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Horospherical flat surfaces in Hyperbolic 3-space

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Abstract. Recently we discovered a new geometry on submanifolds in hyperbolic n-space which is called *horospherical geometry*. Unfortunately this geometry is not invariant under the hyperbolic motions (it is invariant under the canonical action of $SO(n)$), but it has quite interesting features. For example, the flatness in this geometry is a hyperbolic invariant and the total curvatures are topological invariants. In this paper, we investigate the *horospherical flat surfaces* (flat surfaces in the sense of horospherical geometry) in hyperbolic 3-space. Especially, we give a generic classification of singularities of such surfaces. As a consequence, we can say that such a class of surfaces has quite a rich geometric structure.

1. Introduction.

In this paper we investigate a special class of surfaces in hyperbolic 3-space which are called *horospherical flat surfaces*. In the previous theory of surfaces in hyperbolic space, there appeared two kinds of curvatures. One is called the *extrinsic Gauss curvature* K_e and another is the *intrinsic Gauss curvature* K_I (cf., [**1**], [**12**]). The intrinsic Gauss curvature is nothing but the Gauss curvature defined by the induced Riemannian metric on the surface. The relation between these curvatures is known that $K_e = K_I + 1$. In [**14**] we defined a curvature K_h called a *hyperbolic curvature* of the surface by using the hyperbolic Gauss indicatrix which is defined by a slightly modified definition of the hyperbolic Gauss map in [**5**], [**9**], [**24**], [**25**]. This curvature is an extrinsic hyperbolic invariant because we have the relation $K_h = 2 - 2H + K_I$, where H is the mean curvature of the surface. We remark that Kobayashi [**24**], [**25**] had already defined the notion of hyperbolic Gauss curvature under a different framework and studied some basic properties of it from the view point of the theory of Fourier transformations. We also defined another curvature $\widetilde{K}_h$ called the *horospherical Gauss curvature* in [**21**]. The

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horospherical Gauss curvature $\widetilde{K}_h$ is defined for surfaces in the model of hyperbolic space in Minkowski space and it seems that this curvature depends on the choice of the model space. Nevertheless, we can show that it is independent of the choice of the model of hyperbolic space (cf., Section 3). Unfortunately, the horospherical Gauss curvature is not a hyperbolic invariant. However it has very interesting properties. For example, it describes the contact of surfaces with horospheres as a local property. As global properties of this curvature, we showed that the Gauss-Bonnet type theorem [**21**] and the Chern-Lashof type theorem [**6**] hold. We call the geometry related to this curvature the *horospherical geometry* ([**6**], [**14**], [**16**], [**17**], [**18**], [**19**], [**20**], [**21**]). By a direct consequence of the definition, $K_h(p) = 0$ if and only if $\widetilde{K}_h(p) = 0$, so that the horospherical flatness is a hyperbolic invariant. Moreover, there is an important class of surfaces called *linear Weingarten surfaces* which satisfy the relation $aK_I + b(2H - 2) = 0$ $((a, b) \neq (0, 0))$. In [**12**], the Weierstrass-Bryant type representation formulas for such surfaces with $a + b \neq 0$ (called, a *linear Weingarten surface of Bryant type*) was shown. This class of surfaces contains flat surfaces (i.e., $a \neq 0, b = 0$) and CMC-1 (constant mean curvature one) surfaces ($a = 0, b \neq 0$). In the celebrated paper [**5**], Bryant showed the Weierstrass type representation formula for CMC-1 surfaces in hyperbolic space. This is the reason why the class of the surface with $a + b \neq 0$ is called of Bryant type. By using such representation formula, there are a lot of results on such surfaces. We only refer [**12**], [**26**], [**27**], [**32**], [**33**] here. The horospherical flat surface is one of the linear Weingarten surfaces. It is, however, the exceptional case (a *linear Weingarten surface of non-Bryant type*: $a + b = 0$). There are no Weierstrass-Bryant type representation formula for such surfaces so far as we know. Therefore the horospherical flat surfaces are also very important subjects in the hyperbolic geometry.

On the other hand, a *horocyclic surface* is defined to be a one-parameter family of horocycles (cf., Section 4). We call each horocycle a *generating horocycle.* We can show that a horospherical flat surface is (at least locally) parametrized as a horocyclic surface (cf., Theorem 4.4). Therefore, the main subject in this paper is the horospherical flat horocyclic surfaces. In Euclidean space, surfaces with the vanishing Gauss curvature are developable surfaces which belong to a special class of ruled surfaces [**15**]. Therefore, horocyclic surfaces are one of the analogous notions with ruled surfaces in hyperbolic space. In this paper, we study geometric properties and singularities of horospherical flat horocyclic surfaces. Comparing them with ruled surfaces, the situation is quite different. For example, the singularities of ruled surface are at most one point on each ruling in generic. However, the singularities of horocyclic surfaces are at most two points on each generating horocycle in generic. Sometimes they meet or one of them tends to infinity (approaching to the end).

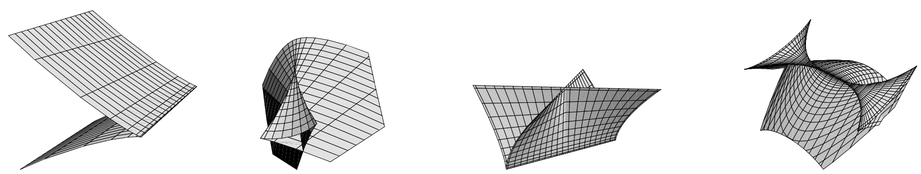
For any smooth curve $A : I \longrightarrow SO_0(3,1)$ in the Lorentzian group, we can define a parametrization F_A of horocyclic surface $M = \mathrm{Image}\, F_A$ in hyperbolic space (it is written by $F_{(\gamma,a_1,a_2)}$ in Section 5). We can easily show that $C = A'A^{-1}$ is a smooth curve in the Lie algebra $\mathfrak{so}(3,1)$ of $SO_0(3,1)$. We can also obtain the curve A in $SO_0(3,1)$ with initial data $A(t_0) = A_0$ from C by the existence theorem of the linear ordinary differential equations. In this sense, $C(t)$ is a hyperbolic invariant of horocyclic surfaces. We remark that $C(t)$ is a matrix of the following form:

$$C(t) = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t) \\ c_1(t) & 0 & c_4(t) & c_5(t) \\ c_2(t) & -c_4(t) & 0 & c_6(t) \\ c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix}.$$

In Section 5 we show that a horocyclic surface $\mathrm{Image} F_A$ is horospherical flat if and only if $c_2(t) = c_1(t) - c_4(t) = 0$. We have a local classification theorem of horospherical flat horocyclic surfaces (cf., Theorem 5.5) which is analogous to the classical classification theorem on developable surfaces in Euclidean space (cf., [**7**], [**15**], [**34**]). However, the situation is quite different from the classification of developable surfaces in $\boldsymbol{R}^3$. It has been known as the Hartman-Nirenberg theorem [**13**] that complete non-singular developable surfaces are cylindrical surfaces. Shiohama and Takagi [**31**] showed that a complete orientable surface with a constant principal curvature in Euclidean space is either totally umbilic or else umbilically free. Moreover, they showed that such surfaces are only sphere or tube of a space curve if the principal curvature is positive. Such a surface is one of the examples of circular surfaces [**22**]. However, there are several examples of complete non-singular horospherical flat horocyclic surfaces. As one of the consequences of the classification, we give an example of the surface with a constant principal curvature which is not umbilically free (Example 5.6). This gives a concrete example of the surface in ([**1**, Example 2.1]) which gives a counter example of the hyperbolic version of the theorem of Shiohama and Takagi [**31**], [**38**]. We can show that a horospherical flat surface with curve singularities is parametrized by F_A which satisfies the equations $c_2(t) = c_1(t) - c_4(t) = c_3(t) = 0$. Therefore we may regard that the space of (parametrizations of) horospherical flat horocyclic surfaces with curve singularities is $C^\infty(I, \mathfrak{hf}_\sigma(3,1))$, where

$$\mathfrak{hf}_\sigma(3,1) = \left\{ C = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ c_1 & 0 & c_4 & c_5 \\ c_2 & -c_4 & 0 & c_6 \\ c_3 & -c_5 & -c_6 & 0 \end{pmatrix} \in \mathfrak{so}(3,1) \;\middle|\; c_2 = c_1 - c_4 = c_3 = 0 \right\}.$$

One of the main results in this paper is a generic classification of singularities of horospherical flat horocyclic surfaces with curve singularities. We say that a singular point (s,t) of F_A is the *cuspidal edge* (respectively *swallowtail*, *cuspidal cross cap* and *cuspidal beaks*) if the germ of the surface $F_A(\boldsymbol{R}\times I)$ at $F_A(s,t)$ is (locally) diffeomorphic to $CE=\{(x_1,x_2,x_3)|{x_1}^2={x_2}^3\}$ (respectively, $SW=\{(x_1,x_2,x_3)|x_1=3u^4+u^2v, x_2=4u^3+2uv, x_3=v\}$, $CCR=\{(x_1,x_2,x_3)\in\boldsymbol{R}^3 \mid x_1=u, x_2=uv^3, x_3=v^2\}$ and $CBK=\{(x_1,x_2,x_3)|x_1=v, x_2=-2u^3+v^2u, x_3=3u^4-v^2u^2\}$).



The cuspidal edge. The swallowtail. The cuspidal cross cap. The cuspidal beaks.

Figure 1.

Our classification theorem is summarized as follows (cf., Theorem 6.2):

Theorem 1.1. *There exists an open and dense subset $\mathscr{O}\subset C^\infty(I,\mathfrak{hf}_\sigma(3,1))$ such that the following properties hold*: *For any $C\in\mathscr{O}$, the germ of the corresponding horospherical flat tangent horospherical surface $F_A(\boldsymbol{R}\times I)$ at a singular point is diffeomorphic to the cuspidal edge, the swallowtail, the cuspidal cross cap or the cuspidal beaks. Moreover, on each generating horocycle, we have the following cases*:

(1) *There are two singular points, both of which are the cuspidal edges.*
(2) *There are two singular points, one of which is the cuspidal edge another is the swallowtail.*
(3) *There is only one singular point which is the cuspidal cross cap.*
(4) *There is only one singular point which is the cuspidal beaks.*

We remark that generic singularities of developable surfaces are the cuspidal edge, the swallowtail or the cuspidal cross cap (cf., [**15**]). In the Beltrami-Klein ball model of hyperbolic space, a plane is a Euclidian plane and a geodesic is a Euclidean line. Therefore we can show that a surface with $K_e\equiv 0$ (we call it an *extrinsic flat surface*) is diffeomorphic to a developable surface in the Euclidean sense in the Beltrami-Klein ball model, so that generic singularities of extrinsic flat surfaces are the same as those of developable surfaces. On the other hand, the cuspidal beaks appear as one of the generic bifurcations of Legendrian singularities (i.e., wave fronts) [**37**]. However, the cuspidal beaks of horospherical flat tangent horocyclic surfaces do not bifurcate under the small perturbation of surfaces in the space of horospherical flat tangent horocyclic surfaces. The cuspidal cross cap and the cuspidal beaks are non-generic singularities of general wave fronts. It has been known [**26**] that generic singularities of flat fronts ($K_I \equiv 0$) are the cuspidal edge or the swallowtail. Therefore horospherical flat surfaces have complicated and interesting singularities compared with other two flat surfaces (i.e., $K_e \equiv 0$, $K_I \equiv 0$). We give the exact recognition conditions for the above singularities of horospherical flat horocyclic surfaces in terms of the invariant $C(t)$ in Theorem 6.2. We can easily show that such the recognition conditions are generic as an application of the ordinary jet-transversality theorem of Thom. Moreover, we have a nice duality relation between horospherical flat tangent horocyclic surfaces and a special class of surfaces in the lightcone (these are called *hyperbolic flat tangent lightcone circular surfaces*). The critical curve of the dual surface in the lightcone draws the shape of the end of the horospherical flat tangent horocyclic surface. We give a generic classification of hyperbolic flat tangent lightcone circular surfaces in Section 8 (cf., Theorem 8.2). Actually the classification list is the same as that in Theorem 6.2. In general the end of horospherical flat horocyclic surface is a point or a curve in ideal boundary if we adopt the Poincaré ball as a model space. In Section 8, we show that the germ of a horospherical flat tangent surface is cuspidal cross cap if and only if the corresponding germ of the end is the ordinary cusp (cf., Corollary 8.2). Here, the ordinary cusp is a plane curve germ diffeomorphic to $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$ (cf., Figure 2).

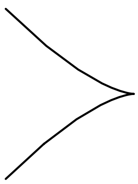


Figure 2. ordinary cusp.

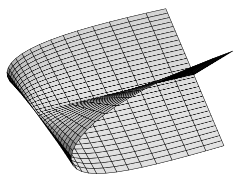


Figure 3. cross cap.

In Appendix A, we give criteria for the recognition of the cuspidal beaks or the cuspidal lips of parametrized surfaces as a byproduct of the proof for Theorem 6.2. Such criteria might be very useful for the study of singular surfaces arising in several areas. We briefly describe a generic classification of singularities for general horocyclic surfaces in Appendix B. As a consequence, any singular point for generic general horocyclic surface is locally diffeomorphic to the cross cap which is the image of $(x_1, x_2) \mapsto (x_1^2, x_2, x_1 x_2)$ (cf., Figure 3). This result indicates that the singularities of horospherical flat horocyclic surfaces are quite different from those of general horocyclic surfaces.

All maps considered here are of class C^∞ unless otherwise stated.

2. Differential geometry in hyperbolic space.

We outline in this section the differential geometry of curves and surfaces in hyperbolic 3-space which are developed in the previous papers [**14**], [**16**]. We adopt the Lorentzian model of the hyperbolic 3-space. Let $\boldsymbol{R}^4 = \{(x_0, x_1, x_2, x_3) \mid x_i \in \boldsymbol{R}\ (i = 0, 1, 2, 3)\}$ be a 4-dimensional vector space. For any $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$, $\boldsymbol{y} = (y_0, y_1, y_2, y_3) \in \boldsymbol{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = -x_0y_0 + \sum_{i=1}^3 x_iy_i$. We call $(\boldsymbol{R}^4, \langle,\rangle)$ *Minkowski space*. We write $\boldsymbol{R}^4_1$ instead of $(\boldsymbol{R}^4, \langle,\rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \boldsymbol{R}^4_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x}\rangle < 0$ respectively. For a vector $\boldsymbol{v} \in \boldsymbol{R}^4_1$ and a real number c, we define *the hyperplane with pseudo normal* $\boldsymbol{v}$ by $HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v}\rangle = c\}$. We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

We now define *hyperbolic* 3*-space* by $H^3_+(-1) = \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1, x_0 \geq 1\}$ and *de Sitter* 3*-space* by $S^3_1 = \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\}$.

For any $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \boldsymbol{R}^4_1$, we define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x^1_0 & x^1_1 & x^1_2 & x^1_3 \\ x^2_0 & x^2_1 & x^2_2 & x^2_3 \\ x^3_0 & x^3_1 & x^3_2 & x^3_3 \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is the canonical basis of $\boldsymbol{R}^4_1$ and $\boldsymbol{x}_i = (x^i_0, x^i_1, x^i_2, x^i_3)$. We can easily show that $\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3\rangle = \det(\boldsymbol{x}\ \boldsymbol{x}_1\ \boldsymbol{x}_2\ \boldsymbol{x}_3)$, so that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is pseudo orthogonal to any $\boldsymbol{x}_i$ $(i = 1, 2, 3)$.

We also define a set $LC^*_+ = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \boldsymbol{R}^4_1 \mid x_0 > 0, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0\}$, which is called the *future lightcone* at the origin. We have three kinds of surfaces in $H^3_+(-1)$ which are given by intersections of $H^3_+(-1)$ and hyperplanes in $\boldsymbol{R}^4_1$. A surface $H^3_+(-1) \cap HP(\boldsymbol{v}, c)$ is called a *sphere*, an *equidistant surface* or a *horosphere* if $H(\boldsymbol{v}, c)$ is spacelike, timelike or lightlike respectively. Especially we write a horosphere as $HS^2(\boldsymbol{v}, c) = H^3_+(-1) \cap HP(\boldsymbol{v}, c)$. If we consider a lightlike vector $\boldsymbol{v}_0 = -\boldsymbol{v}/c$, we have $HS^2(\boldsymbol{v}, c) = HS^2(\boldsymbol{v}_0, -1)$. We call $\boldsymbol{v}_0$ the *polar vector* of $HS^2(\boldsymbol{v}_0, -1)$.

We now construct the extrinsic differential geometry on curves in $H^3_+(-1)$ (cf., [**16**]). Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a regular curve. Since $H^3_+(-1)$ is a Riemannian manifold, we can reparametrize $\boldsymbol{\gamma}$ by the arc-length. Hence, we may assume that $\boldsymbol{\gamma}(s)$ is a unit speed curve. So we have the tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ with $\|\boldsymbol{t}(s)\| = 1$. In the case when $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq -1$, we have a unit vector $\boldsymbol{n}(s) = (\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s))/(\|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|)$. Moreover, define $\boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$. Then we have a pseudo orthonormal frame $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ of $\boldsymbol{R}^4_1$ along $\boldsymbol{\gamma}$. By standard arguments, under the assumption that $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq -1$, we have the following *Frenet-Serre type formulae*:

$$\begin{cases} \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) = \kappa_h(s)\boldsymbol{n}(s) + \boldsymbol{\gamma}(s) \\ \boldsymbol{n}'(s) = -\kappa_h(s)\boldsymbol{t}(s) + \tau_h(s)\boldsymbol{e}(s) \\ \boldsymbol{e}'(s) = -\tau_h(s)\boldsymbol{n}(s) \end{cases}, \tag{1}$$

where $\kappa_h(s) = \|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|$ and $\tau_h(s) = -\frac{\det(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s))}{(\kappa_h(s))^2}$.

We can easily show that the condition $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq -1$ is equivalent to the condition $\kappa_h(s) \neq 0$. We can show that the curve $\boldsymbol{\gamma}(s)$ satisfies the condition $\kappa_h(s) \equiv 0$ if and only if there exists a lightlike vector $\boldsymbol{c}$ such that $\boldsymbol{\gamma}(s) - \boldsymbol{c}$ is a geodesic. Such a curve is called an *equidistant curve*. Moreover $\boldsymbol{\gamma}$ is called a *horocycle* if $\kappa_h(s) \equiv 1$ and $\tau_h(s) \equiv 0$. We can study many properties of hyperbolic space curves by using this fundamental equation.

On the other hand, we give a brief review on the extrinsic differential geometry on surfaces in $H^3_+(-1)$ due to our previous paper [**14**]. Let $\boldsymbol{x} : U \longrightarrow H^3_+(-1)$ be a regular surface (i.e., an embedding), where $U \subset \boldsymbol{R}^2$ is an open subset. We denote that $M = \boldsymbol{x}(U)$ and identify M with U through the embedding $\boldsymbol{x}$. Define a vector

$$\boldsymbol{e}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u)\|}.$$

Then we have $\langle \boldsymbol{e}, \boldsymbol{x}_{u_i}\rangle \equiv \langle \boldsymbol{e}, \boldsymbol{x}\rangle \equiv 0, \langle \boldsymbol{e}, \boldsymbol{e}\rangle \equiv 1$, where $\boldsymbol{x}_{u_i} = \partial \boldsymbol{x}/\partial u_i$. Therefore we have a mapping

$$\boldsymbol{E} : U \longrightarrow S^3_1$$

by $\boldsymbol{E}(u) = \boldsymbol{e}(u)$ which is called the *de Sitter Gauss image* of $\boldsymbol{x}$. Since $\boldsymbol{x}(u) \in H^3_+(-1)$, $\boldsymbol{e}(u) \in S^3_1$ and $\langle \boldsymbol{x}(u), \boldsymbol{e}(u)\rangle = 0$, we can show that $\boldsymbol{x}(u) \pm \boldsymbol{e}(u) \in LC^*_+$. We define a map

$$\boldsymbol{L}^{\pm} : U \longrightarrow LC^*_+$$

by $\boldsymbol{L}^{\pm}(u) = \boldsymbol{x}(u) \pm \boldsymbol{e}(u)$ which is called the *lightcone Gauss image* of $\boldsymbol{x}$. We called $\boldsymbol{L}^{\pm}$ the hyperbolic Gauss indicatrix of $\boldsymbol{x}$ in [**14**]. We change the name of the map $\boldsymbol{L}^{\pm}$ as the above to avoid the confusion. We have shown that $D_v\boldsymbol{L}^{\pm} \in T_pM$ for any $p = \boldsymbol{x}(u_0) \in M$ and $\boldsymbol{v} \in T_pM$, where D_v denotes the *covariant derivative* with respect to the tangent vector $\boldsymbol{v}$. It is easy to show that the surface $\boldsymbol{x}(U) = M$ is a part of a horosphere if and only if one of the lightcone Gauss images $\boldsymbol{L}^{\pm}$ is constant.

Under the identification of U and M, the derivative $d\boldsymbol{x}(u_0)$ can be identified with the identity mapping 1_{T_pM} on the tangent space T_pM, where $p = \boldsymbol{x}(u_0)$. This means that

$$d\boldsymbol{L}^{\pm}(u_0) = 1_{T_pM} \pm d\boldsymbol{E}(u_0).$$

We call the linear transformation $S_p^{\pm} = -d\boldsymbol{L}^{\pm}(u_0) : T_pM \longrightarrow T_pM$ the *hyperbolic shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. We also call $A_p = -d\boldsymbol{E}(u_0) : T_pM \longrightarrow T_pM$ the *de Sitter shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. We denote the eigenvalues of $S_p^{\pm}$ by $\bar{\kappa}_i^{\pm}(p)$ $(i = 1, 2)$ and the eigenvalues of A_p by $\kappa_i(p)$. By the relation $S_p^{\pm} = -1_{T_pM} \pm A_p$, $S_p^{\pm}$ and A_p have same eigenvectors and relations $\bar{\kappa}_i^{\pm}(p) = -1 \pm \kappa_i(p)$. We call $\bar{\kappa}_i^{\pm}(p)$ *hyperbolic principal curvatures* and $\kappa_i(p)$ *de Sitter principal curvatures* (or, simply call *principal curvatures*) of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. We now describe the geometric meaning of the hyperbolic principal curvatures. Let $\boldsymbol{\gamma}(s) = \boldsymbol{x}(u_1(s), u_2(s))$ be a unit speed curve on $M = \boldsymbol{x}(U)$ with $p = \boldsymbol{\gamma}(s_0)$. We consider the *hyperbolic curvature vector* $\boldsymbol{k}(s) = \boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)$ and the *de Sitter normal curvature*

$$\kappa_n^{\pm}(s_0) = \big\langle \boldsymbol{k}(s_0), \boldsymbol{L}^{\pm}(u_1(s_0), u_2(s_0))\big\rangle = \big\langle \boldsymbol{t}'(s_0), \boldsymbol{L}^{\pm}(u_1(s_0), u_2(s_0))\big\rangle + 1$$

of $\boldsymbol{\gamma}(s)$ at $p = \boldsymbol{\gamma}(s_0)$. We can show that the de Sitter normal curvature depends only on the point p and the unit tangent vector of M at p analogous to the Euclidean case. Therefore we have the maximum and the minimum of the de Sitter normal curvature at $p \in M$. We can also show that the de Sitter principal curvatures $\pm\kappa_i(p)$ are equal to the maximum or the minimum of the de Sitter normal curvature at p. Then we have the following hyperbolic Rodoriges type formula: If $\boldsymbol{\gamma}(s) = \boldsymbol{x}(u_1(s), u_2(s))$ is a line of curvature, then $\kappa_n^{\pm}(s)$ is one of the de Sitter principal curvatures at $\boldsymbol{\gamma}(s)$, so that we have

$$-\frac{d\boldsymbol{L}^{\pm}}{ds}(u_1(s), u_2(s)) = \big(\kappa_n^{\pm}(s) - 1\big)\frac{d\boldsymbol{x}}{ds}(u_1(s), u_2(s)).$$

According to the above observations, we define $\bar{\kappa}_n^{\pm}(s) = \kappa_n^{\pm}(s) - 1$ and call it the *hyperbolic normal curvature* of $\boldsymbol{\gamma}(s)$.

The *hyperbolic Gauss curvature* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is defined to be

$$K_h^{\pm}(u_0) = \det S_p^{\pm} = \bar{\kappa}_1^{\pm}(p)\bar{\kappa}_2^{\pm}(p).$$

The *hyperbolic mean curvature* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is defined to be

$$H_h^{\pm}(u_0) = \frac{1}{2}\operatorname{Trace} S_p^{\pm} = \frac{\bar{\kappa}_1^{\pm}(p) + \bar{\kappa}_2^{\pm}(p)}{2}.$$

The *extrinsic* (*de Sitter*) *Gauss curvature* is defined to be

$$K_e(u_0) = \det A_p = \kappa_1(p)\kappa_2(p)$$

and the *de Sitter mean curvature* is

$$H_d(u_0) = \frac{1}{2}\operatorname{Trace} A_p = \frac{\kappa_1(p) + \kappa_2(p)}{2}.$$

We remark that the de Sitter mean curvature is actually the *mean curvature* of M. Therefore we denote it H instead of H_d. We clearly have that $H_h^{\pm}(u) = \pm H(u) - 1$.

We say that a point $u \in U$ or $p = \boldsymbol{x}(u)$ is an *umbilical point* if $\kappa_1(p) = \kappa_2(p)$. Since the eigenvectors of $S_p^{\pm}$ and A_p are the same, the above condition is equivalent to the condition $\bar{\kappa}_1^{\pm}(p) = \bar{\kappa}_2^{\pm}(p)$. We say that $M = \boldsymbol{x}(U)$ is *totally umbilical* if all points on M are umbilical. In [**8**], Cecil and Ryan have characterized totally umbilical submanifolds by using three different functions on hyperbolic space. The following classification theorem of totally umbilical surfaces is well-known (cf., [**17**]):

Proposition 2.1. *Suppose that $M = \boldsymbol{x}(U)$ is totally umbilical. Then $\kappa(p)$ is a constant κ. Under this condition, we have the following classification*:

1) *Suppose that $\kappa^2 \neq 1$.*
   a) *If $\kappa \neq 0$ and $\kappa^2 < 1$, then M is a part of an equidistant surface.*
   b) *If $\kappa \neq 0$ and $\kappa^2 > 1$, then M is a part of a sphere.*
   c) *If $\kappa = 0$, then M is a part of a plane.*
2) *If $\kappa^2 = 1$, then M is a part of a horosphere.*

By definition, $\kappa^2 = 1$ if and only if one of $\bar{\kappa}^{\pm}$ is 0. Therefore, a horosphere is a totally umbilical surface of $\bar{\kappa}^{\pm}$ is 0.

We establish next the hyperbolic (respectively, de Sitter) version of the Weingarten formula. Since $\boldsymbol{x}_{u_i}$ $(i=1,2)$ are spacelike vectors, we have the Riemannian metric (*hyperbolic first fundamental form*) given by $ds^2 = \sum_{i=1}^2 g_{ij}du_idu_j$ on $M=\boldsymbol{x}(U)$, where $g_{ij}(u) = \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}_{u_j}(u)\rangle$ and the *hyperbolic* (respectively, *de Sitter*) *second fundamental invariant* defined by $\bar{h}^{\pm}_{ij}(u) = \langle -\boldsymbol{L}^{\pm}_{u_i}(u), \boldsymbol{x}_{u_j}(u)\rangle$ (respectively, $h_{ij}(u) = -\langle \boldsymbol{e}_{u_i}(u), \boldsymbol{x}_{u_j}(u)\rangle$) for any $u\in U$. They satisfy the relation $\bar{h}^{\pm}_{ij}(u) = -g_{ij}(u)\pm h_{ij}(u)$. In [**14**], [**21**] the following proposition was shown.

PROPOSITION 2.2. *Under the above notations, we have the following formulae*:

(1) $\boldsymbol{L}^{\pm}_{u_i} = -\sum_{j=1}^{2}\left(\bar{h}^{\pm}\right)_i^j \boldsymbol{x}_{u_j}$ (*The hyperbolic Weingarten formula*),

(2) $\boldsymbol{E}_{u_i} = -\sum_{j=1}^{2}\left(h_i^j\right) \boldsymbol{x}_{u_j}$ (*The de Sitter Weingarten formula*),

where $\left((\bar{h}^{\pm})_i^j\right) = (\bar{h}^{\pm}_{ik})(g^{kj})$, $(h_i^j) = (h_{ik})(g^{kj})$ *and* $(g^{kj}) = (g_{kj})^{-1}$.

As a corollary of the above proposition, we have an explicit expression of the hyperbolic (respectively, de Sitter) Gauss curvature in terms of the Riemannian metric and the hyperbolic (respectively, de Sitter) second fundamental invariant.

COROLLARY 2.3. *Under the same notations as in the above proposition, we have the following formulae*:

$$K^{\pm}_h = \frac{\det\left(\bar{h}^{\pm}_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}, \quad K_e = \frac{\det\left(h_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}.$$

We now consider the Riemannian curvature tensor

$$R^{\ell}_{ijk} = \frac{\partial}{\partial u_k}\left\{ {\ell \atop i\ j} \right\} - \frac{\partial}{\partial u_j}\left\{ {\ell \atop i\ k} \right\} + \sum_m \left\{ {m \atop i\ j} \right\}\left\{ {\ell \atop m\ k} \right\} - \sum_m \left\{ {m \atop i\ k} \right\}\left\{ {\ell \atop m\ j} \right\}.$$

We also consider the tensor $R_{ijk\ell} = \sum_m g_{im}R^m_{jk\ell}$. Standard calculations, analogous to those used in the study of the classical differential geometry on surfaces in Euclidean space, lead to the following:

PROPOSITION 2.4. *Under the above notations, we have*

$$K_e = -\frac{R_{1212}}{g} + 1,$$

where $g = g_{11}g_{22} - g_{12}g_{21}$.

We remark that $-R_{1212}/g$ is the *intrinsic Gaussian curvature* of the surface. It is denoted by K_I. Since $\bar{\kappa}_i^{\pm} = -1 \pm \kappa_i$, we deduce the above formula as follows:

PROPOSITION 2.5. *The following relation holds*:

$$K_h^{\pm} = 1 \mp 2H + K_e = 2 \mp 2H + K_I.$$

3. The horospherical geometry in hyperbolic space.

In the previous section we reviewed the properties of lightcone Gauss images and hyperbolic Gauss curvatures. We now consider the notion of hyperbolic Gauss maps introduced by Bryant [**5**] and Epstein [**9**] as follows: If $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$ is a non-zero lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\tilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) \in S_+^2 = \left\{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in LC_+^* \mid x_0 = 1\right\}.$$

We call S_+^2 *the lightcone sphere*. We define a map

$$\widetilde{\boldsymbol{L}}^{\pm} : U \longrightarrow S_+^2$$

by $\widetilde{\boldsymbol{L}}^{\pm}(u) = \widetilde{\boldsymbol{L}^{\pm}(u)}$ and call it *the hyperbolic Gauss map* of $\boldsymbol{x}$. Let T_pM be the tangent space of M at p and N_pM be the pseudo-normal space of T_pM in $T_p\boldsymbol{R}_1^4$. We have the decomposition $T_p\boldsymbol{R}_1^4 = T_pM \oplus N_pM$, so that we also have the Whitney sum $T\boldsymbol{R}_1^4 = TM \oplus NM$. Therefore we have the canonical projection $\Pi : T\boldsymbol{R}_1^4 \longrightarrow TM$. It follows that we have a linear transformation $\Pi_p \circ d\widetilde{\boldsymbol{L}}^{\pm}(u) : T_pM \longrightarrow T_pM$ for $p = \boldsymbol{x}(u)$ by the identification of U and $\boldsymbol{x}(U) = M$ via $\boldsymbol{x}$. We have the following Proposition [**21**]:

PROPOSITION 3.1. *Under the above notation we have the following horospherical Weingarten formula*:

$$\Pi_p \circ \widetilde{\boldsymbol{L}}_{u_i}^{\pm} = -\sum_{j=1}^{2} \frac{1}{\ell_0^{\pm}(u)} \left(\bar{h}^{\pm}\right)_i^j \boldsymbol{x}_{u_j},$$

where $\boldsymbol{L}^{\pm}(u) = (\ell_0^{\pm}(u), \ell_1^{\pm}(u), \ell_2^{\pm}(u), \ell_3^{\pm}(u))$.

We call the linear transformation $\widetilde{S}_p^{\pm} = -\Pi_p \circ d\widetilde{\boldsymbol{L}}^{\pm}$ the *horospherical shape* *operator* of $M = \boldsymbol{x}(U)$. We also define the *horospherical principal curvatures* $\widetilde{\kappa}_i^{\pm}(p)$ $(i = 1, 2)$ as eigenvalues of $\widetilde{S}_p^{\pm}$. By the above proposition, we have $\widetilde{\kappa}_i^{\pm}(p) = (1/\ell_0^{\pm}(p))\bar{\kappa}_i^{\pm}(p)$. The *horospherical Gauss curvature* of $\boldsymbol{x}(U) = M$ is defined to be

$$\widetilde{K}_h^{\pm}(u) = \det \widetilde{S}_p^{\pm} = \widetilde{\kappa}_1^{\pm}(p)\widetilde{\kappa}_2^{\pm}(p).$$

It follows that we have the following relation between the horospherical Gauss curvature and the hyperbolic Gauss curvature:

$$\widetilde{K}_h^{\pm}(u) = \left(\frac{1}{\ell_0^{\pm}(u)}\right)^2 K_h^{\pm}(u).$$

We say that a point $u \in U$ or $p = \boldsymbol{x}(u)$ is a *horo-umbilical point* if $\widetilde{S}_p^{\pm} = \widetilde{\kappa}^{\pm}(p)1_{T_pM}$. By the above proposition, p is a horo-umbilical point if and only if it is an umbilical point. We say that $M = \boldsymbol{x}(U)$ is *totally horo-umbilical* if all points on M are horo-umbilical as usual.

We remark that $\widetilde{\kappa}^{\pm}(p)$ is not invariant under hyperbolic motions but it is an $SO(3)$-invariant. However, we can make sense a point with vanishing horospherical principal curvature as a notion of the hyperbolic differential geometry [**21**].

Proposition 3.2. *For a point $p = \boldsymbol{x}(u)$, $\widetilde{\kappa}_i^{\pm}(p)$ is invariant under hyperbolic motions if and only if $\widetilde{\kappa}_i^{\pm}(p) = 0$.*

Corollary 3.3. *If $M = \boldsymbol{x}(U)$ is totally horo-umbilical and $\widetilde{\kappa}^{\pm}(p) = (1/\ell_0^{\pm}(p))\bar{\kappa}^{\pm}$ is a hyperbolic invariant, then M is a part of a horosphere (i.e., $\widetilde{\kappa}^{\pm} \equiv 0$).*

We now show that the notion of horospherical curvatures is independent of the choice of the model of hyperbolic space. For the purpose, we introduce a smooth function on the unit tangent sphere bundle of hyperbolic space which plays the principal role of the horospherical geometry. Let $SO_0(3, 1)$ be the identity component of the matrix group

$$SO(3, 1) = \left\{g \in GL(4, \boldsymbol{R}) \mid gI_{3,1}{}^t g = I_{3,1}\right\},$$

where

$$I_{3,1} = \left(\begin{array}{c|c} -1 & \mathbf{0} \\ \hline {}^t\mathbf{0} & I_3 \end{array}\right) \in GL(4, \boldsymbol{R}).$$

It is well-known that $SO_0(3,1)$ transitively acts on $H^3_+(-1)$ and the isotropic group at $p=(1,0,0,0)$ is $SO(3)$ which is naturally embedded in $SO_0(3,1)$. Moreover the action induces isometries on $H^3_+(-1)$.

On the other hand, we consider a submanifold $\Delta=\{(\boldsymbol{v},\boldsymbol{w})\mid\langle\boldsymbol{v},\boldsymbol{w}\rangle=0\}$ of $H^3_+(-1)\times S^3_1$ and the canonical projection $\bar{\pi}:\Delta\longrightarrow H^3_+(-1)$. Let $\pi:S(TH^3_+(-1))\longrightarrow H^3_+(-1)$ be the unit tangent sphere bundle over $H^3_+(-1)$. For any $\boldsymbol{v}\in H^3_+(-1)$, we have the local (global) coordinates (v_1,v_2,v_3) of $H^3_+(-1)$ such that $\boldsymbol{v}=(\sqrt{v_1^2+v_2^2+v_3^2+1},v_1,v_2,v_3)$. We can represent the tangent vector $\boldsymbol{w}=\sum_{i=1}^3 w_i\partial/\partial v_i\in T_vH^3_+(-1)$ by

$$\boldsymbol{w}=\left(\frac{1}{v_0}\sum_{i=1}^3 w_iv_i,w_1,w_2,w_3\right)$$

as a vector in Minkowski 4-space. Then $\langle\boldsymbol{w},\boldsymbol{v}\rangle=(-(1/v_0)\sum_{i=1}^3 w_iv_i)v_0+\sum_{i=1}^3 w_iv_i=0$. Therefore $\boldsymbol{w}\in S(T_vH^3_+(-1))$ if and only if $\langle\boldsymbol{w},\boldsymbol{w}\rangle=1$ and $\langle\boldsymbol{v},\boldsymbol{w}\rangle=0$. These conditions are equivalent to the condition $(\boldsymbol{v},\boldsymbol{w})\in\Delta$. This means that we can canonically identify $\pi:S(TH^3_+(-1))\longrightarrow H^3_+(-1)$ with $\bar{\pi}:\Delta\longrightarrow H^3_+(-1)$. Moreover, the linear action of $SO_0(3,1)$ on $\boldsymbol{R}^4_1$ induces the canonical action on Δ (i.e., $g(\boldsymbol{v},\boldsymbol{w})=(g\boldsymbol{v},g\boldsymbol{w})$ for any $g\in SO_0(3,1)$). For any $(\boldsymbol{v},\boldsymbol{w})\in\Delta$, the first component of $\boldsymbol{v}\pm\boldsymbol{w}$ is given by

$$v_0\pm w_0=\sqrt{v_1^2+v_2^2+v_3^2+1}\pm\frac{1}{\sqrt{v_1^2+v_2^2+v_3^2+1}}\sum_{i=1}^3 v_iw_i,$$

so that it can be considered as a function on the unit tangent bundle $S(TH^3_+(-1))$. We now define a function

$$\mathscr{N}_h^{\pm}:\Delta\longrightarrow\boldsymbol{R};\mathscr{N}_h^{\pm}(\boldsymbol{v},\boldsymbol{w})=\frac{1}{v_0\pm w_0}.$$

We call $\mathscr{N}_h^{\pm}$ a *horospherical normalization function* on $H^3_+(-1)$. Since $v_1^2+v_2^2+v_3^2+1$ and $\sum_{i=1}^3 v_iw_i$ are $SO(3)$-invariant functions, $\mathscr{N}_h^{\pm}$ is an $SO(3)$-invariant function. Therefore, $\mathscr{N}_h^{\pm}$ can be considered as a function on the unit tangent sphere bundle over hyperbolic space $SO_0(3,1)/SO(3)$ which is independent of the choice of the model space.

For any embedding $\boldsymbol{x}:U\longrightarrow H^3_+(-1)$, we have the unit normal vector field $\boldsymbol{E}=\boldsymbol{e}:U\longrightarrow S^3_1$, so that $(\boldsymbol{x}(u),\boldsymbol{e}(u))\in\Delta$ for any $u\in U$. It follows that

$$\widetilde{K}_h^{\pm}(u)=\mathscr{N}_h^{\pm}(\boldsymbol{x}(u),\boldsymbol{e}(u))^2K_h^{\pm}(u).$$

The right hand side of the above equality is independent of the choice of the model space.

In the last part of this section we review a global property of the horospherical Gauss curvature. Let M be a closed orientable 2-dimensional manifold and $f : M \longrightarrow H^3_+(-1)$ an immersion. Consider the unit normal $\boldsymbol{E}$ of $f(M)$ in $H^3_+(-1)$. Then we define the lightcone Gauss image in the global

$$\boldsymbol{L}^{\pm} : M \longrightarrow LC^*_+$$

by $\boldsymbol{L}^{\pm}(p) = f(p) \pm \boldsymbol{E}(p)$.

The *global hyperbolic Gauss curvature function* $\mathscr{K}_h : M \longrightarrow \boldsymbol{R}$ is then defined in the usual way in terms of the global lightcone Gauss image $\boldsymbol{L}$. We also define the hyperbolic Gauss map in the global

$$\widetilde{\boldsymbol{L}}^{\pm} : M \longrightarrow S^2_+$$

by $\widetilde{\boldsymbol{L}}^{\pm}(p) = \widetilde{\boldsymbol{L}^{\pm}(p)}$.

We now define a *global horospherical Gauss curvature function*

$$\widetilde{\mathscr{K}_h}^{\pm} : M \longrightarrow \boldsymbol{R}$$

by $\widetilde{\mathscr{K}_h}^{\pm}(p) = \mathscr{N}_h^{\pm}(f(p), \boldsymbol{E}(p))^2 \mathscr{K}_h^{\pm}(p)$. In [**21**] we have shown the following Gauss-Bonnet type theorem.

Theorem 3.4. *If M is a closed orientable 2-dimensional surface in hyperbolic 3-space, then*

$$\frac{1}{2\pi} \int_M \widetilde{\mathscr{K}_h}^{\pm} d\mathfrak{a}_M = \boldsymbol{\chi}(M)$$

where $\boldsymbol{\chi}(M)$ is the Euler characteristic of M, $d\mathfrak{a}_M$ is the area form of M.

We remark that we showed the Gauss-Bonnet type theorem for general even dimensional closed hypersurfaces in hyperbolic n-space [**21**]. Moreover, we defined the notion of horospherical Lipschitz-Killing curvature of submanifold of hyperbolic n-space and showed the Chern-Lashof type inequality for totally absolute horospherical curvatures in [**6**].

4. Horo-flat surfaces.

In this section we consider surfaces with vanishing horospherical (hyperbolic) Gauss curvature. At each point of the surface, we have two different directed light-cone Gauss images $\boldsymbol{L}^{\pm}$. Since the arguments corresponding to the both directions are the similar, we only consider $\boldsymbol{L}^{+} = \boldsymbol{x} + \boldsymbol{e}$ here. We simply write $\boldsymbol{L} = \boldsymbol{L}^{+}$. The other corresponding notations are also written in the similar way (i.e. $\bar{\kappa}$, K_h, $\mathscr{N}_h$, $\widetilde{K}_h$ etc). We say that a surface $M = \boldsymbol{x}(U)$ is a *horospherical flat surface* (briefly, *horo-flat surface*) if $\widetilde{K}_h(p) = 0$ at any point $p \in M$. By definition, $\widetilde{K}_h(p) = 0$ if and only if $K_h(p) = 0$. One of the typical horo-flat surfaces is the horosphere which is the totally umbilical surface with the vanishing horospherical curvature. By Proposition 2.5, a horo-flat surface is a linear Weingarten surface of non-Bryant (non-elliptic) type in the terminology of [**12**]. In this case the surface does not have the Weierstrass-Bryant type parametrization. If we suppose that a surface is umbilically free, then we have the following expression: Let $\boldsymbol{x} : U \longrightarrow H^3_+(-1)$ be a flat horospherical surface without umbilical points, where $U \subset \boldsymbol{R}^2$ is a neighborhood around the origin. In this case, we have two lines of curvature at each point and one of which corresponds to the vanishing hyperbolic principal curvature. We may assume that both the u-curve and the v-curve are the lines of curvature for the coordinate system $(u, v) \in U$. Moreover, we assume that the u-curve corresponds to the vanishing hyperbolic principal curvature. By the hyperbolic Weingarten formula (Proposition 2.2), we have

$$\boldsymbol{L}_u(u, v) = \boldsymbol{0}, \quad \boldsymbol{L}_v(u, v) = -\bar{\kappa}(u, v)\boldsymbol{x}_v(u, v),$$

where $\bar{\kappa}(u, v) \neq 0$. It follows that $\boldsymbol{L}(0, v) = \boldsymbol{L}(u, v)$. We define a function $F : H^3_+(-1) \times (-\varepsilon, \varepsilon) \longrightarrow \boldsymbol{R}$ by

$$F(\boldsymbol{X}, v) = \langle \boldsymbol{L}(0, v), \boldsymbol{X} \rangle + 1,$$

for sufficiently small $\varepsilon > 0$. For any fixed $v \in (-\varepsilon, \varepsilon)$, we have a horosphere $HS^2(\boldsymbol{L}(0, v), -1)$, so that $F = 0$ define a one-parameter family of horospheres. We have the following proposition.

PROPOSITION 4.1. *The surface $M = \boldsymbol{x}(U)$ is the envelope of the family of horospheres defined by $F = 0$.*

PROOF. The envelope defined by $F = 0$ is the surface (might be singular) satisfying the condition $F = F_v = 0$. Here we have

$$F_v(\boldsymbol{X}, v) = \langle \boldsymbol{L}_v(0, v), \boldsymbol{X} \rangle = -\bar{\kappa}(0, v)\langle \boldsymbol{x}_v(0, v), \boldsymbol{X} \rangle.$$

We now consider the function $H(u,v)=F(\boldsymbol{x}(u,v),v)$. Then

$$H(0,v)=F(\boldsymbol{x}(0,v),v)=\langle \boldsymbol{L}(0,v),\boldsymbol{x}(0,v)\rangle+1=-1+1=0.$$

We also have $H_u(u,v)=\langle \boldsymbol{L}(0,v),\boldsymbol{x}_u(u,v)\rangle$. Since $\boldsymbol{L}(0,v)=\boldsymbol{L}(u,v)$, we have $H_u(u,v)=\langle \boldsymbol{L}(u,v),\boldsymbol{x}_u(u,v)\rangle=0$. It follows that $H(u,v)=H(0,v)=0$.

On the other hand, we consider a function $F_v(\boldsymbol{x}(u,v),v)$. By the same reason as the above arguments, we have $\boldsymbol{L}_v(u,v)=\boldsymbol{L}_v(0,v)$, so that

$$\begin{aligned}F_v(\boldsymbol{x}(u,v),v)&=\langle \boldsymbol{L}_v(0,v),\boldsymbol{x}(u,v)\rangle=\langle \boldsymbol{L}_v(u,v),\boldsymbol{x}(u,v)\rangle\\&=-\bar{\kappa}(u,v)\langle \boldsymbol{x}_v(u,v),\boldsymbol{x}(u,v)\rangle.\end{aligned}$$

Since $\langle \boldsymbol{x}(u,v),\boldsymbol{x}(u,v)\rangle=-1$, we have $\langle \boldsymbol{x}_v(u,v),\boldsymbol{x}(u,v)\rangle=0$, so that $F_v(\boldsymbol{x}(u,v),v)=0$. Therefore $\boldsymbol{x}(u,v)$ satisfies the condition

$$F(\boldsymbol{x}(u,v),v)=F_v(\boldsymbol{x}(u,v),v)=0.$$

This means that $M=\boldsymbol{x}(U)$ is the envelope of the family of horospheres defined by $F=0$. $\square$

On the other hand, we consider a surface $\overline{\boldsymbol{x}}:I\times J\longrightarrow H^3_+(-1)$ defined by

$$\overline{\boldsymbol{x}}(s,v)=\boldsymbol{x}(0,v)+s\frac{\boldsymbol{x}_u(0,v)}{\|\boldsymbol{x}_u(0,v)\|}+\frac{s^2}{2}\boldsymbol{L}(0,v),$$

where $I,J\subset\boldsymbol{R}$ are open intervals. We have the following proposition.

PROPOSITION 4.2. *The surface $\overline{M}=\overline{\boldsymbol{x}}(I\times J)$ is the envelope of the family of horospheres defined by $F=0$.*

PROOF. We remind that $\boldsymbol{L}(u,v)=\boldsymbol{x}(u,v)+\boldsymbol{e}(u,v)$ and $\boldsymbol{e}(u,v)$ is the unit spacelike normal of $M=\boldsymbol{x}(U)$ at $\boldsymbol{x}(u,v)$. Since $\langle \boldsymbol{x}(u,v),\boldsymbol{x}(u,v)\rangle=-1$, we have $\langle \boldsymbol{x}(u,v),\boldsymbol{x}_u(u,v)\rangle=0$. It follows that

$$\left\langle \boldsymbol{L}(0,v),\boldsymbol{x}(0,v)+s\frac{\boldsymbol{x}_u(0,v)}{\|\boldsymbol{x}_u(0,v)\|}+\frac{s^2}{2}\boldsymbol{L}(0,v)\right\rangle=-1,$$

so that $F(\overline{\boldsymbol{x}}(s,v),v)=0$. Since $\boldsymbol{L}_v(0,v)=-\bar{\kappa}(0,v)\boldsymbol{x}_v(0,v)$, we have

$$\left\langle \boldsymbol{L}_v(0,v), \boldsymbol{x}(0,v) + s\frac{\boldsymbol{x}_u(0,v)}{\|\boldsymbol{x}_u(0,v)\|} + \frac{s^2}{2}\boldsymbol{L}(0,v) \right\rangle = -\frac{s\bar{\kappa}(0,v)}{\|\boldsymbol{x}_u(0,v)\|}\langle \boldsymbol{x}_v(0,v), \boldsymbol{x}_u(0,v)\rangle.$$

Since both the u-curve and the v-curve are the lines of curvature, $\langle \boldsymbol{x}_v(0,v), \boldsymbol{x}_u(0,v)\rangle = 0$. This means that $F_v(\overline{\boldsymbol{x}}(s,v), v) = 0$. This completes the proof. □

By Propositions 4.1 and 4.2, a horo-flat surface can be reparametrized (at least locally) by

$$\overline{\boldsymbol{x}}(s,v) = \boldsymbol{x}(0,v) + s\frac{\boldsymbol{x}_u(0,v)}{\|\boldsymbol{x}_u(0,v)\|} + \frac{s^2}{2}\boldsymbol{L}(0,v).$$

We now consider the meaning of the above parametrization. If we fix $v = v_0$, we denote that

$$\boldsymbol{a}_0 = \boldsymbol{x}(0,v_0), \quad \boldsymbol{a}_1 = \frac{\boldsymbol{x}_u(0,v_0)}{\|\boldsymbol{x}_u(0,v_0)\|}, \quad \boldsymbol{a}_2 = \boldsymbol{e}(0,v_0).$$

Then we have a curve

$$\boldsymbol{\gamma}(s) = \boldsymbol{a}_0 + s\boldsymbol{a}_1 + \frac{s^2}{2}(\boldsymbol{a}_0 + \boldsymbol{a}_2).$$

Since $\boldsymbol{\gamma}'(s) = \boldsymbol{a}_1 + s(\boldsymbol{a}_0 + \boldsymbol{a}_2)$, we have $\langle \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}'(s)\rangle = \langle \boldsymbol{a}_1, \boldsymbol{a}_1\rangle = 1$. Therefore $\boldsymbol{\gamma}(s)$ has the unit speed. This means that $\boldsymbol{t}(s) = \boldsymbol{a}_1 + s(\boldsymbol{a}_0 + \boldsymbol{a}_2)$. Moreover, $\boldsymbol{t}'(s) = \boldsymbol{a}_0 + \boldsymbol{a}_2$, so that $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle = 0 \neq -1$. We also have

$$\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s) = \boldsymbol{a}_2 - s\boldsymbol{a}_1 - \frac{s^2}{2}(\boldsymbol{a}_0 + \boldsymbol{a}_2).$$

Therefore we have $\langle \boldsymbol{t}'(s) - \boldsymbol{\gamma}(s), \boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\rangle = 1$ which is equivalent to the condition that $\kappa_h(s) = 1$. Moreover $\boldsymbol{\gamma}'''(s) = 0$ implies $\tau_h(s) = 0$. Therefore $\boldsymbol{\gamma}(s)$ is a horocycle. Since $\boldsymbol{\gamma}(0) = \boldsymbol{a}_0$, $\boldsymbol{\gamma}'(0) = \boldsymbol{a}_1$, $\boldsymbol{\gamma}''(0) = \boldsymbol{a}_0 + \boldsymbol{a}_2$, we have the unique solution of the natural equation $\kappa_h(s) = 1$, $\tau_h(s) = 0$ under the above initial data. Therefore we have the following proposition.

PROPOSITION 4.3. *For any $\boldsymbol{a}_0 \in H^3_+(-1)$ and $\boldsymbol{a}_1, \boldsymbol{a}_2 \in S^3_1$ such that $\langle \boldsymbol{a}_i, \boldsymbol{a}_j\rangle = 0$, the unique horocycle with the initial conditions*

$$\boldsymbol{\gamma}(0) = \boldsymbol{a}_0, \quad \boldsymbol{\gamma}'(0) = \boldsymbol{a}_1, \quad \boldsymbol{\gamma}''(0) = \boldsymbol{a}_0 + \boldsymbol{a}_2$$

is given by

$$\boldsymbol{\gamma}(s) = \boldsymbol{a}_0 + s\boldsymbol{a}_1 + \frac{s^2}{2}(\boldsymbol{a}_0 + \boldsymbol{a}_2).$$

Therefore the horo-flat surface is given by the one-parameter family of horocycles. We say that a surface is a *horocyclic surface* if it is locally parametrized by one-parameter families of horocycles around any point. Eventually we have shown the following theorem.

Theorem 4.4. *If $M \subset H^3_+(-1)$ is an umbilically free horo-flat surface, it is a horocyclic surface. Moreover, each horocycle is the line of curvature with the vanishing hyperbolic principal curvature.*

Proof. The first part of the theorem is a simple corollary of Proposition 4.3. For the second part, we assume that $M = \boldsymbol{x}(U)$ and both u-curve and v-curve are the lines of curvature which satisfy $\boldsymbol{L}_u(u,v) = 0$ and $\boldsymbol{L}_v(u,v) = -\bar{\kappa}(u,v)\boldsymbol{x}_v(u,v)$. We now consider the parametrization

$$\overline{\boldsymbol{x}}(s,v) = \boldsymbol{x}(0,v) + s\frac{\boldsymbol{x}_u(0,v)}{\|\boldsymbol{x}_u(0,v)\|} + \frac{s^2}{2}\boldsymbol{L}(0,v)$$

of $M = \boldsymbol{x}(U)$. By a straightforward calculation, we have

$$\overline{\boldsymbol{x}}_s(s,v) = \frac{\boldsymbol{x}_u(0,v)}{\|\boldsymbol{x}_u(0,v)\|} + s\boldsymbol{L}(0,v),$$

$$\overline{\boldsymbol{x}}_v(s,v) = \boldsymbol{x}_v(0,v) + s\left(\frac{\boldsymbol{x}_{uv}(0,v)}{\|\boldsymbol{x}_u(0,v)\|} - \frac{2\langle \boldsymbol{x}_u(0,v), \boldsymbol{x}_{uv}(0,v)\rangle}{\|\boldsymbol{x}_u(0,v)\|^2}\boldsymbol{x}_u(0,v)\right) + \frac{s^2}{2}\boldsymbol{L}_v(0,v).$$

Since $\langle \boldsymbol{L}(0,v), \boldsymbol{x}_u(0,v)\rangle = 0$, we have $\langle \boldsymbol{L}_v(0,v), \boldsymbol{x}_u(0,v)\rangle + \langle \boldsymbol{L}(0,v), \boldsymbol{x}_{uv}(0,v)\rangle = 0$. By the assumption that v-curve is the line of curvature with $\boldsymbol{L}_v(0,v) = -\bar{\kappa}(0,v)\boldsymbol{x}_v(0,v)$, we have $\langle \boldsymbol{L}_v(0,v), \boldsymbol{x}_u(0,v)\rangle = -\bar{\kappa}(0,v)\langle \boldsymbol{x}_v(0,v), \boldsymbol{x}_u(0,v)\rangle = 0$. Therefore we have $\langle \boldsymbol{L}(0,v), \boldsymbol{x}_{uv}(0,v)\rangle = 0$. Since $\boldsymbol{L}(0,v)$ is the lightlike normal vector of $M = \boldsymbol{x}(U)$ at $\boldsymbol{x}(0,v)$, we have $\langle \boldsymbol{L}(0,v), \overline{\boldsymbol{x}}_s(s,v)\rangle = \langle \boldsymbol{L}(0,v), \overline{\boldsymbol{x}}_v(s,v)\rangle = 0$. This means that $\boldsymbol{L}(0,v)$ is the lightlike normal of $M = \boldsymbol{x}(U)$ at $\overline{\boldsymbol{x}}(s,v)$. Therefore we have the lightlike normal $\boldsymbol{L}$ which is constant along the s-curve. Since the s-curve is a horocycle, it is the line of curvature with vanishing hyperbolic principal curvature. □

We remark that horo-flat surfaces are surfaces with one of the (de Sitter) principal curvatures constantly equal to one. The behavior of these surfaces in the regular case is studied in [**30**]. It is known that these surfaces are foliated by horocycles. However, we present here an explicit parametrization of the surface for our purpose.

In the last part of this section we define the end of a surface (or a curve). Let D^3 be the unit ball in $\boldsymbol{R}^3$ with the Poincaré metric. We consider the stereographic projection $P : H^3_+(-1) \longrightarrow D^3$ defined by

$$P(\boldsymbol{x}) = -\boldsymbol{e}_0 + \frac{\boldsymbol{x}+\boldsymbol{e}_0}{1-\langle \boldsymbol{x}, \boldsymbol{e}_0\rangle} = \frac{1}{1+x_0}(0, x_1, x_2, x_3),$$

where $\boldsymbol{e}_0 = (1,0,0,0)$ and $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$. This projection gives the canonical isometry between $H^3_+(-1)$ and D^3. Therefore, the ideal boundary of $H^3_+(-1)$ is identified with the boundary S^2 of D^3 in $\boldsymbol{R}^3$. Moerover, we consider the canonical projection $\pi : \boldsymbol{R}^4_1 \longrightarrow \boldsymbol{R}^3$ defined by $\pi(x_0, x_1, x_2, x_3) = (x_1, x_2, x_3)$. Under this projection S^2_+ is identified with S^2, so that S^2_+ is the ideal boundary of $H^3_+(-1)$. Let $M \subset H^3_+(-1)$ be a surface or a curve. We say that a point $\boldsymbol{y} \in S^2$ is an *end point* of M if $O \cap P(M) \neq \emptyset$ for any open neighborhood O of $\boldsymbol{x}$ in $\boldsymbol{R}^3$. The set of all end points of M is called the *end* of M. For a horocycle

$$\boldsymbol{\gamma}(s) = \boldsymbol{a}_0 + s\boldsymbol{a}_1 + \frac{s^2}{2}\boldsymbol{\ell},$$

we can easily show that

$$\lim_{s\to\infty} P \circ \boldsymbol{\gamma}(s) = \pi \circ \widetilde{\boldsymbol{\ell}},$$

where $\boldsymbol{\ell} = \boldsymbol{a}_0 + \boldsymbol{a}_2$. Under the above identification, $\{\widetilde{\boldsymbol{\ell}}\}$ is the end of the horocycle $\boldsymbol{\gamma}(s)$.

5. Horocyclic surfaces.

In this section we study general properties of horocyclic surfaces. Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a smooth map and $\boldsymbol{a}_i : I \longrightarrow S^3_1$ $(i = 1, 2)$ be smooth mappings from an open interval I with $\langle \boldsymbol{\gamma}(t), \boldsymbol{a}_i(t)\rangle = \langle \boldsymbol{a}_1(t), \boldsymbol{a}_2(t)\rangle = 0$. We define a unit spacelike vector $\boldsymbol{a}_3(t) = \boldsymbol{\gamma}(t) \wedge \boldsymbol{a}_1(t) \wedge \boldsymbol{a}_2(t)$, so that we have a pseudo-orthonormal frame $\{\boldsymbol{\gamma}, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ of $\boldsymbol{R}^4_1$. We now define a mapping

$$F_{(\gamma,a_1,a_2)} : \boldsymbol{R} \times I \longrightarrow H^3_+(-1)$$

by

$$F_{(\gamma,a_1,a_2)}(s,t)=\boldsymbol{\gamma}(t)+s\boldsymbol{a}_1(t)+\frac{s^2}{2}\boldsymbol{\ell}(t),$$

where $\boldsymbol{\ell}(t)=\boldsymbol{\gamma}(t)+\boldsymbol{a}_2(t)$. By Proposition 4.3, we have a horocycle $F_{(\gamma,a_1,a_2)}(s,t_0)$ for any fixed $t=t_0$. We call $F_{(\gamma,a_1,a_2)}$ (or the image of it) a *horocyclic surface*. We also call $\boldsymbol{a}_1(t)$ the *first directrix* and $\boldsymbol{a}_2(t)$ the *second directrix*. Each horocycle $F_{(\gamma,a_1,a_2)}(s,t_0)$ is called a *generating horocycle*. It follows from the arguments in the last part of Section 4 that the end of $F_{(\gamma,a_1,a_2)}$ is the image of $\widetilde{\boldsymbol{\ell}}(t)=\widetilde{\boldsymbol{\gamma}(t)+\boldsymbol{a}_2(t)}$ in S^2_+. By using the above pseudo-orthonormal frame, we define the following fundamental invariants:

$$\begin{aligned}
c_1(t)&=\langle\boldsymbol{\gamma}'(t),\boldsymbol{a}_1(t)\rangle=-\langle\boldsymbol{\gamma}(t),\boldsymbol{a}_1'(t)\rangle, & c_4(t)&=\langle\boldsymbol{a}_1'(t),\boldsymbol{a}_2(t)\rangle=-\langle\boldsymbol{a}_1(t),\boldsymbol{a}_2'(t)\rangle,\\
c_2(t)&=\langle\boldsymbol{\gamma}'(t),\boldsymbol{a}_2(t)\rangle=-\langle\boldsymbol{\gamma}(t),\boldsymbol{a}_2'(t)\rangle, & c_5(t)&=\langle\boldsymbol{a}_1'(t),\boldsymbol{a}_3(t)\rangle=-\langle\boldsymbol{a}_1(t),\boldsymbol{a}_3'(t)\rangle,\\
c_3(t)&=\langle\boldsymbol{\gamma}'(t),\boldsymbol{a}_3(t)\rangle=-\langle\boldsymbol{\gamma}(t),\boldsymbol{a}_3'(t)\rangle, & c_6(t)&=\langle\boldsymbol{a}_2'(t),\boldsymbol{a}_3(t)\rangle=-\langle\boldsymbol{a}_2(t),\boldsymbol{a}_3'(t)\rangle.
\end{aligned}$$

We can show that the following fundamental differential equations for the horocyclic surface:

$$\begin{cases}
\boldsymbol{\gamma}'(t) &= c_1(t)\boldsymbol{a}_1(t)+c_2(t)\boldsymbol{a}_2(t)+c_3(t)\boldsymbol{a}_3(t)\\
\boldsymbol{a}_1'(t) &= c_1(t)\boldsymbol{\gamma}(t)+c_4(t)\boldsymbol{a}_2(t)+c_5(t)\boldsymbol{a}_3(t)\\
\boldsymbol{a}_2'(t) &= c_2(t)\boldsymbol{\gamma}(t)-c_4(t)\boldsymbol{a}_1(t)+c_6(t)\boldsymbol{a}_3(t)\\
\boldsymbol{a}_3'(t) &= c_3(t)\boldsymbol{\gamma}(t)-c_5(t)\boldsymbol{a}_1(t)-c_6(t)\boldsymbol{a}_2(t).
\end{cases}$$

It can be written in the following form:

$$\begin{pmatrix}\boldsymbol{\gamma}'(t)\\ \boldsymbol{a}_1'(t)\\ \boldsymbol{a}_2'(t)\\ \boldsymbol{a}_3'(t)\end{pmatrix}=\begin{pmatrix}0 & c_1(t) & c_2(t) & c_3(t)\\ c_1(t) & 0 & c_4(t) & c_5(t)\\ c_2(t) & -c_4(t) & 0 & c_6(t)\\ c_3(t) & -c_5(t) & -c_6(t) & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}(t)\\ \boldsymbol{a}_1(t)\\ \boldsymbol{a}_2(t)\\ \boldsymbol{a}_3(t)\end{pmatrix}.$$

We remark that

$$C(t)=\begin{pmatrix}0 & c_1(t) & c_2(t) & c_3(t)\\ c_1(t) & 0 & c_4(t) & c_5(t)\\ c_2(t) & -c_4(t) & 0 & c_6(t)\\ c_3(t) & -c_5(t) & -c_6(t) & 0\end{pmatrix}\in\mathfrak{so}(3,1),$$

where $\mathfrak{so}(3,1)$ is the Lie algebra of the Lorentzian group $SO_0(3,1)$. If $\{\boldsymbol{\gamma}(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$ is a pseudo-orthonormal frame field as the above, the 4×4-matrix determined by the frame defines a smooth curve $A : I \longrightarrow SO_0(3,1)$. Therefore we have the relation that $A'(t) = C(t)A(t)$. For the converse, let $A : I \longrightarrow SO_0(3,1)$ be a smooth curve. Then we can show that $A'(t)A(t)^{-1} \in \mathfrak{so}(3,1)$. Moreover, for any smooth curve $C : I \longrightarrow \mathfrak{so}(3,1)$, we apply the existence theorem on the linear systems of ordinary differential equations, so that there exists a unique curve $A : I \longrightarrow SO_0(3,1)$ such that $C(t) = A'(t)A(t)^{-1}$ with an initial data $A(t_0) \in SO_0(3,1)$. Therefore, a smooth curve $C : I \longrightarrow \mathfrak{so}(3,1)$ might be identified with a horocyclic surface in $H^3_+(-1)$. Let $C : I \longrightarrow \mathfrak{so}(3,1)$ be a smooth curve with $C(t) = A'(t)A(t)^{-1}$ and $B \in SO_0(3,1)$, then we have $C(t) = (A(t)B)'(A(t)B)^{-1}$. This means that the curve $C : I \longrightarrow \mathfrak{so}(3,1)$ is a hyperbolic invariant of the pseudo-orthonormal frame $\{\boldsymbol{\gamma}(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$, so that it is a hyperbolic invariant of the corresponding horocyclic surface. We write F_A instead of $F_{(\gamma,a_1,a_2)}$ for a change.

Let $C^\infty(I, \mathfrak{so}(3,1))$ be the space of smooth curves into $\mathfrak{so}(3,1)$ equipped with Whitney C^∞-topology. By the above arguments, we may regard $C^\infty(I, \mathfrak{so}(3,1))$ as the space of horocyclic surfaces, where I is an open interval or the unit circle.

On the other hand, we consider the singularities of horocyclic surfaces. Let $F_{(\gamma,a_1,a_2)} : \boldsymbol{R}\times I \longrightarrow H^3_+(-1)$ be a horocyclic surface defined by

$$F_{(\gamma,a_1,a_2)}(s,t) = \boldsymbol{\gamma}(t) + s\boldsymbol{a}_1(t) + \frac{s^2}{2}\boldsymbol{\ell}(t), \tag{2}$$

where $\boldsymbol{\ell}(t) = \boldsymbol{\gamma}(t) + \boldsymbol{a}_2(t)$. For any curve $\bar{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(t) + s(t)\boldsymbol{a}_1(t) + (s(t)^2/2)\boldsymbol{\ell}(t)$ on the horocyclic surface $F_{(\gamma,a_1,a_2)}$, we define $\bar{\boldsymbol{a}}_1(t) = \boldsymbol{a}_1(t) + s(t)\boldsymbol{\ell}(t)$, $\bar{\boldsymbol{a}}_2(t) = \boldsymbol{\ell}(t) - \bar{\boldsymbol{\gamma}}(t)$. We can show that $\langle \bar{\boldsymbol{a}}_1(t), \bar{\boldsymbol{a}}_1(t)\rangle = \langle \bar{\boldsymbol{a}}_2(t), \bar{\boldsymbol{a}}_2(t)\rangle = 1$ and $\langle \bar{\boldsymbol{a}}_1(t), \bar{\boldsymbol{a}}_2(t)\rangle = \langle \bar{\boldsymbol{a}}_1(t), \bar{\boldsymbol{\gamma}}(t)\rangle = \langle \bar{\boldsymbol{a}}_2(t), \bar{\boldsymbol{\gamma}}(t)\rangle = 0$. By a straightforward calculation, we have

$$F_{(\gamma,a_1,a_2)}(s,t) = \bar{\boldsymbol{\gamma}}(t) + (s - s(t))\bar{\boldsymbol{a}}_1(t) + \frac{(s-s(t))^2}{2}\boldsymbol{\ell}(t).$$

Therefore, if we have a parameter transformation defined by

$$T = t, \quad S = s - s(t), \tag{3}$$

we have $F_{(\gamma,a_1,a_2)}(s,t) = F_{(\bar{\gamma},\bar{a}_1,\bar{a}_2)}(S,T)$. It follows that $\bar{\boldsymbol{\gamma}}(t)$ is the curve on the horocyclic surface $F_{(\gamma,a_1,a_2)}(s,t) = F_{(\bar{\gamma},\bar{a}_1,\bar{a}_2)}(S,T)$ defined by the equation $S = 0$. We call the parameter transformation (3) an *adapted parameter transformation.* By straightforward calculations, we have the following relations:

$$\begin{cases} \bar{c}_1(t) = c_1(t) + \dfrac{s(t)^2}{2}(c_4(t) - c_1(t)) + s(t)c_2(t) + s'(t), \\ \bar{c}_2(t) = c_2(t) + s(t)(c_4(t) - c_1(t)), \\ \bar{c}_3(t) = \left(1 + \dfrac{s(t)^2}{2}\right)c_3(t) + s(t)c_5(t) + \dfrac{s(t)^2}{2}c_6(t), \\ \bar{c}_4(t) = c_4(t) + \dfrac{s(t)^2}{2}(c_4(t) - c_1(t)) + s(t)c_2(t) + s'(t), \\ \bar{c}_5(t) = c_5(t) + s(t)(c_3(t) + c_6(t)), \\ \bar{c}_6(t) = \left(1 - \dfrac{s(t)^2}{2}\right)c_6(t) - s(t)c_5(t) - \dfrac{s(t)^2}{2}c_3(t). \end{cases} \tag{4}$$

It follows that we have $\bar{c}_1(t)-\bar{c}_4(t) = c_1(t)-c_4(t)$. Moreover, we have the following property:

$$\bar{c}_1(t) - \bar{c}_4(t) = \bar{c}_2(t) = 0 \text{ if and only if } c_1(t) - c_4(t) = c_2(t) = 0. \tag{5}$$

PROPOSITION 5.1. *Let $F_{(\gamma,a_1,a_2)}$ be a parameterization of a horocyclic surface of the form*

$$F_{(\gamma,a_1,a_2)}(s,t) = \boldsymbol{\gamma}(t) + s\boldsymbol{a}_1(t) + \frac{s^2}{2}\boldsymbol{\ell}(t)$$

such that $c_1(t) - c_4(t)$ never vanishes. Then Image $F_{(\gamma,a_1,a_2)}$ *has a reparametrization of the form*

$$\bar{F}_{(\bar{\gamma},\bar{a}_1,\bar{a}_2)}(s,t) = \bar{\boldsymbol{\gamma}}(t) + s\bar{\boldsymbol{a}}_1(t) + \frac{s^2}{2}\boldsymbol{\ell}(t),$$

where $\bar{c}_2(t) = \langle \bar{\boldsymbol{\gamma}}', \bar{\boldsymbol{a}}_2\rangle = 0$.

PROOF. Let $\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{a}}_1$ and $\bar{\boldsymbol{a}}_2$ be as those of the previous notations, that is,

$$\bar{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(t) + s(t)\boldsymbol{a}_1(t) + \frac{s(t)^2}{2}\boldsymbol{\ell}(t),$$
$$\bar{\boldsymbol{a}}_1(t) = \boldsymbol{a}_1(t) + s(t)\boldsymbol{\ell}(t), \quad \bar{\boldsymbol{a}}_2(t) = \boldsymbol{\ell}(t) - \bar{\boldsymbol{\gamma}}(t).$$

Since $\langle \bar{\boldsymbol{\gamma}}(t), \bar{\boldsymbol{\gamma}}(t)\rangle = -1$, we have $\langle \bar{\boldsymbol{\gamma}}'(t), \bar{\boldsymbol{\gamma}}(t)\rangle = 0$ and hence $\langle \bar{\boldsymbol{\gamma}}', \bar{\boldsymbol{a}}_2\rangle = \langle \bar{\boldsymbol{\gamma}}'(t), \boldsymbol{\ell}(t) - \bar{\boldsymbol{\gamma}}(t)\rangle = \langle \bar{\boldsymbol{\gamma}}'(t), \boldsymbol{\ell}(t)\rangle$. Taking the derivative of $\bar{\boldsymbol{\gamma}}$, we obtain $\bar{\boldsymbol{\gamma}}'(t) = \boldsymbol{\gamma}'(t) + s'(t)\boldsymbol{a}_1(t) + s(t)\boldsymbol{a}_1'(t) + s(t)s'(t)\boldsymbol{\ell}(t) + (s^2(t)/2)\boldsymbol{\ell}'(t)$. We define $s(t)$ by

$$s(t) = \frac{c_2(t)}{c_1(t) - c_4(t)}.$$

By the second formula of (2), we have $\overline{c}_2(t) = 0$.

Now define $F_{(\bar{\gamma},\bar{a}_1,\bar{a}_2)}(S,T) = F_{(\bar{\gamma},\bar{a}_1,\bar{a}_2)}(s - s(t), t)$, where $S = s - s(t), T = t$. Then

$$\begin{aligned} F_{(\bar{\gamma},\bar{a}_1,\bar{a}_2)}(S,T) &= \bar{\boldsymbol{\gamma}}(t) + (s - s(t))\bar{\boldsymbol{a}}_1(t) + ((s - s(t))^2/2)\boldsymbol{\ell}(t) \\ &= \boldsymbol{\gamma}(t) + s\boldsymbol{a}_1(t) + (s^2/2)\boldsymbol{\ell}(t), \end{aligned}$$

so that $F_{(\bar{\gamma},\bar{a}_1,\bar{a}_2)}$ and $F_{(\gamma,a_1,a_2)}$ have the same image. □

The curve $\bar{\boldsymbol{\gamma}}$ is called the *striction curve* of F if $\langle \bar{\boldsymbol{\gamma}}'(t), \bar{\boldsymbol{a}}_2(t)\rangle = 0$. By Proposition 5.1, we have the unique striction curve under the condition that $c_1(t) - c_4(t) \neq 0$. Then it is given by the equation $S = 0$ after the above adapted parameter transformation. In the case when $c_1(t) - c_4(t) = 0$, there exist striction curves if and only if $c_2(t) = 0$. In the case when $c_1(t) - c_4(t) \neq 0$ or $c_1(t) - c_4(t) = c_2(t) = 0$, we may assume that $\boldsymbol{\gamma}(t)$ is the striction curve of $F_{(\gamma,a_1,a_2)}(s,t)$ which is given by $s = 0$ by an adapted parameter transformation. We can specify the place where the singularities of the horocyclic surface are located. By a straightforward calculation, we have

$$\begin{aligned} \frac{\partial F_{(\gamma,a_1,a_2)}}{\partial s}(s,t) &= \boldsymbol{a}_1(t) + s\boldsymbol{\ell}(t) = s\boldsymbol{\gamma}(t) + \boldsymbol{a}_1(t) + s\boldsymbol{a}_2(t), \\ \frac{\partial F_{(\gamma,a_1,a_2)}}{\partial t}(s,t) &= \boldsymbol{\gamma}'(t) + s\boldsymbol{a}_1'(t) + \frac{s^2}{2}\boldsymbol{\ell}'(t) \\ &= \left(sc_1(t) + \frac{s^2}{2}c_2(t)\right)\boldsymbol{\gamma}(t) + \left(\left(1 + \frac{s^2}{2}\right)c_1(t) - \frac{s^2}{2}c_4(t)\right)\boldsymbol{a}_1(t) \\ &\quad + \left(\left(1 + \frac{s^2}{2}\right)c_2(t) + sc_4(t)\right)\boldsymbol{a}_2(t) \\ &\quad + \left(\left(1 + \frac{s^2}{2}\right)c_3(t) + sc_5(t) + \frac{s^2}{2}c_6(t)\right)\boldsymbol{a}_3(t). \end{aligned}$$

It follows that (s,t) is a singular point of $F_{(\gamma,a_1,a_2)}(s,t)$ if and only if

$$c_2(t) + s(c_4(t) - c_1(t)) = 0, \quad \left(1 + \frac{s^2}{2}\right)c_3(t) + sc_5(t) + \frac{s^2}{2}c_6(t) = 0. \tag{6}$$

By the relations (4), $(s(t),t)$ is a singular point of $F_{(\gamma,a_1,a_2)}$ if and only if $\bar{c}_2(t)=\bar{c}_3(t)=0$ for the adapted parameter transformation $T=t, S=s-s(t)$. The above condition is equivalent to the condition that $S=0$ is a singular point. Then we have the following proposition.

Proposition 5.2. *Let $F_{(\gamma,a_1,a_2)}$ be a horocyclic surface with the striction curve $\boldsymbol{\gamma}$ and $c_1(t)-c_4(t)\neq 0$. If $\boldsymbol{x}_0=F_{(\gamma,a_1,a_2)}(s_0,t_0)$ is a singular value of the horocyclic surface $F_{(\gamma,a_1,a_2)}$, then $s_0=0$, namely, $\boldsymbol{x}_0$ is located on the image of $\boldsymbol{\gamma}$ such that $\boldsymbol{a}_1(t_0)$ is tangent to $\boldsymbol{\gamma}$ at t_0 under the condition $\boldsymbol{\gamma}'(t_0)\neq\mathbf{0}$.*

Proof. If $\boldsymbol{\gamma}'(t_0)=\mathbf{0}$, then $F_{(\gamma,a_1,a_2)}(0,t_0)$ is only singular value on $F_{(\gamma,a_1,a_2)}(s,t_0)$ $(s\in\boldsymbol{R})$. Therefore we assume that $\boldsymbol{\gamma}(t)$ is a unit speed curve. Since c_2 is identically zero and $c_1(t)-c_4(t)\neq 0$, we conclude that if (s_0,t_0) is a singular point of $F_{(\gamma,a_1,a_2)}$, then $s_0=0$ and hence $\boldsymbol{x}_0$ is located in image of $\boldsymbol{\gamma}$. More precisely, singular set is given by the set $\{(0,t_0)\mid c_2(t_0)=c_3(t_0)=0\}$. Therefore $\boldsymbol{\gamma}'(t_0)$ is pseudo-orthogonal to $\boldsymbol{a}_2(t_0),\boldsymbol{a}_3(t_0)$ and $\boldsymbol{\gamma}(t_0)$, so that it is tangent to $\boldsymbol{a}_1(t_0)$. □

On the other hand, by Theorem 4.4, a horo-flat surface is a horocyclic surface $F_{(\gamma,a_1,a_2)}(s,t)$ with the lightlike normal vector $\boldsymbol{\ell}(t)$ around a non-umbilical point. In this case, each horocycle $F_{(\gamma,a_1,a_2)}(s,t_0)$ is a line of curvature. However, at an umbilical point, any direction is a principal direction, so that the tangent direction of the horocycle is also a principal direction. Suppose that $\boldsymbol{\ell}(t)$ is a lightlike normal vector field on $F_{(\gamma,a_1,a_2)}(s,t)$. This means that $\widetilde{\boldsymbol{L}}(s,t)=\widetilde{\boldsymbol{\ell}}(t)$. It follows that $\widetilde{\boldsymbol{L}}_s(s,t)=\widetilde{\boldsymbol{\ell}}_s(t)=\mathbf{0}$. Therefore, the tangent component $\Pi_p\circ\widetilde{\boldsymbol{L}}_s(s,t)$ of $\widetilde{\boldsymbol{L}}_s(s,t)$ is always zero. By Proposition 3.1, the horocyclic surface $F_{(\gamma,a_1,a_2)}(s,t)$ is horo-flat if $\boldsymbol{\ell}(t)$ is a lightlike normal of the surface. We have shown the following proposition.

Proposition 5.3. *An umbilically free horo-flat surface is (at least locally) a horocyclic surface $F_{(\gamma,a_1,a_2)}(s,t)$ with the lightlike normal vector field $\boldsymbol{\ell}(t)$. Conversely, if $F_{(\gamma,a_1,a_2)}(s,t)$ is a horocyclic surface and $\boldsymbol{\ell}(t)$ is a lightlike normal vector field at any (s,t), then it is a horo-flat surface. In this case each horocycle $F_{(\gamma,a_1,a_2)}(s,t_0)$ is a line of curvature.*

We now calculate that

$$\frac{\partial F_{(\gamma,a_1,a_2)}}{\partial s}(s,t)=\boldsymbol{a}_1(t)+s\boldsymbol{\ell}(t),\quad \frac{\partial F_{(\gamma,a_1,a_2)}}{\partial t}(s,t)=\boldsymbol{\gamma}'(t)+s\boldsymbol{a}_1'(t)+\frac{s^2}{2}\boldsymbol{\ell}'(t).$$

Since $\langle\boldsymbol{\ell}(t),\boldsymbol{\ell}(t)\rangle=\langle\boldsymbol{\ell}(t),\boldsymbol{\ell}'(t)\rangle=\langle\boldsymbol{\ell}(t),\boldsymbol{a}_1(t)\rangle=0$, $\boldsymbol{\ell}(t)$ is a lightlike normal at any (s,t) if and only if $c_2(t)+s(c_4(t)-c_1(t))=0$. This condition is equivalent to the condition that $c_2(t) = c_4(t) - c_1(t) = 0$. By the property (5), this condition is invariant under an adapted parameter transformation. Thus, we say that $F_{(\gamma,a_1,a_2)}$ (or, Image $F_{(\gamma,a_1,a_2)}$) is a *horo-flat horocyclic surface* if $c_2(t) = c_4(t) - c_1(t) = 0$ for any t. We also have the following proposition.

PROPOSITION 5.4. *Let $F_{(\gamma,a_1,a_2)}$ be a horocyclic surface with $c_2(t) = 0$. If a horocycle $F_{(\gamma,a_1,a_2)}(s,t_0)$ for each t_0 is one of the lines of curvature, then $F_{(\gamma,a_1,a_2)}$ is horo-flat. Moreover, $\boldsymbol{\ell}(t_0)$ is the lightlike normal along the horocycle $F_{(\gamma,a_1,a_2)}(s,t_0)$.*

PROOF. For any t_0, we consider the horocycle

$$\boldsymbol{\sigma}(s) = F_{(\gamma,a_1,a_2)}(s,t_0) = \boldsymbol{\gamma}(t_0) + s\boldsymbol{a}_1(t_0) + \frac{s^2}{2}\boldsymbol{\ell}(t_0).$$

Since $\boldsymbol{\sigma}(s)$ is a unit speed curve on the horocyclic surface $F_{(\gamma,a_1,a_2)}(s,t)$, $\boldsymbol{t}(s) = \boldsymbol{a}_1(t_0) + s\boldsymbol{\ell}(t_0)$, so that we have the curvature vector $\boldsymbol{k}(s) = \boldsymbol{t}'(s) - \boldsymbol{\sigma}(s) = \boldsymbol{\ell}(t_0) - \boldsymbol{\sigma}(s)$. Let $\boldsymbol{L}(s,t)$ be the lightcone Gauss image of $F_{(\gamma,a_1,a_2)}(s,t)$. Since $\boldsymbol{\sigma}(s)$ is a horocycle, $\kappa_h \equiv 1$ and $\tau_h \equiv 0$. It follows that $\boldsymbol{k}(s) = \boldsymbol{n}(s)$ is the hyperbolic curvature vector of $\boldsymbol{\sigma}(s)$. Therefore the hyperbolic normal curvature of $\boldsymbol{\sigma}(s)$ is $\bar{\kappa}_n(s) = \langle \boldsymbol{n}(s), \boldsymbol{L}(s,t_0)\rangle - 1$.

On the other hand, $\boldsymbol{\sigma}(s)$ is a line of curvature. Then we have

$$-\boldsymbol{L}_s(s,t_0) = \bar{\kappa}_n(s)\frac{\partial F_{(\gamma,a_1,a_2)}}{\partial s}(s,t_0) = \bar{\kappa}_n(s)(\boldsymbol{a}_1(t_0) + s\boldsymbol{\ell}(t_0)).$$

Therefore we have

$$\begin{aligned}\frac{\partial}{\partial s}\langle \boldsymbol{n}(s), \boldsymbol{L}(s,t_0)\rangle &= \frac{\partial}{\partial s}(\langle \boldsymbol{\ell}(t_0), \boldsymbol{L}(s,t_0)\rangle + 1)\\ &= -\bar{\kappa}_n(s)\langle \boldsymbol{a}_1(t_0) + s\boldsymbol{\ell}(t_0), \boldsymbol{\ell}(t_0)\rangle = 0.\end{aligned}$$

It follows that $\bar{\kappa}_n(s) = \langle \boldsymbol{n}(s), \boldsymbol{L}(s,t_0)\rangle - 1$ is constant. Since

$$\frac{\partial F_{(\gamma,a_1,a_2)}}{\partial s}(0,t_0) = \boldsymbol{a}_1(t_0), \quad \frac{\partial F_{(\gamma,a_1,a_2)}}{\partial t}(0,t_0) = \boldsymbol{\gamma}'(t_0) \text{ and } c_2(t) = 0,$$

we have

$$\left\langle \boldsymbol{\ell}(t_0), \frac{\partial F_{(\gamma,a_1,a_2)}}{\partial s}(0,t_0)\right\rangle = \left\langle \boldsymbol{\ell}(t_0), \frac{\partial F_{(\gamma,a_1,a_2)}}{\partial t}(0,t_0)\right\rangle = 0.$$

This means that $\boldsymbol{\ell}(t_0)$ is the lightlike normal of $F_{(\gamma,a_1,a_2)}(s,t)$ at $(0,t_0)$. If necessary, we adopt $-\boldsymbol{a}_2(t)$ instead of $\boldsymbol{a}_2(t)$, $\boldsymbol{L}(0,t_0)$ and $\boldsymbol{\ell}(t_0)$ are parallel, so that $\bar{\kappa}_n(s) = \langle \boldsymbol{n}(0), \boldsymbol{L}(0,t_0)\rangle - 1 = \langle \boldsymbol{\ell}(t_0), \boldsymbol{L}(0,t_0)\rangle + 1 - 1 = 0$. Moreover, we have $\boldsymbol{L}_s(s,t_0) = 0$, then $\boldsymbol{L}(s,t_0)$ is parallel to $\boldsymbol{\ell}(t_0)$. This completes the proof. $\square$

We now consider the space of horo-flat horocyclic surfaces. Remember that $C^\infty(I, \mathfrak{so}(3,1))$ is the space of horocyclic surfaces. We consider a linear subspace of $\mathfrak{so}(3,1)$ defined by

$$\mathfrak{hf}(3,1) = \left\{ C = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ c_1 & 0 & c_4 & c_5 \\ c_2 & -c_4 & 0 & c_6 \\ c_3 & -c_5 & -c_6 & 0 \end{pmatrix} \in \mathfrak{so}(3,1) \;\middle|\; c_2 = c_1 - c_4 = 0 \right\}$$

By the definition of horo-flat horocyclic surfaces, the *space of horo-flat horocyclic surfaces* is defined to be the space $C^\infty(I, \mathfrak{hf}(3,1))$ with Whitney C^∞-topology.

For a horo-flat horocyclic surface $F_{(\gamma,a_1,a_2)}$, the singular points (s,t) are given by the condition that

$$\sigma_C(s,t) = (c_3(t) + c_6(t))s^2 + 2c_5(t)s + 2c_3(t) = 0.$$

Therefore the horo-flat horocyclic surface has singularities at (s,t) if and only if the above quadratic equation has real roots. Under the condition that $c_3(t)+c_6(t) \neq 0$, this condition is equivalent to the condition

$$\delta_C(t) = c_5^2(t) - 2c_3(t)(c_3(t) + c_6(t)) \geq 0.$$

By this inequality, the horo-flat horocyclic surface $F_{(\gamma,a_1,a_2)}$ is non-singular if and only if $c_3(t) \neq 0$ and

$$\begin{cases} c_5(t_0) = 0 & \text{if there exists } t_0 \in I \text{ such that } c_3(t_0) + c_6(t_0) = 0, \\ \delta_C(t) < 0 & \text{if } c_3(t) + c_6(t) \neq 0. \end{cases} \tag{7}$$

We now consider a horo-flat horocyclic suface $F_{(\gamma,a_1,a_2)}$ with singularities.

We start to give a rough classification of singular points. It follows from the above arguments that (s_0,t_0) is a singular point of $F_{(\gamma,a_1,a_2)}$ if one of the following conditions holds:

(1) $c_3(t_0) = 0$ and $c_6(t_0)s_0^2 + 2c_5(t_0)s_0 = 0$.
(2) $c_3(t_0) \neq 0$ and

(a) $c_3(t_0) + c_6(t_0) = 0, c_5(t_0) \neq 0$ and $s_0 = -c_3(t_0)/c_5(t_0)$.
(b) $c_3(t_0)+c_6(t_0) \neq 0, \delta_C(t_0) \geq 0$ and $(c_3(t_0)+c_6(t_0))s_0^2+2c_5(t_0)s_0+2c_3(t_0) = 0$.

In the case (2), we consider the adapted parameter transformation

$$T = t, \quad S = s - s_0,$$

so that $T = t_0, S = 0$ is the singular point and $\overline{c}_3(t_0) = 0$. Therefore we may only consider the case (1) without the loss of generality. Under the condition $c_3(t_0) = 0$, we have the following rough classification:

(α) (s_0, t_0) is a *horocyclic singular point* if $c_5(t_0) = c_6(t_0) = 0$. In this case all points on the horocycle though (s_0, t_0) are singularities.
(β) (s_0, t_0) is a *single singular point* if $c_5(t_0) \neq 0$ and $c_6(t_0) = 0$. In this case $s_0 = 0$ is the only singular point on the horocycle through (s_0, t_0).
(γ) (s_0, t_0) is a *double singular point* if $c_5(t_0) = 0$ and $c_6(t_0) \neq 0$. In this case $s_0 = 0$ is the only singular point on the horocycle through (s_0, t_0).
(δ) (s_0, t_0) is a *separated singular point* if $c_5(t_0) \neq 0$ and $c_6(t_0) \neq 0$. In this case $s_0 = 0$ and $-2c_5(t_0)/c_6(t_0)$ are singular points on the horocycle through (s_0, t_0).

The horocyclic singular points appear as the horocycle through (s_0, t_0), so that these are non-isolated. We consider the single singular point. Since $c_5(t_0) \neq 0$, $c_5(t) \neq 0$ for any t in a neighborhood of t_0. For any neighborhood U of t_0, if there exists a point $t_1 \in U$ such that $c_3(t_1) + c_6(t_1) \neq 0$, then $\delta_C(t_1) > 0$ because $\delta_C(t_0) = c_5^2(t_0) > 0$. This means that (s_1, t_1) is the separated singular point. Therefore, (s_0, t_0) is non-isolated. Moreover, if $c_3(t) + c_6(t) = 0$ near by t_0, (s, t) are singular points for $s = -c_3(t)/c_5(t)$. In this case (s_0, t_0) is also non-isolated. Suppose that $(0, t_0)$ is the double singular point. Since $c_3(t_0)+c_6(t_0) = c_6(t_0) \neq 0$, $c_3(t) + c_6(t) \neq 0$ for sufficiently near by t_0. If $\delta_C(t) < 0$ for $t \neq t_0$, then $(0, t_0)$ is isolated. Otherwise, for any neighborhood U of t_0, there exists a point $t_1 \in U$ such that $\delta_C(t_1) \geq 0$, so that $(0, t_0)$ is non-isolated. It is clear that the separated singular point is non-isolated.

By the arguments in the previous paragraph, we now consider a rough classification of horo-flat horocyclic surfaces. We say that $F_{(\gamma,a_1,a_2)}$ is a *horo-flat horocyclic surface with an isolated singular point* if $c_3(t) + c_6(t) \neq 0$ and there is a point $t_0 \in I$ such that $\delta_C(t_0) = 0$ and $\delta_C(t) < 0$ for any $t \in I \setminus \{t_0\}$. In this case the isolated singular point (s_0, t_0) is the double singular point. By the adapted parameter transformation, we may assume $c_3(t_0) = 0$ and $s_0 = 0$. Therefore, the horo-flat horocyclic surface with an isolated singular point has a parametrization $F_{(\gamma,a_1,a_2)}$ with $c_3(t_0) = c_5(t_0) = 0$, $c_3(t) + c_6(t) \neq 0$ and $\delta_C(t) < 0$ for any

$t \in I \setminus \{t_0\}$.

We consider a horo-flat horocyclic surface with non-isolated singularities. There are some possibilities for the existence of pathological situations in general. For example, if there is a singular point such that it is the limit of the other discrete singularities, then such a point is non-isolated but the situation is very complicated. In order to avoid such a situation, we consider the case when the set of singular points is a union of curves in the parameter space $I \times \boldsymbol{R}$. In this case, $F_{(\gamma,a_1,a_2)}$ has at most two branches of singularities except at the horocyclic singular points. However such branches can pass through the horocyclic singular points. We suppose that one of the branches of the singularities is given by

$$\bar{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(t) + s(t)\boldsymbol{a}_1(t) + \frac{s(t)^2}{2}\boldsymbol{\ell}(t),$$

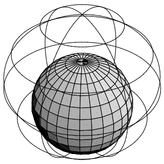
where $s = s(t)$ is one of the solutions of the quadratic equation $\sigma_C(s,t) = 0$ for any t. In this case we can reparametrize the horocyclic surface by $\bar{\boldsymbol{a}}_1(t), \bar{\boldsymbol{a}}_2(t)$ and $S = s - s(t), T = t$, by the adapted parameter transformation, so that one of the branches of the singularities is located on the curve $S = 0$. Therefore, we may assume that one of the branches of singularities are located on $\boldsymbol{\gamma}(t)$. In this case, such singularities satisfy the condition $c_3(t) = 0$. Moreover, the condition $c_3(t) = 0$ is also satisfied at the horocyclic singular points. Therefore, we assume that $c_3(t) = 0$ for any $t \in I$. It follows that $\boldsymbol{\gamma}'(t)$ is parallel to $\boldsymbol{a}_1(t)$ if $\boldsymbol{\gamma}'(t) \neq \boldsymbol{0}$. Moreover, another branch of the singularities is given by the equation $2c_5(t) + sc_6(t) = 0$. If $c_6(t) \neq 0$, we denote that

$$\boldsymbol{\gamma}^\sharp(t) = \boldsymbol{\gamma}(t) + s(t)\boldsymbol{a}_1(t) + \frac{s(t)^2}{2}\boldsymbol{\ell}(t),$$

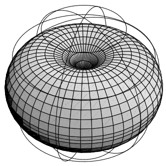
where $s(t) = -2c_5(t)/c_6(t)$. If $c_6(t) = 0$, we have a unique end point $\widetilde{\boldsymbol{\ell}}(t) =$ constant $= \widetilde{\boldsymbol{\ell}}$. In this case $\boldsymbol{\gamma}$ is a curve on a horosphere and Image$F_{(\gamma,a_1,a_2)}$ is a subset of the horosphere.

We call $F_{(\gamma,a_1,a_2)}$ a *generalized horo-cone* if $\boldsymbol{\gamma}(t)$ is constant, $\boldsymbol{a}_1'(t) = c_5(t)\boldsymbol{a}_3(t)$ and $\boldsymbol{a}_2'(t) = c_6(t)\boldsymbol{a}_3(t)$. This condition is equivalent to the condition that $c_1(t) = c_2(t) = c_3(t) = c_4(t) = 0$. Note that a generalized horo-cone is horo-flat. Comparing with developable surfaces in Euclidean 3-space, the notion of generalize horo-cones is the analogous notion of conical surfaces. However, the class of generalized horo-cones contains several different surfaces. We say that a generalized horo-cone $F_{(\gamma,a_1,a_2)}$ is a *horo-cone with a single vertex* if $c_5(t) = 0$ and there are no subinterval $J \subset I$ such that $c_6|J \equiv 0$. In other words, a horocyclic surface $F_{(\gamma,a_1,a_2)}$ is a horo-cone with a single vertex if $c_1(t) = c_2(t) = c_3(t) = c_4(t) = c_5(t) = 0$

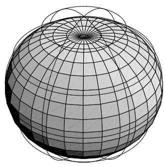
and there are no subinterval $J \subset I$ such that $c_6|J \equiv 0$. In this case, both of $\boldsymbol{\gamma}(t)$ and $\boldsymbol{\gamma}^\sharp(t)$ are constant and $\boldsymbol{\gamma} = \boldsymbol{\gamma}^\sharp$. A generalized horo-cone $F_{(\gamma,a_1,a_2)}$ is called a *horo-cone with two vertices* if both of $\boldsymbol{\gamma}(t)$ and $\boldsymbol{\gamma}^\sharp(t)$ are constant and $\boldsymbol{\gamma} \neq \boldsymbol{\gamma}^\sharp$. By the calculation of the derivative of $\boldsymbol{\gamma}^\sharp(t)$, the above condition is equivalent to the condition that $c_1(t) = c_2(t) = c_3(t) = c_4(t) = 0$, there are no subinterval $J \subset I$ such that $c_5|J \equiv 0$ and there exists a real number λ such that $c_5(t) = \lambda c_6(t)$. If the condition $c_1(t) = c_2(t) = c_3(t) = c_4(t) = c_6(t) = 0$ and there are no subinterval $J \subset I$ such that $c_5|J \equiv 0$, then $\boldsymbol{a}_2(t)$ is constant. It follows that the image of the generalized horo-cone $F_{(\gamma,a_1,a_2)}$ is a part of a horosphere (i.e., we call it a *conical horosphere*). We simply call $F_{(\gamma,a_1,a_2)}$ a *horo-cone* if it is one of the above three cases. We can draw the pictures of horo-cones in the Poincaré ball (Figure 4).



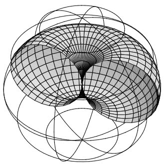
Conical horosphere.



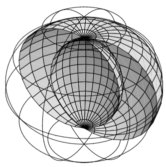
Horo-cone with a single vertex.



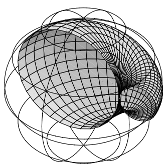
Horo-cone with two vertices.



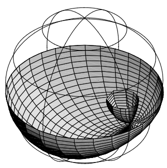
Half cut of horo-cone with a single vertex.



Half cut of horo-cone with two vertices.



Half cut of horo-cone with a shifted single vertex.



Half cut of horo-cone with shifted two vertices.

Figure 4.

We say that a generalized horo-cone $F_{(\gamma,a_1,a_2)}$ is a *semi-horo-cone* if $\gamma^\sharp(t)$ is not constant on I. This condition is equivalent to the conditions that $c_1(t) = c_2(t) = c_3(t) = c_4(t) = 0$, there are no subinterval $J \subset I$ such that $c_5|J \equiv 0$ or $c_6|J \equiv 0$ and $c_5(t)/c_6(t)$ is not a constant on $\{t \in I \mid c_6(t) \neq 0\}$. We remark that if $c_6(t_0) = 0$, then $s(t) = -2c_5(t)/c_6(t)$ tends to ∞ as $t \longrightarrow t_0$. Moreover, we have $\boldsymbol{\ell}'(t_0) = c_6(t_0)\boldsymbol{a}_3(t_0) = \boldsymbol{0}$, so that $\widetilde{\boldsymbol{\ell}}'(t_0) = \boldsymbol{0}$. This means that the end $\widetilde{\boldsymbol{\ell}}(t)$ of $F_{(\gamma,a_1,a_2)}$ has a singular point at t_0. Therefore, we say that semi-horo-cone $F_{(\gamma,a_1,a_2)}$ is a *semi-horo-cone with singular end* if there are zero points of $c_6(t)$. Otherwise, $F_{(\gamma,a_1,a_2)}$ is called a semi-horo-cone with regular end. Of course, we also call $F_{(\gamma,a_1,a_2)}$ a semi-horo-cone if $\boldsymbol{\gamma}(t)$ is not constant and $\gamma^{\sharp}(t)$ is constant. In this case, however, by a suitable adapted parameter transformation, we have the condition that $\boldsymbol{\gamma}(t)$ is constant. The condition that $\boldsymbol{\gamma}(t)$ is not constant is given by $c_1(t) \neq 0$. We can also write the condition that $\boldsymbol{\gamma}^{\sharp}$ is constant in terms of the basic invariant $C(t)$. However it is rather a complicated condition, so that we omit the description here.

Finally, we say that $F_{(\gamma,a_1,a_2)}$ is a *horo-flat tangent horocyclic surface* if both of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}^{\sharp}$ are not constant or $\boldsymbol{\gamma}$ is not constant and $c_6(t) = 0$. In the last case, the end is an isolated point and $F_{(\gamma,a_1,a_2)}$ is a subset of the horosphere (a one parameter family of horocycles which are tangent to $\boldsymbol{\gamma}$ on a horosphere). The above conditons are equivalent to the conditions that $c_2(t) = c_3(t) = c_1(t) - c_4(t) = 0$, there are no subinterval $J \subset I$ such that $c_1|J \equiv 0$ and $\boldsymbol{\gamma}^{\sharp}(t)$ is moving or the set of singular points is equal to $s = 0$ except the horocyclic singular points (i.e., $\boldsymbol{\gamma}^{\sharp}(t)$ cannot exist in $H^3_+(-1)$ anymore).

By the above arguments, we also consider the linear subspace of $\mathfrak{so}(3,1)$ defined by

$$\mathfrak{hf}_\sigma(3,1) = \left\{ C = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ c_1 & 0 & c_4 & c_5 \\ c_2 & -c_4 & 0 & c_6 \\ c_3 & -c_5 & -c_6 & 0 \end{pmatrix} \in \mathfrak{so}(3,1) \;\middle|\; c_2 = c_1 - c_4 = c_3 = 0 \right\}.$$

In order to avoid some pathological situation, we consider the space $C^\infty(I, \mathfrak{hf}_\sigma(3,1))$ with Whitney C^∞-topology. We call it a *space of horo-flat horocyclic surfaces with curve singularities.* In this terminology, one of the branches of the singularities of the horo-flat surface is always located on the image of $\boldsymbol{\gamma}$.

On the other hand, we now consider a local classification of non-singular horo-flat horocyclic surfaces. Let $M \subset H^3_+(-1)$ be a surface patch (i.e., the image of an embedding from an open domain in $\boldsymbol{R}^2$). We say that M is a *horo-flat horocyclic surface patch* if there exists a smooth curve $A : I \longrightarrow SO_0(3,1)$ with $C(t) = A'(t)A^{-1}(t) \in \mathfrak{hf}(3,1)$ and Image $F_A \supset M$, where $F_A = F_{(\gamma,a_1,a_2)} : \boldsymbol{R} \times I \longrightarrow H^3_+(-1)$ is a horo-flat horocyclic surface corresponding to A. We call F_A a *complete parametrization* of M. If we have another complete parametrization $F_{\overline{A}}$ of M by an adapted parameter transformation $T = t, S = s - s(t)$, we call $F_{\overline{A}}$ an *adapted reparametrization.*

Theorem 5.5. *Let M be a horo-flat horocyclic surface patch whose complete parametrization is a horo-flat horocyclic surfaces with curve singularities. Then M is an open subset of the regular part of one of the following horocyclic surfaces*:

(1) *A generalized horo-cone.*
(2) *A horo-flat tangent horocyclic surface.*
(3) *A glue of the above two surfaces.*

Proof. We consider a complete parametrization $F_{(\gamma,a_1,a_2)}$ of the horo-flat horocyclic surface patch. Since $F_{(\gamma,a_1,a_2)}$ is horospherical flat, we have $c_2(t) = c_4(t) - c_1(t) \equiv 0$.

We now assume that $F_{(\gamma,a_1,a_2)}$ has curve singularities. If $\boldsymbol{\gamma}$ is a constant unit vector, we have $\boldsymbol{\gamma}' \equiv \mathbf{0}$, so that the above conditions for horo-flatness are reduced to $c_4(t) = \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_2(t)\rangle = 0$. Moreover, we have the conditions that $c_1(t) = c_2(t) = c_3(t) = 0$. Therefore we have the conditions that $\boldsymbol{a}_1'(t) = c_5(t)\boldsymbol{a}_3(t)$, $\boldsymbol{a}_2'(t) = c_6(t)\boldsymbol{a}_3(t)$. This means that $F_{(\gamma,a_1,a_2)}$ is a generalized horo-cone.

We suppose that $\boldsymbol{\gamma}$ is not constant. It means that $c_2(t) = c_1(t) - c_4(t) = c_3(t) = 0$ and there are no subinterval $J \subset I$ such that $c_1|J \equiv 0$. If $c_6(t) = 0$ on a subinterval $J \subset I$, then we restrict the parameter space on $\boldsymbol{R} \times J$, so that we may assume that $c_6(t) = 0$ on I because we consider the glue of surfaces. In this case, γ is a curve on a horosphere and $F_{(\gamma,a_1,a_2)}$ satisfies one of the conditons for horo-flat tangent horocyclic surfaces.

Therefore we have the condition that there are no subinterval $J \subset I$ such that $c_6|J \equiv 0$. If there is a subinterval $J \subset I$ such that $\boldsymbol{\gamma}^\sharp(t)$ is constant on J, we restrict the parameter space on $\boldsymbol{R}\times J$, so that we have a semi-horo-cone on $\boldsymbol{R}\times J$. Therefore, we may assume that there are no subinterval $J \subset I$ such that $\boldsymbol{\gamma}^\sharp(t)$ is constant on J. This is also one of the conditions for horo-flat tangent horocyclic surfaces. This completes the proof. □

By the above proof, we can show the following proposition. We now consider the class of horo-flat horocyclic surfaces with regular points. For example if $c_3(t) = c_5(t) = c_6(t) = 0$, then any points are singular points. It is actually a horocycle.

Proposition 5.6. *Let $F_{(\gamma,a_1,a_2)} : \boldsymbol{R} \times I \longrightarrow H^3_+(-1)$ be a horo-flat horocyclic surface with regular points. Then there exists an open subset O of I such that $F(\boldsymbol{R}\times O)$ is an open and dense subset of* $\mathrm{Image}\,F_{(\gamma,a_1,a_2)}$ *such that $F(\boldsymbol{R}\times O)$ is a union of the images of the following horo-flat horocyclic surfaces*:

(1) *A regular horo-flat horocyclic surface.*
(2) *A generalized horo-cone.*
(3) *A horo-flat tangent horocyclic surface.*

In Euclidean space, complete non-singular developable surfaces are cylindrical surfaces [**13**]. However, there are various kinds of horo-flat horocyclic surfaces even if these are regular surfaces. Suppose that $F_{(\gamma,a_1,a_2)}$ is a non-singular horo-flat horocyclic surface. We remember that $F_{(\gamma,a_1,a_2)}$ is non-singular if and only if $c_3(t) \neq 0$ and

$$\begin{cases} c_5(t_0) = 0 & \text{if there exists } t_0 \in I \text{ such that } c_3(t_0) + c_6(t_0) = 0, \\ \delta_C(t) < 0 & \text{if } c_3(t) + c_6(t) \neq 0. \end{cases}$$

We say that $F_{(\gamma,a_1,a_2)}$ is a *regular horocylindrical surface* if $c_1(t) = c_2(t) = c_4(t) = c_5(t) = 0$ and $c_3(t)(c_3(t)+c_6(t)) > 0$. This condition is equivalent to the condition that $\boldsymbol{a}_1(t)$ is constant and $F_{(\gamma,a_1,a_2)}$ is non-singular horo-flat horocyclic surface. Moreover, $\boldsymbol{a}_1(t)$ is constant and $c_3(t) + c_6(t) = 0$ if and only if $\boldsymbol{\ell}(t)$ is constant, so that $F_{(\gamma,a_1,a_2)}$ is a part of a horosphere. We also say that $F_{(\gamma,a_1,a_2)}$ is a *secondary regular horocylindrical surface* if $c_1(t) = c_2(t) = c_4(t) = c_6(t) = 0$ and $\delta_C(t) = c_5^2(t) - 2c_3^2(t) < 0$. This condition is equivalent to the condition that $\boldsymbol{a}_2(t)$ is constant and $F_{(\gamma,a_1,a_2)}$ is non-singular horo-flat horocyclic surface. Of course if we remove the condition that $F_{(\gamma,a_1,a_2)}$ is non-singular we simply say it is a *horocylindrical surface* or a *secondary horocylindrical surface* respectively. We can analyze the situation as follows: We define a subspace $\mathfrak{r}(3,1) \subset \mathfrak{hf}(3,1)$ by

$$\begin{aligned} \mathfrak{r}(3,1) = &\big\{C \in \mathfrak{hf}(3,1) \mid c_3 + c_6 = c_5 = 0, c_3 \neq 0\big\} \\ &\cup \big\{C \in \mathfrak{hf}(3,1) \mid c_3 \neq 0, c_3 + c_6 \neq 0, c_5^2 - 2c_3(c_3 + c_6) < 0\big\}. \end{aligned}$$

We also define subspaces $\mathfrak{r}_1(3,1)$ and $\mathfrak{r}_2(3,1)$ of $\mathfrak{r}(3,1)$ by

$$\begin{aligned} \mathfrak{r}_1(3,1) &= \big\{C \in \mathfrak{r}(3,1) \mid c_1 = c_2 = c_4 = c_5 = 0, c_3(c_3 + c_6) > 0\big\}, \\ \mathfrak{r}_2(3,1) &= \big\{C \in \mathfrak{r}(3,1) \mid c_1 = c_2 = c_4 = c_6 = 0, c_5^2 - 2c_3^2 < 0\big\}. \end{aligned}$$

For any $C \in C^\infty(I, \mathfrak{hf}(3,1))$, the corresponding horo-flat horocyclic surface F_A is horocylindrical if $C(I) \subset \mathfrak{r}_1(3,1)$ and secondary horocylindrical if $C(I) \subset \mathfrak{r}_2(3,1)$ respectively. However, $\mathfrak{r}_1(3,1) \cup \mathfrak{r}_2(3,1)$ is a thin set in $\mathfrak{r}(3,1)$, so that there are a lot of non-singular horo-flat horocyclic surfaces which are neither horocylindrical nor secondary horocylindrical. We call such a horo-flat horocyclic surface a *regular horocylindrical surface of general type*. We give some interesting examples of regular horocylindrical surfaces and secondary regular horocylindrical surfaces which suggest that the situation is quite different form the developable surfaces in Euclidean space.

EXAMPLE 5.7. Consider a regular horocylindrical surface $F_{(\gamma,a_1,a_2)}$. Suppose that $\boldsymbol{\gamma}(t)$ is a unit speed curve with $\kappa_h(t) \neq 0$. Then we have the Frenet-type frame $\{\boldsymbol{\gamma}(t), \boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{e}(t)\}$ given in Section 2. By definition, we have $\boldsymbol{\gamma}'(t) \neq \boldsymbol{0}$ and $\boldsymbol{a}_2'(t) = -c_1(t)\boldsymbol{a}_1(t) + c_6(t)\boldsymbol{a}_3(t)$. Suppose that $\boldsymbol{a}_1(t) = \boldsymbol{a}_1$ is constant. It follows that $c_1(t) \equiv 0$, so that $\langle \boldsymbol{t}(t), \boldsymbol{a}_1\rangle = 0$. Taking a derivative of this equation, we have

$$0 = \langle \boldsymbol{t}'(t), \boldsymbol{a}_1\rangle = \langle \kappa_h(t)\boldsymbol{n}(t) + \boldsymbol{\gamma}(t), \boldsymbol{a}_1\rangle = \kappa_h(t)\langle \boldsymbol{n}(t), \boldsymbol{a}_1\rangle.$$

Therefore, we have $\langle \boldsymbol{n}(t), \boldsymbol{a}_1\rangle = 0$. Since $\langle \boldsymbol{\gamma}(t), \boldsymbol{a}_1\rangle = \langle \boldsymbol{t}(t), \boldsymbol{a}_1\rangle = 0$ and $\langle \boldsymbol{a}_1, \boldsymbol{a}_1\rangle = 1$, we have $\boldsymbol{a}_1 = \pm \boldsymbol{e}(t)$. It follows that $\tau_h(t) \equiv 0$. This means that $\boldsymbol{\gamma}(t)$ is a hyperbolic plane curve. If necessary, under a suitable parameter change, we can choose $\boldsymbol{a}_1 = \boldsymbol{e}$ and $\boldsymbol{a}_2(t) = \pm \boldsymbol{n}(t)$. We say that

$$F_{(\gamma,e,\pm n)}(s,t) = \boldsymbol{\gamma}(t) + s\boldsymbol{e} + \frac{s^2}{2}(\boldsymbol{\gamma}(t) \pm \boldsymbol{n}(t))$$

is a *binormal horocyclic surface of a hyperbolic plane curve* $\boldsymbol{\gamma}$. By a straightforward calculation we have

$$\frac{\partial F_{(\gamma,e,\pm n)}}{\partial t}(s,t) = \left\{1 + \frac{s^2}{2}(1 \mp \kappa_h)\right\}\boldsymbol{t}(t), \quad \frac{\partial F_{(\gamma,e,\pm n)}}{\partial s}(s,t) = \boldsymbol{e} + s(\boldsymbol{\gamma}(t) \pm \boldsymbol{n}(t)).$$

Therefore the first fundamental form is given by

$$I_h = ds^2 + \left(1 + \frac{s^2(1 \mp \kappa_h(t))}{2}\right)^2 dt^2.$$

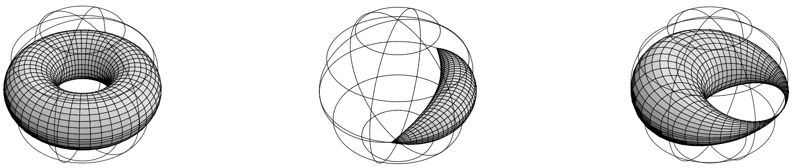
Here, $\boldsymbol{\ell}(t) = \boldsymbol{\gamma}(t) \pm \boldsymbol{n}(t)$ is the lightlike normal vector field along the surface. Then we have

$$\begin{aligned}
-\boldsymbol{\ell}'(t) = -(1 \mp \kappa_h(t))\boldsymbol{t}(t) &= \frac{-1 \pm \kappa_h(t)}{\dfrac{2 + s^2(1 \mp \kappa_h(t))}{2}} \frac{\partial F_{(\gamma,e,\pm n)}}{\partial t}(s,t) \\
&= \frac{-2 \pm 2\kappa_h(t)}{2 + s^2(1 \mp \kappa_h(t))} \frac{\partial F_{(\gamma,e,\pm n)}}{\partial t}(s,t)
\end{aligned}$$

It follows that the de Sitter principal curvatures are

$$1 \quad \text{and} \quad 1 - \frac{2 \mp 2\kappa_h(t)}{2 + s^2(1 \mp \kappa_h(t))}.$$

Since $\kappa_h(t) > 0$, $F_{(\gamma,e,-n)}$ is always umbilically free. We can draw the pictures of such surfaces in the Poincaré ball (cf., Figure 5).



Horo-torus ($\boldsymbol{\gamma}$: circle, $a_1 =$ constant) Banana ($\boldsymbol{\gamma}$: equidistant curve, $a_1 =$ constant) Croissant ($\boldsymbol{\gamma}$: horocycle, $a_1 =$ constant)

Figure 5.

However, $F_{(\gamma,e,n)}$ has umbilical points where $\kappa_h(t) = 1$. This gives a concrete example of the surface with a constant principal curvature which is not umbilically free ([**1**, Example 2.1]). We can draw a horocylindrical surface which has umbilical points along the horocycle through $(0,0,0)$ in Figure 6.

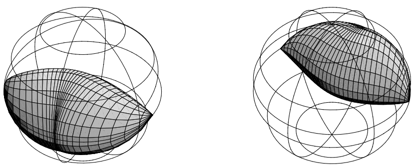


Figure 6. Hips ($\kappa_h(0) = 1$ of $\boldsymbol{\gamma}$, $a_1 =$ constant).

If $\kappa_h \equiv 1$ (i.e., $\boldsymbol{\gamma}(t)$ is a horocycle), then $F_{(\gamma,e,n)}$ is totally umbilical (i.e., a horosphere).

EXAMPLE 5.8. Suppose that $\boldsymbol{a}_2(t) = \boldsymbol{a}_2$ is constant. By the similar calculation as the case Example 5.7, we have $\boldsymbol{a}_2 = \pm \boldsymbol{e}$, so that $\tau_h(t) \equiv 0$. Therefore, $\boldsymbol{\gamma}(t)$ is a hyperbolic plane curve and $\boldsymbol{a}_1(t) = \pm \boldsymbol{n}(t)$. We can also choose $\boldsymbol{a}_1(t) = \boldsymbol{n}(t)$. We say that

$$F_{(\gamma,n,\pm e)}(s,t) = \boldsymbol{\gamma}(t) + s\boldsymbol{n}(t) + \frac{s^2}{2}(\boldsymbol{\gamma}(t) \pm \boldsymbol{e})$$

a *principal normal horocyclic surface of a hyperbolic plane curve* $\boldsymbol{\gamma}$. In this case we have

$$\frac{\partial F_{(\gamma,n,\pm e)}}{\partial t}(s,t) = \left\{1 - \kappa_h(t) + \frac{s^2}{2}\right\}\boldsymbol{t}(t), \quad \frac{\partial F_{(\gamma,n,\pm e)}}{\partial s}(s,t) = \boldsymbol{n}(t) + s(\boldsymbol{\gamma}(t) \pm \boldsymbol{e}).$$

Therefore the first fundamental form is given by

$$I_h = ds^2 + \frac{(2 - 2\kappa_h(t)s + s^2)^2}{4}dt^2.$$

Here, $\boldsymbol{\ell}(t) = \boldsymbol{\gamma}(t) \pm \boldsymbol{e}$ is the lightlike normal vector field along the surface. For a hyperbolic plane curve $\boldsymbol{\gamma}(t)$ with $\kappa_h^2(t) < 2$, we have

$$-\boldsymbol{\ell}'(t) = \frac{-2}{s^2 - 2\kappa_h(t)s + 2}\frac{\partial F_{(\gamma,n,\pm e)}}{\partial t}(s,t).$$

In this case the surface is non-singular and always umbilically free.

In the last part of this section, we consider the singular horo-flat horocyclic surfaces. By the jet-transversality theorem of Thom [**2**], [**28**], there exists an open dense set $\mathscr{O} \subset C^\infty(I, \mathfrak{hf}(3,1))$ such that for any $C \in \mathscr{O}$, it satisfies the condition that $(\delta_C(t), \delta'_C(t)) \neq (0,0)$. If F_A is a horo-flat horocyclic surface with an isolated singular point, there exists $t_0 \in I$ such that $\delta_C(t_0) = \delta'_C(t_0) = 0$, so that $C \notin \mathscr{O}$. This means that the set of horo-flat horocyclic surfaces with an isolated singular point is not generic in the space horo-flat horocyclic surfaces. By the similar arguments as the above, we can also show that the set of generalized horo-cones is not generic in the space of horo-flat horocyclic surfaces. Therefore, we are interested in horo-flat tangent horocyclic surfaces as horo-flat horocyclic surfaces with singularities.

6. Singularities of horo-flat horocyclic surfaces.

In this section we stick to the study of the generic singularities of horo-flat horocyclic surfaces. A horo-flat tangent horocyclic surface is a horocyclic surface $F_{(\gamma,a_1,a_2)}$ which satisfies $c_2(t) = c_4(t) - c_1(t) = c_3(t) = 0$ (see Section 5). Then the space of horo-flat horocyclic surfaces with curve singularities is $C^\infty(I, \mathfrak{hf}_\sigma(3,1))$ with the Whitney C^∞-topology. In this space the condition $c_5(t) = 0$ is a codimension one condition (in the sufficiently higher order jet space $J^\ell(I, \mathfrak{hf}_\sigma(3,1))$. Therefore, we cannot generically avoid the points where $c_5(t) = 0$. Two branches of the singularities meet at such points. This fact suggests us the situation is quite different from the singularities of general wavefront sets or tangent developables in Euclidean space.

In order to study the singularities of horo-flat tangent horocyclic surfaces, we need the criteria for singularities of wavefronts. Since $H^3_+(-1)$ is a Riemannian manifold, we consider the unit tangent sphere bundle $\pi : S(TH^3_+(-1)) \longrightarrow H^3_+(-1)$ which can be identified with the unit cotangent sphere bundle $\widetilde{\pi} : S(T^*H^3_+(-1)) \longrightarrow H^3_+(-1)$ with the canonical contact structure. Let $U^2 \subset \boldsymbol{R}^2$ be an open set. A map $f : U^2 \longrightarrow H^3_+(-1)$ is called a *frontal map* (respectively, *front*) if there exists an isotropic map (respectively, Legendrian immersion) $L_f : U^2 \longrightarrow S(TH^3_+(-1))$ with respect to the canonical contact structure such that $\pi \circ L_f = f$. In this case we say that L_f is the *Legendrian lift* of f. Here, we identify $S(TH^3_+(-1))$ with $\Delta \subset H^3_+(-1) \times S^3_1$, where S^3_1 is the de Sitter 3-space and $\Delta = \{(\boldsymbol{v}, \boldsymbol{w}) \in H^3_+(-1) \times S^3_1 \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$ (see Sections 2 and 3). Under this identification, we denote L_f as $L_f = (f, \nu) : U^2 \to H^3_+(-1) \times S^3_1$. See [**22**] for detail.

Let $f(u,v) : U^2 \longrightarrow H^3_+(-1)$ be a frontal map. We define a function $\lambda(u,v)$ by

$$f(u,v) \wedge f_u(u,v) \wedge f_v(u,v) = \lambda(u,v)\nu(u,v),$$

where $f_u = \partial f/\partial u$ and $f_v = \partial f/\partial v$. We call $\lambda(u,v)$ a *signed area density function* of f. We remark that $p = (u,v)$ is a singular point of f if and only if $\lambda(u,v) = 0$. A singular point $p \in U$ of f is said to be *non-degenerate* if the derivative $d\lambda$ does not vanish at p. By the implicit function theorem, the singular set $S(f)$ is parameterized by a regular curve $\xi(t) : (-\varepsilon, \varepsilon) \longrightarrow U$ in a neighborhood of a non-degenerate singular point p. Since p is non-degenerate, any $\xi(t)$ is non-degenerate for sufficiently small ε. Then there exists a unique direction $\eta(t) \in T_{\xi(t)}U$ up to scalar multiplications such that $df(\eta(t)) = 0$ for each t. We call $\xi'(t)$ the *singular direction* and $\eta(t)$ the *null-direction*. Then we have the following criterion in order to recognize that the singularities are the cuspidal edge, the swallowtail or the cuspidal cross cap.

Proposition 6.1 ([**26**], [**11**]). *Let $f : U^2 \longrightarrow H^3_+(-1)$ be a frontal map and (f, ν) the Legendrian lift of f. Let p be a non-degenerate singular point of f, ξ a regular curve passing through $\xi(0) = p$ such that* Image ξ *is the singular set of f and η a vector field of null-direction along ξ. We set*

$$\varphi(t) := \det\left(\tilde{\xi}, (\tilde{\xi})', D^f_\eta(\nu \circ \xi), \nu \circ \xi\right)(t) \text{ and } \psi(t) := \det(\xi', \eta)(t)$$

where $\tilde{\xi} = f \circ \xi$, D^f is the canonical covariant derivative along a map f induced from the Levi-Civita connection on $H^3_+(-1)$ and $' = d/dt$. Then

(a) *p is a cuspidal edge (that is, f at p is $\mathscr{A}$-equivalent to cuspidal edge) if and only if (f,ν) is an immersion and $\psi(0)\neq 0$, this means the null direction and the singular direction are transversal.*
(b) *p is a swallowtail if and only if (f,ν) is an immersion, $\psi(0)=0$ and $\psi'(0)\neq 0$.*
(c) *p is a cuspidal cross cap if and only if $\psi(0)\neq 0$, $\varphi(0)=0$ and $\varphi'(0)\neq 0$.*

We remark that $\varphi(0)\neq 0$ if and only if (f,ν) is a Legendrian immersion germ at p when $\tilde{\xi}'(0)\neq 0$. We use this criterion to characterize the cuspidal edge, the swallowtail and the cuspidal cross cap of a horo-flat horocyclic surface. For $F=F_{(\gamma,a_1,a_2)}$, we define

$$\nu(s,t):=-\boldsymbol{a}_2(t)+s\boldsymbol{a}_1(t)+\frac{s^2}{2}(\gamma(t)+\boldsymbol{a}_2(t)). \tag{8}$$

We can easily show that $(F_{(\gamma,a_1,a_2)},\nu)$ gives the Legendrian lift, this means that F is a frontal map. It follows from the condition (6) that the singular set of F is $\{(s,t)\mid s(c_5(t)+sc_6(t)/2)=0\}$. By the straightforward calculations, we have

$$\begin{aligned}
F_t(s,t)&=sc_1(t)\boldsymbol{\gamma}(t)+c_1(t)\boldsymbol{a}_1(t)+sc_1(t)\boldsymbol{a}_2(t)+\left(sc_5(t)+\frac{s^2c_6(t)}{2}\right)\boldsymbol{a}_3(t),\\
F_s(s,t)&=s\boldsymbol{\gamma}(t)+\boldsymbol{a}_1(t)+s\boldsymbol{a}_2(t),\\
\nu_t(s,t)&=sc_1(t)\boldsymbol{\gamma}(t)+c_1(t)\boldsymbol{a}_1(t)+sc_1(t)\boldsymbol{a}_2(t)\\
&\quad+\left(-c_6(t)+sc_5(t)+\frac{s^2c_6(t)}{2}\right)\boldsymbol{a}_3(t),\\
\nu_s(s,t)&=s\boldsymbol{\gamma}(t)+\boldsymbol{a}_1(t)+s\boldsymbol{a}_2(t),\\
\lambda(s,t)&=-s\left(c_5(t)+\frac{sc_6(t)}{2}\right).
\end{aligned}\tag{9}$$

Therefore the singular point $(0,t)$ (respectively, $\{(s,t)\mid c_5(t)=-sc_6(t)/2\}$) is non-degenerate if and only if $c_5(t)\neq 0$ (respectively, $c_6(t)\neq 0$). In both cases, we can show that the condition $c_6(t)\neq 0$ is equivalent to the condition that (F,ν) is a Legendrian immersion. Firstly we consider a singular point $(0,t)$. We can see that singular direction is $(0,1)$ and the null direction is $(c_1(t),-1)$. Then we can detect the functions φ and ψ in Proposition 6.1 as follows:

$$\varphi(t)=c_1(t)c_6(t)\text{ and }\psi(t)=c_1(t).$$

Secondly we assume that $c_6(t)\neq 0$ and consider a singular point $(-2c_5(t)/c_6(t),t)$.

By (9), we have

$$\psi(t) = c_1(t) - \left(\frac{2c_5(t)}{c_6(t)}\right)'.$$

By the above arguments, we have the following theorem except the assertion (B).

Theorem 6.2. *Let $F_{(\gamma,a_1,a_2)}$ be a horo-flat tangent horocyclic surface with singularities along $\boldsymbol{\gamma}$.*

(A) *Suppose that $c_5(t_0) \neq 0$ and $c_6(t_0) \neq 0$, then both the points $(0,t_0)$ and $(-s(t_0),t_0)$ are singularities, where $s(t) = 2c_5(t)/c_6(t)$. In this case we have the following*:
 (1) *The point $(0,t_0)$ is the cuspidal edge if and only if $c_1(t_0) \neq 0$.*
 (2) *The point $(0,t_0)$ is the swallowtail if and only if $c_1(t_0) = 0$ and $c_1'(t_0) \neq 0$.*
 (3) *The point $(-s(t_0),t_0)$ is the cuspidal edge if and only if $(c_1 - s')(t_0) \neq 0$.*
 (4) *The point $(-s(t_0),t_0)$ is the swallowtail if and only if*

$$(c_1 - s')(t_0) = 0 \text{ and } (c_1 - s')'(t_0) \neq 0.$$

(B) *Suppose that $c_5(t_0) = 0$ and $c_6(t_0) \neq 0$, then $s(t_0) = 0$, so that $(0,t_0) = (-s(t_0),t_0)$ is a singular point. In this case, the point $(0,t_0)$ is the cuspidal beaks if and only if $c_5'(t_0) \neq 0$, $c_1(t_0) \neq 0$ and $(c_1 - s')(t_0) \neq 0$.*

(C) *Suppose that $c_5(t_0) \neq 0$ and $c_6(t_0) = 0$, then the point $(0,t_0)$ is the cuspidal cross cap if and only if $c_1(t_0) \neq 0$ and $c_6'(t_0) \neq 0$. In this case, $\boldsymbol{\gamma}(t_0)$ is the only singular point on the generating horocycle $F_{(\gamma,a_1,a_2)}(s,t_0)$.*

For the proof of the assertion (B), we need some more arguments and they will be given in Section 7. The following proposition asserts that the conditions in the above theorem is generic in the space of horo-flat tangent horocyclic surfaces, so that the proof of Theorem 1.1 is completed.

Proposition 6.3. *There exists an open dense subset $\mathscr{O} \subset C^\infty(I, \mathfrak{hf}_\sigma(3,1))$ such that any $C(t) \in \mathscr{O}$ satisfies the following conditions*:

(1) *The set of the points $t_0 \in I$ with $c_1(t_0) = 0$, $c_1'(t_0) \neq 0$, $c_5(t_0) \neq 0$ and $c_6(t_0) \neq 0$ is discrete.*
(2) *The set $\{t_0 \in I \mid c_1(t_0) = 0\} \cap \{t_0 \mid c_1'(t_0) = 0,\ c_5(t_0) = 0$ or $c_6(t_0) = 0\}$ is empty.*
(3) *The set of the points $t_0 \in I$ with $(c_1 - s')(t_0) = 0$, $(c_1 - s')'(t_0) \neq 0$, $c_5(t_0) \neq 0$ and $c_6(t_0) \neq 0$ is discrete.*
(4) *The set $\{t_0 \in I \mid (c_1 - s')(t_0) = 0\} \cap \{t_0 \mid (c_1 - s')'(t_0) = 0,\ c_5(t_0) =$*

0 *or* $c_6(t_0) = 0\}$ *is empty.*

(5) *The set of the points* $t_0 \in I$ *with* $c_5(t_0) = 0$, $c_5'(t_0) \neq 0$, $c_6(t_0) \neq 0$, $c_1(t_0) \neq 0$ *and* $(c_1 - s')(t_0) \neq 0$ *is discrete.*

(6) *The set* $\{t_0 \in I \mid c_5(t_0) = 0\} \cap \{t_0 \mid c_5'(t_0) = 0,\ c_6(t_0) = 0,\ c_1(t_0) = 0$ *or* $(c_1 - s')(t_0) = 0\}$ *is empty.*

(7) *The set of the points* $t_0 \in I$ *with* $c_6(t_0) = 0$, $c_6'(t_0) \neq 0$, $c_5(t_0) \neq 0$ *and* $c_1(t_0) \neq 0$ *is discrete.*

(8) *The set* $\{t_0 \in I \mid c_6(t_0) = 0\} \cap \{t_0 \mid c_6'(t_0) = 0,\ c_5(t_0) = 0$ *or* $c_1(t_0) = 0\}$ *is empty.*

For the proof of the above proposition, we only remark that either the submanifolds corresponding to the conditions (1),(3),(5),(7) are codimension one or the conditions (2),(4),(6),(8) are codimension two in $J^1(I, \mathfrak{hf}_\sigma(3,1))$. Therefore the assertion of the proposition follows from the jet-transversality theorem [**2**], [**28**]. Moreover, by the similar arguments of the above proposition we have the following corollary.

Corollary 6.4. *There exists an open dense subset* $\mathscr{O}' \subset C^\infty(I, \mathfrak{hf}_\sigma(3,1))$ *such that any* $C(t) \in \mathscr{O}'$ *satisfies the following conditions*:

(1) $C(t)$ *satisfies the all conditions in Proposition* 6.3.

(2) *The set of the points* $t_0 \in I$ *with* $c_1(t_0) = 0$ *and* $(c_1 - s')(t_0) = 0$ *is empty.*

Corollary 6.4 asserts that there are no point $t_0 \in I$ such that both of two singularities on the generating horocycle through t_0 are swallowtails in generic.

7. The cuspidal beaks.

In this section we give a proof of the assertion (B) in Theorem 6.2. For the purpose, we start to give a brief review on the theory of Legendrian singularities due to Arnol'd-Zakalyukin [**2**], [**36**], [**37**]. Here we only consider local properties, we consider $\boldsymbol{R}^n$ instead of any n-dimensional manifold. Let $\pi : PT^*(\boldsymbol{R}^n) \longrightarrow \boldsymbol{R}^n$ be the projective cotangent bundle over $\boldsymbol{R}^n$. The total space is a contact manifold equipped with the canonical contact structure K on $PT^*(\boldsymbol{R}^n)$. An immersion $i : L \to PT^*(\boldsymbol{R}^n)$ is said to be a *Legendrian immersion* if $\dim L = n$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ the *Legendrian map* and the set $W(i) = \operatorname{image} \pi \circ i$ the *wave front* of i. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$. We remark that each fiber of $\pi : PT^*(\boldsymbol{R}^n) \longrightarrow \boldsymbol{R}^n$ is a Legendrian submanifold. We say that a smooth fiber bundle $\pi : E \longrightarrow M$ is a *Legendrian fibration* if E is a contact manifold and each fiber is a Legendrian submanifold. It is known that all Legendrian fibrations of a fixed dimension are locally fiber preserving contact diffeomorphic ([**2**, Part III]). Therefore we only consider $\pi : PT^*(\boldsymbol{R}^n) \longrightarrow \boldsymbol{R}^n$ here.

The main tool of the theory of Legendrian singularities is the notion of generating families. Let $F : (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}, \boldsymbol{0})$ be a function germ which we call an *unfolding of* $f(q) = F(q, 0)$. We say that F is a *Morse family of hypersurfaces* if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k} \right) : (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0}) \longrightarrow (\boldsymbol{R} \times \boldsymbol{R}^k, \boldsymbol{0})$$

is non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0})$. In this case we have a smooth $(n-1)$-dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0}) \mid F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\Phi_F : (\Sigma_*(F), \boldsymbol{0}) \longrightarrow PT^*\boldsymbol{R}^n$ defined by

$$\Phi_F(q, x) = \left(x, \left[\frac{\partial F}{\partial x_1}(q, x) : \dots : \frac{\partial F}{\partial x_n}(q, x) \right] \right)$$

is a Legendrian immersion germ. The fundamental result of Arnol'd-Zakalyukin [**2**], [**36**] asserts that all Legendrian submanifold germs in $PT^*\boldsymbol{R}^n$ are constructed by the above method. We call F a *generating family* of $\Phi_F(\Sigma_*(F))$. Therefore the wave front of $\Phi_F(\Sigma_*(F))$ is

$$W(\Phi_F) = \Big\{ x \in \boldsymbol{R}^n \mid \exists q \in \boldsymbol{R}^k \text{ such that } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0 \Big\}.$$

We also write $\mathscr{D}_F = W(\Phi_F)$ and call it the *discriminant set* of F.

We now introduce an equivalence relation among Legendrian submanifold germs. Let $i : (L, p) \subset (PT^*\boldsymbol{R}^n, p)$ and $i' : (L', p') \subset (PT^*\boldsymbol{R}^n, p')$ be Legendrian submanifold germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\boldsymbol{R}^n, p) \longrightarrow (PT^*\boldsymbol{R}^n, p')$ such that H preserves fibers of π and that $H(L) = L'$.

Since the Legendrian lift $i : (L, p) \subset (PT^*\boldsymbol{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs:

Proposition 7.1. *Let $i : (L,p) \subset (PT^*\boldsymbol{R}^n, p)$ and $i' : (L',p') \subset (PT^*\boldsymbol{R}^n, p')$ be Legendrian immersion germs such that the representative of both the regular sets of the projections $\pi \circ i$ and $\pi \circ i'$ are dense. Then i and i' are Legendrian equivalent if and only if wave front sets $W(i)$ and $W(i')$ are diffeomorphic as set germs.*

This result has been firstly pointed out by Zakalyukin [**37**]. The assumption in the above proposition is a generic condition for i and i'.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote by $\mathscr{E}_n$ the local ring of function germs $(\boldsymbol{R}^n, \boldsymbol{0}) \longrightarrow \boldsymbol{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathscr{E}_n \mid h(0) = 0\}$. Let $F, G : (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}, 0)$ be function germs. We say that F and G are *P-$\mathscr{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0})$ of the form $\Psi(q,x) = (\psi_1(q,x), \psi_2(x))$ for $(q,x) \in (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0})$ such that $\Psi^*(\langle F\rangle_{\mathscr{E}_{k+n}}) = \langle G\rangle_{\mathscr{E}_{k+n}}$. Here $\Psi^* : \mathscr{E}_{k+n} \longrightarrow \mathscr{E}_{k+n}$ is the pull back $\boldsymbol{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}, \boldsymbol{0})$ be a function germ. We say that F is a *$\mathscr{K}$-versal unfolding of $f = F|\boldsymbol{R}^k \times \{\boldsymbol{0}\}$* if for any unfolding $G : (\boldsymbol{R}^k \times \boldsymbol{R}^m, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}, \boldsymbol{0})$ of f (i.e., $G(q, \boldsymbol{0}) = f(q)$), there exists a map germ $\phi : (\boldsymbol{R}^m, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}^n, \boldsymbol{0})$ such that ϕ^*F and G are P-$\mathscr{K}$-equivalent, where $\phi^*F(q,u) = F(q, \phi(u))$. For an unfolding $F(t,x)$ of a function $f(t)$ of one-variable, we have the following useful criterion on the $\mathscr{K}$-versal unfoldings in (cf., [**4**, 6.10]): We say that f has an *A_r-singularity* at t_0 if $f^{(p)}(t_0) = 0$ for all $1 \le p \le r$, and $f^{(r+1)}(t_0) \ne 0$. We have the following lemma.

Lemma 7.2. *Let F be an unfolding of f and $f(t)$ has an A_r-singularity $(r \ge 1)$ at t_0. We denote the $(r-1)$-jet of the partial derivative $\partial F/\partial x_i$ at t_0 by*

$$j^{(r-1)}\left(\frac{\partial F}{\partial x_i}(t, x_0)\right)(t_0) = \sum_{j=0}^{r-1} \alpha_{ji}(t - t_0)^j$$

for $i = 1, \ldots, n$. Then F is a $\mathscr{K}$-versal unfolding if and only if the $r \times n$ matrix of coefficients (α_{ji}) has rank r $(r \le n)$.

It follows from the above lemma that the function germ defined by

$$t^{r+1} + x_1 t^{r-1} + x_2 t^{r-2} + \cdots + x_{r-1} t + x_r$$

is a $\mathscr{K}$-versal unfolding of $f(t) = t^{r+1}$. One of the main results in the theory of Legendrian singularities is the following theorem:

Theorem 7.3. *Let $F, G : (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}, 0)$ be Morse families of hypersurfaces. Then Φ_F and Φ_G are Legendrian equivalent if and only if F and G are P-$\mathscr{K}$-equivalent.*

Since F, G are function germs on the common space germ $(\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0})$, we do no need the notion of stably P-$\mathscr{K}$-equivalences under this situation (cf., [**2**], [**36**]). As a corollary of Proposition 7.1 and Theorem 7.3, we have the following proposition.

Proposition 7.4. *Let $F, G : (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}, 0)$ be Morse families of hypersurfaces. Suppose that both regular sets of the representative of projections $\pi \circ \Phi_F, \pi \circ \Phi_G$ are dense. Then $(W(\Phi_F), 0)$ and $(W(\Phi_G), 0)$ are diffeomorphic as set germs if and only if F and G are P-$\mathscr{K}$-equivalent.*

On the other hand, Zakalyukin gave a generic classification of one-parameter bifurcations of wave fronts [**37**]. Here we apply his idea to recognize the cuspidal beaks. We now consider the special case when $k = 1, n = 3$. Let $F : (\boldsymbol{R} \times \boldsymbol{R}^3, 0) \longrightarrow (\boldsymbol{R}, 0)$ be an unfolding of $f(t) = F(t, 0)$ such that $f(t)$ is the A_3-type. Let $\widetilde{F} : (\boldsymbol{R} \times \boldsymbol{R}^4, 0) \longrightarrow (\boldsymbol{R}, 0)$ be an unfolding of $f(t)$ defined by $\widetilde{F}(t, \boldsymbol{v}, u) = F(t, \boldsymbol{v}) + ut^2$. Since $f(t)$ is the A_3-type, $f(t)$ is $\mathscr{K}$-equivalent to t^4 (cf., [**4**], [**28**]). Therefore we assume that $f(t) = t^4$. Since $t^4 + x_1 t^2 + x_2 t + x_3$ is a $\mathscr{K}$-versal unfolding of $f(t)$, there exists a map germ $\phi : (\boldsymbol{R}^3, 0) \longrightarrow (\boldsymbol{R}^3, 0)$ such that $F(t, \boldsymbol{v})$ is P-$\mathscr{K}$-equivalent to $t^4 + \phi_1(\boldsymbol{v})t^2 + \phi_2(\boldsymbol{v})t + \phi_3(\boldsymbol{v})$, where $\phi(\boldsymbol{v}) = (\phi_1(\boldsymbol{v}), \phi_2(\boldsymbol{v}), \phi_3(\boldsymbol{v}))$, so that we assume that $F(t, \boldsymbol{v}) = t^4 + \phi_1(\boldsymbol{v})t^2 + \phi_2(\boldsymbol{v})t + \phi_3(\boldsymbol{v})$. Then we have the following proposition.

Proposition 7.5. *Let $F : (\boldsymbol{R} \times \boldsymbol{R}^3, 0) \longrightarrow (\boldsymbol{R}, 0)$ be an unfolding of an A_3-type germ $f(t)$. Then $\widetilde{F}(t, \boldsymbol{v}, u)$ is a $\mathscr{K}$-versal unfolding of $f(t)$ if and only if $F(t, \boldsymbol{v})$ is a Morse family of hypersurfaces.*

Proof. Since both notions are invariant under the P-$\mathscr{K}$-equivalence, we may assume that $F(t, \boldsymbol{v}) = t^4 + \phi_1(\boldsymbol{v})t^2 + \phi_2(\boldsymbol{v})t + \phi_3(\boldsymbol{v})$. Suppose that $F(t, \boldsymbol{v})$ is a Morse family of hypersurfaces. This means that

$$\Delta^*(F) = \left(F, \frac{\partial F}{\partial t}\right) : (\boldsymbol{R} \times \boldsymbol{R}^3, 0) \longrightarrow (\boldsymbol{R}^2, 0)$$

is regular at $\boldsymbol{0}$, so that the rank of the Jacobian matrix of $\Delta^*(F)$,

$$J_{\Delta^* F}(\boldsymbol{0}) = \begin{pmatrix} 0 & \dfrac{\partial \phi_3}{\partial v_1}(0) & \dfrac{\partial \phi_3}{\partial v_2}(0) & \dfrac{\partial \phi_3}{\partial v_3}(0) \\ 0 & \dfrac{\partial \phi_2}{\partial v_1}(0) & \dfrac{\partial \phi_2}{\partial v_2}(0) & \dfrac{\partial \phi_2}{\partial v_3}(0) \end{pmatrix}$$

is two.

On the other hand, we have $\partial\widetilde{F}/\partial v_i = (\partial\phi_1/\partial v_i)t^2 + (\partial\phi_2/\partial v_i)t + \partial\phi_3/\partial v_i$ and $\partial\widetilde{F}/\partial u = t^2$. Therefore we have

$$j^2\left(\frac{\partial\widetilde{F}}{\partial v_i}(t,0)\right)(0) = \frac{\partial\phi_3}{\partial v_i}(0) + \frac{\partial\phi_2}{\partial v_i}(0)t + \frac{\partial\phi_1}{\partial v_i}(0)t^2, \quad (i=1,2,3)$$
$$j^2\left(\frac{\partial\widetilde{F}}{\partial u}(t,0)\right)(0) = t^2$$

By Lemma 7.2, $\widetilde{F}$ is $\mathscr{K}$-versal if and only if the rank of

$$\begin{pmatrix} \dfrac{\partial\phi_3}{\partial v_1}(0) & \dfrac{\partial\phi_3}{\partial v_2}(0) & \dfrac{\partial\phi_3}{\partial v_3}(0) & 0 \\ \dfrac{\partial\phi_2}{\partial v_1}(0) & \dfrac{\partial\phi_2}{\partial v_2}(0) & \dfrac{\partial\phi_2}{\partial v_3}(0) & 0 \\ \dfrac{\partial\phi_1}{\partial v_1}(0) & \dfrac{\partial\phi_1}{\partial v_2}(0) & \dfrac{\partial\phi_1}{\partial v_3}(0) & 1 \end{pmatrix}$$

is three. This condition is equivalent to the condition that the rank of $J_{\Delta^*(F)}(0)$ is two. $\square$

We now assume that F is a Morse family of hypersurfaces, so that the rank of $J_{\Delta^*(F)}(0)$ is two. Therefore the map germ $\widetilde{\phi} : (\boldsymbol{R}^3, 0) \longrightarrow (\boldsymbol{R}^2, 0)$ defined by $\widetilde{\phi}(\boldsymbol{v}) = (\phi_2(\boldsymbol{v}), \phi_3(\boldsymbol{v}))$ is a submersion germ. Without the loss of generality, by the implicit function theorem, there exists a diffeomorphism germ $\psi : (\boldsymbol{R}^3, 0) \longrightarrow (\boldsymbol{R}^3, 0)$ such that $\widetilde{\phi} \circ \psi(\boldsymbol{v}) = (v_2, v_3)$. Therefore we have

$$\psi^* F(t, \boldsymbol{v}) = t^4 + \widetilde{\phi}_1(\boldsymbol{v})t^2 + v_2 t + v_3,$$

for a function germ $\widetilde{\phi}_1 : (\boldsymbol{R}^3, 0) \longrightarrow (\boldsymbol{R}, 0)$. If $(\partial\widetilde{\phi}_1/\partial v_1)(0) \neq 0$, then $\psi^*F(t,\boldsymbol{v})$ is $\mathscr{K}$-versal, so that F is already $\mathscr{K}$-versal. Suppose that $(\partial\widetilde{\phi}_1/\partial v_1)(0) = 0$, then $\widetilde{\psi^* F}(t, \boldsymbol{v}, u) = \psi^*F(t,\boldsymbol{v}) + ut^2$ is a $\mathscr{K}$-versal deformation of $f(t) = t^4$ such that $\psi^*F(t,\boldsymbol{v})$ is P-$\mathscr{K}$-equivalent to F. Suppose that $\widetilde{\phi}_1(v_1, 0, 0)$ has the Morse type singularity at the origin (i.e., $(\partial^2\widetilde{\phi}_1/\partial v_1^2)(0) \neq 0$). By the parametrized Morse lemma, there exists a diffeomorphism germ $\sigma : (\boldsymbol{R}^3, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}^3, \boldsymbol{0})$ such that $\widetilde{\phi}_1 \circ \sigma(\boldsymbol{v}) = g(v_2, v_3) \pm v_1^2$. It follows that

$$\sigma^*\psi^*F(t,\boldsymbol{v}) = t^4 + \big(g(v_2,v_3) \pm v_1^2\big)t^2 + v_2 t + v_3.$$

We now write that $\overline{F}(t,\boldsymbol{v}) = \sigma^*\psi^*F(t,\boldsymbol{v})$. We also define an unfolding $H(t,\boldsymbol{v})$ by

$$H(t,\boldsymbol{v}) = t^4 \pm v_1^2t^2 + v_2t + v_3.$$

Consider a $\mathscr{K}$-versal unfolding $G : (\boldsymbol{R}\times\boldsymbol{R}^3,\boldsymbol{0}) \longrightarrow (\boldsymbol{R},0)$ defined by

$$G(t,v_2,v_3,u) = t^4 + ut^2 + v_2t + v_3.$$

We also consider a $\mathscr{K}$-versal unfolding $\overline{G} : (\boldsymbol{R}\times\boldsymbol{R}^4,\boldsymbol{0}) \longrightarrow (\boldsymbol{R},0)$ defined by $\overline{G}(t,v_1,v_2,v_3,u) = G(t,v_2,v_3,u)$. Then we have $\mathscr{D}_{\overline{G}} = \boldsymbol{R}\times\mathscr{D}_G$. We now define a function germ $\tau : (\boldsymbol{R}^3,\boldsymbol{0}) \longrightarrow (\boldsymbol{R},0)$ by $\tau(v_2,v_3,u) = u - g(v_2,v_3)$. We need the following key lemma ([**37**, Theorem 1.4]).

Lemma 7.6. *Let $\mathscr{F} : (\boldsymbol{R}\times\boldsymbol{R}^k,\boldsymbol{0}) \longrightarrow (\boldsymbol{R},0)$ be a $\mathscr{K}$-versal unfolding defined by*

$$\mathscr{F}(t,\boldsymbol{u}) = t^{k+1} + u_1t^{k-1} + u_2t^{k-2} + \cdots + u_k$$

and $\sigma : (\boldsymbol{R}^k,\boldsymbol{0}) \longrightarrow (\boldsymbol{R},0)$ a function germ with $\partial\sigma/\partial u_1(\boldsymbol{0}) > 0$. Then there exists a diffeomorphism germ $\Phi : (\boldsymbol{R}^k,\boldsymbol{0}) \longrightarrow (\boldsymbol{R}^k,\boldsymbol{0})$ such that $\Phi(\mathscr{D}_{\mathscr{F}}) = \mathscr{D}_{\mathscr{F}}$ and $\sigma\circ\Phi(u_1,\dots,u_k) = u_1$.

We remark that Zakalyukin has shown this lemma for much more general situation than the above case. However, we only need the above simple case in this paper.

We apply the above lemma to G and τ. Then there exists a diffeomorphism germ $\Phi : (\boldsymbol{R}^3,\boldsymbol{0}) \longrightarrow (\boldsymbol{R}^3,\boldsymbol{0})$ such that $\Phi(\mathscr{D}_G) = (\mathscr{D}_G)$ and $\tau\circ\Phi(v_2,v_3,u) = u$.

On the other hand, we define an unfolding $G_g : (\boldsymbol{R}\times\boldsymbol{R}^4,\boldsymbol{0}) \longrightarrow (\boldsymbol{R},0)$ by

$$G_g(t,v_1,v_2,v_3,u) = t^4 + \bigl(u + g(v_2,v_3) \pm v_1^2\bigr)t^2 + v_2t + v_3.$$

Let $\Psi : (\boldsymbol{R}^4,\boldsymbol{0}) \longrightarrow (\boldsymbol{R}^4,\boldsymbol{0})$ be a diffeomorphism germ defined by

$$\Psi(v_1,v_2,v_3,u) = \bigl(v_1,v_2,v_3,u - g(v_2,v_3) \mp v_1^2\bigr).$$

Then we have

$$\Psi^*G_g(t,v_1,v_2,v_3,u) = G_g(t,\Psi(v_1,v_2,v_3,u)) = \overline{G}(t,v_1,v_2,v_3,u)$$

and $\pi\circ\Psi(v_1,v_2,v_3,u) = u - g(v_2,v_3) \mp v_1^2$, where $\pi(v_1,v_2,v_3,u) = u$. We denote that

$$\widetilde{\tau}(v_1, v_2, v_3, u) = u - g(v_2, v_3) \pm v_1^2.$$

Then we have $(1_{\boldsymbol{R}} \times \Phi)(\mathscr{D}_{\overline{G}}) = \mathscr{D}_{\overline{G}}$ and $\widetilde{\tau} \circ (1_{\boldsymbol{R}} \times \Phi)(v_1, v_2, v_3, u) = u \pm v_1^2$.

We also define a diffeomorphism germ $\Theta : (\boldsymbol{R}^4, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}^4, \boldsymbol{0})$ by

$$\Theta(t, v_1, v_2, v_3, u) = \big(t, v_1, v_2, v_3, u \pm v_1^2\big).$$

Then we have $\Theta^* \overline{G} = \overline{H}$, where

$$\overline{H}(t, v_1, v_2, v_3, u) = t^4 + \big(u \pm v_1^2\big)t^2 + v_2 t + v_3.$$

It follows that $\Theta(\mathscr{D}_{\overline{H}}) = \mathscr{D}_{\overline{G}}$ and $\pi \circ \Theta^{-1}(t, v_1, v_2, v_3, u) = u \pm v_1^2$. Therefore, we have a diffeomorphism $\widetilde{\Phi} : (\boldsymbol{R}^4, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}^4, \boldsymbol{0})$ defined by $\widetilde{\Phi} = \Psi \circ (1_{\boldsymbol{R}} \times \Phi) \circ \Theta$. By the above arguments, we have $\widetilde{\Phi}(\mathscr{D}_{\overline{H}}) = \mathscr{D}_{G_g}$ and $\pi \circ \widetilde{\Phi} = \pi$. By Proposition 7.4, there exists a diffeomorphism germ $\widetilde{\Psi} : (\boldsymbol{R}^4, \boldsymbol{0}) \longrightarrow (\boldsymbol{R}^4, \boldsymbol{0})$ of the form

$$\begin{aligned}\widetilde{\Psi}(t, v_1, v_2, v_3, u) = \big(&\psi_0(t, v_1, v_2, v_3, u), \psi_1(v_1, v_2, v_3, u),\\ &\psi_2(v_1, v_2, v_3, u), \psi_3(v_1, v_2, v_3, u), \psi_4(u)\big)\end{aligned}$$

such that $\widetilde{\Psi}^*(\langle \overline{H} \rangle_{\mathscr{E}_{1+4}}) = \langle G_g \rangle_{\mathscr{E}_{1+4}}$. If we restrict the above relation on $u = 0$, then $\sigma^* \psi^* F$ is P-$\mathscr{K}$-equivalent to H. This means that F is P-$\mathscr{K}$-equivalent to H.

On the other hand, for $\psi^* F(t, \boldsymbol{v}) = t^4 + \widetilde{\phi}_1(\boldsymbol{v})t^2 + v_2 t + v_3$, $\Sigma_*(\psi^* F)$ is defined by the equations:

$$\begin{cases} h_1(t, \boldsymbol{v}) = t^4 + \widetilde{\phi}_1(\boldsymbol{v})t^2 + v_2 t + v_3 = 0 \\ h_2(t, \boldsymbol{v}) = 4t^3 + 2\widetilde{\phi}_1(\boldsymbol{v})t + v_2 = 0. \end{cases}$$

We now consider a function germ $\rho : (\Sigma_*(\psi^* F), 0) \longrightarrow \boldsymbol{R}$ defined by

$$\rho(t, v_1) = \left.\frac{\partial^2 \psi^* F}{\partial t^2}\right|_{\Sigma_*(\psi^* F)} = 12t^2 + 2\widetilde{\phi}_1(\boldsymbol{v}),$$

where $(t, \boldsymbol{v}) \in \Sigma_*(\psi^* F)$. Differentiating both functions $h_i(t, \boldsymbol{v}) = 0$ $(i = 1, 2)$ with respect to t and v_1, we have

$$\frac{\partial v_i}{\partial t}(0) = \frac{\partial^2 v_i}{\partial t^2}(0) = \frac{\partial^2 v_i}{\partial t \partial v_1}(0) = 0 \quad (i = 2, 3).$$

Here, we use the fact $(\partial \widetilde{\phi}_1 / \partial v_1)(0) = 0$. It follows that

$$\mathrm{Hess}(\rho)(0) = \begin{pmatrix} 24 & 0 \\ 0 & \dfrac{\partial^2 \widetilde{\phi}_1}{\partial v_1^2}(0) \end{pmatrix},$$

where $\mathrm{Hess}(\rho)(0)$ is a Hessian matrix of ρ at 0. Therefore, $\widetilde{\phi}_1(v_1, 0, 0)$ has the Morse type singularity at 0 if and only if $\rho(t, v_1)$ has the Morse type singularity at 0. We have almost completed the proof of the following recognition lemma.

Lemma 7.7 (Recognition lemma for the cuspidal beaks or the cuspidal lips). *Let $F : (\boldsymbol{R} \times \boldsymbol{R}^3, 0) \longrightarrow (\boldsymbol{R}, 0)$ be a Morse family of hypersurfaces such that $f(t) = F(t, 0)$ is the A_3-type germ. If the function germ $(\partial^2 F / \partial t^2)|_{\Sigma_*(F)}$ has the Morse type singularity at $0 \in \Sigma_*(F)$, then $F(t, \boldsymbol{v})$ is P-$\mathscr{K}$-equivalent to*

$$t^4 \pm v_1^2 t^2 + v_2 t + v_3.$$

Proof. By the previous arguments, it is enough to show the following fact: Suppose that $F, G : (\boldsymbol{R} \times \boldsymbol{R}^n, 0) \longrightarrow (\boldsymbol{R}, 0)$ are the Morse families of hypersurfaces. If F and G are P-$\mathscr{K}$-equivalent, then $(\partial^2 F / \partial t^2)|_{\Sigma_*(F)}$ has the Morse type singularity at the origin if and only if $(\partial^2 G / \partial t^2)|_{\Sigma_*(G)}$ has the Morse type singularity at the origin. This fact follows from definition and straightforward calculations. □

In order to apply the above lemma to our situation, we now consider a family of functions $H : I \times H^3_+(-1) \longrightarrow \boldsymbol{R}$ defined by

$$H(t, \boldsymbol{v}) = \langle \boldsymbol{\ell}(t), \boldsymbol{v} \rangle + 1,$$

where $\boldsymbol{\ell}(t) = \boldsymbol{\gamma}(t) + \boldsymbol{a}_2(t)$. Firstly we consider the derivatives of $H(t, \boldsymbol{v})$ with respect to t. We assume that $c_6(t_0) \neq 0$. Since $\boldsymbol{\ell}'(t) = c_6(t) \boldsymbol{a}_3(t)$, the discriminant set $\mathscr{D}_H$ of H is the horo-flat horocyclic surface

$$F_{(\gamma, a_1, a_2)}(s, t) = \boldsymbol{\gamma}(t) + s \boldsymbol{a}_1(t) + \frac{s^2}{2} \boldsymbol{\ell}(t)$$

around t_0. Suppose that $\boldsymbol{v}_0 = \boldsymbol{\gamma}(t_0)$, then we have

$$\frac{\partial H}{\partial t}(t_0, \boldsymbol{v}_0) = 0, \quad \frac{\partial^2 H}{\partial t^2}(t_0, \boldsymbol{v}_0) = 0, \quad \frac{\partial^3 H}{\partial t^3}(t_0, \boldsymbol{v}_0) = c_5(t_0)c_6(t_0)c_1(t_0),$$

and

$$\begin{aligned}\frac{\partial^4 H}{\partial t^4}(t_0, \boldsymbol{v}_0) = c_5(t_0)\big(c_1'(t_0)c_6(t_0) + 3c_1(t_0)c_6'(t_0)\big) \\ + c_6(t_0)c_1(t_0)\big(2c_5'(t_0) - c_6(t_0)c_1(t_0)\big).\end{aligned}$$

By the above calculations, if we assume that $c_5(t_0) = 0$, $c_6(t_0) \neq 0$, $c_5'(t_0) \neq 0$, $c_1(t_0) \neq 0$ and $(c_1 - s')(t_0) \neq 0$, then $h_{v_0}(t) = H(t, \boldsymbol{v}_0)$ has an A_4-singularity at t_0.

We now define a 4-dimensional unfolding $\widetilde{H} : I \times H^3_+(-1) \times \boldsymbol{R} \longrightarrow \boldsymbol{R}$ by

$$\widetilde{H}(t, \boldsymbol{v}, u) = H(t, \boldsymbol{v}) + u(t - t_0)^2 = \langle \boldsymbol{\ell}(t), \boldsymbol{v} \rangle + u(t - t_0)^2 + 1.$$

Here we consider that $\widetilde{H}$ is a germ at $(t_0, \boldsymbol{v}_0, 0)$.

Lemma 7.8. *We assume that* $c_5(t_0) = 0, c_6(t_0) \neq 0, c_5'(t_0) \neq 0, c_1(t_0) \neq 0$ *and* $(c_1 - s)'(t_0) \neq 0$, *then* $\widetilde{H}$ *is a* $\mathscr{K}$*-versal deformation of* h_{v_0}.

Proof. Since the curve $C(t) \in \mathfrak{so}(3, 1)$ is a hyperbolic invariant, we assume that $\boldsymbol{\gamma}(t_0) = (1, 0, 0, 0)$ by a suitable hyperbolic transformation. Moreover, we assume that $t_0 = 0$ by a parameter transformation. In this case $\widetilde{H}(t, \boldsymbol{v}, u) = \langle \boldsymbol{\ell}(t), \boldsymbol{v} \rangle + ut^2$. If we denote that $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$ and $\boldsymbol{\ell}(t) = (\ell_0(t), \ell_1(t), \ell_2(t), \ell_3(t))$, we have

$$\widetilde{H}(t, \boldsymbol{v}, u) = -\ell_0(t)v_0 + \ell_1(t)v_1 + \ell_2(t)v_2 + \ell_3(t)v_3 + ut^2 + 1.$$

We adopt the local coordinate of $H^3_+(-1)$ by $\boldsymbol{v} = (\sqrt{v_1^2 + v_2^2 + v_3^2 + 1}, v_1, v_2, v_3)$, so that we have $(\partial \widetilde{H}/\partial v_i)(t, \boldsymbol{v}, u) = -\ell_0(t)(v_i/v_0) + \ell_i(t)$, $(i = 1, 2, 3)$. Since $\boldsymbol{v}_0 = \boldsymbol{\gamma}(0) = (1, 0, 0, 0)$, we have

$$j^2\left(\frac{\partial \widetilde{H}}{\partial v_1}(t, \boldsymbol{v}_0, 0)\right)(0) = \ell_1(0) + \ell_1'(0)t + \frac{1}{2}\ell_1''(0)t^2,$$

$$j^2\left(\frac{\partial \widetilde{H}}{\partial v_2}(t, \boldsymbol{v}_0, 0)\right)(0) = \ell_2(0) + \ell_2'(0)t + \frac{1}{2}\ell_2''(0)t^2,$$

$$j^2\left(\frac{\partial \widetilde{H}}{\partial v_3}(t, \boldsymbol{v}_0, 0)\right)(0) = \ell_3(0) + \ell_3'(0)t + \frac{1}{2}\ell_3''(0)t^2,$$

$$j^2\left(\frac{\partial\widetilde{H}}{\partial u}(t,\boldsymbol{v}_0,0)\right)(0)=t^2.$$

It is enough to show that

$$\operatorname{rank}\begin{pmatrix}\ell_1(0) & \ell_1'(0) & \ell_1''(0)\\ \ell_2(0) & \ell_2'(0) & \ell_2''(0)\\ \ell_3(0) & \ell_3'(0) & \ell_3''(0)\\ 0 & 0 & 1\end{pmatrix}=\operatorname{rank}\begin{pmatrix}0 & 0 & 1\\ \ell_1(0) & \ell_1'(0) & 0\\ \ell_2(0) & \ell_2'(0) & 0\\ \ell_3(0) & \ell_3'(0) & 0\end{pmatrix}=3.$$

Since $\langle\boldsymbol{\ell}(t),\boldsymbol{\ell}(t)\rangle=\langle\boldsymbol{\ell}(t),\boldsymbol{\ell}'(t)\rangle=0$, we have

$$\ell_0=\frac{\ell_1^2}{\ell_0}+\frac{\ell_2^2}{\ell_0}+\frac{\ell_3^2}{\ell_0},\quad \ell_0'=\frac{\ell_1\ell_1'}{\ell_0}+\frac{\ell_2\ell_2'}{\ell_0}+\frac{\ell_3\ell_3'}{\ell_0}.$$

It follows that the rank of the last matrix has the same value as the rank of

$$\begin{pmatrix}\ell_0(0) & \ell_0'(0) & 1\\ \ell_1(0) & \ell_1'(0) & 0\\ \ell_2(0) & \ell_2'(0) & 0\\ \ell_3(0) & \ell_3'(0) & 0\end{pmatrix}.$$

Here, $\boldsymbol{\ell}(t)=\boldsymbol{\gamma}(t)+\boldsymbol{a}_2(t)$, then $\boldsymbol{\ell}(0)=\boldsymbol{\gamma}(0)+\boldsymbol{a}_2(0)$ and $\boldsymbol{\ell}'(0)=c_6(0)\boldsymbol{a}_3(0)$. Remember that $\{\boldsymbol{\gamma},\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3\}$ is a pseudo-orthonormal frame at any t. Therefore $\boldsymbol{\ell}(0),\boldsymbol{\ell}'(0),\boldsymbol{\gamma}(0)$ are linearly independent under the condition $c_6(t_0)\neq 0$. Hence the rank of the above matrix is three. This completes the proof. $\square$

By Proposition 7.5, H is a Morse family of hypersurfaces. We can give the proof of the assertion (B) in Theorem 6.2.

Proof of Theorem 6.2, (B). Since $\widetilde{H}$ is a $\mathscr{K}$-versal deformation and H is a Morse family of hypersurfaces, we now calculate $\rho=(\partial^2H/\partial t^2)|\Sigma_*(H)$. Since $\Sigma_*(H)$ is the horo-flat horocyclic surface corresponding to $C(t)\in\mathfrak{so}(3,1)$, we have

$$\begin{aligned}\frac{\partial^2H}{\partial t^2}(t,s)&=\left\langle\ell''(t),\boldsymbol{\gamma}(t)+s\boldsymbol{a}_1(t)+\frac{s^2}{2}\ell(t)\right\rangle\\ &=\left\langle-c_5(t)c_6(t)\boldsymbol{a}_1(t)-c_6(t)^2\boldsymbol{a}_2(t)+c_6'(t)\boldsymbol{a}_3(t),\boldsymbol{\gamma}(t)+s\boldsymbol{a}_1(t)+\frac{s^2}{2}\ell(t)\right\rangle\end{aligned}$$

$$= -sc_5(t)c_6(t) - \frac{s^2}{2}c_6(t)^2.$$

The Hessian matrix of $\rho(s,t) = -sc_5(t)c_6(t) - \frac{s^2}{2}c_6(t)^2$ at $(0, t_0)$ is

$$\mathrm{Hess}(\rho)(0,t_0) = \begin{pmatrix} -c_6^2(t_0) & -c_6'(t_0)c_5(t_0) - c_6(t_0)c_5'(t_0) \\ -c_6'(t_0)c_5(t_0) - c_6(t_0)c_5'(t_0) & 0 \end{pmatrix}.$$

Since $c_5(t_0) = 0, c_5'(t_0) \neq 0$ and $c_6(t_0) \neq 0$, we have $\det \mathrm{Hess}(\rho)(0,t_0) \neq 0$. By Lemma 6.3, H is P-$\mathscr{K}$-equivalent to $t^4 \pm v_1^2t^2 + v_2t + v_3$. The singular set of $F_{(\gamma,a_1,a_2)}$ is given by $\rho(s,t) = 0$. Therefore it consists of two curves transversally intersect at $(0, t_0)$. Therefore the normal form $t^4 - v_1^2t^2 + v_2t + v_3$ is the generating family of the corresponding Legendrian lift. It is nothing but the cuspidal beaks. □

8. Duality between $H_+^3(-1)$ and LC_+^*.

In this section we consider Legendrian dualities between curves and surfaces in $H_+^3(-1)$ or LC_+^*. In [**23**] we have established the duality between pseudo-spheres in Minkowski space. Although there are four dual relations, we only consider the following double fibration:

(a) $H^3(-1) \times LC_+^* \supset \Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -1\}$,
(b) $\pi_{21} : \Delta_2 \longrightarrow H^3(-1)$, $\pi_{22} : \Delta_2 \longrightarrow LC_+^*$,
(c) $\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_2$, $\theta_{22} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_2$.

Here, $\pi_{21}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{22}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$, $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0dv_0 + \sum_{i=1}^3 w_idv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_0dw_0 + \sum_{i=1}^3 v_idw_i$. We remark that $\theta_{21}^{-1}(0)$ and $\theta_{22}^{-1}(0)$ define the same tangent hyperplane field over Δ_2 which is denoted by K_2. In [**23**] we have shown that (Δ_2, K_2) is a contact manifold such that each fibration π_{2i} $(i = 1, 2)$ is a Legendrian fibration. We say that smooth mappings $f : U \longrightarrow H_+^3(-1)$ and $g : U \longrightarrow LC_+^*$ are the *dual* relative to (Δ_2, K_2) if there exists a mapping $\mathscr{L}_{(f,g)} : U \longrightarrow \Delta_2$ such that $\pi_{21} \circ \mathscr{L}_{(f,g)} = f$, $\pi_{22} \circ \mathscr{L}_{(f,g)} = g$ and $\mathscr{L}_{(f,g)}^*\theta_{21} = 0$ (i.e., *integrable* with respect to K_2). If a mapping $f : U \longrightarrow H_+^3(-1)$ is an immersion (i.e., regular surface), we always have the dual of f which is the lightcone Gauss image $\boldsymbol{L}$ of f.

For any pseudo-orthonormal frame $\{\boldsymbol{\gamma}(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$, we have the hyperbolic invariant $C : I \longrightarrow \mathfrak{so}(3,1)$ defined in Section 5. We now define a surface

$$L_{(\gamma,a_2,a_3)} : [0, 2\pi) \times I \longrightarrow LC_+^*$$

by

$$L_{(\gamma,a_2,a_3)}(\theta,t) = \boldsymbol{\gamma}(t) + \cos\theta \boldsymbol{a}_2(t) + \sin\theta \boldsymbol{a}_3(t).$$

We call $L_{(\gamma,a_2,a_3)}(\theta,t)$ a *lightcone circular surface* with respect to $C : I \longrightarrow \mathfrak{so}(3,1)$. For any fixed $t_0 \in I$, we have a circle $\boldsymbol{\gamma}(t_0) + \cos\theta \boldsymbol{a}_2(t_0) + \sin\theta \boldsymbol{a}_3(t_0)$ through $\boldsymbol{\ell}(t_0) = \boldsymbol{\gamma}(t_0) + \boldsymbol{a}_2(t_0)$. We call it a *generating circle.* We have

$$\frac{\partial L_{(\gamma,a_2,a_3)}}{\partial\theta} = -\sin\theta \boldsymbol{a}_2(t) + \cos\theta \boldsymbol{a}_3(t),$$
$$\frac{\partial L_{(\gamma,a_2,a_3)}}{\partial t} = \boldsymbol{\gamma}'(t) + \cos\theta \boldsymbol{a}_2'(t) + \sin\theta \boldsymbol{a}_3'(t).$$

Therefore, $\boldsymbol{\gamma}(t)$ is a (hyperbolic) normal at any regular point (θ,t) if and only if

$$0 = \left\langle \frac{\partial L_{(\gamma,a_2,a_3)}}{\partial t}(\theta,t), \boldsymbol{\gamma}(t) \right\rangle = -\cos\theta c_2(t) - \sin\theta c_3(t)$$

for any θ, which is equivalent to the condition

$$c_2(t) = c_3(t) = 0.$$

Therefore, the regular part of the surface $L_{(\gamma,a_2,a_3)}(\theta,t)$ is flat with respect to the hyperbolic normal if and only if $c_2(t) = c_3(t) = 0$. We call the surface $L_{(\gamma,a_2,a_3)}(\theta,t)$ a *hyperbolic-flat lightcone circular surface* in the sense of [**23**]. If $c_2(t) = c_3(t) = 0$, we have

$$\frac{\partial L_{(\gamma,a_2,a_3)}}{\partial t} = (c_1(t) - c_4(t)\cos\theta - c_5(t)\sin\theta)\boldsymbol{a}_1(t) - c_6(t)\sin\theta \boldsymbol{a}_2(t) + c_6(t)\cos\theta \boldsymbol{a}_3(t).$$

It follows that (θ,t) is a singular point if and only if

$$c_1(t) - c_4(t)\cos\theta - c_5(t)\sin\theta = 0.$$

Therefore, $(0,t)$ is always singular if and only if $c_1(t) - c_4(t) = 0$. In this case, $\boldsymbol{\ell}'(t) = c_6(t)\boldsymbol{a}_3(t)$ and the generating circle is tangent to $\boldsymbol{\ell}(t)$. We call $L_{(\gamma,a_2,a_3)}$ a *hyperbolic-flat tangent lightcone circular surface* if $c_2(t) = c_3(t) = c_1(t) - c_4(t) = 0$. However, the condition $c_2(t) = c_1(t) - c_4(t) = 0$ means that the horocyclic surface $F_{(\gamma,a_1,a_2)}(s,t)$ is horo-flat. Moreover, the condition $c_2(t) = c_1(t) - c_4(t) = c_3(t) = 0$ is equivalent to the condition that $F_{(\gamma,a_1,a_2)}(s,t)$ is a horo-flat tangent horocyclic surface such that one of the branches of singularities is located on the set $(0,t)$.

Therefore we have shown the following proposition.

PROPOSITION 8.1. *For any $C : I \longrightarrow \mathfrak{hf}_\sigma(3,1)$, we have the following:*

(1) *The horocyclic surface $F_{(\gamma,a_1,a_2)}(s,t)$ is a horo-flat tangent horocyclic surface such that one of the branches of the singularities is located on the set $s = 0$ whose image is $\boldsymbol{\gamma}(t)$.*
(2) *The lightcone circular surface $L_{(\gamma,a_2,a_3)}(\theta,t)$ is a hyperbolic-flat tangent lightcone circular surface such that one of the branches of the singularities is located on the set $\theta = 0$ whose image is $\boldsymbol{\ell}(t) = \boldsymbol{\gamma}(t) + \boldsymbol{a}_2(t)$.*

We can show that $\langle F_{(\gamma,a_1,a_2)}(s,t), \boldsymbol{\ell}(t)\rangle = \langle \boldsymbol{\gamma}(t), L_{(\gamma,a_2,a_3)}(\theta,t)\rangle = -1$, so that we have two well defined mappings

$$\mathscr{L}_{(F_{(\gamma,a_1,a_2)},\ell)} : J \times I \longrightarrow \Delta_2,$$
$$\mathscr{L}_{(\gamma,L_{(\gamma,a_2,a_3)})} : [0,2\pi) \times I \longrightarrow \Delta_2.$$

Since $\boldsymbol{\ell}(t)$ (respectively, $\boldsymbol{\gamma}(t)$) is the normal of $F_{(\gamma,a_1,a_2)}(s,t)$ (respectively, $L_{(\gamma,a_2,a_3)}(\theta,t)$), $\mathscr{L}_{(F_{(\gamma,a_1,a_2)},\ell)}$ (respectively, $\mathscr{L}_{(\gamma,L_{(\gamma,a_2,a_3)})}$) is an integrable mapping with respect to K_2. Therefore $F_{(\gamma,a_1,a_2)}(s,t)$ (respectively, $L_{(\gamma,a_2,a_3)}(\theta,t)$) and $\boldsymbol{\ell}(t)$ (respectively, $\boldsymbol{\gamma}(t)$) are the dual relative to (Δ_2, K_2).

On the other hand, S^2_+ is corresponding to the ideal boundary of the Poincaré ball model (or, the Bertlami-Klein model). If we consider $\widetilde{\boldsymbol{\ell}} : I \longrightarrow S^2_+$, then we can interpret that the image of $\widetilde{\boldsymbol{\ell}}$ is the set of end points of the horo-flat horocyclic surface $F_{(\gamma,a_1,a_2)}(s,t)$. We call $\widetilde{\boldsymbol{\ell}}$ the *end curve* of $F_{(\gamma,a_1,a_2)}$. Therefore the singularities of the lightcone circular surface are also an important subject in both of horospherical and hyperbolic geometry. We can show the following theorem.

THEOREM 8.2. *Let $L_{(\gamma,a_2,a_3)}$ be a hyperbolic-flat tangent lightcone circular surface with $c_2(t) = c_3(t) = c_1(t) - c_4(t) = 0$.*

(A) *Suppose that $c_5(t_0) \neq 0$ and $c_1(t_0) \neq 0$, then both the points $(0,t_0)$ and $(\sigma(t_0),t_0)$ are the different singularities, where $\sigma(t)$ is given by the relation $c_1(t)(1-\cos\sigma(t)) = c_5(t)\sin\sigma(t)$. In this case we have the following:*
    (1) *The point $(0,t_0)$ is the cuspidal edge if and only if $c_6(t_0) \neq 0$.*
    (2) *The point $(0,t_0)$ is the swallowtail if and only if $c_6(t_0) = 0$ and $c_6'(t_0) \neq 0$.*
    (3) *The point $(\sigma(t_0),t_0)$ is the cuspidal edge if and only if $(\sigma' + c_6)(t_0) \neq 0$.*
    (4) *The point $(\sigma(t_0),t_0)$ is the swallowtail if and only if*

$$(\sigma' + c_6)(t_0) = 0 \textit{ and } (\sigma' + c_6)'(t_0) \neq 0.$$

(B) *Suppose that* $c_5(t_0) = 0$ *and* $c_1(t_0) \neq 0$. *Then* $\sigma(t_0) = 0$, *so that* $(0, t_0) = (\sigma(t_0), t_0)$ *is a singular point. In this case, the point* $(0, t_0)$ *is the cuspidal beaks if and only if* $c_5'(t_0) \neq 0$, $c_6(t_0) \neq 0$ *and* $(\sigma' + c_6)(t_0) \neq 0$.

(C) *Suppose that* $c_5(t_0) \neq 0$, $c_1(t_0) = 0$ *and* $c_1'(t_0) \neq 0$. *Then we have the followings*:
   (1) *The point* $(0, t_0)$ *is the cuspidal cross cap if and only if* $c_6(t_0) \neq 0$.
   (2) *The point* $(\sigma(t_0), t_0)$ *is the cuspidal cross cap if and only if* $(\sigma' + c_6)(t_0) \neq 0$.

Proof. For the proof of the assertions (A) and (C), we apply the criterion in Proposition 6.1. By the previous calculation, $\{(0, t), (\sigma(t), t) \mid t \in I\}$ is the singular set of $L_{(\gamma, a_2, a_3)}$. Since $c_2 = c_3 = 0$, as shown above,

$$(\gamma, L_{(\gamma, a_2, a_3)}) : [0, 2\pi) \times I \longrightarrow H^3(-1) \times LC^*_+$$

is an isotropic map. Furthermore, if $c_1(t_0) \neq 0$ then (L, γ) is a Legendrian immersion near $(\sigma(t_0), t_0)$.

Since the area density function is

$$\lambda^L(\theta, t) = \det(L, L_\theta, L_t, \gamma),$$

λ^L_θ does not vanish near $(\sigma(t_0), t_0)$ if $c_5(t_0) \neq 0$. We have the singular direction $(-\lambda^L_t, \lambda^L_\theta) = (\sigma'(t), 1)$ and the null direction $(-c_6(t), 1)$ on $(\sigma(t), t)$. So two functions φ^L and ψ^L in Proposition 6.1 are $\varphi^L(t) = c_1(t)(\sigma'(t) + c_6(t))$ and $\psi^L(t) = \sigma'(t) + c_6(t)$ on $(\sigma(t), t)$. Then we get the assertion (A) by Proposition 6.1 (a) and (b). Also we get assertion (C) by Proposition 6.1 (c). One can get easily the case of $(0, t)$ by the similar argument.

On the other hand, for the proof of (B), we also apply the criterion in Lemma 6.3. For the purpose, we consider a family of functions $F : I \times LC^*_+ \longrightarrow \boldsymbol{R}$ defined by $F(t, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(t), \boldsymbol{v} \rangle + 1$. We may suppose that $t_0 = 0$, $\boldsymbol{\gamma}(0) = (1, 0, 0, 0)$, $\boldsymbol{a}_1(0) = (0, 1, 0, 0)$, $\boldsymbol{a}_2(0, 0, 1, 0)$ and $\boldsymbol{a}_3(0) = (0, 0, 0, 1)$ by a suitable hyperbolic motion, so that $\boldsymbol{\ell}(0) = (1, 0, 1, 0)$. By straightforward calculations, we can show that

$$\frac{\partial F}{\partial t}(0, \boldsymbol{\ell}(0)) = 0, \quad \frac{\partial^2 F}{\partial t^2}(0, \boldsymbol{\ell}(0)) = 0, \quad \frac{\partial^3 F}{\partial t^3}(0, \boldsymbol{\ell}(0)) = 0,$$

and

$$\frac{\partial^4 F}{\partial t^4}(0, \boldsymbol{\ell}(0)) = -c_1^2(0) c_6^2(0) - 2c_1(0) c_5'(0) c_6(0).$$

We remark that $(\sigma' + c_6)(0) \neq 0$ if and only if $c_1(0)c_6(0) + 2c_5'(0) \neq 0$. Therefore $f(t) = F(t, 0)$ is the A_4-type germ if and only if $c_5(0) = 0$, $c_1(0) \neq 0$, $c_6(0) \neq 0$ and $(\sigma' + c_6)(0) \neq 0$.

We consider $\widetilde{F}(t, \boldsymbol{v}, u) = F(t, \boldsymbol{v}) + ut^2$, so that

$$\widetilde{F}(t, \boldsymbol{v}, u) = -\gamma_0(t)v_0 + \gamma_1(t)v_1 + \gamma_2(t)v_2 + \gamma_3(t)v_3 + 1 + ut^2,$$

where $\boldsymbol{\gamma}(t) = (\gamma_0(t), \gamma_1(t), \gamma_2(t), \gamma_3(t))$. We take the local coordinate of LC^*_+ which is given by $\boldsymbol{v} = (\sqrt{v_1^2 + v_2^2 + v_3^2}, v_1, v_2, v_3)$. Since $\boldsymbol{v}_0 = \boldsymbol{\ell}(0) = (1, 0, 1, 0)$, we have

$$\begin{aligned}
j^2\left(\frac{\partial \widetilde{F}}{\partial v_1}(t, \boldsymbol{v}_0, 0)\right)(0) &= \gamma_1(0) + \gamma_1'(0)t + \frac{1}{2}\gamma_1''(0)t^2, \\
j^2\left(\frac{\partial \widetilde{F}}{\partial v_2}(t, \boldsymbol{v}_0, 0)\right)(0) &= (-\gamma_0(0) + \gamma_2(0)) + \big(-\gamma_0'(0) + \gamma_2'(0)\big)t \\
&\quad + \frac{1}{2}\big(-\gamma_0''(0) + \gamma_2''(0)\big)t^2, \\
j^2\left(\frac{\partial \widetilde{F}}{\partial v_3}(t, \boldsymbol{v}_0, 0)\right)(0) &= \gamma_3(0) + \gamma_3'(0)t + \frac{1}{2}\gamma_3''(0)t^2, \\
j^2\left(\frac{\partial \widetilde{F}}{\partial u}(t, \boldsymbol{v}_0, 0)\right)(0) &= t^2.
\end{aligned}$$

It is enough to show that

$$\operatorname{rank}\begin{pmatrix} \gamma_1(0) & \gamma_1'(0) & \gamma_1''(0) \\ -\gamma_0(0) + \gamma_2(0) & -\gamma_0'(0) + \gamma_2'(0) & -\gamma_0''(0) + \gamma_2''(0) \\ \gamma_3(0) & \gamma_3'(0) & \gamma_3''(0) \\ 0 & 0 & 1 \end{pmatrix} = 3.$$

Since $\boldsymbol{\gamma}'(0) = c_1(0)\boldsymbol{a}_1(0)$ and $\boldsymbol{\gamma}''(0) = c_1'(0)\boldsymbol{a}_1(0) + c_1^2(0)\boldsymbol{\ell}(0)$, the rank of the above matrix is equal to the rank of the following matrix:

$$\begin{pmatrix} 0 & c_1(0) & c_1'(0) \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is equal to 3 if and only if $c_1(0) \neq 0$. By Proposition 7.5, F is a Morse family of hypersurfaces. By a direct calculation, we have

$$\begin{aligned}\frac{\partial^2 F}{\partial t^2}|\Sigma_*(F) &= \left\langle \boldsymbol{\gamma}''(t), \boldsymbol{\gamma}(t) + \cos\theta \boldsymbol{a}_2(t) + \sin\theta \boldsymbol{a}_3(t)\right\rangle \\ &= -c_1^2(t) + c_1^2(t)\cos\theta + c_1(t)c_5(t)\sin\theta.\end{aligned}$$

We now calculate the Hessian matrix of $\rho(\theta, t) = -c_1^2(t) + c_1^2(t)\cos\theta + c_1(t)c_5(t)\sin\theta$, so that we have

$$\mathrm{Hess}(\rho)(0,0) = \begin{pmatrix} 0 & c_1(0)c_5'(0) \\ c_1(0)c_5'(0) & -c_1^2(0) \end{pmatrix}.$$

This matrix is regular if and only if $c_1(0)c_5'(0) \neq 0$, so that $(\partial^2 F/\partial t^2)|\Sigma_*(F)$ is a Morse function germ at 0. We can easily show that $\rho(\theta, t) = 0$ defines a transversal curve at $(0,0)$ in (θ, t)-plane. This means that the point $(0, t_0) = (\sigma(t_0), t_0)$ is the cuspidal beaks. $\square$

We now compare the results in Theorems 6.2 and 8.2.

Corollary 8.3. *Let $F_{(\gamma,a_1,a_2)}$ be a horo-flat tangent horocyclic surface. Then the germ of the surface at $(0, t_0)$ is the cuspidal cross cap if $c_5(t_0) \neq 0$, $c_6(t_0) = 0$, $c_6'(t_0) \neq 0$ and $c_1(t_0) \neq 0$. In this case the germ of the end curve $\widetilde{\boldsymbol{\ell}} : I \longrightarrow S^2_+$ at t_0 is the ordinary cusp.*

Proof. Since $F_{(\gamma,a_1,a_2)}$ is a horo-flat tangent horocyclic surface, $L_{(\gamma,a_2,a_3)}$ is a hyperbolic-flat tangent lightcone circular surface. By Theorem 6.2, (C), the germ of $F_{(\gamma,a_1,a_2)}$ at $(0, t_0)$ is the cuspidal cross cap if $c_5(t_0) \neq 0$, $c_6(t_0) = 0$, $c_6'(t_0) \neq 0$ and $c_1(t_0) \neq 0$. On the other hand, the germ of $L_{(\gamma,a_2,a_3)}$ is the swallowtail at $(0, t_0)$ by Theorem 8.2, (A). Since $\pi_{22} : \Delta_2 \longrightarrow LC^*_+$ is a Legendrian fibration and the germ of $L_{(\gamma,a_2,a_3)}$ has a Legendrian lift into Δ_2, $L_{(\gamma,a_2,a_3)}$ is a wavefront in LC^*_+. By the general theory of Legendrian and Lagrangian singularities [**2**], $\widetilde{L}_{(\gamma,a_2,a_3)}$ can be regarded as a Lagrangian map. By the relation between wavefronts and caustics, the germ of $L_{(\gamma,a_2,a_3)}$ is the swallowtail if and only if the caustics (critical value set) of $\widetilde{L}_{(\gamma,a_2,a_3)}$ is the ordinary cusp. Here, the critical value set is the image of the end curve $\widetilde{\boldsymbol{\ell}}$. This completes the proof. $\square$

A. Criteria for cuspidal beaks and cuspidal lips.

In this appendix, we shall state criteria for the recognition of the cuspidal beaks or the cuspidal lips as a corollary of arguments in Section 7. Let $L_f = (f, [\nu]) : (U^2, p) \longrightarrow (PT^*(\boldsymbol{R}^3), (f(p), [\nu(p)]))$ be a Legendrian immersion germ. Assume that p is a singular point of f with corank one. Then one can get a non-zero vector field η on U such that $q \in S(f)$ implies $df_q(\eta_q) = 0$. We call this vector field a *null vector field* of L_f. Let λ be the signed area density function as in Section 6. We have the following criteria for the cuspidal beaks or the cuspidal lips.

THEOREM A.1. *Let $L_f = (f, [\nu]) : (U^2, p) \longrightarrow (PT^*(\boldsymbol{R}^3), (f(p), [\nu(p)]))$ be a Legendrian immersion germ and p is a singular point of f with corank one.*
Then following (A) and (A') (respectively, (B) and (B')) are equivalent.

- (A) *f at p is $\mathscr{A}$-equivalent to the cuspidal beaks.*
- (A') *λ has a Morse type singularity of index one at p and $\nabla_\eta \nabla_\eta \lambda(p) \neq 0$. Here, ∇ is the canonical covariant derivative induced by the Levi-Civita connection on $\boldsymbol{R}^3$.*
- (B) *f at p is $\mathscr{A}$-equivalent to the cuspidal lips.*
- (B') *λ has a Morse type singularity of index zero or two at p.*

Here, the *cuspidal lips* is a germ of surface diffeomorphic to $CLP = \{(x_1, x_2, x_3) \mid x_1 = v, x_2 = 2u^3 + v^2 u, x_3 = 3u^4 + u^2 v^2\}$.

PROOF. Obviously the conditions are independent of both of the coordinates and ν. Firstly we take the coordinates (u, v) of U centered at p and (X, Y, Z) of $\boldsymbol{R}^3$ centered at $f(p)$ satisfying:

- The null vector field η is always ∂_v.
- $f(u, v) = (f_1(u, v), f_2(u, v), u)$ and $(f_1)_u = (f_2)_u = (f_1)_{uu} = (f_2)_{uu} = 0$ at $(0, 0)$.
- $\nu(0, 0) = (1, 0, 0)$.

Here $(f_1)_u$ denotes $\partial f_1 / \partial u$, for example. Under these coordinates, we show that (A') (respectively, (B')) implies (A) (respectively, (B)).

We consider a family of plane curves $\Gamma^u(v) = \Gamma(u, v) = (f_1(u, v), f_2(u, v), u)$ in the plane $\Pi_u = \{(X, Y, Z) | Z = u\}$ and show that these are fronts near p. Denote $\nu = (\nu_1, \nu_2, \nu_3)$ and put

$$[N^u(v)] = [N(u, v)] = [(\nu_1(u, v), \nu_2(u, v), 0)].$$

Then $[N^u(v)]$ is well-defined near p. We put $\gamma(u, v) = (f_1(u, v), f_2(u, v))$ and

$n(u,v) = (\nu_1(u,v), \nu_2(u,v))$. Then, since $(\gamma'(u,v)\cdot n(u,v)) \equiv 0$, $(\gamma,[n])$ is an isotropic map for all u, where $'$ denotes $\partial/\partial v$ and $(\,\cdot\,)$ is the canonical inner product of $\boldsymbol{R}^3$. Since $\nu_3'(0)=0$, we have $n'(0)\neq 0$. This implies that for each u near 0, $(\gamma,[n])$ is a Legendrian immersion germ.

We define two functions $\Psi : \boldsymbol{R}\times\boldsymbol{R}^3 \longrightarrow \boldsymbol{R}$ and $\psi : \boldsymbol{R}\longrightarrow\boldsymbol{R}$ as follows:

$$\Psi(v,X_1,X_2,Z) = \nu_1(Z,v)(X_1 - f_1(Z,v)) + \nu_2(Z,v)(X_2 - f_2(Z,v)),$$
$$\psi(v) = \Psi(v,0,0,0).$$

Then we have $W(\Psi)=f(U)$. Hence by Lemma 7.7 and the arguments in Section 7, it is sufficient to prove that ψ has an A_3-singularity, Ψ is a Morse family and $\partial^2\Psi/\partial v^2|_{\Sigma_*(\Psi)}$ has a Morse type singularity with prescribed index at p. In the following context, we put $Z=u$.

Lemma A.2. *It holds that* $f'=f_{uu}=f_u'=f''=0$ *and* $\gamma'=\gamma_{uu}=\gamma_u'=\gamma''=0$ *at* $(0,0)$.

Proof. Since ∂_v is the null vector field, we have $f'(0,0)=0$. It follows that $\gamma'(0,0)=0$. By the conditions on the coordinates of U and $\boldsymbol{R}^3$, we have $f_{uu}(0,0)=0$. Thus $\gamma_{uu}(0,0)=0$. Since $(0,0)$ is a critical point of λ, we have $\det(f_u,f_u',\nu)(0,0)=\lambda_u(0,0)=0$. Hence $f_u'(0,0)\in\operatorname{span}\{f_u(0,0),\nu(0,0)\}$. On the other hand, $(f_u'\cdot f_u)(0,0)=0$ and $(f_u'\cdot\nu)(0,0) = -(f'\cdot\nu_u)(0,0)=0$. It follows that $f_u'(0,0)=0$. This means that $\gamma_u'(0,0)=0$. We can get $f''(0,0)=0$ and $\gamma''(0,0)=0$ by the same arguments on the above. $\square$

First, we show 0 is an A_3-singularity of ψ. Differentiating $(\gamma'\cdot n)=0$ and by Lemma A.2, we have $(\gamma'''\cdot n)(0,0)=0$ and $(\gamma''''\cdot n)(0,0) = -3(\gamma'''\cdot n')(0,0)$. By these formulae and Lemma A.2, we have

$$\psi'(0)=\psi''(0)=\psi'''(0)=0 \ \text{ and } \ \psi''''(0) = -(\gamma'''\cdot n')(0,0). \tag{10}$$

Since $(\gamma'''\cdot n)(0,0)=0$, $\psi''''(0,0)\neq 0$ if and only if $\gamma'''(0,0)\neq 0$.

Now, we assume (A'). Then $\nabla_\eta\nabla_\eta\lambda(p)\neq 0$ if and only if $\lambda''(p)\neq 0$, because the null vector field is ∂_v. Since $f'(0,0)=f''(0,0)=0$, $\det(f_u,f''',\nu)(0,0)\neq 0$, particularly $f'''(0,0)\neq 0$. By the definition of γ, clearly this implies $\gamma'''(0,0)\neq 0$.

On the other hand, we assume (B'). Then $\det\operatorname{Hess}\lambda>0$, so that we have $\lambda''(p)\neq 0$. By the same argument as the case (A'), we have $\gamma'''(0,0)\neq 0$. Therefore ψ has an A_3-singularity at 0.

Second, we prove that Ψ is a Morse family. It is sufficient to prove the matrix

$$\begin{pmatrix} \Psi_{X_1}(0) & \Psi_{X_2}(0) \\ \Psi_{X_1 v}(0) & \Psi_{X_2 v}(0) \end{pmatrix} = \begin{pmatrix} \nu_1(0,0) & \nu_2(0,0) \\ \nu_1'(0,0) & \nu_2'(0,0) \end{pmatrix}$$

is regular. Since vectors $n(0,0), n'(0,0)$ are linearly independent, the matrix is regular.

Third, we prove that $\mathrm{Hess}\,(\partial^2\Psi/\partial v^2|_{\Sigma_*(\Psi)})(0,0)$ is non-degenerate if and only if $\mathrm{Hess}\,\lambda(p)$ is non-degenerate and the indices of the both matrices are the same, under the identification of index 0 and 2. We write $X = (X_1, X_2)$. Since $(\gamma' \cdot n) \equiv 0$, the condition $(v, X_1, X_2, u) \in \Sigma_*(\Psi)$ is equivalent to the condition $((X - \gamma) \cdot n) = ((X - \gamma) \cdot n') = 0$. This is equivalent to $X - \gamma = 0$, that is $\Sigma_*(\Psi) = \{(v, \gamma(u,v), u)\}$. Hence

$$\bar{\lambda}(u,v) := \left.\frac{\partial^2\Psi}{\partial v^2}\right|_{\Sigma_*(\Psi)}(u,v) = 2\,(-\gamma' \cdot n')\,(u,v) + (-\gamma'' \cdot n)\,(u,v). \tag{11}$$

Lemma A.3. *The following holds at* $(0,0)$:

$$(\gamma''_{uu} \cdot n) = -\,(\gamma'_{uu} \cdot n') - 2\,(\gamma''_u \cdot n_u) \quad \textit{and} \quad (\gamma'''_u \cdot n) = -2\,(\gamma''_u \cdot n') - (\gamma''' \cdot n_u)\,.$$

Proof. Differentiate $(\gamma' \cdot n) \equiv 0$ and use Lemma A.2. $\square$

By Lemmata A.2 and A.3, we have $\bar{\lambda}_{uu} = -\,(\gamma'_{uu} \cdot n')$. $\bar{\lambda}'_u = -\,(\gamma''_u \cdot n')$ at $(0,0)$. By the equation (10), $\bar{\lambda}'' = -\,(\gamma''' \cdot n')$ at $(0,0)$. By the definition of γ, we get $(\gamma'_{uu} \cdot n')\,(0,0) = (f'_{uu} \cdot n')\,(0,0)$ and so on. Moreover since $N'(0,0) \perp \nu(0,0)$ and $N'(0,0) \perp f_u(0,0)$, there exists a non zero $k \in \boldsymbol{R}$ such that $N'(0,0) = k f_u(0,0) \times \nu(0,0)$. Hence $-\,(\gamma'_{uu} \cdot n')\,(0,0) = k \det(f_u, f'_{uu}, \nu)(0,0)$ and the other same formulas hold. Since we have $f'(0,0) = f'_u(0,0) = f''(0,0) = 0$,

$$\begin{aligned} \mathrm{Hess}\,\bar{\lambda}(0,0) &= \begin{pmatrix} -\,(\gamma'_{uu} \cdot n') & -\,(\gamma''_u \cdot n') \\ -\,(\gamma''_u \cdot n') & -\,(\gamma''' \cdot n') \end{pmatrix}(0,0) \\ &= \begin{pmatrix} k\det(f_u, f'_{uu}, \nu) & k\det(f_u, f''_u, \nu) \\ k\det(f_u, f''_u, \nu) & k\det(f_u, f''', \nu) \end{pmatrix}(0,0) = \mathrm{Hess}\,k\lambda(p). \end{aligned}$$

This implies that $\mathrm{Hess}\,(\partial^2\Psi/\partial v^2|_{\Sigma_*(\Psi)})(0,0)$ is non-degenerate if and only if $\mathrm{Hess}\,\lambda(p)$ is non-degenerate and the indices of the both matrices are the same, under the identification of index 0 and 2.

The inverse part is obvious since the canonical forms of the cuspidal beaks and the cuspidal lips satisfy the condition and it is independent of the choice of coordinates and ν. □

We now remark on the proofs of Proposition 6.2 (B) and 8.2 (B). For the Proposition 6.2 (B), since the signed area density function of $F_{(\gamma,a_1,a_2)}$ is $-s(c_5+sc_6/2)$, the condition $c_5'(t_0)\neq 0$ is given by $\det\mathrm{Hess}\,\lambda\neq 0$ and the conditions $c_1(t_0)\neq 0$ and $(c_1-s')(t_0)\neq 0$ are given by $\nabla_\eta\nabla_\eta\lambda\neq 0$. For Proposition 8.2 (B), since the signed area density function of $L_{(\gamma,a_2,a_3)}$ is $-c_1+c_1\cos\theta+c_5\sin\theta$, the condition $c_5'(t_0)\neq 0$ is given by the condition $\det\mathrm{Hess}\,\lambda\neq 0$ and the conditions $c_6(t_0)\neq 0$ and $(c_6+\sigma')(t_0)\neq 0$ are given by $\nabla_\eta\nabla_\eta\lambda\neq 0$. Therefore we can also give the proofs as applications of Theorem A.1. We also remark that Theorem A.1 might be very useful for the recognitions of the cuspidal beaks and the cuspidal lips on explicitly parametrized surfaces. We will apply this to various situation in elsewhere.

B. Singularities of general horocyclic surfaces.

In this appendix we consider singularities of general horocyclic surfaces. Let $F=F_{(\gamma,a_1,a_2)}$ be a general horocyclic surface. By the jet-transversality theorem, there is no point $t_0\in I$ with $c_2(t_0)=c_4(t_0)-c_1(t_0)=0$ for a generic $C(t)\in C^\infty(I,\mathfrak{so}(3,1))$. If $c_4(t_0)-c_1(t_0)=0$ and $c_2(t_0)\neq 0$ then F is non-singular at (s,t_0). Therefore we assume that $c_4(t_0)-c_1(t_0)\neq 0$. Suppose (s_0,t_0) is a singular point of F. By the equations (6), we have $s_0=c_2(t_0)/(c_4(t_0)-c_1(t_0))$ and

$$\left(1+\frac{s_0^2}{2}\right)c_3(t_0)+s_0c_5(t_0)+\frac{s_0^2}{2}c_6(t_0)=0. \tag{12}$$

Then $dF((c_1+s^2/2(c_1-c_4))(\partial/\partial s)-\partial/\partial t)(s_0,t_0)=0$. By the characterization of the cross cap (the singular point of semi-regular mapping in [**35**]), (12) and

$$\det\big(F,F_{tt}(c_1+s^2/2(c_1-c_4))^2-2F_{ts}(c_1+s^2/2(c_1-c_4))+F_{ss},\\ F_{tt}(c_1+s^2/2(c_1-c_4))-F_{ts},\ F_t\big)(s_0,t_0)\neq 0,$$

are satisfied if and only if (s_0,t_0) is the cross cap. Suppose that $\boldsymbol{\gamma}$ is the striction curve. By definition, $c_2\equiv 0$, so that $s(t_0)=0$. Moreover, by (12), we have $c_3(t_0)=0$. By a straightforward calculation, this condition is equivalent to the condition

$$\big\{c_1\big(-c_1^2c_5+c_5'+c_4c_6+c_1^3\big(c_4'c_5-c_4c_5'-c_4^2c_6-c_5^2c_6\big)\big)\big\}(t_0)\neq 0. \tag{13}$$

We remark that the condition (13) is a generic condition in $C^\infty(I, \mathfrak{so}(3,1))$. Therefore we have the following theorem.

THEOREM B.1. *There exists an open dense subset $\mathcal{O} \subset C^\infty(I, \mathfrak{so}(3,1))$ such that the germ of the horocyclic surface F_A at any point (s_0, t_0) is an immersion or the cross cap for any $C \in \mathcal{O}$. Here, $A(t) \in SO_0(3,1)$ is the smooth curve corresponding to $C(t) \in \mathfrak{so}(3,1)$.*

We remark that the above theorem and Theorem 1.1 describe how singularities of horo-flat horocyclic surfaces are different from those of general horocyclic surfaces.

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