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Denjoy-Sacksteder theory for groups of diffeomorphisms

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Abstract. We extend the one dimensional Denjoy-Sacksteder theorems to some diffeomorphism groups of smooth compact closed n-dimensional manifolds. More precisely, we show the existence of a non trivial stabilizer for actions of "quasi-conformal groups" admitting an "exceptional" minimal set which is of "strongly decreasing type"; our results include the classical Denjoy-Sacksteder theorems.

Introduction.

Let $\mathfrak{G}$ be a finitely generated group of C^2-diffeomorphisms of the circle. If it admits a Cantor set $\mathscr{E}$ as minimal set, there exist $g \in \mathfrak{G}$ and $x \in \mathscr{E}$ such that

$$g(x) = x \quad \text{and} \quad |g'(x)| < 1.$$

This is the celebrated theorem of Sacksteder for group actions (see [**11**]) and our goal here is to extend it to suitable finitely generated groups of diffeomorphisms of closed manifolds of any dimension.

To do so, we first introduce the notion of "exceptional minimal sets". A minimal set $\mathscr{E}$ for a group $\mathfrak{G}$ acting on a closed manifold M will be called "exceptional" if it has empty interior and is such that the open set $M \backslash \mathscr{E}$ has infinitely many components all of whose closures are pairwise disjoint. Now it happens that, for any $\varepsilon > 0$, McSwiggen constructed in [**7**] infinite cyclic groups of $C^{3-\varepsilon}$-diffeomorphisms of the 2-torus which preserve such an exceptional minimal set but don't admit any non trivial stabilizer. Therefore in order to really extend Sacksteder, we make a double restriction

i) we assume that the minimal sets $\mathscr{E}$ are of "strongly decreasing type" that is the sum of the diameters of the complementary components is bounded (see 1.1),
ii) the group of diffeomorphisms $\mathfrak{G}$ are "quasi-conformal" that is the pointwise

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dilatation of all their elements are uniformly bounded by some constant K (see 1.3).

In this setting, Sacksteder's theorem extends and we provide a proof following quite closely the procedure of proof of the original work of Sacksteder (see 3.2). Indeed Sacksteder's theorem holds under a somewhat weaker differentiability hypothesis and so will it be for our generalization.

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1. Exceptional minimal sets, wandering sequences and quasi conformal groups.

We first introduce and describe the particular class of closed sets which we will choose as minimal sets for our group actions. After that we adapt to our setting the usual notion of wandering domains and finally we introduce the family of quasi-conformal groups.

1.1. Exceptional sets and exceptional minimal sets.

Consider a closed Riemannian manifold $(M, \mathfrak{g})$ and denote by $\delta(A)$ the diameter of any subset $A \subset M$.

Definition 1.1. A closed subset $\mathscr{E}$ of M is called *exceptional* if

i) $\mathscr{E}$ has empty interior,
ii) the complement $M \setminus \mathscr{E}$ of $\mathscr{E}$ in M has infinitely many connected components $\{U_j\}_{j \in \boldsymbol{N}}$ whose closures W_j are pairwise disjoint.

From the metric point of view, we say furthermore that $\mathscr{E}$ is of

1. *decreasing type* if $\lim_{j \to +\infty} \delta(W_j) = 0$,
2. *strongly decreasing type* if $\sum_{j=0}^{+\infty} \delta(W_j) < +\infty$.

Moreover, $\mathscr{E}$ will be called a *Sierpiński-set* if it is of decreasing type and all components W_j are contractible.

Of course an exceptional set of strongly decreasing type is of decreasing type and M being compact, these two notions do not depend on the metric. Indeed they characterize two families of exceptional sets which are of interest.

Example 1.2.

1) A Cantor set embedded in a compact manifold M is exceptional if and only if $M = \boldsymbol{S}^1$ and any such Cantor set is of strongly decreasing type.
2) The triadic Sierpiński carpet is an exceptional set; it is of decreasing but not

strongly decreasing type.

3) In [**14**], Whyburn showed that the standard Sierpiński carpet is homeomorphic to any closed subset of the two dimensional sphere S^2 obtained by removing the interiors U_j of mutually disjoint closed topological disks $\{D_j\}_{j\in\boldsymbol{N}}$ whose diameters tend to 0, provided that it has empty interior. Now if D_j has radius $1/j^2$ for example, the corresponding set is a *Sierpiński set* of strongly decreasing type. It may have strictly positive measure, therefore it is homeomorphic but not diffeomorphic to the standard Sierpiński carpet in general.

More generally, one can extend these constructions and considerations to subsets of any closed manifold (see for example the so-called Sierpiński M-sets introduced in [2]).

Let's come over to Dynamics. Recall that any open equivalence relation on a closed manifold M admits at least one *minimal set*; that is a minimal element in the set of all closed saturated subsets of M ordered by inclusion. Now a natural question is to ask whether an exceptional closed set $\mathscr{E}$ can be a minimal set for some equivalence relation, depending on the regularity of this relation. If so, it will be called an *exceptional minimal set*.

Concerning these exceptional minimal sets, there are several well known facts:

i) In dimension 1, an exceptional set is a Cantor set and the celebrated theorem of Denjoy asserts that a C^2-diffeomorphism of the circle $\boldsymbol{S}^1$ never admits an exceptional minimal set.
ii) If a finitely generated group $\mathfrak{G}$ of C^2-diffeomorphisms of the circle admits an exceptional minimal set $\mathscr{E}$, then by Sacksteder's theorem, there exist a point $x\in\mathscr{E}$ and an element g of the group such that $g(x)=x$ and $|g'(x)|<1$.
iii) It is shown in [**1**], that for a homeomorphism of the torus T^2, any locally connected minimal set without locally separating points either is finite or equals T^2 or is a Sierpiński set.

1.2. Group actions and wandering sequences.

Here we introduce the basic technical tools that we will use all over the paper. We start recalling some notations and conventions concerning finitely generated groups.

Definition 1.3. Let $\mathfrak{G}_*$ be a finite symmetric set of generators of a finitely generated group $\mathfrak{G}$.

i) For any $g\in\mathfrak{G}$, the least integer n such that g can be written as a composition

$$g=\gamma_n\circ\cdots\circ\gamma_1$$

of n generators $\gamma_i \in \mathfrak{G}_*$, is called the *length of* g (with respect to $\mathfrak{G}_*$) and denoted by $l(g)$, the length of the neutral element being equal to 0 by definition.

ii) Further if $\mathfrak{G}$ acts on a space M, any orbit Γ of $\mathfrak{G}$ is naturally endowed with a metric defined by

$$d(u,v) = \min_{v=g(u)} l(g)$$

for any pair of points $u, v \in \Gamma$. Moreover, u and v being fixed, there exists at least one element g such that $v = g(u)$ and $l(g) = d(v,u)$; we call it a *shortcut at* u (or u-*shortcut*). Finally, we call *infinite* u-*shortcut* any sequence $\boldsymbol{g} = \{g_n\}_{n\in\boldsymbol{N}}$ of u-shortcuts of length n such that, for any n, the length $l(g_{n+1} \circ g_n^{-1}) = 1$ which means that $\gamma_{n+1} = g_{n+1} \circ g_n^{-1}$ is an element of $\mathfrak{G}_*$.

Note that if $g_n = \gamma_n \circ \cdots \circ \gamma_1$ is a u-shortcut of length n, then clearly $g_j = \gamma_j \circ \cdots \circ \gamma_1$ is a u-shortcut of length j for any $1 \le j \le n$. Note also that a change of generating set will induce an equivalent distance on the orbits modifying consequently the set of shortcuts. Anyway we fix once and for all the set $\mathfrak{G}_*$.

Lemma 1.4. *For a finitely generated group $\mathfrak{G}$ acting on M and any point $u \in M$, the following conditions are equivalent*:

i) *the orbit $\Gamma(u)$ of u is infinite,*
ii) *there exists an infinite u-shortcut.*

Proof. Of course ii) $\Rightarrow$ i). To prove the converse, take an infinite sequence $\{u_j\}_{j\in\boldsymbol{N}}$ of pairwise distinct points of $\Gamma(u)$; there exists for any j an integer p_j and a u-shortcut g_{p_j} of length p_j such that $u_j = g_{p_j}(u)$. Then because $\mathfrak{G}$ is finitely generated, the set of u-shortcuts of length p_j is finite for any j and therefore it is possible to extract by a diagonal process an infinite subsequence $A \subset \boldsymbol{N}$ such that for any two consecutive elements $r < s$ of A, the length of $\bar{g}_{r,s} = g_{p_r} \circ g_{p_s}^{-1}$ is equal to $p_r - p_s$. For $r \in A$, we write g_{p_r} as a composition of generators and introducing all partial compositions g_n for $0 \le n \le p_r$, we obtain an infinite sequence $\boldsymbol{g} = \{g_n\}_{n\in\boldsymbol{N}}$ which is the wanted infinite u-shortcut. $\square$

Now assume that $\mathfrak{G}$ is a group of homeomorphisms of a closed Riemannian manifold $(M, \mathfrak{g})$ and let $\rho(\mathfrak{G})$ be the associated open equivalence relation. For any subset $C \subset M$, we denote by $\operatorname{sat}(C) = \bigcup_{g\in\mathfrak{G}} g(C)$ the *saturation* of C with respect to $\rho(\mathfrak{G})$.

Remark and Definition 1.5. Assume that the group $\mathfrak{G}$ admits an exceptional minimal set $\mathscr{E}$. If U is a connected component of $M \backslash \mathscr{E}$, any connected component of $\operatorname{sat}(U)$ is of type $g(U)$ for some element $g \in \mathfrak{G}$ and the same claim holds for its closure W.

i) Next defining in the obvious way the *orbit* $\Gamma(W)$ *of* W in the set of all closed subsets of M, we can endow it with a metric similar to that defined in 1.3 for the orbit of a point thus extend the associated notions of shortcuts and define W*-shortcuts* and *infinite* W*-shortcuts*. Statement and Proof of Lemma 1.4 transpose immediately to this new setting.

ii) Finally, W will be called a *wandering domain* if its orbit is infinite and the sequence $\mathscr{W}_{\boldsymbol{g}} = \{g_n(W)\}_{n \in \boldsymbol{N}}$ associated to an infinite W-shortcut $\boldsymbol{g} = \{g_n\}_{n \in \boldsymbol{N}}$ will be called a *wandering sequence initiated* by the wandering domain W and *defined* by the infinite W-shortcut $\boldsymbol{g}$.

It is important to notice that an infinite W-shortcut is also an infinite u-shortcut for any point $u \in W$ but the converse is not true. Finally we introduce some properties of "metric type" for wandering sequences.

Definition 1.6. We say that the wandering sequence $\mathscr{W}_{\boldsymbol{g}} = \{g_n(W)\}_{n \in \boldsymbol{N}}$ is

i) *decreasing* if the sequence of diameters $\{\delta[g_n(W)]\}_{n \in \boldsymbol{N}}$ tends to 0,

ii) *strongly decreasing* if the latter is summable i.e. $\Lambda = \delta(\mathscr{W}_{\boldsymbol{g}}) = \sum_{n=0}^{+\infty} \delta[g_n(W)] < +\infty$.

As M is compact, all metrics on M are equivalent and therefore the fact that a wandering sequence is decreasing [resp. strongly decreasing] does not depend on the metric. Note also that a given closed set W may possibly initiate infinitely many different wandering sequences; this number will reduce essentially to two if the group $\mathfrak{G}$ is cyclic.

Indeed we now show that all components of the complement of an exceptional minimal set are wandering:

Lemma 1.7. *Let $\mathscr{E}$ be an exceptional minimal set for a finitely generated group $\mathfrak{G}$ and let W be the closure of a component U of $M \backslash \mathscr{E}$. Then for any point $u \in \partial W$, there exists a subsequence $\{u_j\}_{j \in \boldsymbol{N}}$ of the orbit $\Gamma(u)$ such that $u_0 = u$, $u_j \notin W$ for $j \geq 1$ and $\lim_{j \to \infty} u_j = u$.*

Proof. Denoting by $A \subset \partial W$ the subset of points u which verify the claim above; we proceed in two steps:

(1) First we claim that either $A = \partial W$ or $A = \emptyset$. Indeed for $u \notin A$, there exists a neighborhood ω of u in M such that $\omega \cap \Gamma(u) \subset \partial W$ which, because $\Gamma(u)$ is dense in $\mathscr{E}$, implies that

$$\omega \cap \mathscr{E} = \omega \cap \partial W$$

and the latter set is open in $\mathscr{E}$. By minimality of $\mathscr{E}$, the saturation $\mathrm{sat}(\omega \cap \partial W)$ is an open dense subset of $\mathscr{E}$ thus equal to $\mathscr{E}$. In particular any point $v \in \partial W$ is mapped into ω by some element $g \in \mathfrak{G}$ and $g(v) \in g(W) \cap W$. But $g(W)$ is the closure of the component $g(U)$ of $M \backslash \mathscr{E}$ and by definition of exceptional sets, we conclude that $g(W) = W$ and $g(\partial W) = \partial W$. The neighborhood $g^{-1}(\omega)$ of v is such that $g^{-1}(\omega) \cap \mathscr{E} = g^{-1}(\omega) \cap \partial W$ which implies that $v \notin A$; we have proved that $A \neq \partial W$ implies $A = \emptyset$.

(2) Now if A is empty, there exists an open neighborhood V of W such that $V \cap \mathscr{E} = \partial W$ showing that ∂W is open in $\mathscr{E}$. Consequently $\bigcup_{g \in \mathfrak{G}} g(\partial W)$ is an open saturated subset of $\mathscr{E}$ which by minimality equals $\mathscr{E}$. It follows that $\{g(V)\}_{g \in \mathfrak{G}}$ is an open cover of the compact set $\mathscr{E}$ from which we can extract a finite subcover $\{g_1(V), g_2(V), \ldots, g_r(V)\}$ so that $\mathscr{E} = \bigcup_{j=1}^{r} g_j(\partial W)$ contradicting the definition of exceptional sets. We conclude that $A = \partial W$ which proves our claim. □

Theorem 1.8. *Let $\mathscr{E}$ be an exceptional minimal set for a finitely generated group $\mathfrak{G}$ of homeomorphisms of a closed compact manifold M. Then the closure W of any connected component U of $M \backslash \mathscr{E}$ is wandering and for any point $u \in \partial W$, there exists a wandering sequence $\mathscr{W}_{\boldsymbol{g}} = \{g_n(W)\}_{n \in \boldsymbol{N}}$ which accumulates on u. In particular $\mathscr{E}$ is the only minimal set for $\mathfrak{G}$.*

Moreover if $\mathscr{E}$ is of decreasing type, the sequence $\{g_n(u)\}_{n \in \boldsymbol{N}}$ accumulates on u.

Proof. The proof is in three steps. We fix a component W, a point $u \in \partial W$ and a subsequence $\{u_j\}$ of $\Gamma(u)$ provided by the previous lemma.

(1) Then for any j, there exists a connected component W_j of $\mathrm{sat}(W)$ such that $u_j \in \partial W_j$. Possibly after selecting an appropriate subsequence, we may assume that these components W_j are mutually distinct which implies immediately that W is wandering. Next as observed in 1.5, we can apply Lemma 1.4 to W and so obtain a wandering sequence $\{g_n(W)\}_{n \in \boldsymbol{N}}$ which accumulates on u thus on $\mathscr{E}$ by choice of u.

(2) Because $M \backslash \mathscr{E}$ is open, any associated sequence $\{g_n(x)\}_{n \in \boldsymbol{N}}$ accumulates on $\mathscr{E}$ as well; it implies that $\mathscr{E}$ is contained in the closure of the orbit of any $x \in M$ thus $\mathscr{E}$ is the only minimal set for $\mathfrak{G}$.

(3) Finally if $\mathscr{E}$ is of decreasing type, the last assertion of (1) above implies that any sequence $\{g_n(x)\}_{n \in \boldsymbol{N}}$ accumulates on u. It is so in particular for the sequence $\{g_n(u)\}_{n \in \boldsymbol{N}}$ itself.

The proof is complete. □

Finally one may ask for examples of exceptional minimal sets of group actions, in particular examples which are not of Sierpiński's type. The following was suggested to us by the referee:

EXAMPLE 1.9.

1) There are well known examples of cyclic groups of diffeomorphisms of the 2-torus which admit exceptional minimal sets; of class C^0 in [**1**] and of class at least C^2 in [**7**], all of Sierpiński's type.
2) To obtain an example which is not of Sierpiński's type, we consider a group $\mathfrak{G}$ of diffeomorphisms of $\boldsymbol{T}^3$ generated by the two diffeomorphisms ($\varphi \times \mathrm{id}$, $\mathrm{id} \times R_\alpha$) where φ is a McSwiggen's diffeomorphism ([**7**]) as before and R_α is an irrational rotation on the circle. Then $\mathfrak{G}$ admits an exceptional minimal set $\mathscr{E}$ whose complement is a union of solid tori.

1.3. Quasi-conformal groups of diffeomorphisms.

In order to extend the theorems of Denjoy and Sacksteder to groups of diffeomorphisms of higher dimensional manifolds, we will have to restrict to a particular class of groups; the so-called "quasi-conformal groups" which we describe now.

Our closed manifold M being still endowed with a Riemannian metric $\mathfrak{g}$, we denote by $|v|$ the norm of a tangent vector v and by T^1M the bundle of unit tangent vectors with fiber T^1_xM over x. For any C^1-diffeomorphism $f : M \to M$, we introduce

i) the *norm of the differential of f at $x \in M$*

$$\|Df_x\| := \sup\bigl\{|Df_x(v)| : v \in T^1_xM\bigr\},$$

ii) the *dilatation of f at $x \in M$* given by

$$H_f(x) := \frac{\sup\{|Df_x(v)| : v \in T^1_xM\}}{\inf\{|Df_x(w)| : w \in T^1_xM\}}.$$

As M is compact, the dilatation of f has an obvious upper bound

$$H_f(x) \le K[f] := \frac{\sup\{|Df(v)| : v \in T^1M\}}{\inf\{|Df(w)| : w \in T^1M\}}$$

and the two notions are related by the following inequalities:

$$\frac{1}{K[f]} \cdot \|Df_x\| \le |Df_x(v)| \le \|Df_x\|,$$

which hold for any $x \in M$ and any unit vector $v \in T^1_x M$.

Now consider a group $\mathfrak{G}$ of C^1-diffeomorphisms of M. In general there is no common upper bound for all the dilatations $K[g]$, $g \in \mathfrak{G}$; to obtain one, we have to restrict to a special class of groups:

Definition 1.10. A group $\mathfrak{G}$ of C^1-diffeomorphisms of a manifold M will be called *quasi-conformal* if there exists a positive constant K such that $K[g] \leq K$ for any element $g \in \mathfrak{G}$.

For example, the group of iterates of a conformal diffeomorphism is quasi-conformal with $K = 1$. Also for any diffeomorphism f of $\boldsymbol{S}^1$, the dilatation $H_f(x)$ at any point $x \in \boldsymbol{S}^1$ equals 1 and therefore any group of C^1-diffeomorphisms of $\boldsymbol{S}^1$ is trivially 1-quasi-conformal. The reader may find other examples of quasi-conformal groups in [**12**], [**13**] or [**6**].

2. Wandering sequences and convergence of differentials.

The mean value theorem plays a crucial role in the proof of the classical Denjoy theorem and its generalization by Sacksteder. Its use strongly relies on the fact that the circle is of dimension 1; here we introduce two analogues of it adapted to our context and in the remainder of the section, we use these analogues for the control of the differentials of short-cuts on wandering domains.

2.1. Mean value inequalities.

Consider again a Riemannian manifold $(M, \mathfrak{g})$ and let ε be its *convexity radius.*

Remark 2.1. Let W be the closure of an open connected domain U of M. If $\delta(W) \leq \varepsilon$, there exists for any $x \in W$, a unique maximal geodesically convex disk C_x centered at x and contained in W, it verifies $\partial C_x \cap \partial W \neq \emptyset$. Finally there is a point $c_W \in W$ such that C_{c_W} has maximal radius which we denote by $\rho(W)$.

Now our *mean value inequalities* will be the following:

Lemma 2.2. *For any W and any diffeomorphism f of M, there exist points $z, y \in W$ such that*:

i) $\rho(W) \cdot \|Df_z\| \leq K[f] \cdot \delta[f(W)]$ *if* $\delta[f(W)] \leq \varepsilon$,
ii) $\delta[f(W)] \leq \delta(W) \cdot \|Df_y\|$ *if* $\delta(W) \leq \varepsilon$.

Proof. (1) To prove claim i), we take the convex disk $C_{f(c_W)} \subset f(W)$ defined above in 2.1. Because $C_{f(c_W)}$ meets $\partial f(W)$, the set $E_* = f^{-1}[C_{f(c_W)}]$ meets ∂W and containing by definition the point c_W, it verifies

$$\delta(E_*) \geq \rho(W).$$

Now take points $a, b \in \partial E_*$ such that $d(a,b) = \delta(E_*)$. As $C_{f(c_W)}$ is convex, there exists a geodesic γ joining $f(a)$ to $f(b)$ in $C_{f(c_W)}$. Let $l(\gamma)$ be its length and let $l(\lambda)$ be the length of the path $\lambda = f^{-1} \circ \gamma$ parametrized by arc length. Then λ joins a to b in E_* thus verifies

$$l(\lambda) \geq \delta(E_*) \geq \rho(W).$$

Finally, according to the usual mean value theorem and the definition of $K[f]$, there exist a point $s \in [0, l(\lambda)]$ and a vector $v \in T_z^1 M$, with $z = \lambda(s)$, such that

$$\delta[f(W)] \geq \delta(C_{f(c_W)}) \geq l(\gamma) = l(\lambda) \cdot |Df_z(v)| \geq l(\lambda) \cdot \frac{1}{K[f]} \cdot \|Df_z\|.$$

Formula i) follows by combination of the two previous inequalities.

(2) To prove claim ii), we choose points $a, b \in \partial W$ such that $d[f(a), f(b)] = \delta[f(W)]$. The assumption $\delta(W) \leq \varepsilon$ implies that W is contained in a convex disk in which there exists a geodesic λ joining a to b with $l(\lambda) \leq \delta(W)$. Then $\gamma = f \circ \lambda$ joins $f(a)$ to $f(b)$ and verifies $l(\gamma) \geq \delta[f(W)]$. Applying again the usual mean value theorem to λ, we see that there exist $s \in [0, l(\lambda)]$ and $v \in T_y^1 M$ with $y = \lambda(s)$ such that, by definition of the norm, we get

$$\delta[f(W)] \leq l(\gamma) = l(\lambda) \cdot |Df_y(v)| \leq \delta(W) \cdot \|Df_y\|.$$

This is claim ii). □

Observe that these inequalities were obtained under the only assumption that f is of class C^1.

2.2. Convergence on wandering domains.

Notations 2.3. In order to simplify our notations, we will from now on write $\|Df(x)\|$ instead of $\|Df_x\|$ for $x \in M$.

We look for an analytical description of wandering sequences. So let $\mathfrak{G}$ be a finitely generated group of diffeomorphisms of M admitting an exceptional minimal set $\mathscr{E}$ and consider

$$\mathscr{W}_{\boldsymbol{g}} = \{g_n(W)\}_{n \in \boldsymbol{N}} = \{W_n\}_{n \in \boldsymbol{N}}$$

a wandering sequence for $\mathfrak{G}$, initiated by the closure W of a component U of $M \backslash \mathscr{E}$.

Our next goal is to control the sequences $\{\|Dg_n(x)\|\}$, $x \in W$, when n tends to ∞. To do so, we have to compare the differentials of the diffeomorphisms g_n at any two different points x and y of W. Here as for the original Denjoy-Sacksteder theorems, we must increase the order of differentiability of $\mathfrak{G}$.

Definition 2.4. We say that a C^1-diffeomorphism g is of *class* C^{1+L} if the continuous function $\operatorname{Log}\|Dg(x)\|$ is Lipschitz and we say that the group $\mathfrak{G}$ is of *class* C^{1+L} if all its elements are of this class.

Then if $\mathfrak{G}$ is finitely generated, we can find a common Lipschitz constant θ for all functions $\operatorname{Log}\|D\gamma(x)\|$ with $\gamma \in \mathfrak{G}_*$; we call it a *Lipschitz constant of* $\mathfrak{G}_*$.

For example, a group of C^2-diffeomorphisms of a compact manifold is of course of class C^{1+L}. It will also be convenient to introduce a particular type of wandering sequences:

Definition and Remark 2.5. The wandering sequence $\mathscr{W}_{\boldsymbol{g}}$ will be called *controlled* if $\delta(W_n) \leq \varepsilon$ for any $n \in \boldsymbol{N}$, where ε is the convexity radius of M; this means in particular that any W_n is contained in a convex closed disk.

For example consider any wandering sequence $\mathscr{W}_{\boldsymbol{g}} = \{W_n\}_{n\in\boldsymbol{N}}$, then any W_q is of course wandering as well and, q being fixed, we define an infinite W_q-shortcut $\boldsymbol{h} = \{h_p\}_{p\in\boldsymbol{N}}$ by setting $h_p = g_{p+q} \circ g_q^{-1}$. The corresponding wandering sequence $\mathscr{V}_{\boldsymbol{h}} = \{h_p(V)\}_{p\in\boldsymbol{N}}$ initiated by $V = W_q$ will be called the *q-tail* of $\mathscr{W}_{\boldsymbol{g}}$ and if $\mathscr{W}_{\boldsymbol{g}}$ is decreasing, there exists an integer q such that its q-tail $\mathscr{V}_{\boldsymbol{h}}$ is decreasing and controlled.

The next lemma is crucial.

Lemma 2.6. *Let $\mathfrak{G}$ be a K-quasi-conformal group of diffeomorphisms of a closed manifold M admitting an exceptional minimal set $\mathscr{E}$ and let $\mathscr{W}_{\boldsymbol{g}}$ be a wandering sequence for $\mathfrak{G}$, initiated by a wandering domain W. Then if $\mathfrak{G}$ is of class C^{1+L} and $\mathscr{W}_{\boldsymbol{g}}$ is strongly decreasing, we obtain the following properties*:

i) *for any two points $x, y \in W$ and any $n \in \boldsymbol{N}$, we get*

$$\|Dg_n(x)\| \leq e^{\beta} \cdot \|Dg_n(y)\|,$$

where θ is a Lipschitz constant for the generating set $\mathfrak{G}_*$, $\Delta = \sum_{n=0}^{+\infty} \delta(W_n)$ and $\beta = \theta \cdot \Delta$,

ii) *if $\mathscr{W}_{\boldsymbol{g}}$ is controlled, then for any $x \in W$ and any $n \in \boldsymbol{N}$,*

$$\|Dg_n(x)\| \leq \frac{K}{\rho(W)} \cdot e^{\beta} \cdot \delta(W_n),$$

iii) *in general, the sequence* $\{\|Dg_n(x)\|\}_{n\in\boldsymbol{N}}$ *converges uniformly to* 0 *on* W *and there exists a positive constant* R *(depending on* $\mathscr{W}_{\boldsymbol{g}}$*) such that*

$$\sum_{n=0}^{+\infty} \|Dg_n(x)\| \le R$$

for any $x \in W$.

PROOF. (1) With the notations of 1.2, we write $g_n = \gamma_n \circ \gamma_{n-1} \cdots \circ \gamma_1$ and by elementary linear algebra, we get the formulas

$$\|Dg_n(x)\| \le \prod_{j=0}^{n-1} \big\|D\gamma_{j+1}[g_j(x)]\big\|$$

and

$$\operatorname{Log} \frac{\|Dg_n(x)\|}{\|Dg_n(y)\|} = \sum_{j=0}^{n-1} \big(\operatorname{Log} \|D\gamma_{j+1}[g_j(x)]\| - \operatorname{Log} \|D\gamma_{j+1}[g_j(y)]\|\big).$$

Next using a Lipschitz constant θ of $\mathfrak{G}_*$, we obtain the following inequalities which imply immediately claim i):

$$\operatorname{Log} \frac{\|Dg_n(x)\|}{\|Dg_n(y)\|} \le \theta \cdot \left(\sum_{j=0}^{n-1} d[g_j(x), g_j(y)]\right) \le \theta \cdot \left(\sum_{j=0}^{n-1} \delta(W_j)\right) \le \theta \cdot \Delta = \beta.$$

(2) Now suppose that the wandering sequence $\mathscr{W}_{\boldsymbol{g}}$ is controlled. Then $\delta[g_n(W)] \le \varepsilon$ for any n and our first mean value inequality: claim i) in 2.2, implies that there exist points $z_n \in W$ such that

$$\|Dg_n(z_n)\| \le \frac{K[g_n]}{\rho(W)} \cdot \delta(W_n) \le \frac{K}{\rho(W)} \cdot \delta(W_n).$$

Claim ii) follows from i) by setting $y = z_n$.

Under the same hypothesis, claim ii) implies claim iii) with the constant $R = (K/\rho(W)) \cdot e^{\beta} \cdot \Delta$.

(3) We return to the general case; as the sequence $\mathscr{W}_{\boldsymbol{g}}$ is decreasing, there exists an integer q such that the q-tail $\mathscr{V}_{\boldsymbol{h}}$ of $\mathscr{W}_{\boldsymbol{g}}$ is controlled thus verifies iii) for some positive constant R_1. Now elementary considerations show that we can find

a constant $R_0 > 0$ such that

$$\sum_{j=0}^{q} \|Dg_j(x)\| \leq R_0$$

for any $x \in W$. The general version of iii) follows immediately with the constant $R = R_0 + R_1$. $\square$

2.3. Extending the convergence.

In a last step we extend the convergence of the sequence $\{\|Dg_n(x)\|\}_{n\in\boldsymbol{N}}$ uniformly over some extension of W. With the notations introduced above, we define two new constants:

$$\hat{\nu} := \frac{1}{2\cdot\theta\cdot e\cdot R} \quad\text{and}\quad \nu := \inf\left(\hat{\nu}, \frac{\varepsilon}{2}\right)$$

and denote by Θ_u the closed convex geodesic disk of radius ν centered at $u \in \partial W$.

Lemma 2.7. *Still assume that $\mathscr{W}_{\boldsymbol{g}}$ is a strongly decreasing wandering sequence for a K-quasi-conformal group $\mathfrak{G}$ of diffeomorphisms of class C^{1+L}. Then for any $u \in \partial W$, any $x \in \Theta_u$ and any $n \in \boldsymbol{N}$, we get*

$$\|Dg_n(x)\| \leq e \cdot \|Dg_n(u)\|.$$

In particular, the sequence $\{\|Dg_n(x)\|\}_{n\in\boldsymbol{N}}$ converges uniformly to 0 on Θ_u and the sequence $\{\delta[g_n(\Theta_u)]\}_{n\in\boldsymbol{N}}$ tends to 0.

Proof. Fix u and x; by exactly the same argument as in the Proof of Lemma 2.6, we obtain

$$\operatorname{Log}\frac{\|Dg_n(x)\|}{\|Dg_n(u)\|} \leq \theta\cdot\left(\sum_{j=0}^{n-1} d[g_j(x), g_j(u)]\right).$$

Now Θ_u being a convex disk, our second mean value inequality: claim ii) of 2.2, gives points $y_j \in \Theta_u$ such that

$$\operatorname{Log}\frac{\|Dg_n(x)\|}{\|Dg_n(u)\|} \leq \theta\cdot 2\nu\cdot\sum_{j=0}^{n-1}\|Dg_j(y_j)\|.$$

Our main claim follows by induction on n, the case $n = 0$ being trivial because g_0

is the identity of M. Indeed using the induction hypothesis, we conclude that

$$\operatorname{Log}\frac{\|Dg_n(x)\|}{\|Dg_n(u)\|} \leq \theta\cdot 2\nu\cdot e\cdot\left(\sum_{j=0}^{n-1}\|Dg_j(u)\|\right) \leq \theta\cdot 2\nu\cdot e\cdot\left(\sum_{j=0}^{+\infty}\|Dg_j(u)\|\right),$$

which implies

$$\operatorname{Log}\frac{\|Dg_n(x)\|}{\|Dg_n(u)\|} \leq 2\theta\cdot\nu\cdot e\cdot R \leq 1$$

by the summability Result of Lemma 2.6 and definition of ν.

The convergence of the sequence $\{\|Dg_n(x)\|\}_{n\in\boldsymbol{N}}$ follows using Lemma 2.6 and that of the sequence $\{\delta[g_n(\Theta_u)]\}_{n\in\boldsymbol{N}}$ using our second mean value inequality (see 2.2). □

3. Sacksteder's type theorems.

We are now in position to apply the previous results and extend the celebrated theorems of Sacksteder (see [**11**]) and Denjoy (see [**3**]).

Definition 3.1. A C^1-diffeomorphism g of M is a *Sacksteder's stabilizer* if there exists $z\in M$ such that

$$g(z)=z \quad\text{and}\quad \|Dg(z)\|<1.$$

The point z will then be called a *contracting fix point.*

Theorem 3.2. *Let $\mathfrak{G}$ be a finitely generated group of C^{1+L}-diffeomorphisms of a closed compact manifold M admitting an exceptional minimal set $\mathscr{E}$. If $\mathfrak{G}$ is quasi-conformal and $\mathscr{E}$ is of strongly decreasing type, then $\mathscr{E}$ is the only minimal set of $\mathfrak{G}$ and*

i) *there exists a Sacksteder's stabilizer $g\in\mathfrak{G}$,*
ii) *the group $\mathfrak{G}$ and any of its orbits have exponential growth.*

Proof. Because $\mathscr{E}$ is of decreasing type, we know by Theorem 1.8 that $\mathscr{E}$ is the only minimal set for $\mathfrak{G}$. Next consider the closure W of some component U of $M\backslash\mathscr{E}$, then again by Theorem 1.8, we know that W initiates a wandering sequence $\mathscr{W}_{\boldsymbol{g}}=\{g_n(W)\}_{n\in\boldsymbol{N}}$ which accumulates on at least one point $u\in\partial W$. On the other hand, as $\mathscr{E}$ is of strongly decreasing type, the neighborhood Θ_u of u provided by Lemma 2.7 is such that the sequence $\{\delta[g_n(\Theta_u)]\}$ tends to 0.

Combining these two observations, we conclude that there exists an integer q such that $g_q(\Theta_u)$ is contained in Θ_u and $\|Dg_q(x)\| < 1$. Then g_q has a contracting fix point $z \in \Theta_u$ by Banach's fix-point theorem; it belongs to the closure of the orbit of u thus to the minimal set $\mathscr{E}$.

In order to prove claim ii), we recall that any orbit of $\mathfrak{G}$ with non exponential growth produces a probability measure μ which is invariant by the action of $\mathfrak{G}$ and whose support is the minimal set $\mathscr{E}$ (see [**10**] or [**4**, t.B, p. 265]). Now if g is a Sacksteder's stabilizer fixing $z \in \mathscr{E}$, there exists a neighborhood V of z such that $g(V) \subset V$ and $\bigcap_{n\in \boldsymbol{N}} g^n(V) = z$ implying $\mu(z) = \mu(V) > 0$ and $\mu(\mathscr{E}) = +\infty$ which is absurd. We conclude that all orbits have exponential growth and the same property holds for $\mathfrak{G}$ whose growth dominates the growth of its orbits. $\square$

As a cyclic group has linear growth, we obtain as corollary the following Denjoy-type theorem:

Theorem 3.3. *Let $\mathfrak{G}$ be a cyclic diffeomorphism group of class C^{1+L}. If it admits an exceptional minimal set $\mathscr{E}$, then*

i) *either $\mathfrak{G}$ is not quasi-conformal,*
ii) *or $\mathscr{E}$ is not of strongly decreasing type.*

We also recover the usual theorems of Denjoy and Sacksteder:

Corollary 3.4. *Let $\mathfrak{H}$ be a finitely generated group of orientation preserving C^{1+L}-diffeomorphisms of the circle $\boldsymbol{S}^1$.*

i) *If $\mathfrak{H}$ is cyclic, it does not admit an exceptional minimal set thus either $\mathfrak{H}$ has a finite orbit or all its orbits are everywhere dense.*
ii) *If $\mathfrak{H}$ admits an exceptional minimal set $\mathscr{E}$, there exists a point $z \in \mathscr{E}$ and an element $h \in \mathfrak{H}$ such that $h(z) = z$ and $|h'(z)| < 1$.*

Proof. We have to check the special assumptions of our Theorem 3.2. Indeed as noticed in Subsection 1.3, any group $\mathfrak{H}$ is trivially 1-quasi-conformal. On the other hand, any exceptional minimal set $\mathscr{E}$ is a Cantor set thus of strongly decreasing type, because here a connected component U of the complement of $\mathscr{E}$ is just an interval whose diameter coincides with its length and the sum of the diameters is just the measure of $\boldsymbol{S}^1 \backslash \mathscr{E}$ which is bounded by 1. $\square$

In the literature, one can find a few examples of exceptional minimal sets but only for cyclic groups of diffeomorphisms. The most relevant are the following:

Example 3.5. McSwiggen's examples.

In [**7**], McSwiggen constructs for every $\varepsilon > 0$, a $C^{3-\varepsilon}$-diffeomorphism g of the 2-torus which has no periodic points, is semi-conjugate to a translation and has a wandering domain with dense orbit; in particular it admits an exceptional minimal set $\mathscr{E}$ but no non trivial stabilizer.

In order to verify the coherence with our result 3.2, we make the following additional observations:

i) The cyclic group $\mathfrak{G}$ generated by any of McSwiggen's diffeomorphisms g is not quasi-conformal; otherwise by a result of Norton and Velling (see Proposition 1 in [**9**]), it would be conjugate to a minimal translation of $\boldsymbol{T}^2$ contradicting the fact that it preserves an exceptional minimal set.
ii) Moreover, by a result of Kwakkel and Markovic in [**5**], one concludes that the wandering domains of $\mathfrak{G}$ do not have "bounded geometry". Although this fact is not directly related to the diameters of the corresponding wandering sequence, one may suspect that the minimal set $\mathscr{E}$ is not of strongly decreasing type.

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