sigma)\rangle-1}\right| \tilde{\chi}_\sigma(y)dy, \end{aligned} \tag{5.3}$$

where $\tilde{\chi}_\sigma(y) = (\chi\circ\pi(\sigma))(y)\chi_\sigma(y)$.

Now, consider the functions $Z_\pm^{(\sigma)}(s)$ for $\sigma \in \Sigma^{(n)}$. We easily see the existence of finite sets of C_0^∞ functions $\{\psi_k : \mathbb{R}^n \to \mathbb{R}_+\}$ and $\{\eta_l : \mathbb{R}^n \to \mathbb{R}_+\}$ satisfying the following conditions.

- The supports of ψ_k and η_l are sufficiently small and $\sum_k \psi_k + \sum_l \eta_l \equiv 1$ on the support of $\tilde{\chi}_\sigma$.
- For each k, f_σ is always positive or negative on the support of ψ_k.
- For each l, the support of η_l intersects the set $\{y \in \mathrm{Supp}(\tilde{\chi}_\sigma); f_\sigma(y) = 0\}$
- The union of the support of η_l for all l contains the set $\{y \in \mathrm{Supp}(\tilde{\chi}_\sigma); f_\sigma(y) = 0\}$

Using the functions ψ_k and η_l, we have

$$Z_\pm^{(\sigma)}(s) = \sum_k I_{\sigma,\pm}^{(k)}(s) + \sum_l J_{\sigma,\pm}^{(l)}(s), \tag{5.4}$$

with

$$I_{\sigma,\pm}^{(k)}(s) = \int_{\mathbb{R}^n} \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} f_\sigma(y)\right)_\pm^s \left(\prod_{j=1}^n y_j^{\langle a^j(\sigma),p\rangle}\right) \left|\prod_{j=1}^n y_j^{\langle a^j(\sigma)\rangle-1}\right| \tilde{\psi}_k(y)dy,$$

$$J_{\sigma,\pm}^{(l)}(s) = \int_{\mathbb{R}^n} \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} f_\sigma(y) \right)_\pm^s \left(\prod_{j=1}^n y_j^{\langle a^j(\sigma),p\rangle} \right) \left| \prod_{j=1}^n y_j^{\langle a^j(\sigma)\rangle -1} \right| \tilde{\eta}_l(y) dy,$$

where $\tilde{\psi}_k(y) = \tilde{\chi}_\sigma(y)\psi_k(y)$ and $\tilde{\eta}_l(y) = \tilde{\chi}_\sigma(y)\eta_l(y)$. If the set $\{y \in \mathrm{Supp}(\tilde{\chi}_\sigma); f_\sigma(y) = 0\}$ is empty, then the functions $J_{\sigma,\pm}^{(l)}(s)$ do not appear.

First, consider the functions $I_{\sigma,\pm}^{(k)}(s)$. Set $\delta(+) = 0$ and $\delta(-) = 1$. For $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{+,-\}^n$, let $\delta(\epsilon) = (\delta(\epsilon_1), \ldots, \delta(\epsilon_n)) \in \{0,1\}^n$. A straightforward computation gives

$$I_{\sigma,\pm}^{(k)}(s) = \sum_{\epsilon \in E(\pm\alpha_k,\sigma)} I_{\sigma,k}^{(\epsilon)}(s) \tag{5.5}$$

with

$$I_{\sigma,k}^{(\epsilon)}(s) = \int_{\mathbb{R}^n} \left(\prod_{j=1}^n (y_j)_{\epsilon_j}^{l(a^j(\sigma))s} \right) \left(\prod_{j=1}^n y_j^{\langle a^j(\sigma),p\rangle} \right) \left| \prod_{j=1}^n y_j^{\langle a^j(\sigma)\rangle -1} \right| |f_\sigma(y)|^s \tilde{\psi}_k(y) dy,$$

where α_k is the sign of f_σ on the support of $\tilde{\psi}_k$ and

$$\begin{aligned} &E(+,\sigma) \text{ (resp. } E(-,\sigma)) \\ &\quad = \left\{ \epsilon \in \{+,-\}^n; \sum_{j=1}^n l(a^j(\sigma))\delta(\epsilon_j) \text{ is even (resp. odd)} \right\}. \end{aligned}$$

We remark that $E(+,\sigma) \cup E(-,\sigma) = \{+,-\}^n$ and that $E(-,\sigma)$ is possibly empty. Moreover, we have

$$I_{\sigma,k}^{(\epsilon)}(s) = (-1)^{g_{\sigma,p}(\epsilon)} \int_{\mathbb{R}^n} \left(\prod_{j=1}^n (y_j)_{\epsilon_j}^{l(a^j(\sigma))s + \langle a^j(\sigma), p+\mathbf{1}\rangle -1} \right) |f_\sigma(y)|^s \tilde{\psi}_k(y) dy, \tag{5.6}$$

where

$$g_{\sigma,p}(\epsilon) = \sum_{i,j=1}^n \delta(\epsilon_j) \cdot a_i^j(\sigma) \cdot p_i. \tag{5.7}$$

The following lemma is useful for analyzing the poles of integrals of the above form.

Lemma 5.2 ([**10**], [**1**]). *Let $\psi(y_1, \ldots, y_n; \mu)$ be a C_0^∞ function of y on $\mathbb{R}^n$*

that is an entire function of the parameter $\mu \in \mathbb{C}$. *Then the function*

$$L(\tau_1, \dots, \tau_n; \mu) = \int_{\mathbb{R}^n} \left(\prod_{j=1}^{n} (y_j)_{\epsilon_j}^{\tau_j} \right) \psi(y_1, \dots, y_n; \mu) dy_1 \cdots dy_n,$$

where ϵ_j *is* $+$ *or* $-$, *can be analytically continued at all the values of* $\tau_1, \dots, \tau_n$ *and* μ *as a meromorphic function. Moreover all its poles are simple and lie on* $\tau_j = -1, -2, \dots$ *for* $j = 1, \dots, n$.

Proof. This is easily obtained by the integration by parts (see [**10**], [**1**]). $\square$

Applying Lemma 5.2 to (5.6), we see that the poles of $I_{\sigma,k}^{(\epsilon)}(s)$ are contained in the set

$$\left\{ -\frac{\langle a^j(\sigma), p + \mathbf{1} \rangle + \nu}{l(a^j(\sigma))}; \nu \in \mathbb{Z}_+, j \in B(\sigma) \right\}, \tag{5.8}$$

where

$$B(\sigma) := \{j; l(a^j(\sigma)) \neq 0\} \subset \{1, \dots, n\}. \tag{5.9}$$

Next, consider the functions $J_{\sigma,\pm}^{(l)}(s)$. Applying Proposition 4.2 and changing the integral variables, we have

$$J_{\sigma,\pm}^{(l)}(s) = \int_{\mathbb{R}^n} \left(y_k \prod_{j \in B_l(\sigma)} y_j^{l(a^j(\sigma))} \right)_{\pm}^{s} \left(\prod_{j \in B_l(\sigma)} y_j^{\langle a^j(\sigma), p \rangle} \right) \left| \prod_{j \in B_l(\sigma)} y_j^{\langle a^j(\sigma) \rangle - 1} \right| \hat{\eta}_l(y) dy, \tag{5.10}$$

where $B_l(\sigma)$ is some subset in $\{1, \dots, n\}$ (with $B_l(\sigma) \neq \{1, \dots, n\}$), $k \in \{1, \dots, n\} \setminus B_l(\sigma)$ and $\hat{\eta}_l \in C_0^\infty(\mathbb{R}^n)$ with $\hat{\eta}_l(0) \neq 0$. In a similar fashion to the case of $I_{\sigma,\pm}^{(k)}(s)$, we have

$$J_{\sigma,\pm}^{(l)}(s) = \sum_{\epsilon \in \tilde{E}(\pm,\sigma)} J_{\sigma,l}^{(\epsilon)}(s),$$

with

$$J_{\sigma,l}^{(\epsilon)}(s) = (-1)^{\tilde{g}_{\sigma,p}(\tilde{\epsilon})} \int_{\mathbb{R}^n} \left((y_k)_{\epsilon_k}^{s} \prod_{j \in B_l(\sigma)} (y_j)_{\epsilon_j}^{l(a^j(\sigma))s + \langle a^j(\sigma), p + \mathbf{1} \rangle - 1} \right) \hat{\eta}_l(y) dy, \tag{5.11}$$

where $\epsilon = (\epsilon_k, \tilde{\epsilon})$ with $\tilde{\epsilon} = (\epsilon_j)_{j \in B_l(\sigma)}$, $\tilde{g}_{\sigma,p}(\tilde{\epsilon}) = \sum_{j \in B_l(\sigma)} \sum_{i=1}^n \delta(\epsilon_j) \cdot a_i^j(\sigma) \cdot p_i$ and

$$\begin{aligned} &\tilde{E}(+,\sigma) \text{ (resp. } \tilde{E}(-,\sigma)) \\ &\quad = \left\{ \epsilon = (\epsilon_k, \tilde{\epsilon}); \delta(\epsilon_k) + \sum_{j \in B_l(\sigma)} l(a^j(\sigma)) \delta(\epsilon_j) \text{ is even (resp. odd)} \right\}. \end{aligned}$$

We remark that ${}^{\#}\tilde{E}(+,\sigma) = {}^{\#}\tilde{E}(-,\sigma)$ and, in particular, both $\tilde{E}(+,\sigma)$ and $\tilde{E}(-,\sigma)$ are nonempty.

By applying Lemma 5.2 to (5.11), the poles of $J_{\sigma,l}^{(\epsilon)}(s)$ are contained in the set

$$\left\{ -\frac{\langle a^j(\sigma), p + \mathbf{1} \rangle + \nu}{l(a^j(\sigma))}; \nu \in \mathbb{Z}_+, j \in \tilde{B}_l(\sigma) \right\} \cup (-\mathbb{N}), \tag{5.12}$$

where $\tilde{B}_l(\sigma) = \{ j \in B_l(\sigma); l(a^j(\sigma)) \neq 0 \}$.

Finally, the union of the sets (5.8) and (5.12) for all σ, ϵ equals the set in the theorem. It is easy to show the case of $Z(s)$ by using the relationship: $Z(s) = Z_+(s) + Z_-(s)$. □

Remark 5.3. We explain in more detail the reason why the set $(-\mathbb{N})$ in (5.12) is necessary to express the poles of $J_{\sigma,l}^{(\epsilon)}(s)$, namely, each $J_{\sigma,l}^{(\epsilon)}(s)$ possibly has a pole on $(-\mathbb{N})$. From Proposition 4.2, the nondegenerate condition of f implies that f_σ is nonsingular at the zero set of f_σ. By choosing an appropriate coordinate system near the zero set of f_σ, f can be locally expressed by $y_k(\prod_{j \in B_l(\sigma)} y_j^{l(a^j(\sigma))})$ as in (5.10). Moreover, the existence of $(y_k)_{\epsilon_k}^s$ in (5.11) induces the poles on $(-\mathbb{N})$ by Lemma 5.2.

For $p \in \mathbb{Z}_+^n$, we define

$$\beta(p) = \max \left\{ -\frac{\langle a, p + \mathbf{1} \rangle}{l(a)}; a \in \tilde{\Sigma}^{(1)} \right\}. \tag{5.13}$$

If $s = \beta(p)$ is a pole of $Z_\pm(s)$, $Z(s)$, then we denote by $\eta_\pm(p)$, $\hat{\eta}(p)$ the order of its pole, respectively. For $\sigma \in \Sigma^{(n)}$, let

$$A_p(\sigma) = \left\{ j \in B(\sigma); \beta(p) = -\frac{\langle a^j(\sigma), p + \mathbf{1} \rangle}{l(a^j(\sigma))} \right\} \subset \{1, \ldots, n\}.$$

The following proposition shows the relationship between "the values of $\beta(p)$, $\eta_\pm(p)$, $\hat{\eta}(p)$" and "the geometrical conditions of $\Gamma_+(f)$ and the point p".

Proposition 5.4. *Let $q = (q_1, \ldots, q_n)$ be the point of the intersection of $\partial\Gamma_+(f)$ with the line joining the origin and the point $p + \mathbf{1} = (p_1 + 1, \ldots, p_n + 1)$. Then*

$$-\beta(p) = \frac{p_1 + 1}{q_1} = \cdots = \frac{p_n + 1}{q_n} = \frac{\langle p \rangle + n}{\langle q \rangle} = \frac{1}{d(f, x^p)},$$

$$\eta_\pm(p), \hat{\eta}(p) \leq \begin{cases} \rho_f(q) & \text{if } 1/d(f,x^p) \text{ is not an integer,} \\ \min\{\rho_f(q)+1, n\} & \text{otherwise,} \end{cases}$$

where ρ_f and $d(\cdot,\cdot)$ are as in Section 2.1. Note that $m(f, x^p) = \rho_f(q) = \rho_f(d(f, x^p)(p + \mathbf{1}))$.

Remark 5.5. In the case when $n = 2$ or 3, $\rho_f(q)$ is equal to $\min\{\hat{m}_p, n\}$, where $\hat{m}_p$ is the number of the $(n-1)$-dimensional faces of $\Gamma_+(f)$ containing the point q. This, however, does not generally hold for $n \geq 4$.

Proof. For $a \in \Sigma^{(1)}$, we denote by $q(a)$ the point of the intersection of the hyperplane $H(a, l(a))$ with the line $\{t \cdot (p + \mathbf{1}); t \in \mathbb{R}\}$, where $H(\cdot,\cdot)$ is as in (3.1). Then it is easy to see

$$q(a) = \frac{l(a)}{\langle a, p + \mathbf{1} \rangle} \cdot (p + \mathbf{1}). \tag{5.14}$$

From (5.14), the condition that $-\langle a, p + \mathbf{1} \rangle / l(a)$ takes the maximum is equivalent to the geometrical condition that $q(a)$ is as far as possible from the origin. To be more precise, we have the following equivalences: For $a \in \tilde{\Sigma}^{(1)}$,

$$\beta(p) = -\frac{\langle a, p + \mathbf{1} \rangle}{l(a)} \iff q = q(a) \iff q \in H(a, l(a)). \tag{5.15}$$

From (5.14) and (5.15), we have $-\beta(p) = (p_1 + 1)/q_1 = \cdots = (p_n + 1)/q_n = (\langle p \rangle + n)/\langle q \rangle$. From the definition of $d(\cdot,\cdot)$, the above value equals $1/d(f, x^p)$. Note that $q = d(f, x^p)(p + \mathbf{1})$.

Next, consider the orders of the poles of $Z_\pm(s), Z(s)$ at $s = \beta(p)$. From the proof of Theorem 5.1, it suffices to analyze the poles of $I_{\sigma,k}^{(\epsilon)}(s)$, $J_{\sigma,l}^{(\epsilon)}(s)$. Applying Lemma 5.2 to the integrals (5.6), (5.11), we see the upper bounds of orders of the poles at $s = \beta(p)$ of these functions as follows.

$I^{(\epsilon)}_{\sigma,k}(s)$	${}^{\#}A_p(\sigma)$
$J^{(\epsilon)}_{\sigma,l}(s)$	$\min\{{}^{\#}A_p(\sigma), n-1\}$ if $\beta(p) \notin (-\mathbb{N})$
	$\min\{{}^{\#}A_p(\sigma)+1, n\}$ if $\beta(p) \in (-\mathbb{N})$

From these estimates of orders, in order to obtain the estimates in the proposition, it suffices to show $\rho_f(q) = \max\{{}^{\#}A_p(\sigma); \sigma \in \Sigma^{(n)}\}$. From the definition of $A_p(\sigma)$ and (5.15), we have

$$\begin{aligned} {}^{\#}A_p(\sigma) &= {}^{\#}\{j; q \in H(a^j(\sigma), l(a^j(\sigma)))\} \\ &= {}^{\#}\{j; \gamma \subset H(a^j(\sigma), l(a^j(\sigma)))\}, \end{aligned}$$

where γ is the face of $\Gamma_+(f)$ whose relative interior contains the point q (i.e., $q \in (\gamma \setminus \partial\gamma)$.) From the definition of ρ_f, the codimension of γ is $\rho_f(q)$. Since $a^1(\sigma), \dots, a^n(\sigma)$ are linearly independent for each $\sigma \in \Sigma^{(n)}$, ${}^{\#}\{j; \gamma \subset H(a^j(\sigma), l(a^j(\sigma)))\}$ is not larger than $\rho_f(q)$ for any $\sigma \in \Sigma^{(n)}$. On the other hand, the closure of γ^* is a cone belonging to the fan Σ_0 constructed from $\Gamma_+(f)$ (see Section 4.2) and the dimension of this cone is $\rho_f(q)$. There exists an n-dimensional cone $\hat{\sigma}$ in a simplicial subdivision Σ of Σ_0 whose $\rho_f(q)$-dimensional face is contained in the closure of γ^*. This means ${}^{\#}\{j; \gamma \subset H(a^j(\hat{\sigma}), l(a^j(\hat{\sigma})))\} = \rho_f(q)$. Hence, we see $\rho_f(q) = \max\{{}^{\#}A_p(\sigma); \sigma \in \Sigma^{(n)}\}$.

Lastly, it follows from $\Gamma_0 \supset \{p\}$ that $m(f, x^p) = \rho_f(d(f, x^p)(p + \mathbf{1})) = \rho_f(q)$. □

Next, let us consider the coefficients of the Laurent expansions of $Z_+(s)$ and $Z_-(s)$ at the poles. The following lemma is useful for computing the coefficients explicitly.

Lemma 5.6. *Let ψ be a C^∞ function on $\mathbb{R}$ and $k \in \mathbb{N}$. Then*

$$\lim_{s \to -k} (s+k) \int_{-\infty}^{\infty} y_{\pm}^s \psi(y) dy = \frac{(\pm 1)^{k-1}}{(k-1)!} \psi^{(k-1)}(0).$$

In particular, we have

$$\lim_{s \to -1} (s+1) \int_{-\infty}^{\infty} y_{\pm}^s \psi(y) dy = \psi(0).$$

Proof. The above formula is easily obtained by the integration by parts. □

When $d(f,x^p)>1$, we compute the coefficients of $(s-\beta(p))^{-m(f,x^p)}$ in the Laurent expansions of $Z_\pm(s)$, $Z(s)$. Let

$$C_\pm=\lim_{s\to\beta(p)}(s-\beta(p))^{m(f,x^p)}Z_\pm(s),\quad C=\lim_{s\to\beta(p)}(s-\beta(p))^{m(f,x^p)}Z(s),$$

respectively.

THEOREM 5.7. *Suppose that* (i) *f is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron,* (ii) *$\varphi(x)=x^p$, where every component of $p\in\mathbb{Z}_+^n$ is even, and* (iii) *$d(f,x^p)>1$. If the support of χ is contained in a sufficiently small neighborhood of the origin, then C_+ and C_- are nonnegative and $C=C_++C_-$ is positive.*

PROOF. Let

$$\Sigma_p^{(n)}=\{\sigma\in\Sigma^{(n)};{}^\#A_p(\sigma)=m(f,x^p)\}.$$

First, we consider the case when $m(f,x^p)<n$. For $\sigma\in\Sigma_p^{(n)}$, considering the equations (5.4) and applying Lemma 5.6 to (5.6), (5.11) with respect to each y_j for $j\in A_p(\sigma)$, we have

$$\lim_{s\to\beta(p)}(s-\beta(p))^{m(f,x^p)}Z_\pm^{(\sigma)}(s)=\sum_k G_\pm^{(k)}(\sigma)+\sum_l H_\pm^{(l)}(\sigma),\tag{5.16}$$

with

$$G_\pm^{(k)}(\sigma)=\sum_{\epsilon\in E(\pm\alpha_k,\sigma)}\frac{(-1)^{g_{\sigma,p}(\epsilon)}}{\prod_{j\in A_p(\sigma)}l(a^j(\sigma))}\cdot\int_{D(\sigma)}\left(\prod_{j\notin A_p(\sigma)}(y_j)_{\epsilon_j}^{l(a^j(\sigma))\beta(p)+\langle a^j(\sigma),p+\mathbf{1}\rangle-1}\right)|f_\sigma(\hat{y})|^{\beta(p)}\tilde{\psi}_k(\hat{y})d\hat{y},\tag{5.17}$$

$$H_\pm^{(l)}(\sigma)=\sum_{\tilde{\epsilon}\in\tilde{E}(\pm,\sigma)}\frac{(-1)^{\tilde{g}_{\sigma,p}(\tilde{\epsilon})}}{\prod_{j\in A_p(\sigma)}l(a^j(\sigma))}\cdot\int_{D(\sigma)}(y_k)_{\epsilon_k}^{\beta(p)}\left(\prod_{j\in B_l(\sigma)\setminus A_p(\sigma)}(y_j)_{\epsilon_j}^{l(a^j(\sigma))\beta(p)+\langle a^j(\sigma),p+\mathbf{1}\rangle-1}\right)\tilde{\eta}_l(\hat{y})d\hat{y},\tag{5.18}$$

where the summations in (5.16) are taken for all k, l satisfying $T_{A_p(\sigma)} \cap \mathrm{Supp}(\psi_k) \neq \emptyset$ and $A_p(\sigma) \subset B_l(\sigma)$, $\hat{y}$ is defined by $\hat{y}_j = 0$ for $j \in A_p(\sigma)$, $\hat{y}_j = y_j$ for $j \notin A_p(\sigma)$, $d\hat{y} = \prod_{j \notin A_p(\sigma)} dy_j$, $D(\sigma) = \{y \in \mathbb{R}^n; y_j = 0 \text{ for } j \in A_p(\sigma)\} (\approx \mathbb{R}^{n-m(f,x^p)})$ and the other symbols are the same as in (5.6), (5.11). Note that the integrals in (5.18) are convergent and interpreted as improper integrals. In (5.17), (5.18), we deform the cut-off functions ψ_k and η_l as the volume of the support of η_l tends to zero for all l. Then it is easy to see that the limit of $H_{\pm}^{(l)}(\sigma)$ is zero, while that of $\sum_k G_{\pm}^{(k)}(\sigma)$ is $G_{+,\pm}(\sigma) + G_{-,\mp}(\sigma)$, respectively, with

$$G_{u,v}(\sigma) = \sum_{\epsilon \in E(u,\sigma)} \frac{(-1)^{g_{\sigma,p}(\epsilon)}}{\prod_{j \in A_p(\sigma)} l(a^j(\sigma))} \cdot \int_{D_v(\sigma)} \left(\prod_{j \notin A_p(\sigma)} (y_j)_{\epsilon_j}^{l(a^j(\sigma))\beta(p) + \langle a^j(\sigma), p+\mathbf{1}\rangle - 1} \right) |f_\sigma(\hat{y})|^{\beta(p)} \tilde{\chi}_\sigma(\hat{y}) d\hat{y}, \tag{5.19}$$

where $u, v \in \{+, -\}$ and

$$D_v(\sigma) = \{y \in \mathrm{Supp}(\tilde{\chi}_\sigma); v f_\sigma(y) > 0 \text{ and } y_j = 0 \text{ for } j \in A_p(\sigma)\}.$$

Note that the above integral is also improper, if f_σ has a zero on $D(\sigma)$. As a result, we have

$$C_{\pm} = \sum_{\sigma \in \Sigma_p^{(n)}} (G_{+,\pm}(\sigma) + G_{-,\mp}(\sigma)), \tag{5.20}$$

respectively.

Next, we consider the case when $m(f, x^p) = n$. Noticing $\Sigma_p^{(n)} = \{\sigma; A_p(\sigma) = B(\sigma) = \{1, \dots, n\}\}$, we obtain the corresponding coefficients as

$$C_{\pm} = \sum_{\sigma \in \Sigma_p^{(n)}} \sum_{\epsilon \in E(\pm\alpha,\sigma)} \frac{(-1)^{g_{\sigma,p}(\epsilon)} |f_\sigma(0)|^{\beta(p)}}{\prod_{j=1}^n l(a^j(\sigma))}, \tag{5.21}$$

where α is the sign of $f_\sigma(0)$.

Now, let us assume that every component of p is even. By the definition (5.7), $g_{\sigma,p}(\epsilon)$ is also even. From (5.19), (5.20), (5.21), we see the nonnegativity of the coefficients C_+, C_-. Moreover, since $E(+,\sigma)$ is nonempty, the coefficient $C = C_+ + C_-$ is positive. □

The following proposition is concerned with the poles of $Z_+(s)$ and $Z_-(s)$, which are induced by the set of zeros of f_σ.

PROPOSITION 5.8. *Suppose that the conditions* (i), (ii) *in Theorem* 5.1 *are satisfied and* (iii) $d(f, x^p) < 1$. *Let* $1, \dots, k_*$ *be all the natural numbers strictly smaller than* $-\beta(p) = 1/d(f, x^p)$. *If the support of* χ *is contained in a sufficiently small neighborhood of the origin, then* $Z_+(s)$ *and* $Z_-(s)$ *have at* $s = -1, \dots, -k_*$ *poles of order not higher than* 1 *and do not have other poles in the region* $\mathrm{Re}(s) > \beta(p)$. *Moreover, let* a_k^+, a_k^- *be the residues of* $Z_+(s)$, $Z_-(s)$ *at* $s = -k$, *respectively, then we have* $a_k^+ = (-1)^{k-1} a_k^-$ *for* $k = 1, \dots, k_*$.

PROOF. For $j = 2, \dots, n$, let l_j, m_j be positive integers such that $m_j/l_j > k_*$. Let $\epsilon_j = +$ or $-$, $j = 2, \dots, n$, be arbitrarily fixed. Let $C_k^\pm$ be the residues at $s = -k$ of the functions

$$g_\pm(s) := \int_{\mathbb{R}^n} \left((y_1)_\pm^s \prod_{j \in B} (y_j)_{\epsilon_j}^{l_j s + m_j - 1} \right) \eta(y) dy,$$

respectively, where B is a subset in $\{2, \dots, n\}$ and $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta(0) \neq 0$.

By carefully observing the analysis of $J_{\sigma,\pm}^{(l)}(s)$ in the proof of Theorem 5.1, it suffices to show the following.

(a) $g_+(s)$ and $g_-(s)$ have at $s = -1, \dots, -k_*$ poles of order 1 and they do not have other poles in $\mathrm{Re}(s) > \beta(p)$;
(b) $C_k^+ = (-1)^{k-1} C_k^-$ for $k = 1, \dots, k_*$.

From Lemma 5.2, (a) is easy to see. By using Lemma 5.6, we obtain

$$C_k^\pm = \frac{(\pm 1)^{k-1}}{(k-1)!} \int_{\mathbb{R}^{n-1}} \left(\prod_{j \in B} (y_j)_{\epsilon_j}^{-l_j k + m_j - 1} \right) \frac{\partial^{k-1} \eta}{\partial y_1^{k-1}} (0, y_2, \dots, y_n) dy_2 \cdots dy_n.$$

This expression implies (b). □

REMARK 5.9. We can easily generalize the results in this subsection as follows. The same assertions in Theorems 5.1 and 5.7, and Proposition 5.8 can be obtained, even if x^p is replaced by $x^p \tilde{\varphi}(x)$ where $\tilde{\varphi} \in C^\infty(U)$ with $\tilde{\varphi}(0) \neq 0$. Here, in the case of Theorem 5.7, when $\tilde{\varphi}(0) < 0$, "positive" and "nonnegative" must be changed to "negative" and "nonpositive", respectively.

5.2. The convenient case.

Next, let us consider the poles of $Z_\pm(s)$ in (5.1) and $Z(s)$ in (5.2) in the case that f or φ is convenient, i.e., the associated Newton polyhedron intersects all the coordinate axes.

Theorem 5.10. *Suppose that* (i) f *is nondegenerate over* $\mathbb{R}$ *with respect to its Newton polyhedron and* (ii) *at least one of the following conditions is satisfied*:

(a) f *is convenient*;
(b) φ *is convenient*;
(c) φ *is real analytic on a neighborhood of the origin.*

If the support of χ *is contained in a sufficiently small neighborhood of the origin, then the poles of the functions* $Z_+(s)$, $Z_-(s)$ *and* $Z(s)$ *are contained in the set*

$$\left\{-\frac{1}{d(f,\varphi)}-\frac{\nu}{l(a)};\nu\in\mathbb{Z}_+,\ a\in\tilde{\Sigma}^{(1)}\right\}\cup(-\mathbb{N}),$$

where $l(a)$ *is as in* (4.1) *and* $\tilde{\Sigma}^{(1)}$ *is as in Theorem* 5.1. *Moreover, for each* $Z_+(s)$, $Z_-(s)$ *and* $Z(s)$, *if* $s=-1/d(f,\varphi)$ *is a pole, then its order is not larger than*

$$\begin{cases} m(f,\varphi) & \text{if } 1/d(f,\varphi) \text{ is not an integer},\\ \min\{m(f,\varphi)+1,n\} & \text{otherwise}.\end{cases}$$

Proof. Let Σ_f and Σ_φ be the fans constructed from the Newton polyhedra of f and φ, respectively. Define $\Sigma_0=\{\sigma\cap\tilde{\sigma};\sigma\in\Sigma_f,\ \tilde{\sigma}\in\Sigma_\varphi\}$. Then it is easy to see that Σ_0 is also a fan. Fix a simplicial subdivision Σ of Σ_0 and let (Y_Σ,π) be the real resolution associated with Σ as in Section 4.

First, let us compute the form of $\varphi\circ\pi$.

Lemma 5.11. *Let* φ *be a* C^∞ *function defined on a neighborhood of the origin. When* φ *is convenient or real analytic near the origin, define* $\tilde{l}(a)=\min\{\langle a,\alpha\rangle;\alpha\in\Gamma_+(\varphi)\}$ *for* $a\in\mathbb{Z}_+^n$. *Otherwise, define* $\tilde{l}(a)=\min\{\langle a,\alpha\rangle;\alpha\in\Gamma_+(\varphi)\}$ *for* $a\in\mathbb{N}^n$ *and* $\tilde{l}(a)=0$ *for* $a\in\mathbb{Z}_+^n\setminus\mathbb{N}^n$. *Then, for* $\sigma\in\Sigma^{(n)}$, $\varphi\circ\pi(\sigma)$ *can be expressed as*

$$\varphi(\pi(\sigma)(y))=\left(\prod_{j=1}^{n}y_j^{\tilde{l}(a^j(\sigma))}\right)\varphi_\sigma(y), \tag{5.22}$$

where φ_σ *is a* C^∞ *function defined on a neighborhood of the origin.* (*Needless to say, if* φ *is real analytic, so is* φ_σ.)

Proof of Lemma 5.11. Let us consider the case that φ is a C^∞ function. By using Taylor's formula, φ can be expressed as (3.4) in Section 3. Substituting

$x = \pi(\sigma)(y)$ with (4.2) into (3.4), we have

$$\varphi(\pi(\sigma)(y)) = \sum_{\alpha \in S_\varphi \cap U_N} c_\alpha \left(\prod_{j=1}^n y_j^{\langle a^j(\sigma), \alpha \rangle} \right) + \sum_{p \in \mathbb{Z}_+^n, \langle p \rangle = N} \left(\prod_{j=1}^n y_j^{\langle a^j(\sigma), p \rangle} \right) \varphi_p(\pi(\sigma)(y)).$$

Here, take a sufficiently large $N \in \mathbb{N}$ such that the union of the hypersurfaces $H(a, \tilde{l}(a)) \cap \mathbb{R}_+^n$ for all $a \in \Sigma^{(1)} \cap \mathbb{N}^n$ is contained in the set $U_N = \{\alpha \in \mathbb{R}_+^n; \langle \alpha \rangle \le N\}$. If $a \in \Sigma^{(1)} \cap \mathbb{N}^n$, then we see that $\langle a, \alpha \rangle \ge \tilde{l}(a)$ for $\alpha \in \Gamma_+(\varphi)$ and $\langle a, p \rangle \ge \tilde{l}(a)$ for $p \in \mathbb{Z}_+^n$ with $\langle p \rangle = N$. Therefore, we can get

$$\varphi(\pi(\sigma)(y)) = \left(\prod_{j \in C(\sigma)} y_j^{\tilde{l}(a^j(\sigma))} \right) \varphi_\sigma(y),$$

where $C(\sigma) = \{j; a^j(\sigma) \in \mathbb{N}^n\} \subset \{1, \ldots, n\}$ and φ_σ is a C^∞ function defined on a neighborhood of the origin. Notice that if φ is convenient, then $\tilde{l}(a^j(\sigma)) = 0$ for $a \in \mathbb{Z}_+^n \setminus \mathbb{N}^n$. Thus, we obtain the expression (5.22) for every C^∞ function φ.

The case of real analytic φ is easier, so the proof is omitted. $\square$

Using the map $\pi : Y_\Sigma \to \mathbb{R}^n$ with $x = \pi(y)$ and the cut-off functions $\{\chi_\sigma; \sigma \in \Sigma^{(n)}\}$ in the proof of Theorem 5.1 and substituting (5.22), we have

$$\begin{aligned} Z_\pm(s) &= \int_{\mathbb{R}^n} f(x)_\pm^s \varphi(x)\chi(x)dx \\ &= \int_{Y_\Sigma} ((f \circ \pi)(y))_\pm^s (\varphi \circ \pi)(y)(\chi \circ \pi)(y)|J_\pi(y)|dy \\ &= \sum_{\sigma \in \Sigma^{(n)}} Z_\pm^{(\sigma)}(s), \end{aligned}$$

with

$$\begin{aligned} Z_\pm^{(\sigma)}(s) &= \int_{\mathbb{R}^n} ((f \circ \pi(\sigma))(y))_\pm^s (\varphi \circ \pi(\sigma))(y)(\chi \circ \pi(\sigma))(y)\chi_\sigma(y)|J_{\pi(\sigma)}(y)|dy \\ &= \int_{\mathbb{R}^n} \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} f_\sigma(y) \right)_\pm^s \left(\prod_{j=1}^n y_j^{\tilde{l}(a^j(\sigma))} \varphi_\sigma(y) \right) \left| \prod_{j=1}^n y_j^{\langle a^j(\sigma) \rangle - 1} \right| \tilde{\chi}_\sigma(y)dy, \end{aligned} \tag{5.23}$$

where $\tilde{\chi}_\sigma(y) = (\chi \circ \pi(\sigma))(y)\chi_\sigma(y)$. By an argument similar to that in the proof of Theorem 5.1, we see that the poles of $Z_\pm^{(\sigma)}(s)$ are contained in the set

$$\left\{ -\frac{\tilde{l}(a^j(\sigma)) + \langle a^j(\sigma)\rangle + \nu}{l(a^j(\sigma))}; j \in B(\sigma),\ \nu \in \mathbb{Z}_+ \right\} \cup (-\mathbb{N}), \tag{5.24}$$

where $B(\sigma)$ is as in (5.9). Note that the set $(-\mathbb{N})$ is necessary to express the poles because of the same reason as in Remark 5.3.

Next, consider a geometrical meaning of the largest element of the union of the first set in (5.24) for all $\sigma \in \Sigma^{(n)}$. If φ is convenient or real analytic near the origin, then $\tilde{l}(a) = \min\{\langle a, \alpha\rangle; \alpha \in \Gamma_+(\varphi)\}$ if $a \in \mathbb{Z}_+^n$. Therefore, from the definitions of $\beta(\cdot)$ in (5.13) and Proposition 5.4, we have

$$\begin{aligned}
&\max\left\{ -\frac{\tilde{l}(a) + \langle a\rangle}{l(a)}; a \in \tilde{\Sigma}^{(1)} \right\} \\
&= \max\left\{ -\frac{\langle a, \alpha + \mathbf{1}\rangle}{l(a)}; \alpha \in \Gamma_+(\varphi), a \in \tilde{\Sigma}^{(1)} \right\} \\
&= \max\{\beta(\alpha); \alpha \in \Gamma_+(\varphi)\} \\
&= \max\left\{ -\frac{1}{d(f, x^\alpha)}; \alpha \in \Gamma_+(\varphi) \right\} \\
&= \max\left\{ -\frac{1}{d(f, x^\alpha)}; \alpha \in \Gamma(\varphi, f) \right\} = -\frac{1}{d(f,\varphi)}.
\end{aligned} \tag{5.25}$$

On the other hand, when f is convenient, it is easy to see $\tilde{\Sigma}^{(1)} = \Sigma^{(1)} \cap \mathbb{N}^n$, which implies the first equality in (5.25). Note that $\tilde{\Sigma}^{(1)} \supset \Sigma^{(1)} \cap \mathbb{N}^n$ in general case. From (5.24) and (5.25), we see that the poles of $Z_\pm(s)$ are contained in the set in the theorem.

Finally, consider the orders of the poles of $Z_\pm(s)$ at $s = -1/d(f,\varphi)$. Considering the construction of simplicial subdivision in the beginning of the proof, we see that for each $\sigma \in \Sigma^{(n)}$ there exists a vertex α_σ of $\Gamma_+(\varphi)$ such that $\langle a^j(\sigma), \alpha_\sigma\rangle = \tilde{l}(a^j(\sigma))$ for $j = 1, \ldots, n$. Therefore, for $\sigma \in \Sigma^{(n)}$, we have

$$\begin{aligned}
&{}^{\#}\left\{ j; -\frac{\tilde{l}(a^j(\sigma)) + \langle a^j(\sigma)\rangle}{l(a^j(\sigma))} = -\frac{1}{d(f,\varphi)} \right\} \\
&= {}^{\#}\{ j; d(f,\varphi)(\langle a^j(\sigma), \alpha_\sigma\rangle + \langle a^j(\sigma)\rangle) = l(a^j(\sigma))\} \\
&= {}^{\#}\{ j; \langle a^j(\sigma), d(f,\varphi)(\alpha_\sigma + \mathbf{1})\rangle = l(a^j(\sigma))\}
\end{aligned}$$

$$= {}^{\#}\{j; d(f,\varphi)(\alpha_\sigma + \mathbf{1}) \in H(a^j(\sigma), l(a^j(\sigma)))\}$$
$$= \rho_f(d(f,\varphi)(\alpha_\sigma + \mathbf{1})). \tag{5.26}$$

The last equality follows from the last part of the proof of Proposition 5.4. Since $\rho_f(d(f,\varphi)(\alpha_\sigma + \mathbf{1})) \le m(f,\varphi)$, we obtain the estimate of the orders of the poles in Theorem 5.10 by the same argument as that in Proposition 5.4.

It is easy to show the case of $Z(s)$ by using the relationship: $Z(s) = Z_+(s) + Z_-(s)$. □

REMARK 5.12. In the beginning of the above proof, we constructed a simplicial subdivision Σ from a more complicated process. Before estimating the orders of poles, the usual subdivision as in Theorem 5.1 is sufficient.

Next, when $d(f,\varphi) > 1$, we consider the coefficients of $(s + 1/d(f,\varphi))^{-m(f,\varphi)}$ in the Laurent expansions of $Z_\pm(s)$ and $Z(s)$. Let

$$\begin{aligned} C_\pm &= \lim_{s \to -1/d(f,\varphi)} (s + 1/d(f,\varphi))^{m(f,\varphi)} Z_\pm(s), \\ C &= \lim_{s \to -1/d(f,\varphi)} (s + 1/d(f,\varphi))^{m(f,\varphi)} Z(s), \end{aligned} \tag{5.27}$$

respectively.

THEOREM 5.13. *Suppose that* (i) f *is convenient and nondegenerate over* $\mathbb{R}$ *with respect to its Newton polyhedron,* (ii) φ_{Γ_0} *is nonnegative* (*resp. nonpositive*) *on a neighborhood of the origin and* (iii) $d(f,\varphi) > 1$. *If the support of* χ *is contained in a sufficiently small neighborhood of the origin, then* C_+ *and* C_- *are nonnegative* (*resp. nonpositive*) *and* $C = C_+ + C_-$ *is positive* (*resp. negative*).

PROOF. We only show the theorem in the nonnegative case in the assumption (ii).

Let Σ_0 be the fan constructed from the Newton polyhedron of f. Fix a simplicial subdivision Σ of Σ_0 and let (Y_Σ, π) be the real resolution associated with Σ as in Section 4.

Notice that $\Gamma(\varphi, f)$ and Γ_0 are compact sets, because f is convenient. Recall that the essential set Γ_0 is expressed as the disjoint union of some finite faces $\gamma_1, \dots, \gamma_l$ of $\Gamma_+(\varphi)$ and that the nonnegativity of φ is equivalent to that of φ_{γ_μ} for $\mu = 1, \dots, l$ (see Section 3).

We define the functions $\varphi_1, \dots, \varphi_{l+2}$ as follows.

$$\varphi_\mu(x) = \varphi_{\gamma_\mu}(x) \quad \text{for } \mu = 1, \dots, l,$$
$$\varphi_{l+1}(x) = \varphi_{\Gamma(\varphi,f)\setminus\Gamma_0}(x), \qquad \varphi_{l+2}(x) = \varphi(x) - \sum_{\mu=1}^{l+1} \varphi_\mu(x). \tag{5.28}$$

Substituting (5.28) into (5.1), we obtain

$$Z_\pm(s) = \sum_{\mu=1}^{l+2} Z_{\mu,\pm}(s),$$

where

$$Z_{\mu,\pm}(s) = \int_{\mathbb{R}^n} f(x)_\pm^s \varphi_\mu(x)\chi(x)dx, \quad \mu = 1, \dots, l+2.$$

From Theorem 5.10, we see that the poles of $Z_{l+2,\pm}(s)$ are contained in the set

$$\left\{ -\frac{1}{d(f,\varphi_{l+2})} - \frac{\nu}{l(a)}; \nu \in \mathbb{Z}_+, a \in \tilde{\Sigma}^{(1)} \right\} \cup (-\mathbb{N}).$$

Since $d(f,\varphi_{l+2}) < d(f,\varphi)$, $Z_{l+2,\pm}(s)$ can be extended analytically to the region $\mathrm{Re}(s) \geq -1/d(f,\varphi) - \delta$ with small $\delta > 0$. Moreover, the order of the poles of $Z_{l+1,\pm}(s)$ at $s = -1/d(f,\varphi)$ is less than $m(f,\varphi)$ from Theorem 5.7 (or Theorem 5.10). It suffices to consider the poles of $Z_{\mu,\pm}(s)$ at $s = -1/d(f,\varphi)$ for $\mu = 1, \dots, l$.

For each $\sigma \in \Sigma^{(n)}$ and $\mu \in \{1, \dots, l\}$, there exists a set $A_\mu(\sigma) \subset \{1, \dots, n\}$ such that

$$A_\mu(\sigma) = \left\{ j \in B(\sigma); \frac{\langle a^j(\sigma), \alpha + \mathbf{1} \rangle}{l(a^j(\sigma))} = \frac{1}{d(f,\varphi)} \right\} \subset \{1, \dots, n\}, \tag{5.29}$$

for all $\alpha \in \gamma_\mu \cap \mathbb{Z}_+^n$, where $B(\sigma)$ is as in (5.9). For μ, define

$$\Sigma_\mu^{(n)} = \{\sigma \in \Sigma^{(n)}; {}^\# A_\mu(\sigma) = m(f,\varphi)\}.$$

By applying the argument in the proof of Theorem 5.1 with the condition $d(f,\varphi) > 1$, it suffices to consider the pole at $s = -1/d(f,\varphi)$ of the function

$$I_{\mu,\sigma,\epsilon}(s) = \int_{\mathbb{R}^n} \Phi_{\mu,\epsilon}(y,s)|f_\sigma(y)|^s \psi(y)dy, \tag{5.30}$$

with

$$\Phi_{\mu,\epsilon}(y,s) = \left(\prod_{j=1}^{n} (y_j)_{\epsilon_j}^{l(a^j(\sigma))s}\right)\left(\sum_{\alpha\in\gamma_\mu} c_\alpha \prod_{j=1}^{n} y_j^{\langle a^j(\sigma),\alpha\rangle}\right)\left|\prod_{j=1}^{n} y_j^{\langle a^j(\sigma)\rangle-1}\right|, \quad (5.31)$$

where $\sigma \in \Sigma_\mu^{(n)}$, $\epsilon \in \{+,-\}^n$, and $\psi(y)$ is a cut-off function on $\mathbb{R}(\sigma)^n$ satisfying that $\mathrm{Supp}(\psi) \subset \mathrm{Supp}(\tilde{\chi}_\sigma)$, where $\tilde{\chi}_\sigma$ is as in (5.3), and its support does not intersect the set $\{y \in \mathrm{Supp}(\tilde{\chi}_\sigma); f_\sigma(y) = 0\}$. Indeed, if $\sigma \in \Sigma^{(n)} \setminus \Sigma_\mu^{(n)}$, then the order of the pole of $I_{\mu,\sigma,\epsilon}(s)$ is less than $m(f,\varphi)$.

A simple computation gives

$$\begin{aligned}\Phi_{\mu,\epsilon}(y,s) &= \sum_{\alpha\in\gamma_\mu} (-1)^{g_{\sigma,\alpha}(\epsilon)} c_\alpha \left(\prod_{j=1}^{n} (y_j)_{\epsilon_j}^{l(a^j(\sigma))s+\langle a^j(\sigma),\alpha+\mathbf{1}\rangle-1}\right)\\ &= \left(\prod_{j\in A_\mu(\sigma)} (y_j)_{\epsilon_j}^{l(a^j(\sigma))(s+1/d(f,\varphi))-1}\right)\Phi_{\mu,\epsilon}(\tilde{y},s), \qquad (5.32)\end{aligned}$$

where $g_{\sigma,\alpha}(\epsilon)$ is as in (5.7) and $\tilde{y}$ is defined by $\tilde{y}_j = \epsilon_j 1 \in \{\pm 1\}$ for $j \in A_\mu(\sigma)$; $\tilde{y}_j = y_j$ for $j \notin A_\mu(\sigma)$. Substituting (5.32) into (5.30), we have

$$I_{\mu,\sigma,\epsilon}(s) = \int_{\mathbb{R}^n} \left(\prod_{j\in A_\mu(\sigma)} (y_j)_{\epsilon_j}^{l(a^j(\sigma))(s+1/d(f,\varphi))-1}\right)\Phi_{\mu,\epsilon}(\tilde{y},s)|f_\sigma(y)|^s\psi(y)dy.$$

First, we consider the case when $m(f,\varphi) < n$. By applying Lemma 5.6, the coefficient of $(s+1/d(f,\varphi))^{-m(f,\varphi)}$ in the Laurent expansion of $I_{\mu,\sigma,\epsilon}(s)$ is

$$\frac{1}{\prod_{j\in A_\mu(\sigma)} l(a^j(\sigma))} \int_{\mathbb{R}^{n-m(f,\varphi)}} \Phi_{\mu,\epsilon}(\tilde{y},-1/d(f,\varphi))|f_\sigma(\hat{y})|^{-1/d(f,\varphi)}\psi(\hat{y})d\hat{y}, \quad (5.33)$$

where $\hat{y}$ is defined by $\hat{y}_j = 0$ for $j \in A_\mu(\sigma)$, $\hat{y}_j = y_j$ for $j \notin A_\mu(\sigma)$ and $d\hat{y} = \prod_{j\notin A_\mu(\sigma)} dy_j$. From (5.31), the nonnegativity of φ_{γ_μ} implies that of $\Phi_{\mu,\epsilon}$. Moreover, since φ_{γ_μ} is a polynomial, there is an open set in $\mathrm{Supp}(\psi)$ such that $\Phi_{\mu,\epsilon}(\cdot,-1/d(f,\varphi))$ is positive there. Consider the case that the support of ψ contains the origin. Then we can see that (5.33) is positive for any $\sigma \in \Sigma_\mu^{(n)}$ and $\epsilon \in \{+,-\}^n$.

Next, we consider the case when $m(f,\varphi) = n$. Then, $A_\mu(\sigma) = B(\sigma) = \{1,\dots,n\}$. Since the essential set Γ_0 is a set of vertices of $\Gamma_+(\varphi)$, the nonnegative assumption of φ_{Γ_0} implies that each φ_μ can be expressed as a monomial of the form: $\varphi_\mu(x) = c_\alpha x^\alpha$, where c_α is a positive constant and every component of α is even. From an easy computation, the corresponding coefficient in this case is obtained as

$$\frac{c_\alpha |f_\sigma(0)|^{-1/d(f,\varphi)}}{\prod_{j=1}^n l(a^j(\sigma))}. \tag{5.34}$$

Of course, (5.34) is positive for any $\sigma \in \Sigma_\mu^{(n)}$ and is independent of $\epsilon \in \{-,+\}^n$.

Finally, by the same argument as that in the proof of Theorem 5.7, we can see the nonnegativity of C_+ and C_- and the positivity of $C = C_+ + C_-$ in the theorem. □

Proposition 5.14. *Suppose that the conditions* (i), (ii) *in Theorem* 5.10 *are satisfied and* (iii) $d(f,\varphi) < 1$. *Let* $1, \ldots, k_*$ *be all the natural numbers strictly smaller than* $1/d(f,\varphi)$. *If the support of* χ *is contained in a sufficiently small neighborhood of the origin, then* $Z_+(s)$ *and* $Z_-(s)$ *have at* $s = -1, \ldots, -k_*$ *poles of order not higher than 1 and they do not have other poles in the region* $\mathrm{Re}(s) > -1/d(f,\varphi)$. *Moreover, let* a_k^+, a_k^- *be the residues of* $Z_+(s)$, $Z_-(s)$ *at* $s = -k$, *respectively, then we have* $a_k^+ = (-1)^{k-1} a_k^-$ *for* $k = 1, \ldots, k_*$.

Proof. The difference between (5.3) and (5.23) does not essentially affect the argument in the proof of Proposition 5.8. The details are left to the readers. □

5.3. Remarks.

In this subsection, let us consider Theorem 5.7 (with Remark 5.9) and Theorem 5.13 under the additional assumption: f is nonnegative or nonpositive near the origin. The following theorem shows that the same assertions can be obtained without the assumption: $d(f,\varphi) > 1$.

Theorem 5.15. *Suppose that* (i) f *is nondegenerate over* $\mathbb{R}$ *with respect to its Newton polyhedron,* (ii) f *is nonnegative or nonpositive on a neighborhood of the origin and* (iii) *at least one of the following condition is satisfied*:

(a) φ *is expressed as* $\varphi(x) = x^p \tilde{\varphi}(x)$ *on a neighborhood of the origin, where every component of* $p \in \mathbb{Z}_+^n$ *is even and* $\tilde{\varphi}(0) > 0$ (*resp.* $\tilde{\varphi}(0) < 0$);

(b) f *is convenient and* φ_{Γ_0} *is nonnegative* (*resp. nonpositive*) *on a neighborhood of the origin.*

If the support of χ *is contained in a sufficiently small neighborhood of the origin, then* C_+ *and* C_- *are nonnegative* (*resp. nonpositive*) *and* $C = C_+ + C_-$ *is positive* (*resp. negative*), *where* $C_\pm$, C *are as in* (5.27).

PROOF. We only show the case that f is nonnegative in the assumption (ii).

Let Σ_0 be the fan constructed from the Newton polyhedron of f. Fix a simplicial subdivision Σ of Σ_0 and let (Y_Σ, π) be the real resolution associated with Σ as in Section 4. Let $\sigma \in \Sigma^{(n)}$. First, from Proposition 4.2 and the assumptions (i), (ii), there exists a neighborhood U_σ of the origin such that $f(x) \geq 0$ for $x \in U_\sigma$ and

$$f(\pi(\sigma)(y)) = \left(\prod_{j=1}^{n} y_j^{l(a^j(\sigma))} \right) f_\sigma(y) \quad \text{for } y \in \pi(\sigma)^{-1}(U_\sigma),$$

where f_σ is real analytic and $f_\sigma(0) > 0$. It is easy to see that all $l(a^j(\sigma))$ are even and that $f_\sigma(y) \geq 0$ for $y \in \pi(\sigma)^{-1}(U_\sigma)$.

Now, let us assume that there exists a point $y_0 \in T_I \cap \pi(\sigma)^{-1}(U_\sigma)$ with nonempty $I \subset \{1, \ldots, n\}$ such that $f_\sigma(y_0) = 0$. (See (4.3) for the definition of T_I.) Since f is nondegenerate, Proposition 4.2 implies that there is a point $y_* \in \pi(\sigma)^{-1}(U_\sigma)$ near y_0 such that $f_\sigma(y_*) < 0$. This contradicts the nonnegativity of f on U_σ, so we see that $\{y \in \pi(\sigma)^{-1}(U_\sigma); f_\sigma(y) = 0\} \subset (\mathbb{R} \setminus \{0\})^n$.

Therefore, there exists a small neighborhood $V_\sigma \subset U_\sigma$ of the origin such that $\{y; f_\sigma(y) = 0\} \cap \pi(\sigma)^{-1}(V_\sigma) = \emptyset$. Then f_σ is positive on $\pi(\sigma)^{-1}(V_\sigma)$ for each $\sigma \in \Sigma^{(n)}$.

Let us investigate the properties of poles of $Z_\pm(s)$ in (5.1), where the cut-off function χ has a support contained in the set $V := \bigcap_{\sigma \in \Sigma^{(n)}} V_\sigma$. We apply the arguments in the proofs of Theorems 5.7 and 5.13 to these cases. In the process of analysis, the decomposition similar to (5.4) is obtained, but the functions corresponding to $J_{\sigma,\pm}^{(k)}(s)$ do not appear because f_σ is always positive on V. Thus, since it suffices to consider the poles of $I_{\sigma,\pm}^{(k)}(s)$, we can easily obtain the assertions of this theorem. $\square$

5.4. Certain symmetrical properties.

We denote by $\beta_\pm(f, \varphi)$, $\hat{\beta}(f, \varphi)$ the largest poles of $Z_\pm(s)$, $Z(s)$ and by $\eta_\pm(f, \varphi)$, $\hat{\eta}(f, \varphi)$ their orders, respectively.

THEOREM 5.16. *Let f, φ be nonnegative or nonpositive real analytic functions defined on a neighborhood of the origin. Suppose that f and φ are convenient and nondegenerate over $\mathbb{R}$ with respect to their Newton polyhedra. If the support of χ is contained in a sufficiently small neighborhood of the origin, then we have*

$$\beta_\pm(x^{\mathbf{1}} f, \varphi)\beta_\pm(x^{\mathbf{1}}\varphi, f) \leq 1 \ \text{and} \ \hat{\beta}(x^{\mathbf{1}} f, \varphi)\hat{\beta}(x^{\mathbf{1}}\varphi, f) \leq 1 \tag{5.35}$$

Moreover, the following two conditions are equivalent:

(i) *The equality holds in each estimate in* (5.35);
(ii) *There exists a positive rational number d such that* $\Gamma_+(x^{\mathbf{1}}f) = d \cdot \Gamma_+(x^{\mathbf{1}}\varphi)$.

If the condition (i) *or* (ii) *is satisfied, then we have* $\eta_\pm(x^{\mathbf{1}}f,\varphi) = \eta_\pm(x^{\mathbf{1}}\varphi, f) = \hat{\eta}(x^{\mathbf{1}}\varphi, f) = n$.

Proof. Admitting Lemmas 5.17 and 5.19 below, we prove the theorem as follows. From the assumptions in the theorem and Lemma 5.17 with $p = \mathbf{1}$, Theorem 5.13 implies the equations $\beta(x^{\mathbf{1}}f,\varphi) = -1/d(x^{\mathbf{1}}f,\varphi)$ and $\beta(x^{\mathbf{1}}\varphi, f) = -1/d(x^{\mathbf{1}}\varphi, f)$. By Lemma 5.19, these equations imply the inequality (5.35) and the equivalence of (i) and (ii). The condition (ii) in the theorem implies $\Gamma(\varphi, x^{\mathbf{1}}f) = \Gamma(\varphi)$. Thus, Γ_0 is the set of the vertices (i.e., zero-dimensional faces) of $\Gamma(\varphi)$. This means $\eta_\pm(x^{\mathbf{1}}f,\varphi) = m(x^{\mathbf{1}}f,\varphi) = n$. Similarly, we see $\eta_\pm(x^{\mathbf{1}}\varphi, f) = n$. □

Lemma 5.17. *Let $p \in \mathbb{Z}_+^n$ and g be a C^∞ function defined on a neighborhood of the origin in $\mathbb{R}^n$ such that $\Gamma(g) \neq \emptyset$. Then, g is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron if and only if so is $x^p g$.*

Proof. First, notice that if a C^∞ function ψ has a quasihomogeneous property: $\psi(t^{m_1}x_1,\ldots,t^{m_n}x_n) = t^c\psi(x)$ for $t > 0$ and $x \in \mathbb{R}^n$, then $\nabla\psi(x) = 0$ implies $\psi(x) = 0$. In fact, this follows from Euler's identity:

$$m_1 x_1 \frac{\partial\psi}{\partial x_1}(x) + \cdots + m_n x_n \frac{\partial\psi}{\partial x_n}(x) = c\psi(x)$$

for $x \in \mathbb{R}^n$.

Now, we assume that $h(x) := x^p g(x)$ is not nondegenerate in the above sense. Then, there exist a face γ of $\Gamma_+(h)$ and a point $x_0 \in (\mathbb{R}\setminus\{0\})^n$ such that $\nabla h_\gamma(x_0) = 0$ and $h_\gamma(x_0) = 0$. It is easy to see the existence of the face $\tilde{\gamma}$ of $\Gamma_+(g)$ such that $h_\gamma(x) = x^p g_{\tilde{\gamma}}(x)$. Note that $\tilde{\gamma} + p = \gamma$. The assumption on h implies $\nabla g_{\tilde{\gamma}}(x_0) = 0$, which means that g is not nondegenerate.

The above argument is also available for $p \in (-\mathbb{Z}_+)^n$, so the converse can be shown similarly. □

Remark 5.18. By observing the above proof, it might be expected that the above lemma can be generalized as follow. "Let g be the same as above and let a C^∞ function ρ be positive on $(\mathbb{R} \setminus \{0\})^n$. Then g is nondegenerate in the above sense if and only if so is $\rho(x)g(x)$." But, this claim is not true. In fact, consider the two-dimensional case: $g(x,y) = xy$ and $\rho(x,y) = (x-y)^2 + x^4$.

Lemma 5.19. *Let g,h be C^∞ functions defined near the origin, then we have*

$$d(x^{\mathbf{1}}g, h)d(x^{\mathbf{1}}h, g) \geq 1.$$

The equality holds in the above, if and only if there exists a positive rational number d such that $\Gamma_+(x^{\mathbf{1}}g) = d \cdot \Gamma_+(x^{\mathbf{1}}h)$.

Proof. From the definition of $d(\cdot, \cdot)$,

$$d(x^{\mathbf{1}}g, h) \cdot \Gamma_+(x^{\mathbf{1}}h) \subset \Gamma_+(x^{\mathbf{1}}g), \tag{5.36}$$

$$d(x^{\mathbf{1}}h, g) \cdot \Gamma_+(x^{\mathbf{1}}g) \subset \Gamma_+(x^{\mathbf{1}}h). \tag{5.37}$$

Putting (5.36),(5.37) together, we have

$$(d(x^{\mathbf{1}}g, h)d(x^{\mathbf{1}}h, g)) \cdot \Gamma_+(x^{\mathbf{1}}g) \subset \Gamma_+(x^{\mathbf{1}}g). \tag{5.38}$$

This implies $d(x^{\mathbf{1}}g, h)d(x^{\mathbf{1}}h, g) \geq 1$.

If there exists a positive number d such that $\Gamma_+(x^{\mathbf{1}}g) = d \cdot \Gamma_+(x^{\mathbf{1}}h)$, then it is clear that $d = d(x^{\mathbf{1}}g, h) = 1/d(x^{\mathbf{1}}h, g)$. On the other hand, if $\Gamma_+(x^{\mathbf{1}}g) \neq d \cdot \Gamma_+(x^{\mathbf{1}}h)$ for any $d > 0$, then the inclusions in (5.36), (5.37) are in the strict sense, therefore so is the inclusion in (5.38). Then we see that $d(x^{\mathbf{1}}g, h)d(x^{\mathbf{1}}h, g) > 1$. $\square$

6. Proofs of the theorems in Section 2.

6.1. Relationship between $I(\tau)$ and $Z_\pm(s)$.

It is known (see [**17**], [**19**], [**1**], etc.) that the study of the asymptotic behavior of the oscillatory integral $I(\tau)$ in (1.1) can be reduced to an investigation of the poles of the functions $Z_\pm(s)$ in (5.1). Here, we explain an outline of these situations. Let f, φ, χ satisfy the conditions (A), (B), (C) in Section 2.2. Suppose that the support of χ is sufficiently small.

Define the Gelfand-Leray function: $K : \mathbb{R} \to \mathbb{R}$ as

$$K(t) = \int_{W_t} \varphi(x)\chi(x)\omega, \tag{6.1}$$

where $W_t = \{x \in \mathbb{R}^n; f(x) = t\}$ and ω is the surface element on W_t which is determined by $df \wedge \omega = dx_1 \wedge \cdots \wedge dx_n$. $I(\tau)$ and $Z_\pm(s)$ can be expressed by using $K(t)$: Changing the integral variables in (1.1), (5.1), we have

$$I(\tau) = \int_{-\infty}^{\infty} e^{i\tau t}K(t)dt = \int_0^{\infty} e^{i\tau t}K(t)dt + \int_0^{\infty} e^{-i\tau t}K(-t)dt, \tag{6.2}$$

$$Z_\pm(s) = \int_0^{\infty} t^s K(\pm t)dt, \tag{6.3}$$

respectively. Applying the inverse formula of the Mellin transform to (6.3), we have

$$K(\pm t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z_{\pm}(s) t^{-s-1} ds, \tag{6.4}$$

where $c > 0$ and the integral contour follows the line $\mathrm{Re}(s) = c$ upwards. Recall that $Z_+(s)$ and $Z_-(s)$ are meromorphic functions and their poles exist on the negative part of the real axis. By deforming the integral contour as c tends to $-\infty$ in (6.4), the residue formula gives the asymptotic expansions of $K(t)$ as $t \to \pm 0$. Substituting these expansions of $K(t)$ into (6.2), we can get an asymptotic expansion of $I(\tau)$ as $\tau \to +\infty$.

Through the above calculation, we see more precise relationship for the coefficients. If $Z_+(s)$ and $Z_-(s)$ have the Laurent expansions at $s = -\lambda$:

$$Z_{\pm}(s) = \frac{B_{\pm}}{(s+\lambda)^{\rho}} + O\left(\frac{1}{(s+\lambda)^{\rho-1}}\right),$$

respectively, then the corresponding part in the asymptotic expansion of $I(\tau)$ has the form

$$B\tau^{-\lambda}(\log \tau)^{\rho-1} + O(\tau^{-\lambda}(\log \tau)^{\rho-2}).$$

Here a simple computation gives the following relationship:

$$B = \frac{\Gamma(\lambda)}{(\rho-1)!}\left[e^{i\pi\lambda/2}B_+ + e^{-i\pi\lambda/2}B_-\right], \tag{6.5}$$

where Γ is the Gamma function.

6.2. Proofs of Theorems 2.2, 2.7 and 2.12.

Applying the above argument to the results relating to $Z_{\pm}(s)$ in Section 5, we obtain the theorems in Section 2.

Proof of Theorem 2.2. This theorem follows from Theorem 5.1 with Remark 5.9 and Theorem 5.10. Notice that Propositions 5.8 and 5.14 and the relationship (6.5) induce the cancellation of the coefficients of the term, whose orders are larger than $-1/d(f, \varphi)$. The estimate in Remark 2.3 is also obtained by using the estimates of orders of the poles of $Z_{\pm}(s), Z(s)$ in Proposition 5.4 and Theorem 5.10.

Proof of Theorem 2.7. This theorem follows from Theorem 5.7 with Remark 5.9 and Theorems 5.13 and 5.15. Notice that the relationship (6.5) gives the information about the coefficient of the leading term of $I(\tau)$.

Proof of Theorem 2.12. This theorem follows from Theorem 5.16.

7. Examples.

In this section, we give some examples of the phase and the amplitude in the integral (1.1), which clarifies the subtlety of our results in Sections 2 and 6. Throughout this section, we always assume that f, φ, χ satisfy the conditions (A), (B), (C) in Section 2. (In Examples 1, 2, each f, φ, φ_t satisfies the respective condition.)

7.1. The one-dimensional case.

Let us compute the asymptotic expansion of $I(\tau)$ in (1.1) as $\tau \to +\infty$ in the one-dimensional case by using our analysis in this paper. As mentioned in Section 2, the results below can also be obtained by using the analysis in [**26**]. Note that the computation below is valid for C^∞ phases. From the assumptions $\Gamma_+(f), \Gamma_+(\varphi) \neq \emptyset$, f, φ can be expressed as

$$f(x) = x^q \tilde{f}(x), \quad \varphi(x) = x^p \tilde{\varphi}(x),$$

where $q, p \in \mathbb{Z}_+$, $q \geq 2$ and $\tilde{f}, \tilde{\varphi}$ are C^∞ functions defined on a neighborhood of the origin with $\tilde{f}(0)\tilde{\varphi}(0) \neq 0$. Suppose that the support of χ is so small that $\tilde{f}, \tilde{\varphi}$ do not have any zero on the support.

It is easy to see that f is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron, $\Gamma_+(f) = [q, \infty)$, $\Gamma_+(\varphi) = [p, \infty)$, $d(f, \varphi) = q/(p+1)$ and $m(f, \varphi) = 1$. Let α be the sign of $\tilde{f}(x)$ on the support of χ. From a simple computation, for even q

$$\begin{aligned} Z_\alpha(s) &= \int_0^\infty x^{qs+p} \big\{ |\tilde{f}(x)|^s \tilde{\varphi}(x)\chi(x) + (-1)^p |\tilde{f}(-x)|^s \tilde{\varphi}(-x)\chi(-x) \big\} dx, \\ Z_{-\alpha}(s) &= 0, \end{aligned} \tag{7.1}$$

and for odd q

$$\begin{aligned} Z_\alpha(s) &= \int_0^\infty x^{qs+p} |\tilde{f}(x)|^s \tilde{\varphi}(x)\chi(x) dx, \\ Z_{-\alpha}(s) &= (-1)^p \int_0^\infty x^{qs+p} |\tilde{f}(-x)|^s \tilde{\varphi}(-x)\chi(-x) dx. \end{aligned} \tag{7.2}$$

By using Lemma 5.2, we can see that the poles of $Z_\pm(s)$ are simple and they are contained in the set $\{-(p+1+\nu)/q; \nu \in \mathbb{Z}_+\}$. By using Lemma 5.6, we can compute the explicit values of the coefficients of the term $(s+(p+1)/q)^{-1}$ in the Laurent expansions of $Z_+(s)$ and $Z_-(s)$.

Next, applying the argument in Section 6.1, we have

$$I(\tau) \sim \tau^{-(p+1)/q} \sum_{j=0}^{\infty} C_j \tau^{-j/q} \quad \text{as } \tau \to \infty.$$

The relationship (6.5) gives the values of the coefficient C_0. As a result, we can see all the cases that $\beta(f,\varphi) = -1/d(f,\varphi)$ holds.

(i) (q: even; p: even) $C_0 = (2/q)\Gamma((p+1)/q)|\tilde{f}(0)|^{-(p+1)/q}\tilde{\varphi}(0)e^{\alpha i((p+1)/2q)\pi} \neq 0$, which implies $\beta(f,\varphi) = -1/d(f,\varphi)$;
(ii) (q: even; p: odd) $C_0 = 0$, which implies $\beta(f,\varphi) < -1/d(f,\varphi)$;
(iii) (q: odd; p: even) $C_0 = (2/q)\Gamma((p+1)/q)|\tilde{f}(0)|^{-(p+1)/q}\tilde{\varphi}(0)\cos(((p+1)/2q)\pi)$, which implies that $\beta(f,\varphi) = -1/d(f,\varphi)$ is equivalent to $(p+1)/2q \notin \mathbb{N}+1/2$;
(iv) (q: odd; p: odd) $C_0 = \alpha(2i/q)\Gamma((p+1)/q)|\tilde{f}(0)|^{-(p+1)/q}\tilde{\varphi}(0)\sin(((p+1)/2q)\pi)$, which implies that $\beta(f,\varphi) = -1/d(f,\varphi)$ is equivalent to $(p+1)/2q \notin \mathbb{N}$.

Let us compare the conditions (a), (b), (c), (d) in Theorem 2.7 with the condition of p, q. That q (resp. p) is even is equivalent to the condition (b) (resp. (c), (d)). The condition (a) is equivalent to the inequality: $((p+1)/2q)\pi < \pi/2$, which implies $C_0 \neq 0$ in (iii).

7.2. Example 1.

Consider the following two-dimensional example:

$$f(x_1,x_2) = x_1^4,$$
$$\varphi(x_1,x_2) = x_1^2 x_2^2 + e^{-1/x_2^2} \ (=: \varphi_1(x_1,x_2) + \varphi_2(x_1,x_2)),$$

and χ is radially symmetric about the origin. It is easy to see that f is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron, $\Gamma_+(f) = \{(4,0)\} + \mathbb{R}_+^2$, $\Gamma_+(\varphi) = \Gamma_+(\varphi_1) = \{(2,2)\} + \mathbb{R}_+^2$, $\Gamma_+(\varphi_2) = \emptyset$, $d(f,\varphi) = 4/3$, $m(f,\varphi) = 1$. Define

$$Z_\pm^{(j)}(s) = \int_{\mathbb{R}^2} (f(x))_\pm^s \varphi_j(x)\chi(x)dx \quad j = 1,2.$$

Note $Z_-(s) = 0$. A simple computation gives

$$Z_+^{(1)}(s) = 4\int_0^\infty \int_0^\infty x_1^{4s+2} x_2^2 \chi(x_1, x_2) dx_1 dx_2.$$

By Lemma 5.2, we see that the poles of $Z_+^{(1)}(s)$ are simple and they are contained in the set $\{-3/4, -4/4, -5/4, \ldots\}$. Similarly, the poles of

$$Z_+^{(2)}(s) = 4\int_0^\infty \int_0^\infty x_1^{4s} e^{-1/x_2^2} \chi(x_1, x_2) dx_1 dx_2$$

are simple and contained in the set $\{-1/4, -2/4, -3/4, \ldots\}$. Moreover, Lemma 5.6 implies that the coefficient of $(s + 1/4)^{-1}$ is

$$\int_0^\infty e^{-1/x_2^2} \chi(0, x_2) dx_2 > 0.$$

Therefore, we have $\beta_+(f, \varphi) = \beta(f, \varphi) = -1/4$. As a result, $\beta(f, \varphi) > -1/d(f, \varphi)$ $(= -3/4)$.

This example does not satisfy the condition (d) in Theorem 2.2. Noticing that $\Gamma_+(\varphi) = \{(2, 2)\} + \mathbb{R}_+^2$, we see that the information of the Newton polyhedron is not sufficient to understand the behavior of oscillatory integrals in the case of C^∞ amplitudes.

7.3. Example 2.

Consider the following two-dimensional example with a real parameter t:

$$f(x_1, x_2) = x_1^5 + x_1^6 + x_2^5,$$
$$\varphi_t(x_1, x_2) = x_1^2 + t x_1 x_2 + x_2^2.$$

It is easy to see that f is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron, $(\varphi_t)_{\Gamma_0}(x) = \varphi_t(x)$, $d(f, \varphi_t) = 5/4$, and $m(f, \varphi_t) = 1$. $(\varphi_t)_{\Gamma_0}(x)$ is nonnegative on $\mathbb{R}^2$, if and only if $|t| \leq 2$. Thus, Theorem 5.7 implies that $\beta(f, \varphi_t) = -1/d(f, \varphi_t) = -4/5$ if $|t| \leq 2$. In this example, we understand the situation in more detail from the explicit computation below.

By applying the computation in Section 5, we see the properties of poles of the functions $Z_+(s)$ and $Z_-(s)$ in the following. The poles of the functions $Z_+(s)$ and $Z_-(s)$ are contained in the set $\{-4/5, -5/5, -6/5, \ldots\}$ and their order is at most one. Let $C_+(t)$, $C_-(t)$ be the coefficients of $(s - 4/5)^{-1}$ in the Laurent expansions of $Z_+(s)$ and $Z_-(s)$. Then, we have $C_+(t) = C_-(t) = A + tB$ with

$$A := \frac{1}{5}\int_{-\infty}^{\infty} |u^5+1|^{-4/5}(u^2+1)du, \quad B := \frac{1}{5}\int_{-\infty}^{\infty} |u^5+1|^{-4/5}udu.$$

Note that A is positive and B is negative.

Next, applying the argument in Section 6.1, $I(\tau)$ has the asymptotic expansion of the form:

$$I(\tau) \sim \tau^{-4/5}\sum_{j=0}^{\infty} C_j(t)\tau^{-j/5} \quad \text{as } \tau \to +\infty. \tag{7.3}$$

The relationship (6.5) gives $C_0(t) = 2\Gamma(4/5)\cos(2/5\pi)(A+tB)$.

Set $t_0 = -A/B(>0)$. From the above value of $C_0(t)$, if $t \neq t_0$, then the equation $\beta(f,\varphi_t) = -1/d(f,\varphi_t)$ holds. This means that the condition (d) in Theorem 2.7 is not necessary to satisfy the above equation. Furthermore, this example shows that the oscillation index is determined by not only the geometry of the Newton polyhedra but also the values of the coefficients of x^α for $\alpha \in \Gamma_0$ in the Taylor expansion of the amplitude.

Note 7.1. The existence of the term x_1^6 in f produces infinitely many non-zero coefficients $C_j(t)$ in the asymptotic expansion (7.3) for any t.

7.4. Comments on results in [1].

As mentioned in the Introduction, there have been studies in [**1**] in a similar direction to our investigations. In our language, their results can be stated as follows.

"Theorem" 7.1 (Theorem 8.4 in [**1**, p. 254]). *If f is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron, then*

(i) $\beta(f,\varphi) \leq -1/d(f,\varphi)$;
(ii) *If* $d(f,\varphi) > 1$ *and* $\Gamma_+(\varphi) = \{p\} + \mathbb{R}_+^n$ *with* $p \in \mathbb{Z}_+^n$, *then* $\beta(f,\varphi) = -1/d(f,\varphi)$.

Unfortunately, more additional assumptions are necessary to obtain the above assertions (i), (ii). Indeed, it is easy to see that Example 1 violates (i), (ii). As for (ii), even if φ is real analytic, the one-dimensional case in Section 7.1 indicates that at least some condition on the power p is needed. (It is easy to find counterexamples in higher dimensional case.) The same case shows that the evenness of p is not always necessary to satisfy $\beta(f,\varphi) = -1/d(f,\varphi)$.

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References

[1] V. I. Arnol'd, S. M. Guseĭn-Zade and A. N. Varchenko, Singularities of Differentiable Maps. II, Monogr. Math., **83**, Birkhäuser, 1988.

[2] B.-Y. Chen, J. Kamimoto and T. Ohsawa, Behavior of the Bergman kernel at infinity, Math. Z., **248** (2004), 695–708.

[3] K. Cho, J. Kamimoto and T. Nose, Asymptotics of the Bergman function for semipositive holomorphic line bundles, Kyushu J. Math., **65** (2011), 349–382.

[4] J. Denef, A. Laeremans and P. Sargos, On the largest nontrivial pole of the distribution $|f|^s$, Sûrikaisekikenkyûsho Kôkyûroku, **999** (1997), 1–9.

[5] J. Denef, J. Nicaise and P. Sargos, Oscillating integrals and Newton polyhedra, J. Anal. Math., **95** (2005), 147–172.

[6] J. Denef and P. Sargos, Polyèdre de Newton et distribution f_+^s. I, J. Analyse Math., **53** (1989), 201–218.

[7] J. Denef and P. Sargos, Polyèdre de Newton et distribution f_+^s. II, Math. Ann., **293** (1992), 193–211.

[8] M. V. Fedorjuk, Non-homogeneous generalized functions of two variables. (Russian), Mat. Sb. (N.S.), **49** (1959), 431–446.

[9] W. Fulton, Introduction to Toric Varieties, Ann. of Math. Stud., **131**, Princeton University Press, Princeton, NJ, 1993.

[10] I. M. Gel'fand and G. E. Shilov, Generalized Functions. I, Academic Press, New York, 1964.

[11] M. Greenblatt, A direct resolution of singularities for functions of two variables with applications to analysis, J. Anal. Math., **92** (2004), 233–257.

[12] M. Greenblatt, An elementary coordinate-dependent local resolution of singularities and applications, J. Funct. Anal., **255** (2008), 1957–1994.

[13] M. Greenblatt, The asymptotic behavior of degenerate oscillatory integrals in two dimensions, J. Funct. Anal., **257** (2009), 1759–1798.

[14] M. Greenblatt, Oscillatory integral decay, sublevel set growth, and the Newton polyhedron, Math. Ann., **346** (2010), 857–895.

[15] M. Greenblatt, Resolution of singularities, asymptotic expansions of oscillatory integrals, and related phenomena, J. Anal. Math., **111** (2010), 221–245.

[16] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II, Ann. of Math. (2), **79** (1964), 109–326.

[17] J. Igusa, Forms of Higher Degree. Tata Inst. Fund. Res. Lectures on Math. and Phys., **59**, New Delhi, 1978.

[18] J. Kamimoto, Newton polyhedra and the Bergman kernel, Math. Z., **246** (2004), 405–440.

[19] A. Kaneko, Newton diagrams, Singular points and Oscillatory integrals, Lecture Note at Sophia University, **11** (in Japanese), 1981.

[20] G. Kempf, F. F. Knudsen, D. Mumford and B. Saint-Donat, Toroidal Embeddings. I, Lecture Notes in Math., **339**, Springer-Verlag, 1973.

[21] B. Malgrange, Intégrales asymptotiques et monodromie, Ann. Sci. École Norm. Sup. (4), **7** (1974), 405–430.

[22] T. Oda, Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties, Ergeb. Math. Grenzgeb. (3), **15**, Springer-Verlag, 1988.

[23] M. Oka, Non-Degenerate Complete Intersection Singularity, Actualites Math., Hermann, Paris, 1997.

[24] M. Pramanik and C. W. Yang, Decay estimates for weighted oscillatory integrals in $\mathbb{R}^2$,

Indiana Univ. Math. J., **53** (2004), 613–645.

[25] H. Schulz, Convex hypersurfaces of finite type and the asymptotics of their Fourier transforms, Indiana Univ. Math. J., **40** (1991), 1267–1275.

[26] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Math. Ser., **43**, Princeton University Press, Princeton, NJ, 1993.

[27] V. A. Vasilíev, Asymptotic behavior of exponential integrals in the complex domain, Funct. Anal. Appl., **13** (1979), 239–247.

[28] A. N. Varchenko, Newton polyhedra and estimation of oscillating integrals, Funct. Anal. Appl., **10** (1976), 175–196.

[29] G. M. Ziegler, Lectures on Polytopes, Grad. Texts in Math., **152**, Springer-Verlag, New York, 1995.

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