

Calabi-Yau structures and Einstein-Sasakian structures on crepant resolutions of isolated singularities

By Ryushi GOTO

(Received June 22, 2010)
(Revised Feb. 11, 2011)

Abstract. Let X_0 be an affine variety with only normal isolated singularity at p . We assume that the complement $X_0 \setminus \{p\}$ is biholomorphic to the cone $C(S)$ of an Einstein-Sasakian manifold S of real dimension $2n - 1$. If there is a resolution of singularity $\pi : X \rightarrow X_0$ with trivial canonical line bundle K_X , then there is a Ricci-flat complete Kähler metric for every Kähler class of X . We also obtain a uniqueness theorem of Ricci-flat conical Kähler metrics in each Kähler class with a certain boundary condition. We show there are many examples of Ricci-flat complete Kähler manifolds arising as crepant resolutions.

Introduction.

Let X be a Kähler manifold of complex dimension n with trivial canonical line bundle K_X and Ω a nowhere vanishing holomorphic n -form on X . If a Kähler form ω satisfies the following equation,

$$\Omega \wedge \bar{\Omega} = c_n \omega^n,$$

for a constant c_n , then the Ricci curvature of ω vanishes, that is, ω is a Ricci-flat Kähler metric, where $\bar{\Omega}$ is the complex conjugate of Ω . The well-known Calabi-Yau theorem, due to Yau on a compact Kähler manifold with the first Chern class $c_1 = 0$ was proved by solving the Monge-Ampère equation which shows that there exists a unique Ricci-flat Kähler metric in each Kähler class. On a non-compact complete Kähler manifold X with $c_1(X) = 0$, it is an outstanding problem whether there exist Ricci-flat Kähler metrics. We need to impose suitable asymptotic conditions on the boundary. There are many remarkable results on Calabi-Yau theorem on non-compact complete Kähler manifolds with $c_1 = 0$. Tian

2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53C55.

Key Words and Phrases. Ricci-flat Kähler metric, Einstein-Sasakian metric, Monge-Ampère equation, Calabi-Yau structures.

The author was partly supported by the Grant-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science.

and Yau [36], [37], Bando-Kobayashi [3] and Joyce [19] solved the Monge-Ampère equation under various boundary conditions. On the other hand, the hyperKähler quotient construction [17], [20] produces many Ricci-flat Kähler manifolds in a simple and algebraic way, some of which are not obtained by the analytic method [13]. Recently rapid developments occur in the Einstein-Sasakian geometry which yield a new view point of the problem of Ricci-flat Kähler metrics. If we have a positive Einstein-Sasakian manifold S , then the cone $C(S) = \mathbf{R}_{>0} \times S$ admits a Ricci-flat Kähler cone metric. Boyer-Galicki [4] constructed a family of positive Einstein-Sasakian metrics on the links on hypersurfaces with isolated singularities, which includes interesting examples such as homology spheres. Martelli and Sparks [26] constructed explicitly a Sasaki-Einstein metric on the sphere bundle of the canonical line bundle of the blown up CP^2 at one point which is a irregular Sasaki manifold. Futaki, Ono and Wang [11] showed that a sphere bundle of the canonical line bundle on every toric compact Fano manifold admits an Einstein-Sasakian metric. These results imply that there are many notable examples of Ricci-flat cone metrics which are constructed by Einstein-Sasakian manifolds. In the present paper, we introduce conical Kähler metrics which are complete Kähler metrics with the certain boundary condition (see the Definition 1.3 in Section 1). We apply an existence theorem of Ricci-flat Kähler metrics to the class of conical Kähler metrics. Let ω be a conical Kähler metric on X with $\Omega \wedge \overline{\Omega} = c_n F \omega^n$ for a positive function F . If F satisfies

$$\|e^{(2+\delta)t}(F-1)\|_{C^k} < \infty,$$

for $0 < \delta < 2n - 2$ and $k \geq 2$, then there exists a Ricci-flat conical Kähler metric ω_u on X (see Theorem 1.5 in Section 1 for more detail).

The existence theorem can be also deduced from the arguments by Bando-Kobayashi in [3] and Tian-Yau [37]. Our boundary conditions in the theorem are modified for conical Kähler manifolds. The author gives a proof of the theorem for the sake of readers in Section 2.

As an application, we discuss the existence of Ricci-flat Kähler metrics on resolution X with trivial K_X of an affine variety X_0 with only normal isolated singularity $\{p\}$.

THEOREM 5.1. *Let X_0 be an affine variety with only normal isolated singularity at p . We assume that the complement $X_0 \setminus \{p\}$ is biholomorphic to the cone $C(S)$ of an Einstein-Sasakian manifold S of real dimension $2n - 1$. If there is a resolution of singularity $\pi : X \rightarrow X_0$ with trivial canonical line bundle K_X , then there is a Ricci-flat complete Kähler metric for every Kähler class of X .*

Our theorem covers the crucial case where a Kähler class does not belong to the compactly supported cohomology group. We use a vanishing theorem on X and the Hodge and the Lefschetz decomposition theorems on a Sasakian manifold to construct a suitable initial Kähler metric in every Kähler class which the existence theorem can be applied. We show that a Ricci-flat Kähler conical metric is unique in each Kähler class if we impose a certain boundary condition on metrics (see Theorem 1.8).

In Section 1, we introduce the class of conical Kähler metrics and show the existence theorem and the uniqueness theorem of Ricci-flat Kähler metrics on them. In Section 2 we give a proof of the existence theorem and the uniqueness theorem. In Section 3, we will give a short explanation of Sasakian metrics and Kähler cone metrics. In Section 4 we discuss a one to one correspondence between Einstein-Sasakian structures and Ricci-flat Kähler cone metrics. In Section 5, as an application of Section 1, we obtain Ricci-flat Kähler conical metrics on crepant resolutions of normal isolated singularities as above. In Section 6, we construct several families of Ricci-flat Kähler conical metrics on crepant resolutions of normal isolated singularities. Our examples include: resolutions of the isolated quotient singularities, the total spaces of canonical line bundles of Kähler-Einstein Fano manifolds, the total space of the canonical line bundle of every toric Fano manifold and small resolutions of ordinary double points of dimension 3.

Some of these examples are already known. Joyce [19] showed the Calabi-Yau theorem on resolutions of the isolated quotient singularities which is called asymptotically locally Euclidean (ALE). Calabi [6] used the bundle construction to obtain Ricci-flat Kähler metrics on the total spaces of the canonical line bundles of Kähler-Einstein Fano manifolds. It must be noted that the Kähler classes of these Ricci-flat Kähler metrics lie in the compactly support cohomology group. Van Coevering [40] and Santoro [31] constructed Ricci-flat Kähler metrics on crepant resolutions whose Kähler classes also belong to the compactly support cohomology group.

Our method provides wider classes of complete Ricci-flat Kähler metrics (see Examples in Section 6) and Theorem 5.1 shows that the conjecture discussed in [25] and [40] on the existence of complete Ricci-flat Kähler metrics on resolutions of cones is affirmative.

After the author posted this paper in the Arxiv, Van Coevering [41] submitted a paper which covers the existence theorem 1.5 by the different method.

The author would like to thank Professor A. Futaki and Professor R. Kobayashi for suggestive discussions about Ricci-flat Kähler metrics. He thanks Professor S. Bando for his kind and remarkable comments about the existence theorem in [3]. He is grateful to Professor A. Fujiki for his helpful advises. He also thanks the referee for his careful reading of the paper.

1. Existence theorem of Ricci-flat conical Kähler metrics.

Let (S, g_S) be a compact Riemannian manifold of dimension $2n-1$ and $C(S) = \mathbf{R}_{>0} \times S$ the product of the positive real number $\mathbf{R}_{>0}$ and S with a coordinate $r \in \mathbf{R}_{>0}$, which is called the cone of S . By changing the coordinate $t = \log r$, we regard the cone $C(S)$ as the cylinder $\mathbf{R} \times S$ with the cylinder parameter $t \in \mathbf{R}$.

DEFINITION 1.1. The cylinder metric g_{cyl} on $C(S)$ is the product metric,

$$dt^2 + g_S$$

and the cone metric g_{cone} is given by

$$g_{\text{cone}} = dr^2 + r^2 g_S.$$

Let ∇_{cyl} be the Levi-Civita connection with respect to the cylinder metric g_{cyl} and $|\alpha|_{g_{\text{cyl}}}$ the point-wise norm of a tensor α by g_{cyl} . The C^k -norm of a tensor α is given by

$$\|\alpha\|_{C^k} = \sum_{i=0}^k \sup |\nabla_{\text{cyl}}^i \alpha|_{g_{\text{cyl}}}.$$

We also have the Hölder norm $\|\alpha\|_{C^{k,\alpha}}$, for $0 < \alpha < 1$. In this paper we use the C^k -norm and $C^{k,\alpha}$ -norm with respect to the cylinder metric g_{cyl} unless it is mentioned.

Since $r = e^t$, we have the relation

$$g_{\text{cone}} = r^2 g_{\text{cyl}} \tag{1.1}$$

which follows from

$$\begin{aligned} r^2 g_{\text{cyl}} &= r^2 (dt)^2 + r^2 g_S \\ &= (dr)^2 + r^2 g_S \\ &= g_{\text{cone}}. \end{aligned}$$

DEFINITION 1.2. A manifold X has a *cylindrical boundary* if there is a compact set K of X such that the complement $X \setminus K$ is diffeomorphic to the cylinder $C(S) = \mathbf{R} \times S$. We identify $X \setminus K$ with the cone $C(S)$. A Riemannian

metric \bar{g} on a manifold X with a cylindrical boundary $C(S)$ is a *cylindrical metric* if \bar{g} satisfies the following condition on $X \setminus K \cong C(S)$,

$$\|e^{\delta t}(\bar{g} - g_{\text{cyl}})\|_{C^k} < \infty,$$

for some $\delta > 0$ and an integer $k > 4$. In other words, the difference between \bar{g} and g_{cyl} decays exponentially with order $O(e^{-\delta t})$, including their higher order derivatives up to k

$$\sum_{i=0}^k |\nabla_{\text{cyl}}^i(\bar{g} - g_{\text{cyl}})|_{g_{\text{cyl}}} = O(e^{-\delta t}).$$

We always extend the cylinder parameter t as a C^∞ function on X and identify the complement $X \setminus K$ with the cone $C(S)$.

DEFINITION 1.3. A Riemannian metric g on a manifold with a cylindrical boundary $C(S)$ is a *conical metric* if $r^{-2}g = e^{-2t}g$ is a cylindrical metric, that is, g satisfies

$$\|e^{(-2+\delta)t}(g - g_{\text{cone}})\|_{C^k} < \infty.$$

A conical Kähler metric on a complex manifold X is a conical Riemannian metric which is Kählerian. A conical Kähler form ω is a Kähler form with the associate Riemannian metric is conical and a manifold with a conical Kähler form is called a conical Kähler manifold.

DEFINITION 1.4. Let (X, ω) be a conical Kähler manifold with trivial canonical line bundle K_X and Ω a nowhere vanishing holomorphic n -form Ω on X . If a pair (Ω, ω) satisfies the following equation

$$\Omega \wedge \bar{\Omega} = c_n \omega^n,$$

for a constant c_n , then (Ω, ω) is called a Calabi-Yau structure whose Kähler form ω gives the Ricci-flat Kähler metric.

The following theorem can also deduced from the arguments by Bando-Kobayashi [3] and Tian-Yau [37]. For the completeness of the paper, the author gives a proof of existence theorem in Section 2.

THEOREM 1.5. *Let (X, ω) be a conical Kähler manifold of complex dimen-*

sion n with trivial canonical line bundle K_X and Ω a nowhere vanishing holomorphic n -form Ω on X which defines a positive function F by

$$\Omega \wedge \bar{\Omega} = c_n F \omega^n.$$

If F satisfies the following condition

$$\|e^{(2+\delta)t}(F-1)\|_{C^{k,\alpha}} < \infty,$$

for $0 < \delta < 2n-2$ and $k \geq 2$, $0 < \alpha < 1$, then there exists a smooth solution u of the Monge-Ampère equation $\Omega \wedge \bar{\Omega} = c_n \omega_u^n$, such that

$$\omega_u = \omega + \sqrt{-1} \partial \bar{\partial} u \quad (1.2)$$

is a conical Ricci-flat Kähler form with the condition,

$$\|e^{\delta t} u\|_{C^{k+2,\alpha}} < \infty.$$

(Note that ω_u is a complete Ricci-flat Kähler metric.)

REMARK 1.6. In Theorem 1.5, the decay order of $F-1$ is crucial for the existence of Ricci-flat Kähler metrics. Note that we estimate it by the cylinder metric g_{cyl} . In order to obtain a solution of the Monge-Ampère equation, we need to solve the equation of the Laplacian with respect to the conical metric g , $\Delta_g u = v$ on X , which is a linearization of the Monge-Ampère equation. The equation $\Delta_g u = v$ has a unique solution u with the decay order $O(e^{-\delta t})$ if v decays with order $O(e^{-(2+\delta)t})$ for $0 < \delta < 2n-2$. Thus we require that F decays with the order $O(e^{-(2+\delta)t})$.

We show that a Ricci-flat conical Kähler metric in the form as in Theorem 1.5 is unique.

THEOREM 1.7. *If there are two Ricci-flat conical Kähler metrics ω and ω' satisfying*

$$\omega' = \omega + \sqrt{-1} \partial \bar{\partial} u,$$

where u is a function with $\|e^{\delta t} u\|_{C^k} < \infty$, for a positive δ and $k \geq 0$. Then $\omega = \omega'$.

There is the action of the automorphism group of X on Ricci-flat Kähler metrics, in this sense, the uniqueness theorem does not hold on non-compact

Kähler manifolds. However if we impose the following boundary condition on metrics, we obtain the uniqueness theorem of Ricci-flat conical Kähler metrics in each Kähler class

THEOREM 1.8. *Let ω and ω' be two Ricci-flat conical Kähler metrics on X of dimension n with $[\omega] = [\omega'] \in H^2(X)$. We assume that X has a cylindrical boundary $C(S) = \mathbf{R} \times S$ with $H^1(S) = \{0\}$. If $\|e^{\lambda t}(\omega - \omega')\|_{C^k} < \infty$, for a constant $\lambda > n$, then $\omega = \omega'$.*

2. Proof of the existence theorem.

2.1. The Laplacian on conical Riemannian manifolds.

Let (X, \bar{g}) be a $2n$ dimensional cylindrical Riemannian manifold with cylindrical boundary $C(S)$ and g_{cyl} the cylinder metric on $C(S) = \mathbf{R} \times S$ as in Section 1, $g_{\text{cyl}} = dt^2 + g_S$, where $t \in \mathbf{R}$ is the cylinder parameter on $C(S)$ and g_S is a Riemannian metric on the manifold S of dimension $2n - 1$. We denote by $\|f\|_{L_k^p}$ the Sobolev norm of a function f on X with respect to the cylindrical metric \bar{g} ,

$$\|f\|_{L_k^p} = \sum_{i=0}^k \left(\int_X |\nabla^i f|^p \text{vol}_{\text{cyl}} \right)^{1/p},$$

where vol_{cyl} is the volume form with respect to \bar{g} . We define a weighted Sobolev norm on X by using the exponential function $e^{\delta t}$,

$$\|f\|_{L_{k,\delta}^p} = \|e^{\delta t} f\|_{L_k^p}. \quad (2.1)$$

The weighted Sobolev space is the completion of C^∞ functions on X with compact support with respect to the weighted Sobolev norm. Note that a cylindrical Riemannian manifold is complete. We also define a weighted Hölder norm as $\|f\|_{C_\delta^{k,\alpha}} = \|e^{\delta t} f\|_{C^{k,\alpha}}$ for k and $0 < \alpha < 1$ with respect to the cylindrical metric. Recall that a Riemannian metric g is conical if $r^{-2}g = e^{-2t}g$ is a cylindrical metric on X . On a conical Riemannian manifold (X, g) , we use the weighted Sobolev norm $\|f\|_{L_{k,\delta}^p}$ with respect to the cylindrical Hermitian metric $r^{-2}g$ and the weighted Hölder norm is also defined in terms of the cylindrical metric $r^{-2}g$. If a function f lies in $C_\delta^{k,\alpha}$ implies that f decays exponentially with order $O(e^{-\delta t})$ together with its derivatives. (We call δ the weight.) Then the Laplacian Δ_g gives the bounded linear operator from $L_{k+2,\delta}^p$ to $L_{k,\delta+2}^p$.

REMARK 2.1. Let vol_{cone} be the volume form with respect to the conical metric g . Then note that

$$\mathrm{vol}_{\mathrm{cone}} = r^{-2n} \mathrm{vol}_{\mathrm{cyl}}.$$

Let $*_g$ be the Hodge star operator with respect to the conical metric g and $*_{\mathrm{cyl}}$ the Hodge star operator with respect to the cylindrical metric $r^{-2}g$.

LEMMA 2.2. *We define a differential operator P by*

$$P = r^2 \Delta_g.$$

Then the operator P is given by

$$P = r^2 \Delta_g = \Delta_{\mathrm{cyl}} - (2n - 2) \frac{\partial}{\partial t} \quad (2.2)$$

on the cylinder boundary $C(S) = X \setminus K \cong \mathbf{R} \times S$, where t is the cylinder parameter and Δ_{cyl} the Laplacian with respect to the cylindrical metric $r^{-2}g$.

PROOF. Since $\Delta_g = *_g d *_g d$ and $\Delta_{\mathrm{cyl}} = *_{\mathrm{cyl}} d *_{\mathrm{cyl}} d$, comparing two Hodge star operators $*_g$ and $*_{\mathrm{cyl}}$, we have (2.2). \square

The cylinder metric $g_{\mathrm{cyl}} = dt^2 + g_S$ on $C(S)$ gives the Laplacian $\Delta_{g_{\mathrm{cyl}}} = (-i(\partial/\partial t))^2 + \Delta_S$ which is invariant under the translation of cylinder parameter t , where Δ_S is the Laplacian of (S, g_S) .

Let $I(P)$ be the translation-invariant operator given by

$$I(P) = \Delta_{g_{\mathrm{cyl}}} - (2n - 2) \frac{\partial}{\partial t} \quad (2.3)$$

$$= \left(-i \frac{\partial}{\partial t} \right)^2 + \Delta_S - (2n - 2) \frac{\partial}{\partial t}. \quad (2.4)$$

Then the all coefficients of the operators $P - I(P)$ decays exponentially together with their derivatives, which implies that we can apply the theory of elliptic differential operators on cylindrical Riemannian manifolds developed by [23], [24]. (Note that these operators are called b -operators in [24].) We can select suitable weighted Sobolev spaces for the operator P to be a Fredholm operator. Substituting λ for $-i(\partial/\partial t)$ into (2.4), we define a family of the operators $I(P, \lambda)$ on S given by

$$I(P, \lambda) = \lambda^2 - (2n - 2)\lambda i + \Delta_S$$

which is called the indicial family parametrized by $\lambda \in \mathbf{C}$. The operator

$$I(P, \lambda) : L_{k+2}^p(S) \rightarrow L_k^p(S)$$

is an isomorphism for all $\lambda \in \mathcal{C} \setminus \text{Spec}(P)$, that is, the operator $I(P)$ does not admit a bounded inverse for $\lambda \in \text{Spec}(P)$. The properties of the set $\text{Spec}(P)$ are developed in [23] and [24]. In our cases, $\text{Spec}(P)$ is explicitly described as

LEMMA 2.3.

$$\text{Spec}(P) = \{0, (2n-2)\sqrt{-1}, \mu_j^+ \sqrt{-1}, \mu_j^- \sqrt{-1} \mid j = 1, 2, \dots\}$$

where $\mu_j^\pm \sqrt{-1}$ are two solutions of the quadratic equation $x^2 - (2n-2)\sqrt{-1}x + \lambda_j = 0$ and λ_j is the j -th eigenvalue of the Laplacian Δ_S on S .

PROOF. It follows from $I(P, \lambda) = \lambda^2 - (2n-2)\lambda i + \Delta_S$. \square

Let $\text{ImSpec}(P)$ be the set consisting of the imaginary parts of elements of $\text{Spec}(P)$. Then $\text{ImSpec}(P)$ is given by

$$\dots < \mu_2^- < \mu_1^- < 0 < 2n-2 < \mu_1^+ < \mu_2^+ < \dots$$

We select the weighted Sobolev space $L_{k,\delta}^p$ with weight $\delta \notin \text{ImSpec}(P)$. Then from Theorem 1.1 in [23] and [24], $P : L_{k+2,\delta}^p \rightarrow L_{k,\delta}^p$ is a Fredholm operator with index $\text{ind}(P, \delta)$. Further for $0 < \delta < n-1$, since the Laplacian Δ_g is self-adjoint, applying the Theorem 7.4 (see p. 436) in [23], we obtain

$$\text{ind}(P, \delta + n - 1) + \text{ind}(P, -\delta + n - 1) = 0.$$

There is the formula for the index of the cylindrical operators depending on weights δ in [23]. Applying Theorem 1.2 in [23], we have

$$-\text{ind}(P, \delta + n - 1) + \text{ind}(P, -\delta + n - 1) = 0.$$

Thus we obtain $\text{ind}(P, \delta) = 0$ for $0 < \delta < 2(n-1)$.

By the result, we have the solvability of the equation of the Laplacian Δ_g .

PROPOSITION 2.4. *Let (X, g) be a $2n$ -dimensional conical Riemannian manifold and Δ_g the Laplacian with respect to the conical metric g . For a weight δ with $0 < \delta < 2n-2$, the Laplacian Δ_g gives an isomorphism between $L_{k+2,\delta}^p$ and $L_{k,\delta+2}^p$, for positive integers p, k with $1/p < (k-\alpha)/2n$ for $0 < \alpha < 1$.*

PROOF. It follows from the Sobolev embedding theorem that $L_{k+2,\delta}^p \subset C_\delta^{2,\alpha}$.

By the Maximum principle, a harmonic function attains its Maximum and Minimum at the boundary. Since a function in $C_\delta^{2,\alpha}$ decays exponentially, we have $\ker \Delta_g = \{0\}$ which implies that $\ker P = \ker \Delta_g = \{0\}$. Since $\text{ind}(P, \delta) = 0$, we have an isomorphism $P : L_{k+2,\delta}^p \rightarrow L_{k,\delta}^p$. Since Δ_g is the composition $r^{-2}P$, $\Delta_g : L_{k+2,\delta}^p \rightarrow L_{k,\delta+2}^p$ is an isomorphism also. \square

Then we have the followings,

PROPOSITION 2.5. *Let (X, g) be a $2n$ -dimensional conical Riemannian manifold and Δ_g the Laplacian with respect to the conical metric g . For a weight δ with $0 < \delta < 2n - 2$, the Laplacian Δ_g gives an isomorphism between $C_\delta^{k+2,\alpha}$ and $C_{\delta+2}^{k,\alpha}$. In other words, there is a unique solution $u \in C_\delta^{k+2,\alpha}$ of the equation $\Delta_g u = f$ for all $f \in C_{\delta+2}^{k,\alpha}$, where $k \geq 0$.*

REMARK 2.6. If (X, g) is an ALE space, then Proposition 2.5 implies Theorem 8.3.5 (a) in [19]. The Hölder space $C_\beta^{k,\alpha}$ in [19] coincides with our Hölder space $C_\delta^{k,\alpha}$ with $\delta = -\beta$.

We need the following lemma for a proof of Proposition 2.5.

LEMMA 2.7. *Let (X, g) be a conical Riemannian manifold with the cylinder parameter t and Δ_g the Laplacian with respect to the conical metric g . We take a smaller weight $\tilde{\delta}$ with $0 < \tilde{\delta} < \delta < 2n - 2$. We assume that a function $u \in C_{\tilde{\delta}}^k$ satisfies*

$$\Delta_g u = h,$$

and we already have a bound of C^0 -norm $\|u\|_{C^0} < C_1$ and a weighted C^0 -norm $\|e^{(2+\delta)t}h\|_{C^0} < C_2$. Then there is a constant $C > 0$ depending only on C_1, C_2 and g such that

$$\|e^{\delta t}u(x)\|_{C^0} < C.$$

PROOF OF LEMMA 2.7. Let Δ_{cone} be the Laplacian with respect to the cone metric g_{cone} on the cone $C(S)$. Then we see that

$$\Delta_{\text{cone}} = e^{-2t} \left(- \left(\frac{\partial}{\partial t} \right)^2 - (2n-2) \frac{\partial}{\partial t} + \Delta_S \right).$$

Thus we have

$$\Delta_{\text{cone}} e^{-\delta t} = \delta(2n - 2 - \delta)e^{-(2+\delta)t} > 0$$

on the cone $C(S)$, for $0 < \delta < 2n - 2$. Then from the definition of the conical metric, there are constants $C_3 > 0$ and $T > 0$ such that

$$\Delta_g e^{-\delta t} \geq C_3 e^{-(2+\delta)t}, \quad (2.5)$$

on the region $D_T := \{t > T\}$. We take a constant C satisfying the two inequalities,

$$C_3 C > C_2 \quad (2.6)$$

$$C e^{-\delta T} - C_1 > 0. \quad (2.7)$$

Then from (2.5) and (2.6), we have

$$\Delta_g (C e^{-\delta t} \pm u) > C C_3 e^{-(2+\delta)t} \pm h \quad (2.8)$$

$$> C C_3 e^{-(2+\delta)t} - C_2 e^{-(2+\delta)t} \quad (2.9)$$

$$> 0. \quad (2.10)$$

From (2.7), we also have an inequality on the compact set $\{t \leq T\}$

$$(C e^{-\delta t} \pm u)(x) \geq C e^{-\delta T} - C_1 \geq 0, \quad (2.11)$$

for all $x \in \{t \leq T\}$. If the function $C e^{-\delta t} \pm u$ have its Minimum at $x_0 \in D_T$, then $\Delta_g (C e^{-\delta t} \pm u)(x_0) \leq 0$. Thus from (2.10), $C e^{-\delta t} \pm u$ can not have the Minimum on D_T . Since $\lim_{t \rightarrow \infty} u(x) = 0$, it follows from (2.11) that $(C e^{-\delta t} \pm u)(x) \geq 0$, for all $x \in X$. Hence we have $\|e^{\delta t} u\|_{C^0} < C$. \square

PROOF OF PROPOSITION 2.5. Since $u \in C_\delta^{k+2,\alpha}$ decays exponentially, it follows that $\ker \Delta_g = \{0\}$. For a smaller weight $0 < \tilde{\delta} < \delta$, any function $f \in C_{2+\tilde{\delta}}^{k,\alpha}$ is a function in $L_{k,\tilde{\delta}+2}^p$. Then it follows from Proposition 2.4 that there is a function $u \in L_{k+2,\tilde{\delta}}^p$ such that $\Delta_g u = f$. Since u is a function in $C_{\tilde{\delta}}^{k+2,\alpha}$, it follows from Lemma 2.7 that $u \in C_\delta^0$. We have the following equation,

$$\Delta_g (e^{\delta t} u) = H,$$

where $H \in C_\delta^{k,\alpha}$ is a function consisting of the terms $(\Delta_g e^{\delta t})u$, $e^{\delta t} f$ and $(\nabla_g e^{\delta t})(\nabla_g u)$. Thus from the Schauder estimate, we have $e^{\delta t} u \in C^{k+2,\alpha}$. Hence

we have a unique solution $u \in C_\delta^{k+2,\alpha}$ of the equation $\Delta_g u = f$, for $f \in C_\delta^{k+2,\alpha}$. \square

2.2. The Sobolev inequality for conical Riemannian manifolds.

The following theorem is already known for geometers, cf. Proposition 3.2 in [37], Theorem 1.2 (weighted Sobolev inequalities) in [16] and Theorem 2.7 for conical Riemannian manifolds in [41] (see also [28] for Riemannian manifolds with nonnegative Ricci curvature which is based on the method of [15]).

PROPOSITION 2.8. *Let (X, g) be a conical Riemannian manifold of dimension $2n$. Then every function $f \in L^{2\varepsilon} \cap L_1^2$ satisfies:*

$$\|f\|_{L^{2\varepsilon}} < S \|df\|_{L^2},$$

where $\varepsilon = n/(n-1)$ and S is a constant which depends only on n and g . Note that the point-wise norm and L^2 -norm are defined by the cone metric g in this proposition.

Note that the curvature of a conical Riemannian manifold decays with the order $O(e^{-2t})$. In the proposition, we do not assume the condition:

$$\int_X f \, \text{vol} = 0,$$

which is necessary for the Sobolev inequality for compact Riemannian manifolds.

2.3. The inequality of solutions of the Monge-Ampère equation.

Let (X, g, ω) be a conical Kähler manifold. We use the same notation as in Theorem 1.5. We assume that there is a C^∞ function u on X which satisfies the equation

$$\omega_u^n = F \omega^n,$$

where

$$\omega_u = \omega + dd^c u$$

is a Kähler form on X and a C^∞ positive function F satisfies $F - 1 \in C_{2+\delta}^{k,\alpha}$ and $u \in C_\delta^{k+2,\alpha}$ and $d^c = (1/2\sqrt{-1})(\partial - \bar{\partial})$. We select a weight δ with $0 < \delta < 2n - 2$ and a natural number $p > 2$ with $\delta p + 2 > 2n$. Then since $\omega^n = e^{2nt} \text{vol}_{\text{cyl}}$, the function $u|u|^{p-2}(1-F)$ is integrable with respect to the volume form ω^n . Thus

we have

$$\begin{aligned}\int_X u|u|^{p-2}(1-F)\omega^n &= \int_X u|u|^{p-2}(\omega^n - \omega_u^n) \\ &= \int_X u|u|^{p-2}(-dd^c u) \wedge (\omega^{n-1} + \cdots + \omega_u^{n-1}).\end{aligned}$$

Since $\delta p + 2 > 2n$ and $u|u|^{p-2}d^c u \wedge (\omega^{n-1} + \cdots + \omega_u^{n-1})$ decays exponentially with respect to the cylinder parameter t , we can apply the Stokes theorem,

$$\int d(u|u|^{p-2}d^c u \wedge (\omega^{n-1} + \cdots + \omega_u^{n-1})) = 0.$$

Substituting $d(u|u|^{p-2}) = (p-1)|u|^{p-2}du$, we obtain

$$\int_X u|u|^{p-2}(1-F)\omega^n = (p-1) \int_X |u|^{p-2}du \wedge d^c u \wedge (\omega^{n-1} + \cdots + \omega_u^{n-1}). \quad (2.12)$$

Since $du \wedge d^c u \wedge (\omega^i \wedge \omega_u^{n-1-i}) \geq 0$ at each point on X , we have

$$du \wedge d^c u \wedge (\omega^{n-1} + \cdots + \omega_u^{n-1}) \geq du \wedge d^c u \wedge \omega^n.$$

Substituting it into (2.12), we have

$$\int_X u|u|^{p-2}(1-F)\omega^n \geq (p-1) \int_X |u|^{p-2}du \wedge d^c u \wedge \omega^{n-1} \quad (2.13)$$

$$= \frac{p-1}{n} \int_X |u|^{p-2}|du|_g^2 \omega^n \quad (2.14)$$

$$= \frac{4(p-1)}{p^2 n} \int_X |(d|u|^{p/2})|_g^2 \omega^n \quad (2.15)$$

where we are using

$$\frac{1}{4}p^2|u|^{p-2}|du|_g^2 = |(d|u|^{p/2})|_g^2$$

and $|(d|u|^{p/2})|_g^2$ is the norm of the 1-form $d|u|^{p/2}$ with respect to the conical metric g . Hence we obtain the inequality,

PROPOSITION 2.9.

$$\int_X |(d|u|^{p/2})|_g^2 \omega^n \leq \frac{p^2 n}{4(p-1)} \int_X u|u|^{p-2} (1-F) \omega^n,$$

where $p > 2$ with $p\delta > 2n - 2$. For simplicity, we denote by K the constant $pn/4(p-1)$.

2.4. An inequality for the induction.

LEMMA 2.10. Let $A(p)$ be a positive function of one variable $p \in \mathbf{R}$ which satisfies the following inequality,

$$A(p\varepsilon)^p \leq c_1 p (A(p-1))^{p-1} \quad (2.16)$$

where c_1 is a constant and $\varepsilon = n/(n-1) > 1$. We assume that there is a natural number $N > 1$ such that $A(p) < c_0$ for all $p \in [N-1, N\varepsilon]$. Then $A(p)$ satisfies the following inequality

$$A(p) \leq s_1 (s_2 p)^{-s_3/p}, \quad (2.17)$$

where constants s_1, s_2 depend only on c_0, c_1 and N and $s_3 = 2\varepsilon/(\varepsilon-1) = 2n$. In particular, we have

$$\lim_{p \rightarrow \infty} A(p) \leq s_1$$

which implies that $A(p)$ is bounded by a constant which does not depend on p .

PROOF. We take a constant s_2 which satisfies

$$2^{s_3} c_1 p^{-1} \varepsilon^{2(n-1)} \leq s_2^2, \quad (2.18)$$

for all $p > N$. (It is possible because the left hand side is bounded.) And then we choose a constant $s_1 > 1$ with

$$c_0 \leq s_1 (s_2 p)^{-s_3/p}, \quad (2.19)$$

for all p . (It is also possible since $\lim_{p \rightarrow \infty} (s_2 p)^{-s_3/p} = 1$.) Then we shall show that $A(p)$ satisfies the inequality (2.17) by the induction on m . The inequality (2.17) holds for $p \in [N-1, N\varepsilon]$. We assume that $A(p)$ satisfies the inequality (2.17) for all $p \in [N-1, N\varepsilon^m]$. Then we shall show that the inequality holds

for $p\varepsilon \in [N-1, N\varepsilon^{m+1}]$. From our assumption, $A(p)$ and $A(p-1)$ satisfies the inequality (2.17) and then applying (2.16), we have

$$A(p\varepsilon)^p \leq c_1 p s_1^{p-1} (s_2(p-1))^{-s_3} \leq (2^{s_3} c_1 p) s_1^p (s_2 p)^{-s_3} \quad (2.20)$$

by using $(p/(p-1))^{s_3} < 2^{s_3}$. If we have the following inequality

$$(2^{s_3} c_1 p) s_1^p (s_2 p)^{-s_3} \leq s_1^p (s_2 p \varepsilon)^{-s_3/\varepsilon}, \quad (2.21)$$

then $A(p\varepsilon)$ satisfies the inequality (2.17). Thus it suffices to show (2.21). The inequality (2.21) is equivalent to

$$(2^{s_3} c_1 p) p^{-s_3+s_3/\varepsilon} \leq (\varepsilon)^{-s_3/\varepsilon} s_2^{s_3/n}. \quad (2.22)$$

Since $-s_3 + s_3/\varepsilon = -s_3/n = -2$ and $s_3/\varepsilon = 2(n-1)$, (2.22) is

$$2^{s_3} c_1 p^{-1} \leq (\varepsilon)^{-2(n-1)} s_2^2. \quad (2.23)$$

This is the inequality which s_2 satisfies. Thus the result follows from the induction. \square

2.5. The openness.

Our proof of the theorem relies on the continuity method. We consider a family of the Monge-Ampère equation parametrised by $s \in [0, 1]$. For $F_s = 1 - s + sF$, we define

$$\omega_{u_s}^n = F_s \omega^n. \quad (2.24)$$

Let S be a subset of $[0, 1]$ defined by

$$S = \{s \in [0, 1] \mid \text{the equation (2.24) has a solution } u_s \in C_\delta^{k+2, \alpha}\}.$$

Note that $F - 1 \in C_\delta^{k, \alpha}$, for all $\tilde{\delta} \leq \delta$. Since $F_0 = 1$, the equation (2.24) has the trivial solution 0 for $s = 0$. If we show that S is open and closed, then there is a solution u_1 for $s = 1$ which gives the solution of the Monge-Ampère equation in the theorem. In order to show that S is open, taking the derivative of the equation (2.24) at u_s , we have the equation of the Laplacian

$$-\Psi_s \triangle_s u_s = \dot{F}_s, \quad (2.25)$$

where $u_s \in C_\delta^{k+2,\alpha}$ and $F_s \in C_{\delta+2}^{k,\alpha}$ and Ψ_s is a bounded function given by

$$\Psi_s = \frac{1}{2} \frac{\omega_{u_s}^n}{\omega^n}$$

and Δ_s is the Laplacian with respect to the Kähler metric ω_{u_s} . Since $u_s \in C_\delta^{k+2,\alpha}$, the metric ω_{u_s} is a conical Kähler metric. Then since $F-1 \in C_{2+\delta}^{k,\alpha}$, it follows from Proposition 2.5 there is a unique solution u_s of the equation (2.25). We define two Banach manifolds $\mathcal{M} = \{u \in C_\delta^{k+2,\alpha} \mid \omega_u > 0\}$. $\mathcal{N} = \{f+1 \mid f \in C_{2+\delta}^{k,\alpha}\}$. Then the smooth map $\Psi : \mathcal{M} \rightarrow \mathcal{N}$ is defined by

$$\Psi(u) = \frac{\omega_u^n}{\omega^n}.$$

The differential of Ψ at u is the map $d\Psi_u : C_\delta^{k+2,\alpha} \rightarrow C_\delta^{k,\alpha}$ which is given by

$$d\Psi_u(u) = -\Psi_u \Delta_u,$$

where Δ_u is the Laplacian with respect to the Kähler metric ω_u . Then from Proposition 2.5, $d\Psi_u$ is isomorphism and it follows from the implicit function theorem that Φ is locally an isomorphism. Then S is an open set.

2.6. C^0 -estimates.

We shall show the C^0 -estimate of the Monge-Ampère equation $\omega_{u_s}^n = F_s \omega^n$. For simplicity, we write u for u_s and F for F_s . We use ω^n as the (conical) volume form of the integrant in this subsection. Recall that $u \in C_\delta^{k+2,\alpha}$ and $F-1 \in C_{2+\delta}^{k,\alpha}$, for $0 < \delta < 2n-2$. Substituting $f = |u|^{p/2}$ into the Sobolev inequality (Proposition 2.8), we have

$$\left(\int_X |u|^{p\varepsilon} \omega^n \right)^{1/\varepsilon} \leq S^2 \int_X |d|u|^{p/2}|_g^2 \omega^n.$$

Applying the inequality (Proposition 2.9), we have

$$\left(\int_X |u|^{p\varepsilon} \omega^n \right)^{1/\varepsilon} \leq S^2 K p \int_X u |u|^{p-2} (1-F) \omega^n. \quad (2.26)$$

We set $A(p)$ by

$$A(p) = \left(\int_X |u|^p \omega^n \right)^{1/p}$$

and $c_1 = S^2 K \|1 - F\|_{C^0}$ by the C^0 -norm of $1 - F$. Then the inequality (2.26) implies

$$A(p\varepsilon)^p \leq c_1 p (A(p-1))^{p-1}. \quad (2.27)$$

Then we shall show an estimate of the C^0 -norm by using Lemma 2.10. Recall $p\delta > 2n - 2$ in Proposition 2.9. We set $r > 1$ by $(p-1)r = p\varepsilon$ and define $s > 1$ by $1/s + 1/r = 1$. Then it follows

$$(\delta + 2)s > 2n. \quad (2.28)$$

Applying the Hölder inequality to the right hand side of (2.26) with the exponents r, s , we have

$$A(p\varepsilon)^p \leq S^2 K p (A(p\varepsilon))^{p\varepsilon/r} \left(\int_X |1 - F|^s \omega^n \right)^{1/s}. \quad (2.29)$$

Since $p - p\varepsilon/r = 1$, we have

$$A(p\varepsilon) \leq S^2 K p \left(\int_X |1 - F|^s \omega^n \right)^{1/s}.$$

Note that from (2.28), $\left(\int_X |1 - F|^s \omega^n \right) < \infty$. Thus for sufficiently large N , we see that $A(p) < c_0$ for all $p \in [N-1, N\varepsilon]$. Then together with (2.27), we can apply Lemma 2.10 to obtain

$$A(p) = \left(\int_X |u|^p \omega^n \right)^{1/p} < C,$$

where C does not depend on p . Thus we obtain an estimate of C^0 -norm of u .

2.7. C^2 -estimate.

There are many good references for an estimate of C^2 -norm of a solution u of the Monge-Ampère equation [1], [19], [29], [33]. Bando-Kobayashi [3] developed a way for the C^2 -estimate to apply the Schwartz lemma which makes a comparison of two metrics ω and ω_u by using their curvature, which is clear from a geometric

point of view.

We shall give a quick view of the C^2 -estimate on our conical Kähler manifolds. Let $\mathrm{tr}_{\omega_u} \omega$ be the trace of ω by the Kähler metric ω_u . It follows that $\mathrm{tr}_{\omega_u} \omega > 0$ is the norm of ω by ω_u which gives the C^2 -norm of u . According to the Bando-Kobayashi's result, there is a constant C depending only on the curvature of ω and the Ricci curvature of ω_u such that the following inequality holds,

$$-\square_u \log \mathrm{tr}_{\omega_u} \omega \geq -C(1 + \mathrm{tr}_{\omega_u} \omega)$$

where \square_u is the complex Laplacian with respect to the Kähler metric ω_u . Since u is a solution of the Monge-Ampère equation, the Ricci curvature of ω_u is already bounded. Then by using,

$$-\square_u u = n - \mathrm{tr}_{\omega_u} \omega$$

we have

$$-\square_u (\log \mathrm{tr}_{\omega_u} \omega - (C+1)u) \geq \mathrm{tr}_{\omega_u} \omega - (C+1)n - C.$$

Since the function u decays exponentially with order $O(e^{-\delta t})$, we may assume the function $(\log \mathrm{tr}_{\omega_u} \omega - (C+1)u)$ takes its maximum at a point $x_0 \in X$. Then

$$0 \geq -\square_u (\log \mathrm{tr}_{\omega_u} \omega - (C+1)u)(x_0) \tag{2.30}$$

$$\geq (\mathrm{tr}_{\omega_u} \omega - (C+1)n - C)(x_0). \tag{2.31}$$

Hence

$$(C+1)n + C \geq \mathrm{tr}_{\omega_u} \omega(x_0).$$

It follows that

$$(\log \mathrm{tr}_{\omega_u} \omega - (C+1)u)(x_0) \geq (\log \mathrm{tr}_{\omega_u} \omega - (C+1)u)(x), \quad \forall x \in X.$$

Thus we have the upper bound for $\mathrm{tr}_{\omega_u} \omega$,

$$\sup_X \log \mathrm{tr}_{\omega_u} \omega < \log((C+1)n + C) + 2(C+1)\|u\|_{C^0}.$$

By using the Monge-Ampère equation $\omega_u^n = F\omega^n$, we have C^2 -estimate of u and we obtain an a priori constant $C > 0$ such that

$$C^{-1}\omega < \omega_u < C\omega.$$

2.8. C^0_δ -estimate.

In this subsection, we shall give an estimate of the weighted C^0 -norm u of a solution of the Monge-Ampère equation. In the process of our estimate, we need to choose a smaller weight $\tilde{\delta}$ with $0 < \tilde{\delta} < \delta$. We can obtain a way of the weighted C^0 -estimate by applying the method by Joyce, (see Section 8 in [19]). Our notation are the same as in the previous subsection.

At first we have the following weighted inequality,

LEMMA 2.11. *For a constant $q > 0$ satisfying*

$$q < p\delta - 2n + 2$$

we have

$$\int_X r^q |(d|u|^{p/2})_g|^2 \omega^n \leq \frac{p^2 m}{4(p-1)} \int_X r^q u |u|^{p-2} (1-F) \omega^n + q^2 C \int_X r^{q-2} |u|^p \omega^n$$

where $r = e^t$ and a constant C depends only on the C^2 -norm of u given in the previous section.

PROOF. The condition

$$q < p\delta - 2n + 2$$

implies that the Stokes theorem holds,

$$\int_X d(r^q u |u|^{p-2} d^c u \wedge (\omega^{n-1} + \cdots + \omega_u^{n-1})) = 0.$$

Then the result follows as in the way of Proposition 2.9. \square

The following lemma is a slight generalization of Lemma 2.10

LEMMA 2.12. *Let $B(p)$ be a positive function with one variable $p \geq 0$. We assume that $B(p)$ satisfies the following,*

$$B(p\varepsilon)^p \leq c_1 p B(p-1)^{p-1} + (c_2 p^2 + c_3) B(p)^p \quad (2.32)$$

where $\varepsilon = n/(n-1)$ and there are a natural number N and a positive c_0 such that $B(p) < c_0$ for all $p \in [N-1, N\varepsilon]$. Then there are constants s_1, s_2 depending only

on c_0, c_1, c_2, c_3 such that for all $p > N$, $B(p)$ satisfies the inequality,

$$B(p) \leq s_1(s_2 p)^{-s_3/p}$$

where $s_3 = 2\varepsilon/(\varepsilon - 1)$. In particular, we have

$$\lim_{p \rightarrow \infty} B(p) = s_1.$$

Thus $B(p)$ is bounded by a constant which does not depend on p .

PROOF. The proof is essentially same as the one in Lemma 2.10. \square

We shall start our $C_{\tilde{\delta}}^0$ -estimate, paying our attention on the weight. We choose a weight $\tilde{\delta}$ satisfying $0 < \tilde{\delta} < \delta$ and try to obtain a $C_{\tilde{\delta}}^0$ -norm of u . We set

$$q = p\tilde{\delta} - 2n + 2.$$

Then q satisfies

$$-p\delta + 2n - 2 + q < 0 \quad (2.33)$$

which is the condition of Lemma 2.11. We define $B(p)$ by the weighted L^p -norm of u with respect to the cylindrical metric.

$$B(p) = \|u\|_{L_{0,\tilde{\delta}}^p} = \left(\int_X |e^{\tilde{\delta}t} u|^p \text{vol}_{\text{cyl}} \right)^{1/p}.$$

Note $\omega^n = e^{2nt} \text{vol}_{\text{cyl}}$. Then we have

$$\int_X e^{\varepsilon qt} |u|^{p\varepsilon} \omega^n = \int_X e^{\varepsilon p\tilde{\delta}t} |u|^{p\varepsilon} e^{(-2n+2)\varepsilon t} e^{2nt} \text{vol}_{\text{cyl}}. \quad (2.34)$$

Since $\varepsilon = n/(n-1)$, we have $(-2n+2)\varepsilon + 2n = 2(1-n)n/(n-1) + 2n = 0$. Then it follows

$$\int_X e^{\varepsilon qt} |u|^{p\varepsilon} \omega^n = \int_X |e^{\tilde{\delta}t} u|^{p\varepsilon} \text{vol}_{\text{cyl}} = B(p\varepsilon)^{p\varepsilon}. \quad (2.35)$$

We substitute $e^{(q/2)t} |u|^{p/2}$ into the Sobolev inequality in Proposition 2.8. Since in

the Sobolev inequality, the norm is given by the conical metric ω and the volume form is also ω^n , the left hand side of the inequality is

$$\|e^{(q/2)t}|u|^{p/2}\|_{L^{2\varepsilon}}^2 = \left(\int_X e^{q\varepsilon t}|u|^{p\varepsilon}\omega^n \right)^{1/\varepsilon} = B(p\varepsilon)^p \quad (2.36)$$

and using the norm by the cone metric, we find that the right hand side is

$$\|d(e^{(q\tilde{\delta}/2)t}|u|^{p/2})\|_{L^2}^2 \leq C \int_X e^{qt}|d|u|^{q/2}|^2\omega^n + Cq\tilde{\delta} \int_X e^{(q-2)t}|u|^p\omega^n. \quad (2.37)$$

The second term is given by

$$\int_X e^{(q-2)\tilde{\delta}t}|u|^p\omega^n = \int_X |e^{\tilde{\delta}t}u|^p e^{-2nt}e^{2nt}\text{vol}_{\text{cyl}} = B(p)^p. \quad (2.38)$$

Since $e^{(2+\tilde{\delta})t}|1-F| < C$, we find that

$$\int_X e^{qt}|u|^{p-1}|1-F|\omega^n = \int_X e^{-\delta t}e^{(q-2)t}|u|^{p-1}e^{(2+\tilde{\delta})t}|1-F|\omega^n \quad (2.39)$$

$$\leq C \int_X e^{-\tilde{\delta}t}e^{p\tilde{\delta}t}|u|^{p-1}\text{vol}_{\text{cyl}} \quad (2.40)$$

$$\leq C \int_X |e^{\tilde{\delta}t}u|^{p-1}\text{vol}_{\text{cyl}} \quad (2.41)$$

$$= CB(p-1)^{p-1}. \quad (2.42)$$

Substituting these into the Sobolev inequality and combining the inequality in Proposition 2.11 we obtain

$$\|u_s\|_{L_{0,\tilde{\delta}}^{p\varepsilon}}^p < C_1p\|u_s\|_{L_{0,\tilde{\delta}}^{p-1}}^{p-1} + (C_2p^2 + C_2)\|u_s\|_{L_{0,\tilde{\delta}}^p}^p. \quad (2.43)$$

$$(2.44)$$

The inequality implies

$$B(p\varepsilon)^p < C_1pB(p-1)^{p-1} + (C_2p^2 + C_2)B(p)^p. \quad (2.45)$$

We apply a simple trick to show a bound of $B(p)$ from $A(p)$ with respect to the weight. Recall

$$A(p) = \left(\int_X |u|^p \omega^n \right)^{1/p} = \left(\int_X |u|^p e^{2nt} \text{vol}_{\text{cyl}} \right)^{1/p}.$$

For $p\delta > 2n - 2$, we already have a bound C of $A(p)$ in the subsection of C^0 -estimate. On the other hand,

$$B(p) = \left(\int_X |u|^p e^{p\tilde{\delta}t} \text{vol}_{\text{cyl}} \right)^{1/p}.$$

Thus for p satisfying $2n - 2 < p\tilde{\delta} < 2n$, we have $B(p) < A(p) < C$, where $0 < \tilde{\delta} < \delta < 2n - 2$, that is, $B(p) < A(p) < C$ if p satisfies $(2n - 2)/\tilde{\delta} < p < 2n/\tilde{\delta}$. Thus if we take sufficiently small $\tilde{\delta} > 0$, there is a $N > 0$ such that $B(p) < C$ holds for all $p \in [N - 1, N\varepsilon]$. Applying Lemma 2.12 to our $B(p)$, We obtain the estimate $B(p) < C$, where C does not depend on p . Thus it follows that the C^0 -norm of $e^{\tilde{\delta}t}u$ is bounded.

2.9. C_δ^0 -estimate by the maximum (minimum) principle.

We shall obtain an C_δ^0 -estimate by using the maximum (minimum) principle.

PROPOSITION 2.13. *There is a constant C which does not depend on $s \in S$ such that $\|e^{\delta t}u_s\|_{C^0} < C$.*

PROOF. We shall use the equation (2.25) in Subsection 2.5 to obtain a C_δ^0 -estimate of a solution of Monge-Ampère equation,

$$-\Psi_s \triangle_s \dot{u}_s = \dot{F}_s, \quad (2.46)$$

where $\dot{u}_s = (d/dt)u_s$ and $\dot{F}_s = F - 1$. Note that the family of solutions $\{u_s\}_{s \in S}$ is differentiable with respect to the parameter s from the implicit function theorem. It suffices to show a uniform C_δ^0 estimate of \dot{u}_s in order to obtain a uniform C^0 -estimate of u_s for $s \in S$. We already have the C_δ^0 -estimate. Note that Ψ_s is bounded. Then as in compact Kähler manifolds, we have $C_\delta^{k,\alpha}$ -estimate, for $0 < \tilde{\delta} < \delta$. It implies that we have the $C_\delta^{k,\alpha}$ -estimate of a solution \dot{u}_s of the equation (2.25) which does not depend on s . The difference of every coefficients of the Laplacians $\triangle_s - \triangle$ decay exponentially with the order $e^{-(2+\tilde{\delta})t}$. Then we have constants C_3 and $T > 0$ such that

$$\triangle_s e^{-\delta t} \geq C_3 e^{-(2+\delta)t},$$

on the region $\{t > T\}$. We take a constant C which satisfies the two inequalities,

(2.6) and (2.7), where $\|\dot{u}_s\|_{C^0} < C_1$ and $\|e^{(2+\delta)t}\Psi_s^{-1}\dot{F}_s\|_{C^0} < C_2$. Then we can apply the same method as in Lemma 2.7 to obtain

$$\|e^{\delta t}\dot{u}_s\|_{C^0} < C,$$

where C does not depend on s . Hence we have a C^0 -estimate of $e^{\delta t}\dot{u}_s$. Thus we obtain a C_δ^0 -estimate of u_s . \square

PROOF OF THE EXISTENCE THEOREM 1.5. We obtain a $C^{2,\alpha}$ -estimate of u by applying the general method of the 2-nd order elliptic differential equations to our conical Kähler manifolds as in the case of compact Kähler manifolds. This method was developed in [10], [21] (for instance, see [39]). Successively we apply the Schauder estimate to obtain the $C^{k+2,\alpha}$ -estimate of u (see Theorem 6.2 and Theorem 17.15 in [14]). This procedure is explained in page 89 [32]. We have the equation,

$$\Delta_s e^{\delta t}\dot{u}_s = H \quad (2.47)$$

where the $C^{k,\alpha}$ -norm of H is bounded. The $C^{k,\alpha}$ -norms of coefficients of the Laplacian Δ_s are bounded. We already have the bound of C_δ^0 -norm of u . Then applying the Schauder estimate to the equation (2.47), we obtain an estimate of the weighted norms $C_\delta^{k+2,\alpha}$ of u . Hence it follows that the set S is closed by the Ascoli-Arzelà lemma. Since S is open, it follows that $S = [0, 1]$. Thus we have a solution of the Monge-Ampère which gives the Ricci-flat conical Kähler metric on X . \square

PROOF OF THEOREM 1.7. Since $\omega' = \omega_u = \omega + \sqrt{-1}\partial\bar{\partial}u$, and both ω and ω' satisfy the Monge-Ampère equation, then we have $\Omega \wedge \bar{\Omega} = c_n \omega^n = c_n \omega_u^n$. It follows that $F \equiv 1$. Then apply the inequality of Proposition 2.9 for $\delta p > 2n - 2$, we have

$$\int_X |(d|u|^{p/2})|_g^2 \omega^n = 0.$$

Hence $u \equiv \text{constant}$ and it follows $\omega = \omega'$. \square

PROOF OF THEOREM 1.8. We set $\alpha = \omega - \omega'$. Then the d -exact 2-form α decays with the order $O(e^{-\lambda t})$, i.e., $\alpha \in C_\lambda^k$ with respect to the cylindrical metric. Let f be the real function given by the contraction $\wedge_\omega \alpha$ in terms of the Kähler form ω , where the real function f lies in $C_{\lambda+2}^k$. For $n - 2 < \lambda < 2n - 2$, we put $\lambda_1 = \lambda$. We have a solution $u \in C_{\lambda_1}^{k+2}$ such that $(1/2)\Delta_\omega u = f$, where

Δ_ω denotes the Laplacian with respect to ω . In the cases of $\lambda \geq 2n - 2$, then we also have a function $u \in C_{\lambda_1}^{k+2}$ such that $(1/2)\Delta_\omega u = f$ for $n - 2 < \lambda_1 < 2n - 2$ since $f \in C_\lambda^k \subset C_{\lambda_1}^k$ for $\lambda_1 < \lambda$. Put $\beta = \alpha - (1/\sqrt{-1})\partial\bar{\partial}u$. Then using $(1/\sqrt{-1})\wedge_\omega\partial\bar{\partial}u = \square_\omega u = (1/2)\Delta_\omega u$, we have $\wedge_\omega\beta = 0$. Thus β is a real primitive form of type $(1, 1)$ and we have

$$\beta \wedge \beta \wedge \frac{\omega^{n-2}}{(n-2)!} = -|\beta|^2 \frac{\omega^n}{n!}. \quad (2.48)$$

We denote by $H_\lambda^\bullet(X)$ the cohomology group which are defined by

$$H_\lambda^p(X) = \frac{\{a \in C_\lambda^k(\wedge^p) \mid da = 0\}}{\{db \mid b \in C_\lambda^{k+1}(\wedge^{p-1})\}},$$

where $C_\lambda^k(\wedge^p)$ denotes p -forms which decay with the order $O(e^{-\lambda t})$. Then it turns out that $H_\lambda^p(X) \cong H_{\text{cpt}}^p(X)$ for $\lambda > 0$, where $H_{\text{cpt}}^p(X)$ is the compactly supported cohomology group. Since $H^1(S) = \{0\}$, we have the exact sequence,

$$0 \rightarrow H_{\text{cpt}}^2(X) \rightarrow H^2(X) \rightarrow H^2(S) \rightarrow \cdots$$

Hence the map $H_{\text{cpt}}^2(X) \cong H_\lambda^2(X) \rightarrow H^2(X)$ is injective for $\lambda > 0$. Since β is d -exact which lies in $C_{\lambda_1}^k(\wedge^2)$ for $n - 2 < \lambda_1$, it follows that there is a 1-form $\eta \in C_{\lambda_1}^{k+1}(\wedge^1)$ such that $d\eta = \beta$. Thus we have $\beta \wedge \beta \wedge \omega^{n-2}/(n-2)! = d\eta \wedge \beta \wedge \omega^{n-2}/(n-2)!$. Since ω lies in C_{-2}^∞ , we see that $\eta \wedge \beta \wedge \omega^{n-2}$ decays with the order $O(r^{-2(\lambda_1-n+2)})$ for $\lambda_1 > n - 2$. Then it follows from the Stokes theorem that we have

$$\int_X \beta \wedge \beta \wedge \frac{\omega^{n-2}}{(n-2)!} = \int_X d\eta \wedge \beta \wedge \frac{\omega^{n-2}}{(n-2)!} = 0.$$

Then from (2.48), we see that $\beta = 0$. Thus $\alpha = (1/\sqrt{-1})\partial\bar{\partial}u$ for $u \in C_{\lambda_1}^{k+2}$. Since $\lambda_1 > 0$, the result follows from Theorem 1.7. \square

3. Sasakian structures and Kähler structures on the cone.

We will give a brief explanation of Sasakian manifolds from a view point of a correspondence to Kähler structures on the cones. A notion of Sasaki manifolds was introduced by Sasaki and now there is a good reference on Sasaki geometry [4] in which the material in this section can be found. Let S be a compact manifold

of dimension $2n - 1$. Note that S does not have a boundary. The cone of S is the product $\mathbf{R}_{>0} \times S$ with $r = e^t \in \mathbf{R}_{>0}$. A Riemannian metric g_S on S yields the cone metric g on $C(S)$ by

$$g = dr^2 + r^2 g_S. \quad (3.1)$$

In this section a metric on $C(S)$ is always the cone metric g which is given by a metric g_S on S .

DEFINITION 3.1. A $(2n - 1)$ dimensional Riemannian manifold (S, g_S) is a Sasakian manifold if there is a complex structure J on the cone such that $(C(S), g, J)$ is a Kähler manifold.

This is a relevant definition of Sasakian manifolds focusing on the relation to Kähler geometry, however which is different from the ordinary definition.

We shall explain the correspondence to the ordinary one in which interesting geometric structures are included. The Kähler structure $(C(S), g, J, \omega)$ as in Definition 3.1 gives geometric structures on S . At first we regard S as the hypersurface $\{x \in C(S) \mid r(x) = 1\}$ in $C(S)$. We denote by $r(\partial/\partial r) = \partial/\partial t$ the vector field on $C(S)$ which is defined by the translation of the cone in terms of the cylinder parameter t . The following lemma is known, for instance, see appendix [25].

LEMMA 3.2. *Let $(C(S), g, J, \omega)$ be the Kähler manifold of a Sasakian manifold S . Then we have $\mathcal{L}_{\partial/\partial t} J = 0$.*

The Lie derivative $\mathcal{L}_{r(\partial/\partial r)} J = 0$ implies that J is invariant under the translations with respect to t , in other words, the vector field $\partial/\partial t$ is the real part of a holomorphic vector field.

PROOF. Let ∇ be the Levi-Civita connection of the cone metric g . Then we see that $\nabla_u(\partial/\partial t) = u$, for all vector field u . For any vector fields u, v , we have

$$\begin{aligned} (\mathcal{L}_u J)v &= [u, Jv] - J[u, v] \\ &= \nabla_u(Jv) - \nabla_{Jv}u - J\nabla_u v + J\nabla_v u \\ &= (\nabla_u J)v - \nabla_{Jv}u + J\nabla_v u. \end{aligned}$$

Since g is a Kähler metric, we have $\nabla J = 0$. Then substituting $u = \partial/\partial t$, we obtain

$$(\mathcal{L}_{\partial/\partial t} J)v = -\nabla_{Jv} \frac{\partial}{\partial t} + J\nabla_v \frac{\partial}{\partial t} = 0. \quad \square$$

We define a vector field $\xi = Jr(\partial/\partial r) = J(\partial/\partial t)$ on $C(S)$. Since g is a Hermitian metric, ξ is orthogonal to $\partial/\partial t$ which implies that ξ is a vector field along the hypersurface S . The restriction of ξ to S is denoted by

$$\xi_S = \xi|_S.$$

We have the complex structure J^* which acts on 1-forms by $(J^*\theta)(v) = \theta(Jv)$, for $\theta \in T^*C(S)$ and $v \in TC(S)$. Then a 1-form η on $C(S)$ is given by

$$\eta = -J^*\frac{dr}{r} = -J^*dt$$

and we denote by η_S the restriction of η to S . (For simplicity, we write J for J^* from now on.) Then it follows from the definition that $\eta(\xi) = \eta_S(\xi_S) = 1$. The Kähler form ω is defined by $\omega(u, v) = g(Ju, v)$, for $u, v \in TC(S)$. The Lie derivative $\mathcal{L}_{\partial/\partial t}$ is induced from the translation of t -direction on $C(S)$. The group of one parameter transformations f_λ on $C(S)$ is given by

$$f_\lambda(r, y) = (\lambda r, y), \quad y \in S$$

and

$$\frac{d}{d\lambda} f_\lambda^*|_{\lambda=0} = \mathcal{L}_{r(\partial/\partial r)}.$$

From (3.1), we have $f_\lambda^*g = \lambda^2g$ and $\mathcal{L}_{r(\partial/\partial r)}g = 2g$. Since $\mathcal{L}_{r(\partial/\partial r)}J = 0$, we have $\mathcal{L}_{r(\partial/\partial r)}\omega = 2\omega$. Since $d\omega = 0$, applying the formula of the Lie derivative $\mathcal{L}_u = i_ud + di_u$, we have

$$di_{r(\partial/\partial r)}\omega = 2\omega. \quad (3.2)$$

Then we have

LEMMA 3.3.

$$i_{r(\partial/\partial r)}\omega = i_\xi g = r^2\eta.$$

PROOF. Since $\xi = J(\partial/\partial t)$, and $\eta = -J^*dt$, we have $\eta(\xi) = 1$. Then $i_\xi g(\xi) = r^2\eta(\xi) = r^2$. Since $J(\partial/\partial t) \in TS$, we obtain $\eta(\partial/\partial t) = -dt(J(\partial/\partial t)) = 0$. Let $\langle \partial/\partial t, J(\partial/\partial t) \rangle^\perp$ be the orthogonal subspace to the space spanned by two

vector fields $\partial/\partial t$ and $J(\partial/\partial t)$. Then $\langle \partial/\partial t, J(\partial/\partial t) \rangle^\perp$ is invariant under the action of J and we have $\eta(u) = -dt(Ju) = 0$ for $u \in \langle \partial/\partial t, J(\partial/\partial t) \rangle^\perp$. Thus $i_\xi g = r^2 \eta$. \square

Applying Lemma 3.3 to the equation (3.2), since $\eta = -J^*(dr/r)$, we have

$$2\omega = dr^2 \eta = -d(J^* r dr) = -\frac{1}{2} dJ^* dr^2. \quad (3.3)$$

Hence

$$\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} r^2.$$

Since $2\omega = 2rdr \wedge \eta + r^2 d\eta$ is a symplectic structure on $C(S)$,

$$(2\omega)^n = 2nr^{2n} dt \wedge \eta \wedge (d\eta)^{n-1} \neq 0.$$

It implies that $\eta_S \wedge (d\eta_S)^{n-1} \neq 0$. We see that η_S is a contact structure on S and ξ_S is the Reeb vector field. We define a distribution D of dimension $2n-2$ by $D = \ker \eta_S = \{u \in TS \mid \eta_S(u) = 0\}$, then $D = \langle \partial/\partial t, J(\partial/\partial t) \rangle^\perp$ and D is invariant under the action of J . We define a section $\Phi_S \in \text{End}(TS)$ by

$$\Phi_S(v) = \begin{cases} Jv & (v \in D) \\ 0 & (v = \xi). \end{cases}$$

Together with the contact structure η_S and the Reeb vector field ξ_S on S , $\Phi_S \in \text{End}(TS)$ gives the Riemannian metric on S by

$$g_S(u, v) = \eta_S \otimes \eta_S(u, v) + d\eta_S(u, \Phi_S v), \quad (3.4)$$

for $u, v \in TS$. By the same procedure, an almost Kähler structure on $C(S)$ as in Definition 3.1 gives the structure (η_S, ξ_S, Φ_S) on S . When J is integrable, the corresponding structure on S admits suitable properties.

LEMMA 3.4. *If an almost complex structure J on $C(S)$ is integrable, we have*

$$\mathcal{L}_\xi J = J \mathcal{L}_{\partial/\partial t} J.$$

PROOF. Since J is integrable, the Nijenhuis tensor vanishes,

$$\left[J \frac{\partial}{\partial t}, Ju \right] = J \left[J \frac{\partial}{\partial t}, u \right] + J \left[\frac{\partial}{\partial t}, Ju \right] + \left[\frac{\partial}{\partial t}, u \right] \quad (3.5)$$

where $u \in TC(S)$. Since $\xi = J(\partial/\partial t)$, we have

$$\mathcal{L}_\xi J(u) = \left[J \frac{\partial}{\partial t}, Ju \right] - J \left[J \frac{\partial}{\partial t}, u \right] \quad (3.6)$$

$$\mathcal{L}_{\partial/\partial t} J(u) = \left[\frac{\partial}{\partial t}, Ju \right] - J \left[\frac{\partial}{\partial t}, u \right]. \quad (3.7)$$

Thus from (3.5), we have $\mathcal{L}_\xi J = J\mathcal{L}_{\partial/\partial t}J$. \square

Then $\partial/\partial t - \sqrt{-1}J(\partial/\partial t) = \partial/\partial t - \sqrt{-1}\xi$ is a holomorphic vector field on $C(S)$. In particular, $[\partial/\partial t, \xi] = [\partial/\partial t, J(\partial/\partial t)] = 0$.

LEMMA 3.5. *If $\mathcal{L}_\xi J = 0$, then $\mathcal{L}_{\xi_S}\eta_S = 0$.*

PROOF. It follows from $\mathcal{L}_\xi J = 0$ that we have $\mathcal{L}_\xi \eta = -\mathcal{L}_\xi(Jdt) = -J(\mathcal{L}_\xi dt)$. Since $i_\xi dt = i_{J(\partial/\partial t)}dt = 0$, we have $\mathcal{L}_\xi \eta = 0$. Then the result follows from $i_S^* \mathcal{L}_\xi \eta = \mathcal{L}_{\xi_S} \eta_S$. \square

Thus if J is integrable, then $\mathcal{L}_\xi \eta = 0$. This also implies that $i_\xi d\eta = 0$ and $d\eta_S$ is a basic form on S . Since $\mathcal{L}_{\xi_S} \Phi_S = 0$, (3.4) yields $\mathcal{L}_{\xi_S} g_S = 0$ and we see that ξ_S is a Killing vector field on (S, g_S) .

Hence we obtain the structure (η_S, ξ_S, Φ_S) on S from a Kähler structure with $\mathcal{L}_{\partial/\partial t}J = 0$ and a cone metric. Conversely we shall construct the Kähler structure as in Definition 3.1 on $C(S)$ from the structure (η_S, ξ_S, Φ_S) on S . Let η_S be a contact structure on S . Since the Darboux's theorem gives the standard form of a contact structure, it follows that $D = \ker \eta_S$ is a $2n - 2$ dimensional distribution on which $d\eta_S$ is a non-degenerate 2-form.

Then a section $\Phi_S \in \text{End}(TS)$ is defined as an almost complex structure on D with $\Phi_S(\xi_S) = 0$. Such a section Φ_S is compatible with η_S if the following condition hold

- $d\eta_S(\Phi_S u, \Phi_S v) = d\eta_S(u, v), \quad u, v \in D$
- $d\eta_S(u, \Phi_S u) > 0, \quad (u \neq 0 \in D).$

These conditions imply that a pair $(d\eta_S, \Phi_S)$ is a Hermitian structure on D . Thus a compatible pair (η_S, Φ_S) gives a Riemannian metric g_S by (3.4) on S . Then we have the cone metric g on the cone $C(S) = \mathbf{R}_{>0} \times S$ by

$$g = dr^2 + r^2 g_S.$$

The tangent bundle $TC(S)$ is decomposed into $T\mathbf{R} \times TS$ and we regarded $\xi_S \in TS$ as the vector field ξ on $C(S)$. An almost complex structure J on $C(S)$ is given by

$$J\left(r\frac{\partial}{\partial r}\right) = \xi, \quad J|_D = \Phi_S.$$

Then g is a Hermitian metric with respect to J and the corresponding 2-form ω is a symplectic form,

$$2\omega = d(r^2\eta_S) = 2rdr \wedge \eta_S + r^2d\eta_S$$

that is, (g, J, ω) is an almost Kähler structure on $C(S)$. Since Φ_S is a section of $\text{End}(TS)$, the induced J is invariant under the translation with respect to t . Thus $\mathcal{L}_{\partial/\partial t}J = 0$. Hence we obtain the almost Kähler structure as in Definition 3.1 from the structure (η_S, ξ_S, Φ_S) .

DEFINITION 3.6. Let S be a manifold of dimension $2n - 1$ which admits a contact structure η_S and a compatible structure Φ_S to η_S . A compatible pair (η_S, Φ_S) is a Sasakian structure on S if the corresponding almost Kähler structure $(C(S), g, J, \omega)$ is Kählerian, that is, J is integrable.

Hence our argument is reduced to the following,

PROPOSITION 3.7. *There is a one to one correspondence between Sasakian structures (η_S, ξ_S, Φ_S) on S and Kähler structures (g, J, ω) on $C(S)$ consisting of a cone metric g and a translation-invariant complex structure J .*

4. Einstein-Sasakian structures and Calabi-Yau cone structures.

As in Section 1, let S be a compact manifold of dimension $2n - 1$ and $C(S)$ the cone $\mathbf{R}_{>0} \times S$ with $r = e^t \in \mathbf{R}_{>0}$. We assume that S is simply connected in this section. This is relevant since S has the Ricci positive metric and it follows from Myer's theorem that a finite cover of S is simply connected. Thus we replace S with the finite cover if necessary.

Let $(C(S), J, g, \omega)$ be a Kähler structure on $C(S)$ corresponding to a Sasakian structure on S by Proposition 3.7. Then as in Section 1, we have

$$\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} r^2 \tag{4.1}$$

$$g = dr^2 + r^2 g_S \tag{4.2}$$

$$\mathcal{L}_{\partial/\partial t}J = 0. \tag{4.3}$$

We also see in Section 1

$$2\omega = d(r^2\eta_S) = 2rdr \wedge \eta_S + r^2d\eta_S. \quad (4.4)$$

We assume that the canonical line bundle of $(C(S), J)$ is trivial and there is a nowhere-vanishing holomorphic n -form Ω . Note that ω and g satisfy

$$\mathcal{L}_{\partial/\partial t}\omega = 2\omega \quad (4.5)$$

$$\mathcal{L}_{\partial/\partial t}g = 2g. \quad (4.6)$$

A pair (Ω, ω) consisting of a holomorphic n -form Ω and a Kähler form ω induces a Kähler structure (J, g, ω) on $C(S)$.

DEFINITION 4.1. A pair consisting of a d -closed, holomorphic n -form Ω and a Kähler form ω on $C(S)$ is a *Calabi-Yau cone structure* if the induced Kähler metric g of the Kähler structure (J, g, ω) is a cone metric which corresponds to a Sasakian structure on S as in Proposition 3.7 and satisfies the Monge-Ampère equation,

$$\Omega \wedge \sigma(\bar{\Omega}) = \frac{(2\sqrt{-1})^n}{n!} \omega^n, \quad (4.7)$$

where $\bar{\Omega}$ is the complex conjugate of Ω and $\sigma(\bar{\Omega})$ is given by

$$\sigma(\bar{\Omega}) = \begin{cases} +\bar{\Omega}, & n \equiv 0, 1 \pmod{4}, \\ -\bar{\Omega}, & n \equiv 2, 3 \pmod{4}. \end{cases}$$

We regard S as the hypersurface $\{t = 0\} = \{r = 1\}$ as before. We denote by $r(\partial/\partial r) = \partial/\partial t$ the vector field on $C(S)$ which is defined by the translation of the cone in terms of the cylinder parameter t . The Lie derivative $\mathcal{L}_{r(\partial/\partial r)}J = 0$ implies that J is invariant under the translations with respect to t , in other words, the vector field $\partial/\partial t$ is the real part of a holomorphic vector field.

Let $i_S : S \rightarrow C(S)$ be the inclusion of S into $C(S)$ and $p_S : C(S) \rightarrow S$ the projection to S . For a Calabi-Yau cone structure (Ω, ω) , using the interior product by the vector field $\partial/\partial t$, we define

$$\psi = i_{\partial/\partial t}\Omega, \quad (4.8)$$

$$r^2\eta = i_{\partial/\partial t}\omega, \quad (4.9)$$

and restricting to S , we have $\psi_S := i_S^* \psi$, $A\eta_S = i_S^* \eta$. Since the vector field $\partial/\partial t$ is the real part of a holomorphic vector field, it follows that ψ is a holomorphic $(n-1)$ -form on $C(S)$, however ψ is not d -closed. Since $(dt - \sqrt{-1}\eta) = dt + \sqrt{-1}Jdt$ is a form of type $(1,0)$ it follows from (4.8) that

$$\Omega = (dt - \sqrt{-1}\eta) \wedge \psi. \quad (4.10)$$

Since (Ω, ω) satisfies the Monge-Ampère equation, substituting (4.10), we have

$$\begin{aligned} \Omega \wedge \sigma(\bar{\Omega}) &= (dt - \sqrt{-1}\eta) \wedge \psi \wedge \sigma(\bar{\psi}) \wedge (dt + \sqrt{-1}\eta) \\ &= 2\sqrt{-1}dt \wedge \eta \wedge \psi \wedge \sigma(\bar{\psi}). \end{aligned}$$

It also follows from (4.4) that

$$\frac{(2\sqrt{-1})^n}{n!} \omega^n = \frac{(\sqrt{-1})^n}{n!} 2nr^{2n} dt \wedge \eta \wedge (d\eta)^{n-1}.$$

Then by applying the interior product $i_{\partial/\partial t}$ to both sides of the Monge-Ampère equation, we have

$$\eta \wedge \psi \wedge \sigma(\bar{\psi}) = c_{n-1} r^{2n} \eta \wedge \left(\frac{1}{2} d\eta \right)^{n-1}, \quad (4.11)$$

where c_{n-1} is a constant. By restricting to $S = \{r = 1\}$, we have

$$\eta_S \wedge \psi_S \wedge \sigma(\bar{\psi}_S) = c_{n-1} \eta_S \wedge \left(\frac{1}{2} d\eta_S \right)^{n-1}. \quad (4.12)$$

Let D be the subbundle of TS given by the kernel of $d\eta_S$. Then ψ_S defines a section Φ_S on $\text{End}(D)$ which is an almost complex structure and then ψ_S is a nowhere vanishing section of the canonical line bundle K_D of D . Since ω is Kählerian, we see that $d\eta_S$ is a Hermitian form on D and then (η_S, Φ_S) is an Sasakian structure on S . An Einstein-Sasakian structure on S is a Sasakian structure whose metric g is an Einstein metric on S .

PROPOSITION 4.2. *Let (Ω, ω) be a Calabi-Yau cone structure on $C(S)$. Then (Ω, ω) corresponds to an Einstein-Sasakian structure on S under the correspondence as in Proposition 3.7. Conversely, an Einstein-Sasakian structure on S gives a Calabi-Yau cone structure (Ω, ω) on $C(S)$, that is, there is a one to one*

correspondence between Einstein-Sasakian structures on S and Calabi-Yau cone structures on $C(S)$.

As though a correspondence as in Proposition 4.2 is already known among experts in Sasakian geometry (see [4] for instance), our point of view may be rather relevant since we focus on Calabi-Yau cone structures on $C(S)$ which exactly correspond to Einstein-Sasakian structures on S . In the appendix we shall give a proof.

5. Ricci-flat conical Kähler metrics on crepant resolutions of normal isolated singularities.

Let X_0 be an affine variety of complex dimension n with only normal isolated singularity at $p \in X_0$. In this section, we assume that the complement $X_0 \setminus \{p\}$ is biholomorphic to the cone $C(S)$ of an Einstein-Sasakian manifold S of real dimension $2n - 1$. As we see in Sections 3 and 4, there is the weighted Calabi-Yau structure (Ω, ω_0) on the cone $C(S)$ and the Ricci-flat Kähler cone metric ω_0 which is given by

$$\omega_0 = \frac{\sqrt{-1}}{2} \partial \bar{\partial} r^2. \quad (5.1)$$

THEOREM 5.1. *Let X_0 be an affine variety with only normal isolated singularity at p . We assume that the complement $X_0 \setminus \{p\}$ is biholomorphic to the cone $C(S)$ of an Einstein-Sasakian manifold S of real dimension $2n - 1$. If there is a resolution of singularity $\pi : X \rightarrow X_0$ with trivial canonical line bundle K_X , then there is a Ricci-flat conical Kähler metric for every Kähler class of X .*

REMARK 5.2. Van Coevering [40] showed that there is a Ricci-flat conical Kähler metric in the Kähler class which belongs to the compactly supported cohomology group $H_{\text{cpt}}^2(X)$ of X . Our theorem shows that this kind of restricted condition is not necessary and implies that the conjecture on the existence of complete Ricci-flat Kähler metrics in [25] is affirmative.

The Ricci-flat Kähler cone metric ω_0 is written as

$$\omega_0 = e^{2t} (2dt \wedge d^c t + dd^c t), \quad (5.2)$$

where $dd^c t$ is the transversal Kähler metric on the Sasakian manifold S and the Reeb vector field ξ_S is given by $J(\partial/\partial t)$ which gives the Reeb foliation on S . Let $H_B^p(S)$ be the basic cohomology group on S with respect to the Reeb foliation.

There are the Hodge decomposition of the basic cohomology groups and the Lefschetz decomposition as in Kähler geometry. Thus the second basic cohomology group $H_B^2(S)$ is decomposed into the followings,

$$H_B^2(S, \mathbf{C}) = H_B^{1,1}(S) \oplus H_B^{2,0}(S) \oplus H_B^{0,2}(S), \quad (5.3)$$

$$H_B^{1,1}(S, \mathbf{R}) = \mathbf{R}dd^c t \oplus P_B^{1,1}(S, \mathbf{R}), \quad (5.4)$$

where $H_B^{p,q}(S)$ denote the basic Dolbeault cohomology group of type (p, q) and $\mathbf{R}dd^c t$ is the one dimensional space generated by the transversal Kähler form $dd^c t$ and $P_B^{1,1}(S)$ is the basic primitive cohomology group of type $(1, 1)$ with respect to $dd^c t$. Since the transversal complex structure on S admits the positive first Chern class which implies that the anti-canonical line bundle on S is positive. Then by the same procedure as in Kähler manifolds, we obtain the vanishing theorem of Kodaira and the Serre duality of the basic cohomology groups. The basic cohomology groups inherit many of the same properties as the ordinary Dolbeault cohomology groups of the Kähler manifolds [9], [22], [4]. Then we have the vanishing of cohomology groups,

LEMMA 5.3. *Let S be a compact positive Sasaki manifold of real dimension $2n - 1$. Then we have*

$$H_B^{q,0}(S) = H_B^{0,q}(S) = \{0\}, \quad \forall q > 0.$$

In particular, $H_B^{2,0}(S) = H_B^{0,2}(S) = \{0\}$.

PROOF. Since the transversal complex structure on S admits the positive first basic Chern class which is represented by a positive real basic form of type $(1, 1)$, there is a Sasakian structure with the same underlying transversal complex structure whose transversal Ricci curvature is positive from the transverse Calabi-Yau theorem [9], [5]. Let η be the contact form on the $(2n - 1)$ dimensional Sasakian manifold S . Then we have the pairing $\tilde{\Psi}$ of the basic forms α, β by

$$\tilde{\Psi}(\eta \wedge \alpha \wedge \beta) := \int_S \eta \wedge \alpha \wedge \beta.$$

Since $d\eta$ is a basic 2-form, we have the integration by part. By using the basic Hodge star operator $*_B$, we have the L^2 -metric $\|\alpha\|_{L^2}^2 = \int_S \eta \wedge \alpha \wedge *_B \alpha = \int_S (\alpha, \alpha) \text{vol}_S$, (see Boyer-Galicki, [4] Sasakian Geometry, page 217) where (\cdot, \cdot) is the metric on forms and vol_S is the volume form on the Sasakian manifold (S, g) . Then the transversal Kähler structure on S gives a basic, Hermitian connection A

on the transversal holomorphic line bundle K^{-1} such that the curvature form F_A is a positive, basic $(1, 1)$ form on S . We denote by $\bar{\partial}_B$ the $\bar{\partial}$ -operator acting on K^{-1} -valued basic forms. Let $\square_B^{\bar{\partial}}$ be the basic Laplacian of $\bar{\partial}$ -operator acting on K^{-1} -valued basic forms and \square_B^{∂} the complex conjugate of $\square_B^{\bar{\partial}}$. Then we have the Kodaira-Nakano identity,

$$\square_B^{\bar{\partial}} = \square_B^{\partial} + [\sqrt{-1}F_A, \wedge].$$

(The identity follows from a local calculation as in Kähler geometry, see for instance Griffith-Harris, Principle of algebraic geometry, pages 154–155.) Let ϕ be a K^{-1} -valued basic form of type $(n-1, q)$, $q > 0$. Then we have the inequality using the basic Hermitian metric on K^{-1} ,

$$\|\bar{\partial}_B \phi\|_{L^2}^2 + \|\bar{\partial}_B^* \phi\|_{L^2}^2 \geq \int_X ([\sqrt{-1}F_A, \wedge] \phi, \phi) \text{vol}_S \geq 2\pi q \|\phi\|_{L^2}^2.$$

Since K^{-1} admits a basic Hermitian connection, we have the de Rham-Hodge theorem of K^{-1} -valued basic forms of type $(n-1, q)$ with respect to the Dolbeault Laplacian $\square_B^{\bar{\partial}}$ (see [9]). Thus every element of cohomology group $H_B^{n-1, q}(S, K^{-1})$ is represented by the K^{-1} -valued basic harmonic forms w.r.t. $\square_B^{\bar{\partial}}$ (note that S is a transversally complex manifold of dimension $n-1$). If ϕ is harmonic, it follows from the inequality that $\phi = 0$. Then we have the vanishing $H_B^{n-1, q}(S, K^{-1}) = \{0\}$. There is an isomorphism $H_B^{n-1, q}(S, K^{-1}) \cong H_B^{0, q}(S, K \otimes K^{-1})$. Since $K \otimes K^{-1}$ is a trivial complex line bundle, we have the vanishing of the cohomology groups

$$H_B^{0, q}(S) = \{0\}, \quad \forall q > 0.$$

By using the basic Dolbeault decomposition (see also page 217 in [4]), we have $H_B^{q, 0}(S) = \overline{H_B^{0, q}(S)} = \{0\}$, ($q > 0$). \square

REMARK. Note that the transverse Dolbeault theorem is not necessary to show the vanishing of the cohomology groups of Lemma 5.3. The vanishing of the cohomology groups on Lemma 5.3 was already shown in [5] when the positive Sasaki manifolds are regular or quasi-regular. This vanishing theorem on positive Sasaki manifolds can be proved by the different method (Bochner technique) [8].

LEMMA 5.4.

$$H_B^2(S) = H_B^{1, 1}(S) = \mathbf{R}dd^c t \oplus H^2(S, \mathbf{R}).$$

PROOF. It is shown that the de Rham cohomology group $H^i(S)$ of the Sasakian manifold S coincides with the basic primitive cohomology groups $P_B^i(S)$, for $1 \leq i \leq n-1$ (for instance, see Proposition 7.4.13, pp. 233 in [4]). It follows from Lemma 5.3 that

$$H_B^2(S) = H_B^{1,1}(S) = \mathbf{R}dd^c t \oplus P_B^2(S) = \mathbf{R}dd^c t \oplus H^2(S). \quad \square$$

LEMMA 5.5. *Let $\pi : X \rightarrow X_0$ be a resolution of an affine variety X_0 with only normal isolated singularity with trivial K_X . Then we have the vanishing*

$$H^i(X, \mathcal{O}_X) = \{0\}, \quad (\forall i > 0). \quad (5.5)$$

Further let $C(S)$ be the complement $X \setminus E$ where E is the exceptional set of the resolution X . Then we also have

$$H^i(C(S), \mathcal{O}_{C(S)}) = \{0\}, \quad (n-1 > \forall i > 0). \quad (5.6)$$

In particular, if $n \geq 3$, we have

$$H^1(C(S), \mathcal{O}_{C(S)}) = \{0\},$$

where $\dim_{\mathbf{C}} X = n$.

PROOF. Since X_0 is an affine variety and $H^p(X_0, R^q \pi_* \mathcal{O}_X) = \{0\}$ for $p > 0$, the first vanishing (5.5) follows from the Grauert-Riemenschneider vanishing theorem,

$$R^i \pi_* K_X = R^i \pi_* \mathcal{O}_X = 0, \quad (i > 0).$$

Let $H_E^i(X, \mathcal{O}_X)$ be the local cohomology groups with supports in E , and coefficients in the structure sheaf \mathcal{O}_X . Then the short exact sequence,

$$0 \rightarrow \Gamma_E(X, \mathcal{O}_X) \rightarrow \Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(C(S), \mathcal{O}_{C(S)}) \rightarrow 0$$

yields the long exact sequence,

$$\begin{aligned} \cdots \rightarrow H_E^i(X, \mathcal{O}_X) &\rightarrow H^i(X, \mathcal{O}_X) \\ &\rightarrow H^i(C(S), \mathcal{O}_{C(S)}) \rightarrow H_E^{i+1}(X, \mathcal{O}_X) \rightarrow \cdots \end{aligned} \quad (5.7)$$

By applying the local duality theorem of the cohomology groups to the resolution

$\pi : X \rightarrow X_0$, we have

$$\dim H_E^j(X, \mathcal{O}_X) = \dim R^{n-j}\pi_* K_X = \dim R^{n-j}\pi_* \mathcal{O}_X,$$

(see for instance, Corollary 2.5.15, p.60 [18]). Then it follows from the Grauert-Riemenschneider vanishing theorem that

$$H_E^j(X, \mathcal{O}_X) = \{0\}$$

for all $n - j > 0$. Thus from the exact sequence (5.7) and (5.5), we obtain

$$H^i(C(S), \mathcal{O}_{C(S)}) = \{0\}, \quad (n - 1 > \forall i > 0).$$

It implies that $H^1(C(S), \mathcal{O}_{C(S)}) = \{0\}$ if $n \geq 3$. □

The resolution $\pi : X \rightarrow X_0$ gives the identification $\pi|_{X \setminus E} : X \setminus E \cong X_0 \setminus \{p\} = C(S)$. We denote by $\{t > T\}$ the subset of the cone $C(S) = \mathbf{R} \times S$ defined by

$$\{t > T\} := \{(t, s) \in \mathbf{R} \times S \mid t > T\},$$

for a constant T . The subset $\{t > T\}$ is regarded as the subset of X by the identification $\pi|_{X \setminus E} : X \setminus E \cong X_0 \setminus \{p\} = C(S)$. Then the set $\{t > T\}$ is an unbounded region of X and we also denote by $\{t \leq T\}$ the complement of $\{t > T\}$ in X which is a compact set including the exceptional set E . Let p_S be the projection from $C(S) = \mathbf{R} \times S$ to S . We use the same notation p_S for the restricted projection $\{t > T\} \rightarrow S$.

LEMMA 5.6. *Let κ be an arbitrary Kähler form on X with the Kähler class $[\kappa] \in H^2(X, \mathbf{R})$. We assume $\dim_{\mathbf{C}} X \geq 3$. Then there is a real form $\tilde{\kappa}$ of type $(1, 1)$ on X which satisfies the followings,*

- (i) $[\tilde{\kappa}] = [\kappa] \in H^2(X, \mathbf{R})$.
- (ii) *For a sufficiently large constant T_+ , the restriction $\tilde{\kappa}$ to the subset $\{t > T_+\}$ is given by the pull back of a d -closed, primitive basic form $\hat{\kappa}$ of type $(1, 1)$ on S , i.e.,*

$$\tilde{\kappa}|_{\{t > T\}} = p_S^* \hat{\kappa}.$$

PROOF. It follows from Lemmas 5.3 and 5.4 that we have

$$H^2(C(S)) \cong H^2(S) \cong P_B^2(S) \cong P_B^{1,1}(S).$$

Then there is a d -closed, primitive basic form $\hat{\kappa} \in P_B^{1,1}(S)$ and a 1-form θ on $C(S)$ such that

$$\kappa|_{C(S)} = p_S^* \hat{\kappa} - d\theta. \quad (5.8)$$

Then $d\theta$ is a form of type $(1, 1)$ on $C(S)$ where $\theta = \theta^{1,0} + \theta^{0,1}$ and we have

$$d\theta = \bar{\partial}\theta^{1,0} + \partial\theta^{0,1}, \quad \partial\theta^{1,0} = 0, \quad \bar{\partial}\theta^{0,1} = 0.$$

It follows from Lemma 5.5 that there is a function ϕ on $C(S)$ such that $\theta^{0,1} = \bar{\partial}\phi$. Since θ is a real form, it follows from $\theta^{1,0} = \bar{\theta}^{0,1}$ that

$$\begin{aligned} d\theta &= \partial\theta^{0,1} + \overline{\partial\theta^{0,1}} \\ &= \partial\bar{\partial}\phi + \bar{\partial}\partial\bar{\phi} \\ &= 2\sqrt{-1}\partial\bar{\partial}\phi^{Im} \end{aligned}$$

where $\phi^{Im} = 1/2\sqrt{-1}(\phi - \bar{\phi})$ is the imaginary part of ϕ . Let ρ_T be a smooth function such that

$$\rho_T(x) = \begin{cases} 1 & x \in \{T < t\} \\ 0 & x \in \{t \leq T - 1\} \end{cases}$$

and $0 \leq \phi_T(x) \leq 1$ on X , i.e., ρ_T is a cut off function which takes the value 0 on a neighborhood of E and 1 at the infinity. We define $\tilde{\kappa}$ by

$$\tilde{\kappa} = \kappa + 2\sqrt{-1}\partial\bar{\partial}\rho_{T_+}\phi^{Im}.$$

Then we see that $\tilde{\kappa}$ is a form of type $(1, 1)$ on X which is the Kähler form κ on $\{t \leq T_+ - 1\}$ and $p_S^* \hat{\kappa}$ on $\{t > T_+\}$ from (5.8). Thus $\tilde{\kappa}$ satisfies the conditions. \square

LEMMA 5.7. *Let κ be an arbitrary Kähler form on X . We assume $\dim_{\mathbf{C}} X \geq 3$. Then there exists a Kähler form $\omega_{\kappa,0}$ which satisfies the followings,*

- (i) $[\omega_{\kappa,0}] = [\kappa] \in H^2(X, \mathbf{R})$.
- (ii) *There are constants T_+ and T_- such that $\omega_{\kappa,0}$ coincides with $c\omega_0 + \tilde{\kappa}$ on the region $\{1 + T_- < t\}$, where c is a positive constant and ω_0 is the Kähler cone metric on $C(S)$ as before and $\tilde{\kappa}$ is the form in Lemma 5.6, where $1 + T_- < T_+$.*

PROOF. We see that there is a positive function $\psi(x)$ on \mathbf{R} which satisfies

$$\psi(x) = \begin{cases} x & \text{if } x > e^{2(1+T_-)} \\ \text{constant} & \text{if } e^{2T_-} > x \end{cases}$$

and $\psi'(x) \geq 0$ and $\psi''(x) \geq 0$ on the region $\{e^{2T_-} \leq x \leq e^{2(1+T_-)}\}$. Then the composite function $\psi(r^2)$ is a function on X since ψ is a constant on the region $\{t < T_-\}$, where $r = e^t$ and we have

$$\sqrt{-1}\partial\bar{\partial}\psi(r^2) = \sqrt{-1}\psi''(r^2)\partial r^2 \wedge \bar{\partial} r^2 + \sqrt{-1}\psi''(r^2)\partial\bar{\partial} r^2.$$

It implies that $\sqrt{-1}\partial\bar{\partial}\psi(r^2)$ is semi-positive on the region $\{t \leq 1 + T_-\}$ which is the Kähler cone metric ω_0 on the region $\{1 + T_- < t\}$. Thus we define $\omega_{\kappa,0}$ by

$$\omega_{\kappa,0} = c\sqrt{-1}\partial\bar{\partial}\psi(r^2) + \tilde{\kappa}.$$

Then $\omega_{\kappa,0}$ satisfies both conditions (i) and (ii). So it suffices to show that $\omega_{\kappa,0}$ is a Kähler form for constants T_+ , T_- and c , which are taken by the following. We divide X into the following five regions:

- (1) $\{t < T_-\}$ (2) $\{T_- \leq t \leq 1 + T_-\}$ (3) $\{1 + T_- < t < T_+ - 1\}$,
 (4) $\{T_+ - 1 \leq t \leq T_+\}$ (5) $\{T_+ < t\}$.

On the region (1), $\omega_{\kappa,0}$ is the Kähler form κ and on the region (3), $\omega_{\kappa,0}$ is $c\omega_0 + k$ which is also Kählerian. On the region (2), $\omega_{\kappa,0} = k + \sqrt{-1}\partial\bar{\partial}\psi(r^2)$ is positive since $\sqrt{-1}\partial\bar{\partial}\psi(r^2)$ is semi-positive and κ is positive. On the region (5), $\omega_{\kappa,0}$ is a sum of the Kähler cone form ω_0 and the bounded form $p_S^*\hat{\kappa}$ with respect to the cylinder metric. Since the cone metric ω_0 grows with order $O(e^{2t})$, there is a sufficiently large $T_+ = T_+(c_0)$ for a positive c_0 such that $c_0\sqrt{-1}\partial\bar{\partial}\psi(r^2) + \tilde{\kappa}$ becomes positive. Finally since the region (4) is compact, there is a positive c with $c > c_0$ such that $\omega_{\kappa,0}$ is a Kähler form on the region (4). We can see that $\omega_{\kappa,0}$ is still positive for $c > c_0$ on (5). Hence $\omega_{\kappa,0}$ is a Kähler form. \square

PROOF OF THEOREM 5.1. In the case $\dim_{\mathbf{C}} X = 2$, X is a minimal resolution of the ordinary double point (see [20]). Thus every Kähler class is represented by the class of the exceptional divisors which lies in compactly supported cohomology groups and we can have a Kähler form $\omega_{\kappa,0}$ as in Lemma 5.7 with $\hat{k} = 0$. Hence we have the initial Kähler form $\omega_{\kappa,0}$ as in Lemma 5.7 for every Kähler class on X of dimension $n \geq 2$. We recall that a positive function F is defined by

$$\Omega \wedge \overline{\Omega} = c_n F_k \omega_{k,0}^n$$

with $F_k = e^{f_k}$. We shall show that the function F_k satisfies

$$\|e^{(2+\delta)t}(F_k - 1)\|_{C^{k,\alpha}} < \infty, \quad \delta > 0, \quad k \geq 3, \quad 0 < \alpha < 1$$

with respect to the cylindrical metric $dt \wedge d^c t + dd^c t = e^{-2t} \omega_0$. Then the result follows from Theorem 1.5.

Since ω_0 is a Ricci-flat Kähler cone metric, we have

$$\Omega \wedge \overline{\Omega} = c_n \omega_0^n.$$

Then $\omega_0^n = e^{f_k} \omega_{k,0}^n$. Since $\omega_{k,0} = \omega_0 + \tilde{k} = \omega_0 + p_S^* \hat{k}$ on the region $\{t > T_+\}$, we have

$$e^{-f_k} = \frac{\omega_{k,0}^n}{\omega_0^n} = \frac{(\omega_0 + \tilde{k})^n}{\omega_0^n} \quad (5.9)$$

$$= 1 + n \frac{\tilde{k} \wedge \omega_0^{n-1}}{\omega_0^n} \quad (5.10)$$

$$+ \sum_{i=2}^n \frac{n!}{i!(n-i)!} \frac{\tilde{k}^i \wedge \omega_0^{n-i}}{\omega_0^n}. \quad (5.11)$$

Since $\tilde{k} = p_S^* \hat{k}$ is the pullback of a primitive, basic $(1,1)$ -form on S with respect to $dd^c t$, we have $\tilde{k} \wedge (dd^c t)^{n-2} = 0$. Thus the second term of (5.10) vanishes since we have

$$(n-1)^{-1} \tilde{k} \wedge \omega_0^{n-1} = \tilde{k} \wedge e^{2(n-1)t} (2dt \wedge d^c t \wedge (dd^c t)^{n-2}) \quad (5.12)$$

$$= e^{2(n-1)t} (2dt \wedge d^c t \wedge \hat{k} \wedge (dd^c t)^{n-2}) = 0. \quad (5.13)$$

Each term of (5.11) is given by

$$(n-i)^{-1} \frac{\tilde{k}^i \wedge \omega_0^{n-i}}{\omega_0^n} = \frac{\tilde{k}^i \wedge e^{2(n-i)t} (2dt \wedge d^c t \wedge (dd^c t)^{n-i-1})}{e^{2nt} (2dt \wedge d^c t \wedge (dd^c t)^n)} \quad (5.14)$$

$$= e^{-2it} \frac{\tilde{k}^i \wedge (2dt \wedge d^c t \wedge (dd^c t)^{n-i-1})}{(2dt \wedge d^c t \wedge (dd^c t)^n)}. \quad (5.15)$$

Since the 1-form dt and $d^c t$ and the 2-form $dd^c t$, \tilde{k} are bounded with respect to

the cylinder metric, we have

$$\frac{\tilde{k}^i \wedge \omega_0^{n-i}}{\omega_0^n} = O(e^{-2it}) = O(r^{-2i}).$$

Note that $\tilde{\kappa} = p_S^* \hat{\kappa}$ for a basic 2-form $\hat{\kappa}$. It follows from $i \geq 2$ that the every term in (5.11) decays with the order $O(e^{-4t})$. Hence we have $e^{-f_\kappa} - 1 = O(e^{-4t})$. Since e^{-f_κ} is bounded, we obtain

$$e^{f_\kappa} - 1 = O(e^{-4t}).$$

We also have the estimate of the higher order derivative of F_k since we use the C^k -norm with respect to the cylinder metric. Note that the C^k -norm of e^{-4t} is just estimated the derivative by the cylinder parameter t which also decays with the order $O(e^{-4t})$.

Hence there is a solution u of the Monge-Ampère equation

$$\Omega \wedge \bar{\Omega} = c_n(\omega_{k,0} + dd^c u)^n$$

from the existence theorem 1.5. Then the Kähler form $\omega_k = \omega_{k,0} + dd^c u$ gives a Ricci-flat Kähler metric on X . \square

REMARK 5.8. If the basic $(1,1)$ -form $\hat{\kappa}$ is not primitive, then the function $F - 1$ defined by the Kähler $\omega_{\kappa,0} = \omega_0 + p_S^* \hat{\kappa}$ in the region $\{t > T_+\}$ only decays with the order $O(e^{-2t})$, which we can not apply the existence theorem 1.5. Thus it is intriguing that $\hat{\kappa}$ is primitive for our construction of Ricci-flat Kähler metrics.

6. Examples of Calabi-Yau structures on crepant resolutions.

We construct examples of Ricci-flat conical Kähler metrics. Some of them are already known, however there are new Ricci-flat conical Kähler metrics included whose Kähler classes do not belong to the compactly supported cohomology groups. We start with the trivial example.

EXAMPLE 6.1. The complement $\mathbf{C}^n \setminus \{0\}$ is the cone of the sphere of dimension $2n - 1$ and the standard Kähler metric ω_{st} on \mathbf{C}^n is a conical Ricci-flat Kähler metric with $r^2 = \sum_{i=1}^n |z_i|^2$. The induced metric on the sphere is an Einstein-Sasakian metric.

EXAMPLE 6.2. Let Γ be a finite subgroup of the special unitary group $SU(n)$ which freely acts on $\mathbf{C}^n \setminus \{0\}$. Then the quotient $X_0 = \mathbf{C}^n / \Gamma$ has a normal isolated

singularity at the origin 0 and the complement $\mathbf{C}^n \setminus \{0\}/\Gamma$ is the cone $C(S)$ of the Einstein-Sasakian manifold $S := S^{2n-1}/\Gamma$. The induced metric on the cone $C(S)$ is a Ricci-flat Kähler cone metric. Then Theorem 5.1 shows that a crepant resolution of the isolated quotient singularity \mathbf{C}^n/Γ has a Ricci-flat conical Kähler metric. This class is already obtained by Kronheimer [20] for $n = 2$ and by Joyce [19] for $n > 2$.

EXAMPLE 6.3. Let Z be a compact Kähler-Einstein manifold with positive 1-st Chern class $c_1(Z)$, which is called a Fano manifold (we assume that $\dim_{\mathbf{C}} Z = n - 1 \geq 2$). Then the sphere bundle S of the canonical line bundle K_Z of the Fano manifold Z admits an Einstein-Sasakian metric and the complement of the zero-section $K_Z \setminus \{0\}$ is the cone $C(S)$ which has the Ricci-flat Kähler cone metric. Then we apply Theorem 5.1 to $X = K_Z$ and obtain a Ricci-flat conical Kähler metric in every Kähler class of K_Z . Calabi [6] already constructed a Ricci flat Kähler metric on K_Z by the bundle construction whose Kähler class lies in the compactly supported cohomology group, i.e., the anti-canonical class. Since we have the vanishing of the cohomology groups $H^1(S) = \{0\}$ and $H_{\text{cpt}}^3(X) \cong H^3(X, X \setminus Z) \cong H^1(Z) = \{0\}$ by the duality theorem. Then we obtain the exact sequence,

$$0 \rightarrow H_{\text{cpt}}^2(X) \rightarrow H^2(X) \rightarrow H^2(S) \rightarrow 0.$$

We also have $H_{\text{cpt}}^2(X) \cong H^2(X, X \setminus Z) \cong H^0(Z) \cong \mathbf{R}$. Thus $\dim H^2(X) = \dim H^2(S) + 1$. If $b_2(Z) = \dim H^2(Z) = \dim H^2(X)$ is greater than 1, it follows from the exact sequence that there is a Kähler class which does not belong to the compactly supported cohomology group. Thus in the cases, we obtain a new family of Ricci-flat Kähler metrics on K_Z .

EXAMPLE 6.4. Recently Futaki-Ono-Wang [11] obtained an Einstein-Sasakian metric on the sphere bundle of the canonical line bundle of every toric Fano manifold which gives the Ricci-flat Kähler cone metric on the complement $K_Z \setminus \{0\}$. Thus Theorem 5.1 shows that there exists a Ricci-flat conical Kähler metric in every Kähler class on the total space of the canonical line bundle on an arbitrary toric Fano manifold. If the canonical line bundle is the m -th tensor of a line bundle L on a toric Fano manifold, then we can apply the theorem to the total space L to obtain a Ricci-flat conical Kähler metric on L in each Kähler class. Let $\widehat{CP^2}$ be the blown up CP^2 at one point. Then $\widehat{CP^2}$ is a toric Fano manifold which does not admit a Kähler-Einstein metric. However it is shown that the canonical line bundle $K_{\widehat{CP^2}}$ of $\widehat{CP^2}$ admits a family of Ricci-flat conical Kähler metrics which is parametrized by the open set $H^2(\widehat{CP^2})$, i.e., the Kähler cone of Z . Note that $\dim H^2(\widehat{CP^2}) = 2$ and $\dim H_{\text{cpt}}^2(K_{\widehat{CP^2}}) = 1$. Sasaki-Einstein

metrics on the blown up \mathbf{CP}^2 at one point was explicitly constructed by Martelli and Sparks [26] and Oota-Yasui [30] described a Ricci-flat metric on a resolved Calabi-Yau cone.

EXAMPLE 6.5. We denote by Σ_k a blown up \mathbf{CP}^2 at k points in general position ($0 \leq k \leq 8$). Then it is known that Σ_k admits an Einstein-Kähler metric for $3 \leq k \leq 8$ [38], [34]. Since Σ_k for $k = 1, 2$ is a toric Fano surface, there is a Einstein-Sasakian metric on the sphere bundle of the canonical line bundle on Σ_k [11], [27]. Hence we obtain a complete Ricci-flat Kähler metrics in every Kähler class of the canonical line bundle of a blown up \mathbf{CP}^2 at generic k points for all $0 \leq k \leq 8$.

EXAMPLE 6.6. Let X_0 be the 3 dimensional hypersurface in \mathbf{C}^4 defined by $z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0$. The hypersurface X_0 has a singularity at the origin 0 which is the ordinary double point and there is a small resolution $X \rightarrow X_0$ with trivial K_X . The small resolution X is the total space $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on \mathbf{CP}^1 and X_0 is obtained by the contraction of the zero-section to one point. The complement $X_0 \setminus \{0\}$ is the cone $C(S)$ of the sphere bundle S over the product $\mathbf{CP}^1 \times \mathbf{CP}^1$. Thus the sphere bundle S has an Einstein-Sasakian metric which induces the Ricci-flat Kähler cone metric on $C(S)$. Then we obtain a one dimensional family of Ricci-flat conical Kähler metrics on the resolution, since $\dim H^2(X) = 1$ and $H_{\text{cpt}}^2(X) = 0$. Candelas and de la Ossa [7] described a Ricci-flat metric on a conifold and it is also keen to show that the Ricci-flat metric in [7] coincides with the one in our family constructed by Theorem 5.1.

7. Appendix.

We shall show the following Lemma 7.2 and Lemma 7.3 to prove Proposition 4.2. As in Section 4, we assume that S is simply connected. Let Ric_{g_S} be the Ricci curvature of a Sasakian manifold (S, g_S) . We see that $d\eta_S$ is a Kähler form on the distribution D which satisfies $\mathcal{L}_{\xi_S} d\eta_S = 0$. Then $d\eta_S$ gives the transversal Riemannian metric g_S^T on D with $\mathcal{L}_{\xi_S} g_S^T = 0$. Let $\text{Ric}_{g_S^T}$ be the Ricci curvature of the transversal Riemannian metric g_S^T . Then there is a relations between $\text{Ric}_{g_S^T}$ and Ric_{g_S} .

PROPOSITION 7.1 (Boyer-Galicki, p. 224, Theorem 7.3.12).

$$\text{Ric}_{g_S}(u, v) = \text{Ric}_{g_S^T}(u, v) - 2g_S(u, v), \quad u, v \in D$$

$$\text{Ric}_{g_S}(u, \xi) = 2(n-1)\eta_S(u), \quad \forall u \in TS.$$

Thus a Sasakian metric g_S is Einstein if and only if the transversal metric g_S^T

is Einstein with Einstein constant $2n$.

LEMMA 7.2. *Let (Ω, ω) be a Calabi-Yau cone structure on $C(S)$ and $(\eta_S, g_S, \xi_S, \psi_S)$ the corresponding Sasakian structure on S . Let g_S^T be the transversal Riemannian metric given by the transversal Kähler structure $d\eta_S$ on S . Then g_S^T is Einstein with Einstein constant $2n$, that is, $\text{Ric}_{g_S^T} = 2ng_S^T$.*

PROOF. We take an open covering $\{U_\alpha\}_\alpha$ of S such that each U_α admits coordinates $(x, z_1^\alpha, \dots, z_{n-1}^\alpha)$ which is compatible with transversal holomorphic structure on S , that is, $(z_1^\alpha, \dots, z_{n-1}^\alpha)$ gives the transversal complex structures and $\partial/\partial x = \xi_S$. (Such coordinates are called foliation coordinates.) Then we have a d -closed, local holomorphic n -form $\Omega_{S,\alpha} = dz_1^\alpha \wedge \dots \wedge dz_{n-1}^\alpha$ which is a basic section of $K_D|_{U_\alpha}$. For simplicity, we write $\Omega_{S,\alpha}$ for its pullback $p_S^* \Omega_{S,\alpha}$ on $C(S)$. Since $\psi = i_{\partial/\partial t} \Omega$ is a holomorphic section of $p_S^* K_D$, there is a holomorphic function f_α on $p_S^{-1}(U_\alpha)$ such that

$$\psi = e^{f_\alpha} \Omega_{S,\alpha}.$$

Substituting $\psi = e^{f_\alpha} \Omega_{S,\alpha}$ into (4.11), we have

$$e^{f_\alpha + \bar{f}_\alpha} \eta \wedge \Omega_{S,\alpha} \wedge \sigma(\bar{\Omega}_{S,\alpha}) = c_{n-1} r^{2n} \eta \wedge \left(\frac{1}{2} d\eta\right)^{n-1}. \quad (7.1)$$

We define a function k_α locally by

$$\Omega_{S,\alpha} \wedge \sigma(\bar{\Omega}_{S,\alpha}) = e^{k_\alpha} c_{n-1} \left(\frac{1}{2} d\eta_S\right)^{n-1}. \quad (7.2)$$

Then the Ricci form of the transversal Kähler form g_S^T on an open set U_α of S is given by $\text{Ric}_{g_S^T} = \sqrt{-1} \partial_B \bar{\partial}_B k_\alpha$, where $\bar{\partial}_B$ is the $\bar{\partial}$ -operator with respect to the transversal complex structure on S and ∂_B is its complex conjugate. By multiplying $\eta = \eta_S$ to both sides of (7.2) and pulling back by p_S to $C(S)$, we have

$$\eta \wedge \Omega_{S,\alpha} \wedge \sigma(\bar{\Omega}_{S,\alpha}) = e^{p_S^* k_\alpha} c_{n-1} \eta \wedge \left(\frac{1}{2} d\eta_S\right)^{n-1}. \quad (7.3)$$

Comparing to (7.1), we have

$$p_S^* e^{k_\alpha} = r^{2n} e^{-(f_\alpha + \bar{f}_\alpha)} = e^{2nt} e^{-(f_\alpha + \bar{f}_\alpha)}. \quad (7.4)$$

A function $p_S^* \kappa_\alpha$ on $C(S)$ is basic with respect to both ξ and $\partial/\partial t$. Thus we have $p_S^* \partial_B \bar{\partial}_B k_\alpha = \partial \bar{\partial} p_S^* \kappa_\alpha$. Then it follows that

$$\mathrm{Ric}_{g_S^T} = \sqrt{-1} \partial_B \bar{\partial}_B k_\alpha = i_S^* \sqrt{-1} \partial \bar{\partial} p_S^* \kappa_\alpha,$$

where $\bar{\partial}$ is the $\bar{\partial}$ -operator with respect to the complex structure on $C(S)$. From (7.4), we have

$$\mathrm{Ric}_{g_S^T} = 2ni_S^* \sqrt{-1} \partial \bar{\partial} t - i_S^* \sqrt{-1} \partial \bar{\partial} (f_\alpha + \bar{f}_\alpha). \quad (7.5)$$

Since $\bar{\partial} f_\alpha = 0$, we have $\partial \bar{\partial} (f_\alpha + \bar{f}_\alpha) = 0$ and the transversal Kähler form is $(1/2)d\eta_S = \sqrt{-1} \partial \bar{\partial} t|_S$. Thus we obtain

$$\mathrm{Ric}_{g_S^T} = 2n \sqrt{-1} \partial \bar{\partial} t = 2n \left(\frac{1}{2} d\eta_S \right). \quad (7.6)$$

Then it follows that $\mathrm{Ric}_{g_S^T} = 2n g_S^T$. \square

Conversely the following lemma shows that an Einstein-Sasakian structure on S gives the Calabi-Yau cone structure on $C(S)$.

LEMMA 7.3. *An Einstein-Sasakian structure on S corresponds to the Calabi-Yau cone structure on $C(S)$ under the correspondence in Proposition 3.7.*

PROOF. As in Proposition 7.1, an Einstein-Sasaki structure gives the transversal Einstein-Kähler metric with scalar curvature $2n$, $\mathrm{Ric}_{g_S^T} = 2n g_S^T$. Thus the 1-st Chern class $c_B^1(S)$ of the transversal canonical line bundle K_D is represented by the form $2n((1/2)d\eta_S) = n d\eta_S$. There is an exact sequence on S ,

$$\rightarrow H^0(S) \xrightarrow{i} H_B^2(S) \xrightarrow{j} H^2(S) \rightarrow$$

in which we have $i(\alpha) = \alpha d\eta_S$ for $\alpha \in H^0(S)$ and $j(c_B^1(S)) = nj(d\eta_S) = j \circ i(n) = 0$. Thus $c^1(S) = c^1(K_D)$ vanishes. Since S is simply connected, the line bundle K_D is trivial. Let $\{U_\alpha\}$ be an open covering of S such that each U_α admits a foliation coordinates $(x^\alpha, z_1^\alpha, \dots, z_{n-1}^\alpha)$. We denote by $p_S : C(S) \rightarrow S$ the projection. We take U_α sufficiently small such that every d -closed holomorphic 1-form on $p_S^{-1}(U_\alpha) \cong \mathbf{R} \times U_\alpha$ is written as a d -exact 1-form of a holomorphic function, that is, the holomorphic de Rham theorem holds on U_α . Taking a d -closed, basic, holomorphic $(n-1)$ -form $\Omega_{S,\alpha} = dz_1^\alpha \wedge \dots \wedge dz_{n-1}^\alpha$ on U_α , we define a basic function k_α by

$$\Omega_{S,\alpha} \wedge \sigma(\overline{\Omega}_{S,\alpha}) = c_{n-1} e^{\kappa_\alpha} \left(\frac{1}{2} d\eta_S \right)^{n-1}. \quad (7.7)$$

Then the transversal Ricci form $\text{Ric}_{g_S^T}$ is given by

$$\text{Ric}_{g_S^T} = \sqrt{-1} \partial_B \bar{\partial}_B \kappa_\alpha. \quad (7.8)$$

Since the transversal Kähler form $(1/2)d\eta_S$ is Einstein-Kähler, we have

$$\sqrt{-1} \partial_B \bar{\partial}_B \kappa_\alpha = 2n \left(\frac{1}{2} d\eta_S \right) = 2ni_S^* (\sqrt{-1} \partial \bar{\partial} t). \quad (7.9)$$

Pulling back them by the projection $p_s : C(S) \rightarrow S$ to $C(S)$, since κ_α is a basic function, we have

$$p_S^* \text{Ric}_T \left(\frac{1}{2} d\eta_S \right) = p_S^* \sqrt{-1} \partial_B \bar{\partial}_B \kappa_\alpha \quad (7.10)$$

$$= \sqrt{-1} \partial \bar{\partial} p_S^* \kappa_\alpha = 2n (\sqrt{-1} \partial \bar{\partial} t). \quad (7.11)$$

Then on $p_S^{-1}(U_\alpha) \subset C(S)$, we have

$$\sqrt{-1} \partial \bar{\partial} (p_S^* \kappa_\alpha - 2nt) = 0.$$

Since U_α is sufficiently small such that every d -closed holomorphic 1-form is a d exact form of a holomorphic function, there is a holomorphic function q_α on $p^{-1}(U_\alpha)$ satisfying

$$\kappa_\alpha - 2nt = q_\alpha + \bar{q}_\alpha.$$

Then from (7.7), we have

$$(e^{-q_\alpha} \Omega_{S,\alpha}) \wedge \sigma(e^{-\bar{q}_\alpha} \overline{\Omega}_{S,\alpha}) = c_{n-1} r^{2n} \left(\frac{1}{2} d\eta_S \right)^{n-1}. \quad (7.12)$$

Note that $r = e^t$. We define ψ by $e^{-q_\alpha} \Omega_{S,\alpha}$. Then the pair (η_S, ψ) satisfies (4.12). If we set $\Omega_\alpha = (dt + \sqrt{-1} J dt) \wedge (e^{-q_\alpha} \Omega_{S,\alpha})$, then Ω_α is a local holomorphic n -form on $C(S)$ which satisfies

$$\Omega_\alpha \wedge \sigma(\overline{\Omega}_\alpha) = c_n \omega^n$$

where $c_n = (2\sqrt{-1})^n/n!$. Hence the Ricci curvature of the Kähler metric ω on $C(S)$ vanishes. Further since S is simply connected, the canonical line bundle on $C(S)$ is trivial. Then there is a holomorphic n -form Ω on $C(S)$ satisfying the Monge-Ampère equation,

$$\Omega \wedge \sigma(\overline{\Omega}) = c_n \omega^n.$$

Thus (Ω, ω) is a Calabi-Yau cone structure on $C(S)$. Hence we have the result. \square

PROOF OF THEOREM 4.2. The theorem follows from Lemma 7.2 and Lemma 7.3 and Proposition 7.1. \square

References

- [1] T. Aubin, Nonlinear Analysis on Manifolds, Monge-Ampère equations, Grundlehren Math. Wiss., **252**, A Series of Comprehensive Studies on Mathematics, Springer-Verlag, 1982.
- [2] S. Bando and R. Kobayashi, Ricci-flat Kähler metrics on affine algebraic manifolds, Geometry and Analysis on Manifolds, In: (ed. T. Sunada), Lecture Notes in Math., **1339**, Springer-Verlag, 1988, pp. 20–31.
- [3] S. Bando and R. Kobayashi, Ricci-flat Kähler metrics on affine algebraic manifolds, II, *Math. Ann.*, **287** (1990), 175–180.
- [4] C. P. Boyer and K. Galicki, Sasakian Geometry, Oxford Math. Monogr., Oxford University Press, Oxford, 2008.
- [5] C. P. Boyer, K. Galicki and M. Nakamaye, On positive Sasakian geometry, *Geom. Dedicata*, **101** (2003), 93–102.
- [6] E. Calabi, Métriques kählériennes et fibrés holomorphes, *Ann. Sci. École Norm. Sup.* (4), **12** (1979), 269–294.
- [7] P. Candelas and Xenia C. de la Ossa, Comments on conifolds, *Nuclear Phys. B*, **342** (1990), 246–268.
- [8] R. J. Conlon, A vanishing theorem for positive Sasaki manifolds, preprint, 2011.
- [9] A. El Kacimi-Alaoui, Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications, [Transversely elliptic operators on a Riemannian foliation, and applications], *Compositio Math.*, **73** (1990), 57–106.
- [10] L. C. Evans, Classical solutions of fully nonlinear, convex, second-order elliptic equations, *Comm. Pure Appl. Math.*, **35** (1982), 333–363.
- [11] A. Futaki, H. Ono and G. Wang, Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds, *J. Differential Geom.*, **83** (2009), 585–635.
- [12] S. Gallot, A Sobolev inequality and some geometric applications, *Spectra of Riemannian manifolds*, Kaigai Publications, Tokyo, 1983, pp. 45–55.
- [13] R. Goto, On hyper-Kähler manifolds of type A_∞ and D_∞ , *Comm. Math. Phys.*, **198** (1998), 469–491.
- [14] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Grundlehren Math. Wiss., **224**, A Series of Comprehensive Studies in Mathematics,

- Springer-Verlag, 1983.
- [15] A. Grigor'yan and L. Saloff-Coste, Stability results for Harnack inequalities, *Ann. Inst. Fourier (Grenoble)*, **55** (2005), 825–890.
 - [16] H.-J. Hein, Weighted Sobolev inequalities under lower Ricci curvature bounds, *Proc. Amer. Math. Soc.*, **139** (2011), 2943–2955.
 - [17] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Hyper-Kähler metrics and supersymmetry, *Comm. Math. Phys.*, **108** (1987), 535–589.
 - [18] S. Ishii, Introduction to singularities (Tokuiten nyumon), Springer-Verlag, Tokyo, 1997.
 - [19] D. D. Joyce, Compact Manifolds with Special Holonomy, Oxford Math. Monogr., Oxford University Press, Oxford, 2000.
 - [20] P. B. Kronheimer, The construction of ALE spaces as Hyper-Kähler quotients, *J. Differential Geom.*, **29** (1989), 665–683.
 - [21] N. V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations, *Izv. Akad. Nauk SSSR Ser. Mat.*, **46** (1982), 487–523 [Russian], English translation in *Math. USSR-Izv.*, **20** (1983), 459–492.
 - [22] F. W. Kamber and P. Tondeur, de Rham-Hodge theory for Riemannian foliations, *Math. Ann.*, **277** (1987), 415–431.
 - [23] R. B. Lockhart and R. C. McOwen, Elliptic differential operators on noncompact manifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **12** (1985), 409–447.
 - [24] R. B. Melrose, Atiyah-Patodi-Singer Index Theorem, Res. Notes in Math., **4**, A K Peters Ltd., Wellesley, MA, 1993.
 - [25] D. Martelli and J. Sparks, Symmetry-breaking vacua and baryon condensates in AdS/CFT correspondence, *Phys. Rev D*, **79** (2009), 065009.
 - [26] D. Martelli and J. Sparks, Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals, *Comm. Math. Phys.*, **262** (2006), 51–89.
 - [27] D. Martelli, J. Sparks and S.-T. Yau, Sasaki-Einstein manifolds and volume minimisation, *Comm. Math. Phys.*, **280** (2008), 611–673.
 - [28] V. Minerbe, Weighted Sobolev inequality and Ricci flat manifolds, *Geom. Funct. Anal.*, **18** (2009), 1696–1749.
 - [29] H. Nakajima, Nonlinear Analysis and Complex Geometry, Gendai suugaku no tenkai (in Japanese), Iwanami Shoten, 1999.
 - [30] T. Oota and Y. Yasui, Explicit toric metric on resolved Calabi-Yau cone, *Phys. Lett. B*, **639** (2006), 54–56.
 - [31] B. Santoro, Existence of complete Kähler Ricci-flat metrics on crepant resolutions, *math.DG/0902.0595*
 - [32] Y. T. Siu, Lectures on Hermitian-Einstein Metrics for Stable Bundles and Kähler-Einstein Metrics, DMV Seminar, **8**, Birkhäuser, 1987.
 - [33] G. Tian, Canonical Metrics in Kähler Geometry, Lectures in Math. ETH Zürich, Birkhäuser, 2000.
 - [34] G. Tian, On Calabi's conjecture for complex surfaces with positive first Chern class, *Invent. Math.*, **101** (1990), 101–172.
 - [35] G. Tian and S.-T. Yau, Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry, Mathematical Aspects of String Theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., **1**, World Scientific Publishing, 1987, pp. 574–628.
 - [36] G. Tian and S.-T. Yau, Complete Kähler manifolds with zero Ricci curvature, I, *J. Amer. Math. Soc.*, **3** (1990), 579–609.
 - [37] G. Tian and S.-T. Yau, Complete Kähler manifolds with zero Ricci curvature, II, *Invent. Math.*, **106** (1991), 27–60.

- [38] G. Tian and S.-T. Yau, Kähler-Einstein metrics on complex surfaces with $C_1 > 0$, [Comm. Math. Phys.](#), **112** (1987), 175–203.
- [39] N. S. Trudinger, Lectures on nonlinear elliptic equations of second order, Lectures in Mathematical Sciences, The University of Tokyo, **9**, 1995.
- [40] C. van Coevering, Ricci-flat Kähler metrics on crepant resolutions of Kähler cones, [Math. Ann.](#), **347** (2010), 581–611.
- [41] C. van Coevering, Regularity of asymptotically conical Ricci-flat Kähler metrics, arXiv:0912.3946

Ryushi GOTO

Department of Mathematics

Graduate School of Science

Osaka University

Toyonaka, Osaka 560-0043

Japan

E-mail: goto@math.sci.osaka-u.ac.jp