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Invariant subspaces and reducing subspaces of weighted Bergman space over bidisk

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Abstract. In this paper, we study the invariant subspace and reducing subspace of the weighted Bergman space over bidisk. The minimal reducing subspace of Toeplitz operator $T_{z^N} = T_{z_1^N z_2^N}$ is completely described, and Beurling-type theorem of some invariant subspace of the weighted Bergman space over bidisk is also obtained.

1. Introduction.

Let dA denote Lebesgue area measure on the unit disk D, normalized so that the measure of D equals 1. For $\alpha > -1$, we denote the measure dA_α by $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. The weighted Bergman space $A_\alpha^2(D)$ consists of analytic functions f

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

in the unit disk D such that

$$\|f\|_\alpha^2 = \sum_{n=0}^{\infty} \omega_n |a_n|^2 < \infty,$$

where $\omega_n = \frac{n!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n)}$. If $e_n(z) = \sqrt{\frac{1}{\omega_n}} z^n$, then $\{e_n(z)\}$ is an orthonormal basis for $A_\alpha^2(D)$.

It is easy to see that $A_\alpha^2(D)$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_D f(z)\overline{g(z)} dA_\alpha(z), \qquad f, g \in A_\alpha^2(D).$$

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Let Q be the Bergman orthogonal projection from $L^2(D)$ onto $A^2_\alpha(D)$. For a bounded measurable function $f \in L^\infty(D)$, the Toeplitz operator with symbol f is defined by $T_f h = Q(fh)$, for $h \in A^2_\alpha(D)$. It is clear that $T_f : A^2_\alpha(D) \to A^2_\alpha(D)$ is a bounded linear operator.

The unit bidisk D^2 and the torus T^2 are the cartesian products of two copies of D and of T, respectively. Observe that T^2 is only a small part of the boundary ∂D^2. T^2 is usually called the distinguished boundary of D^2. The weighted Bergman space $A^2_\alpha(D^2)$ is then the space of all holomorphic functions in $L^2(D^2, dv_\alpha)$, where $dv_\alpha(z) = dA_\alpha(z_1)dA_\alpha(z_2)$. For multi-index $\beta = (\beta_1, \beta_2)$, let

$$e_\beta = \sqrt{\frac{1}{w_{\beta_1} w_{\beta_2}}} z^\beta,$$

then $\{e_\beta\}_\beta$ is an orthnormal basis for $A^2_\alpha(D^2)$.

Let P be the Bergman orthogonal projection from $L^2(D^2)$ onto $A^2_\alpha(D^2)$. For a bounded measurable function $f \in L^\infty(D^2)$, the Toeplitz operator with symbol f is defined by $T_f h = P(fh)$, for $h \in A^2_\alpha(D^2)$. It is clear that $T_f : A^2_\alpha(D^2) \to A^2_\alpha(D^2)$ is a bounded linear operator.

For the general theory of the weighted Bergman space on the unit disk and bidisk, readers refer to [**3**], [**9**] and [**6**].

One of the reasons that invariant subspaces in Bergman spaces A^2_α have attracted so much attention in recent years is that they are closely related to an old open problem in Operator Theory. More specifically, the invariant subspace problem (of whether every bounded linear operator on a separable Hilbert space of infinite dimension has a nontrivial invariant subspace) is equivalent to the following question about invariant subspaces of the Bergman space A^2_α: Given two invariant subspaces I and J of A^2_α with $I \subset J$ and $\dim(J \ominus I) = \infty$, does there exist another invariant subspace M of A^2_α lying strictly between I and J? See [**4**] for an explanation and references.

It is well known that the multiplication operator M_z on $A^2_\alpha(D)$ possesses a very rich structure theory, although its definition seems simple-minded. It poses many serious questions to be answered, such as the understanding of its invariant subspace. We mention here the work [**1**]. The study of invariant subspace of general analytic multiplication operators has also picked up momentum, see [**5**] for example. One of the problems we will be concerned with in this paper is Beurling-type theorem of weighted Bergman space over bidisk.

Besides the structure of the invariant subspaces, the understanding of invariant subspace lattice is also helpful to the invariant subspace problem. In [**10**], Kehe Zhu got a complete description of the reducing subspaces of multiplication operators on Bergman space induced by z^2 and by Blaschke products with two zeros in D. In [**7**], Michael Stessin and Kehe Zhu extended the result in [**10**] to the reducing subspaces of weighted unilateral shift operators of finite multiplicity. In [**2**] and [**8**], Kunyu Guo, Shunhua Sun, Dechao Zheng and Changyong Zhong developed a machinery and completely classified nontrivial minimal reducing subspaces of the multiplication operator by a Blaschke product with order three and four zeros respectively, on the Bergman space of the unit disk via the Hardy space of the bidisk.

Motivated by [**10**], [**7**], [**2**] and [**8**], in this paper we investigate reducing subspace lattice of Toeplitz operator $T_{z^N} = T_{z_1^N z_2^N}$ in $A^2_\alpha(D^2)$ and obtain a complete description of the minimal reducing subspaces of T_{z^N} in $A^2_\alpha(D^2)$.

Let us begin the study by doing some preparations.

We let E be a separable Hilbert space of infinite dimension, and $\{\delta_j : j \geq 0\}$ be the orthonormal basis for E, and we let $L^2(E)$ denote the E-valued weighted Bergman space on the unit disk D, i.e.

$$L^2(E) = \left\{ f : D \to E \mid f = \sum_{n=0}^{\infty} x_n z^n,\ \|f\|^2_{L^2(E)} = \sum_{n=0}^{\infty} \omega_n \|x_n\|^2_E < \infty \right\}.$$

In order to make a study of the weighted Bergman space $A^2_\alpha(D^2)$, we identify the space E with another copy of the Bergman space. Then $L^2(E) = A^2_\alpha(D) \otimes E$ will be identified with $A^2_\alpha(D) \otimes A^2_\alpha(D) = A^2_\alpha(D^2)$. We do this in the following way.

Let u be the unitary map from E to $A^2_\alpha(D)$ such that

$$u(\delta_j) = e_j(z_2), \quad j \geq 0.$$

Then $U = I \otimes u$ is a unitary from $A^2_\alpha(D) \otimes E$ to $A^2_\alpha(D) \otimes A^2_\alpha(D)$ such that

$$U(e_i(z_1)\delta_j) = e_i(z_1)e_j(z_2), \quad i, j \geq 0,$$

where I is the identity operator on $A^2_\alpha(D)$.

A closed subspace M of $A^2_\alpha(D^2)$ is called an invariant subspace of the operator A, if $AM \subseteq M$.

A closed subspace M of $A^2_\alpha(D^2)$ is called a reducing subspace of the operator A, if M is an invariant subspace of both A and its adjoint A^*.

In Section 2, we study the minimal reducing subspace of $T_{z_1^N}$, $T_{z_2^N}$ and T_{z^N} over bidisk. And then, in Section 3, Beurling-type theorem of some special kind of invariant subspace over bidisk is obtained.

We can now state our main result.

Theorem 1.1. *Suppose M is a reducing subspace of T_{z^N} in $A^2_\alpha(D^2)$, then there exist nonnegative integers a,b,k,m with $0 \le m \le N-1$ and $a,b \in \{0,1\}$ such that*

$$Span\{(az_1^k + bz_2^k)(z_1z_2)^{m+lN} : l = 0,1,2,\dots\} \subseteq M.$$

In particular, M is minimal, if and only if,

$$M = Span\{(az_1^k + bz_2^k)(z_1z_2)^{m+lN} : l = 0,1,2,\dots\}.$$

Theorem 1.2. *Suppose $-1 < \alpha \le 0$ and for any $i = 1,2$, M is an invariant subspace of T_{z_i} in $A^2_\alpha(D^2)$. Then M is generated by $M \ominus T_{z_i}M$, that is*

$$M = [M \ominus T_{z_i}M].$$

2. The reducing subspace of the weighted Bergman space over bidisk.

Throughout this section we fix an integer $N > 1$, and consider the complete description of the reducing subspaces of the operators T_{z^N} and $T_{z_i^N}(i = 1,2)$ in the weighted Bergman space $A^2_\alpha(D^2)$.

Note that for any $f \in A^2_\alpha(D^2)$,

$$f = \sum_{n=0}^{\infty} z_2^n g_n(z_1), \quad (z_1,z_2) \in D^2,$$

where $\{g_n\}_n$ are holomorphic functions in $A^2_\alpha(D)$. It is the unique decomposition with respect to

$$A^2_\alpha(D^2) = \sum_{n=0}^{\infty} \oplus z_2^n A^2_\alpha(D).$$

Let the closed subspace $z_2^n A^2_\alpha(D)$ be denoted by $X_n^{(1)}$, then we have

$$A^2_\alpha(D^2) = \sum_{n=0}^{\infty} \oplus X_n^{(1)}.$$

Since $T_{z_1^N}$ is an operator on $A^2_\alpha(D^2)$ and $X_n^{(1)}$ are its invariant subspaces, $T_{z_1^N}$ is the direct sum of its restrictions to $X_n^{(1)}(n = 0, 1, 2, \ldots)$, i.e.

$$T_{z_1^N} = \sum_{n=0}^{\infty} \oplus T_{z_1^N} | X_n^{(1)}.$$

Let S_n be the restriction of the operator $T_{z_1^N}$ to the closed subspace $X_n^{(1)}$. First, we will give the description of the reducing subspaces of S_n in $A_\alpha^2(D^2)$, and that is based on the following result, see Theorem 14 in [**7**] for details.

Suppose that M_{z^N} is the weighted unilateral shift operator on $A_\alpha^2(D)$, then $X_n = Span\{z^{n+kN} : k = 0, 1, 2, \ldots\}$ $(0 \leq n \leq N-1)$ are the only minimal reducing subspaces of M_{z^N} in $A_\alpha^2(D)$. In particular, there are exactly 2^N reducing subspaces of M_{z^N} in $A_\alpha^2(D)$, and they are simply the direct sum of these minimal reducing subspaces.

Here and throughout the paper we use Span to denote the closed linear span of a set in a Hilbert space.

Theorem 2.1. *For any $n = 0, 1, 2, \ldots$, $X_n^{(1)} = z_2^n A_\alpha^2(D)$ is a closed subspace of $A_\alpha^2(D^2)$. Then*

$$Span\big\{z_2^n z_1^{n_1+\alpha_1 N} : \alpha_1 = 0, 1, 2, \ldots\big\} \ (0 \leq n_1 \leq N-1)$$

are the only minimal reducing subspaces of S_n. In particular, there are exactly 2^N reducing subspaces of S_n in $X_n^{(1)}$, and they are simply the direct sum of these minimal reducing subspaces.

Proof. Let $M \subseteq z_2^n A_\alpha^2(D)$ be a closed subspace in $X_n^{(1)}$, and

$$M_0 = \big\{f(z_1) \in A_\alpha^2(D) : z_2^n f(z_1) \in M\big\},$$

it is easy to see that M_0 is a closed subspace in $A_\alpha^2(D)$, and $z_2^n M_0 = M$.

If M is a reducing subspace of S_n, for any $f(z_1) \in M_0$,

$$z_2^n M_{z_1^N} f(z_1) = z_2^n z_1^N f(z_1) = S_n(z_2^n f(z_1)) \in M,$$
$$z_2^n M_{z_1^N}^* f(z_1) = P(z_2^n \overline{z}_1^N f(z_1)) = S_n^*(z_2^n f(z_1)) \in M,$$

so M_0 is a reducing subspace of $M_{z_1^N}$ in $A_\alpha^2(D)$.

Conversely, if M_0 is a reducing subspace of $M_{z_1^N}$ in $A_\alpha^2(D)$, similarly, M is a reducing subspace of S_n in $X_n^{(1)}$.

If M is minimal, we assume that M_0' is a nonzero proper reducing subspace contained in M_0. Then $z_2^n M_0' \subseteq z_2^n M_0 = M$. It is a contradiction, since M is minimal. So M_0 is minimal in $A^2_\alpha(D)$.

Conversely, if M_0 is minimal, similarly, M is minimal.

Thus M is a minimal reducing subspace of S_n in $X_n^{(1)}$, if and only if, M_0 is a minimal reducing subspace of $M_{z_1^N}$ in $A^2_\alpha(D)$.

By Theorem 14 in [**7**], the result is proved. □

Throughout this paper, we denote $Span\{z_1^{n_1+\alpha_1 N} : \alpha_1 = 0, 1, 2, \ldots\}$ by $M_{n_1}^{(1)}$, and $Span\{z_2^{n_2+\alpha_2 N} : \alpha_2 = 0, 1, 2, \ldots\}$ by $M_{n_2}^{(2)}$.

LEMMA 2.1. *Let M be a reducing subspace of $T_{z_1^N}$ in $A^2_\alpha(D^2)$. If $f \in M$, $g \in M^\perp$,*

$$f(z_1, z_2) = \sum_{p=0}^{\infty} f_p(z_2) z_1^p, \quad g(z_1, z_2) = \sum_{q=0}^{\infty} g_q(z_2) z_1^q,$$

then for any $p, q \geq 0$, $f_p(z_2) z_1^p \in M$, $g_q(z_2) z_1^q \in M^\perp$.

PROOF. Assume that M is a reducing subspace of $T_{z_1^N}$. For $m, n \geq 0$, we firstly consider the orthogonal decomposition of $z_2^n z_1^m$ with respect to M. Let

$$z_2^n z_1^m = \alpha(z_1, z_2) + \beta(z_1, z_2),$$

where $\alpha(z_1, z_2) \in M$, $\beta(z_1, z_2) \in M^\perp$, and $\alpha(z_1, z_2) = \sum_{k=0}^{\infty} \alpha_k(z_2) z_1^k$ be the multiple Fourier series of α. Let P_M be the orthogonal projection from $A^2_\alpha(D^2)$ onto M. Then we have

$$\begin{aligned}
T^*_{z_1^N} T_{z_1^N}\big(z_2^n z_1^m\big) &= P\big(z_2^n z_1^{m+N} \overline{z}_1^N\big) = z_2^n Q\big(z_1^{m+N} \overline{z}_1^N\big) \\
&= z_2^n \sum_{k=0}^{\infty} \left\langle Q\big(z_1^{m+N} \overline{z}_1^N\big), \frac{z_1^k}{\|z_1^k\|} \right\rangle \frac{z_1^k}{\|z_1^k\|} \\
&= z_2^n \sum_{k=0}^{\infty} \frac{1}{\omega_k} \big\langle z_1^{m+N}, z_1^{k+N} \big\rangle z_1^k \\
&= \frac{\omega_{m+N}}{\omega_m} z_2^n z_1^m \\
&= \frac{\omega_{m+N}}{\omega_m} (\alpha + \beta),
\end{aligned}$$

$$P_M T^*_{z_1^N} T_{z_1^N}\left(z_2^n z_1^m\right) = P_M\left(\frac{\omega_{m+N}}{\omega_m}(\alpha+\beta)\right) = \frac{\omega_{m+N}}{\omega_m}\alpha,$$

and

$$\begin{aligned} P_M T^*_{z_1^N} T_{z_1^N}(\alpha+\beta) &= P_M T^*_{z_1^N} T_{z_1^N}\alpha + P_M T^*_{z_1^N} T_{z_1^N}\beta \\ &= T^*_{z_1^N} T_{z_1^N}\alpha \\ &= P\left(\sum_{k=0}^{\infty}\alpha_k(z_2) z_1^{k+N}\overline{z}_1^N\right) \\ &= \sum_{k=0}^{\infty}\alpha_k(z_2) Q\left(z_1^{k+N}\overline{z}_1^N\right) \\ &= \sum_{k=0}^{\infty}\frac{\omega_{k+N}}{\omega_k}\alpha_k(z_2) z_1^k. \end{aligned}$$

It follows that

$$\frac{\omega_{m+N}}{\omega_m}\alpha = \sum_{k=0}^{\infty}\frac{\omega_{k+N}}{\omega_k}\alpha_k(z_2) z_1^k,$$

or

$$\sum_{k\neq m}\left(\frac{\omega_{k+N}}{\omega_k} - \frac{\omega_{m+N}}{\omega_m}\right)\alpha_k(z_2) z_1^k = 0.$$

Since $\frac{\omega_{k+N}}{\omega_k} \neq \frac{\omega_{m+N}}{\omega_m}$ when $k \neq m$, we get $\alpha_k(z_2) = 0$, $\forall k \neq m$.

That is $\alpha(z_1, z_2) = \alpha_m(z_2) z_1^m$, and $\beta(z_1, z_2) = (z_2^n - \alpha_m(z_2)) z_1^m$. Since $\|\alpha\|^2 + \|\beta\|^2 = \|z_2^n z_1^m\|^2$, it is easy to see that $\|\alpha_m(z_2)\|^2 \leq \|z_2^n\|^2$.

Therefore there exists a sequence of functions $\{\alpha_{n,m}(z_2)\}_{n,m} \subseteq A^2_\alpha(D)$ such that

$$\|\alpha_{n,m}(z_2)\|^2 \leq \|z_2^n\|^2,$$

and

$$z_2^n z_1^m = \alpha_{n,m}(z_2) z_1^m + \left(z_2^n - \alpha_{n,m}(z_2)\right) z_1^m$$

is the unique orthogonal decomposition of $z_2^n z_1^m$ with respect to M.

For any function $f \in M$, $f = \sum_{p=0}^{\infty} f_p(z_2) z_1^p$, it is easy to check that

$$f_p(z_2) z_1^p = \sum_{n=0}^{\infty} a_{p,n} z_2^n z_1^p = \sum_{n=0}^{\infty} a_{p,n} \alpha_{n,p}(z_2) z_1^p + \sum_{n=0}^{\infty} a_{p,n} (z_2^n - \alpha_{n,p}(z_2)) z_1^p.$$

Let $h_p(z_2) = \sum_{n=0}^{\infty} a_{p,n} \alpha_{n,p}(z_2)$, then

$$\|h_p(z_2)\|^2 \le \sum_{n=0}^{\infty} |a_{p,n}|^2 \|\alpha_{n,p}(z_2)\|^2 \le \sum_{n=0}^{\infty} |a_{p,n}|^2 \|z_2^n\|^2 = \|f_p(z_2)\|^2 < \infty,$$

and

$$f_p(z_2) z_1^p = h_p(z_2) z_1^p + (f_p(z_2) - h_p(z_2)) z_1^p,$$

where $h_p(z_2) z_1^p \in M$, and $(f_p(z_2) - h_p(z_2)) z_1^p \in M^{\perp}$.

So f has the unique orthogonal decomposition with respect to M:

$$f = \sum_{p=0}^{\infty} h_p(z_2) z_1^p + \sum_{p=0}^{\infty} (f_p(z_2) - h_p(z_2)) z_1^p.$$

Since $f \in M$, $\sum_{p=0}^{\infty} (f_p(z_2) - h_p(z_2)) z_1^p = 0$. Then for any $p = 0, 1, \dots$, $f_p(z_2) = h_p(z_2)$, that implies $f_p(z_2) z_1^p \in M$.

Similarly, for any function $g \in M^{\perp}$, $g = \sum_{q=0}^{\infty} g_q(z_2) z_1^q$, then $g_q(z_2) z_1^q \in M^{\perp}$, $\forall q = 0, 1, \dots$. So the proof is completed. $\square$

Theorem 2.2. *For any function $f = f(z_2) \in A_\alpha^2(D)$, and each integer n_1 with $0 \le n_1 \le N-1$, let*

$$f(z_2) M_{n_1}^{(1)} = Span\{f(z_2) z_1^{n_1 + \alpha_1 N} : \alpha_1 = 0, 1, 2, \dots\},$$

then $\{f(z_2) M_{n_1}^{(1)}\}$ are the only minimal reducing subspaces of $T_{z_1^N}$ in $A_\alpha^2(D^2)$. Every reducing subspace of $T_{z_1^N}$ in $A_\alpha^2(D^2)$ contains a minimal reducing subspace.

Proof. By Theorem 14 in [**7**], it is obvious that for any nonnegative integer n_1 with $0 \le n_1 \le N-1$, and any $f(z_2) \in A_\alpha^2(D)$, $f(z_2) M_{n_1}^{(1)}$ is a reducing subspace of $T_{z_1^N}$ in $A_\alpha^2(D^2)$.

In the following, we are going to prove that for any reducing subspace M of

$T_{z_1^N}$, there exist a function $f(z_2) \in A^2_\alpha(D)$ and a nonnegative integer n_1 such that $f(z_2)M^{(1)}_{n_1} \subseteq M$.

For any nonzero function $f(z_1, z_2) \in M$,

$$f(z_1, z_2) = \sum_{n=0}^{\infty} f_n(z_2) z_1^n,$$

by Lemma 2.1, for any n, $f_n(z_2)z_1^n \in M$. For any $n = 0, 1, 2, \ldots$, there are two nonnegative integers n_1, α_1 such that

$$n = n_1 + \alpha_1 N, \quad (0 \le n_1 \le N - 1).$$

Since $f(z_1, z_2) \ne 0$, there exists a nonnegative integer n such that $f_n(z_2) \ne 0$ and $f_n(z_2) z_1^{n_1+\alpha_1 N} \in M$. We know that M is invariant under the operators $T_{z_1^N}$ and $T^*_{z_1^N}$, so

$$f_n(z_2) Span\{z_1^{n_1+lN} : l = 0, 1, 2, \ldots\} \subseteq M,$$

i.e. $f_n(z_2)M^{(1)}_{n_1} \subseteq M$.

Assume that M is a minimal reducing subspace of $T_{z_1^N}$ in $A^2_\alpha(D^2)$. As is stated above, there exists a reducing subspace $f(z_2)M^{(1)}_{n_1}$ of $T_{z_1^N}$ such that $f(z_2)M^{(1)}_{n_1} \subseteq M$. It forces that $M = f(z_2)M^{(1)}_{n_1}$.

Finally, we will prove that if $M^{(1)}_{n_1}$ is a minimal reducing subspace of $M_{z_1^N}$ in $A^2_\alpha(D)$, $0 \le n_1 \le N - 1$, then for any $f(z_2) \in A^2_\alpha(D)$, $f(z_2)M^{(1)}_{n_1}$ is a minimal reducing subspace of $T_{z_1^N}$ in $A^2_\alpha(D^2)$.

Assume that M is a nonzero proper minimal reducing subspace of $T_{z_1^N}$ contained in $f(z_2)M^{(1)}_{n_1}$ for some n_1. Let a closed subspace $M' = \{g(z_1) \in A^2_\alpha(D) : f(z_2)g(z_1) \in M\}$. It is obvious that $M = f(z_2)M'$ and M is minimal reducing subspace of $T_{z_1^N}$ in $A^2_\alpha(D^2)$, if and only if, M' is a minimal reducing subspace of $M_{z_1^N}$ in $A^2_\alpha(D)$. We can see that $M' \subset M^{(1)}_{n_1}$. So it is a contradiction, since $M^{(1)}_{n_1}$ is minimal. Thus $f(z_2)M^{(1)}_{n_1}$ is minimal.

In conclusion, we obtain that M is a minimal reducing subspace of $T_{z_1^N}$ in $A^2_\alpha(D^2)$, if and only if, for some function $f(z_2) \in A^2_\alpha(D)$ and nonnegative integer n_1 with $0 \le n_1 \le N - 1$, $M = f(z_2)M^{(1)}_{n_1}$. Thus the proof is completed. □

THEOREM 2.3. *For any function $f = f(z_1) \in A^2_\alpha(D)$, and each integer n_2 with $0 \le n_2 \le N - 1$,*

$$Span\{f(z_1)z_2^{n_2+\alpha_2 N} : \alpha_2 = 0, 1, 2, \dots\} = f(z_1)M_{n_2}^{(2)}$$

is the only minimal reducing subspace of $T_{z_2^N}$ in $A_\alpha^2(D^2)$. Every reducing subspace of $T_{z_2^N}$ in $A_\alpha^2(D^2)$ contains a minimal reducing subspace.

Proof. The proof follows from the symmetry of z_1, z_2 and Theorem 2.2. □

Theorem 2.4. *Suppose M is a reducing subspace of both $T_{z_1^N}$ and $T_{z_2^N}$ in $A_\alpha^2(D^2)$, then there exist nonnegative integers n_1, n_2 with $0 \le n_1, n_2 \le N-1$ such that $M_{n_1}^{(1)} \otimes M_{n_2}^{(2)} \subseteq M$. In particular, M is minimal, if and only if, $M = M_{n_1}^{(1)} \otimes M_{n_2}^{(2)}$. And there are N^2 minimal reducing subspaces in $A_\alpha^2(D^2)$.*

Proof. If M is a reducing subspace of $T_{z_1^N}$, for $f = \sum_{k=0}^{\infty} f_k(z_2)z_1^k \in M$, by Lemma 2.1, for any k, $f_k(z_2)z_1^k \in M$. Since M is a reducing subspace of $T_{z_2^N}$, for $f_k(z_2)z_1^k = \sum_{l=0}^{\infty} a_{k,l} z_2^l z_1^k$, by Lemma 2.1, for any k, l, $a_{k,l} z_2^l z_1^k \in M$. There are nonnegative integers n_1, n_2, n_1', n_2' with $0 \le n_1, n_2 \le N-1$ such that $k = n_1 + n_1'N, l = n_2 + n_2'N$, then

$$Span\{z_1^{n_1+n_1'N} z_2^{n_2+n_2'N} : n_1', n_2' = 0, 1, 2, \dots\} \subseteq M, \quad i.e. \quad M_{n_1}^{(1)} \otimes M_{n_2}^{(2)} \subseteq M.$$

It is obvious that $M_{n_1}^{(1)} \otimes M_{n_2}^{(2)}$ is a reducing subspace of both $T_{z_1^N}$ and $T_{z_2^N}$. If M is minimal, then $M = M_{n_1}^{(1)} \otimes M_{n_2}^{(2)}$.

It is easy to see that $M_{n_1}^{(1)} \otimes M_{n_2}^{(2)}$ is a minimal reducing subspace. So there are N^2 minimal reducing subspaces of both $T_{z_1^N}$ and $T_{z_2^N}$ in $A_\alpha^2(D^2)$. □

Lemma 2.2. *For any nonnegative integers m_1, m_2, l, N with $l \ge 1$, $N > 1$. If $J \ne (m_1, m_2)$ or $J \ne (m_2, m_1)$, where $J = (j_1, j_2)$ is a multi-index, then $\frac{\omega_{j_1+lN}\omega_{j_2+lN}}{\omega_{m_1+lN}\omega_{m_2+lN}} - 1 \ne 0$.*

Proof. Without loss of generality, we might as well let $m_1 \ge m_2$.

Let $\Delta = \frac{\omega_{j_1+lN}\omega_{j_2+lN}}{\omega_{m_1+lN}\omega_{m_2+lN}}$.

For the sequence $\{\omega_n\}$ is decreasing, if $j_1, j_2 > m_1$, then $\Delta - 1 < 0$; if $m_2 > j_1, j_2$, then $\Delta - 1 > 0$.

If $j_1 > m_1 \ge m_2 > j_2$, and $j_1 - m_1 = m_2 - j_2$, then

$$\Delta = \frac{(j_1 + lN)\cdots(m_1 + 1 + lN)(m_2 + 1 + \alpha + lN)\cdots(j_2 + 2 + \alpha + lN)}{(j_1 + 1 + \alpha + lN)\cdots(m_1 + 2 + \alpha + lN)(m_2 + lN)\cdots(j_2 + 1 + lN)}.$$

It is easy to calculate that for the function $f(x, y) = \frac{(\alpha+1+x)y}{(\alpha+1+y)x} - 1$,

$$\begin{cases} f(x,y) > 0, & \text{if } y > x; \\ f(x,y) < 0, & \text{if } y < x, \end{cases}$$

where $\alpha > -1$. So $\Delta > 1$.

If $j_1 > m_1 \geq m_2 > j_2$, and $j_1 - m_1 \neq m_2 - j_2$, we will prove it by contradiction.

Suppose that $\Delta - 1 = 0$. Let

$$H_1(\lambda) = \frac{(j_1+\lambda)\cdots(m_1+1+\lambda)(1+\alpha+m_2+\lambda)\cdots(2+\alpha+j_2+\lambda)}{(1+\alpha+j_1+\lambda)\cdots(2+\alpha+m_1+\lambda)(m_2+\lambda)\cdots(j_2+1+\lambda)} - 1,$$

then $H_1(\lambda)$ is rational and holomorphic at infinity, and

$$\lim_{|\lambda|\to\infty} H_1(\lambda) = 0.$$

Let $H_2(\lambda) = H_1(\frac{1}{\lambda})$, we can choose ρ small enough, for $|\lambda| < \rho$, $H_2(\lambda)$ is holomorphic, and

$$\lim_{|\lambda|\to 0} H_2(\lambda) = 0.$$

Then 0 is the removable singular point of $H_2(\lambda)$.

Since $H_2(\lambda)$ vanishes at all the points $\frac{1}{lN}$ $(l = 1, 2, \ldots)$ whose limit point is 0, 0 is the essential singular point of $H_2(\lambda)$. It is a contradiction. So $H_2(\lambda) \equiv 0$, for $|\lambda| < \rho$. Then $H_1(\lambda) \equiv 0$, for $|\lambda| > \frac{1}{\rho}$.

$H_1(\lambda) \equiv 0$, if and only if, for any $0 \leq n \leq j_1 - j_2 - m_1 + m_2$, the term λ^n's coefficient in the numerator of $H_1(\lambda)$ is zero. For the term $\lambda^{j_1-j_2-m_1+m_2-1}$, its coefficient is

$$(1+\alpha)(j_1 - m_1 - m_2 + j_2) = 0,$$

then $j_1 - m_1 = m_2 - j_2$. It is a contradiction. So $\triangle - 1 \neq 0$.

Similarly, we can prove the case: $m_1 > j_1 > j_2 > m_2$, $\triangle - 1 \neq 0$. So if $J \neq (m_1, m_2)$, then $\triangle - 1 \neq 0$. And for the same reason, if $J \neq (m_2, m_1)$, then $\triangle - 1 \neq 0$. □

Lemma 2.3. *Suppose M is a reducing subspace of T_{z^N} in $A^2_\alpha(D^2)$ and P_M is the orthogonal projection from $A^2_\alpha(D^2)$ onto M. For any nonnegative integers k, m,*

$$P_M\big(z_1^k(z_1z_2)^m\big) = \big(az_1^k + bz_2^k\big)(z_1z_2)^m,$$

where $a \in \boldsymbol{R}$, $a - a^2 = |b|^2$, *and* $0 \le a$, $|b| \le 1$.

Proof. If M is a reducing subspace of T_{z^N} in $A^2_\alpha(D^2)$, the orthogonal decomposition of $z_1^k (z_1 z_2)^m$ with respect to M is

$$z_1^k (z_1 z_2)^m = f + g, \quad f \in M, \; g \in M^\perp.$$

Let $f = \sum_J a_J z^J$ be the multiple Fourier series of f. For any $l = 1, 2, \ldots$,

$$\begin{aligned} P_M T^*_{z^{lN}} T_{z^{lN}} (f+g) = T^*_{z^{lN}} T_{z^{lN}} f &= \sum_J a_J Q\big(z_1^{j_1+lN} \overline{z}_1^{lN}\big) Q\big(z_2^{j_2+lN} \overline{z}_2^{lN}\big) \\ &= \sum_J a_J \frac{\omega_{j_1+lN}\omega_{j_2+lN}}{\omega_{j_1}\omega_{j_2}} z_1^{j_1} z_2^{j_2}, \end{aligned}$$

and

$$\begin{aligned} P_M T^*_{z^{lN}} T_{z^{lN}} \big(z_1^k (z_1 z_2)^m\big) &= P_M \Big(P\big(z_1^{k+m+lN} \overline{z}_1^{lN} z_2^{m+lN} \overline{z}_2^{lN}\big)\Big) \\ &= P_M \left(\frac{\omega_{k+m+lN}\omega_{m+lN}}{\omega_{k+m}\omega_m} z_1^k (z_1 z_2)^m \right) \\ &= P_M \left(\frac{\omega_{k+m+lN}\omega_{m+lN}}{\omega_{k+m}\omega_m} (f+g) \right) \\ &= \frac{\omega_{k+m+lN}\omega_{m+lN}}{\omega_{k+m}\omega_m} f. \end{aligned}$$

It follows that

$$\frac{\omega_{k+m+lN}\omega_{m+lN}}{\omega_{k+m}\omega_m} f = \sum_J a_J \frac{\omega_{j_1+lN}\omega_{j_2+lN}}{\omega_{j_1}\omega_{j_2}} z_1^{j_1} z_2^{j_2},$$

then

$$f = \sum_J a_J \frac{\omega_{j_1+lN}\omega_{j_2+lN}}{\omega_{k+m+lN}\omega_{m+lN}} \frac{\omega_{k+m}\omega_m}{\omega_{j_1}\omega_{j_2}} z_1^{j_1} z_2^{j_2},$$

or

$$\sum_J a_J \left(\frac{\omega_{j_1+lN}\omega_{j_2+lN}}{\omega_{k+m+lN}\omega_{m+lN}} \frac{\omega_{k+m}\omega_m}{\omega_{j_1}\omega_{j_2}} - 1 \right) z_1^{j_1} z_2^{j_2} = 0.$$

Let $\Delta = \frac{\omega_{j_1+lN}\omega_{j_2+lN}}{\omega_{k+m+lN}\omega_{m+lN}} \frac{\omega_{k+m}\omega_m}{\omega_{j_1}\omega_{j_2}}$. If $\Delta - 1 = 0$, let

$$H_1(\lambda) = \frac{\omega_{j_1+\lambda}\omega_{j_2+\lambda}}{\omega_{k+m+\lambda}\omega_{m+\lambda}} \frac{\omega_{k+m}\omega_m}{\omega_{j_1}\omega_{j_2}} - 1,$$

then $H_1(\lambda)$ is rational and holomorphic at infinity, and

$$\lim_{|\lambda|\to\infty} H_1(\lambda) = \frac{\omega_{k+m}\omega_m}{\omega_{j_1}\omega_{j_2}} - 1.$$

Let

$$H_2(\lambda) = \begin{cases} \frac{\omega_{k+m}\omega_m}{\omega_{j_1}\omega_{j_2}} - 1, & \text{if } \lambda = 0; \\ H_1(\frac{1}{\lambda}), & \text{if } \lambda \neq 0, \end{cases}$$

we can choose ρ small enough, for $|\lambda| < \rho$, $H_2(\lambda)$ is holomorphic, and

$$\lim_{|\lambda|\to 0} H_2(\lambda) = \frac{\omega_{k+m}\omega_m}{\omega_{j_1}\omega_{j_2}} - 1.$$

Then 0 is the removable singular point of $H_2(\lambda)$.

Since $H_2(\lambda)$ vanishes at all the points $\frac{1}{lN}(l = 1, 2, \ldots)$ whose limit point is 0, 0 is the essential singular point of $H_2(\lambda)$. It is a contradiction. So $H_2(\lambda) \equiv 0$, for $|\lambda| < \rho$. And $\frac{\omega_{k+m}\omega_m}{\omega_{j_1}\omega_{j_2}} = 1$. Thus $\Delta - 1 = \frac{\omega_{j_1+lN}\omega_{j_2+lN}}{\omega_{k+m+lN}\omega_{m+lN}} - 1 = 0$. By Lemma 2.2, $J = (k+m, m)$ or $J = (m, k+m)$.

So for any $J \neq (k+m, m)$ or $J \neq (m, k+m)$, $\Delta - 1 \neq 0$, then $a_J = 0$.

Thus $f = az_1^k(z_1z_2)^m + bz_2^k(z_1z_2)^m$, i.e. $P_M(z_1^k(z_1z_2)^m) = (az_1^k + bz_1^k)(z_1z_2)^m$.

Since $\langle f, z_1^k(z_1z_2)^m - f\rangle = 0$, $\|f\|^2 = \langle z_1^k(z_1z_2)^m, f\rangle$,

$$|a|^2\omega_{k+m}\omega_m + |b|^2\omega_{k+m}\omega_m = \overline{a}\omega_{k+m}\omega_m,$$

$$(|a|^2 + |b|^2)\omega_{k+m}\omega_m = \overline{a}\omega_{k+m}\omega_m,$$

for $\omega_{k+m}\omega_m \neq 0$, then $a \in \boldsymbol{R}$, $a - a^2 = |b|^2$, and $0 \leq a, |b| \leq 1$. □

Lemma 2.4. *Suppose M is a reducing subspace of T_{z^N} in $A^2_\alpha(D^2)$. For any function*

$$f = \sum_J a_J z^J \in M,$$

if there are nonnegative integers $p, q > 0$, $a_{p,q} \neq 0$, then

$$\big(z_1^{|p-q|} + z_2^{|p-q|}\big)(z_1 z_2)^{\min\{p,q\}} \in M, \quad or \quad z_1^p z_2^q \in M.$$

Proof. For any function $f = \sum_J a_J z^J \in M$,

$$f = \sum_{j_1 > j_2} a_{j_1,j_2} z_1^{j_1 - j_2}(z_1 z_2)^{j_2} + \sum_{j_1 = j_2} a_{j_1,j_2}(z_1 z_2)^{j_1} + \sum_{j_1 > j_2} a_{j_2,j_1} z_2^{j_1 - j_2}(z_1 z_2)^{j_2}.$$

Case 1: $p = q$.

According to Lemma 2.3, let $k = 0$, $m = p$, $P_M((z_1 z_2)^p) = (a+b)(z_1 z_2)^p$, then $(z_1 z_2)^p \in M$ or $(z_1 z_2)^p \in M^{\perp}$.

If $(z_1 z_2)^p \in M^{\perp}$, then $\langle f, (z_1 z_2)^p\rangle = a_{p,p}\omega_p^2 = 0$, so $a_{p,p} = 0$, it is a contradiction with $a_{p,p} \neq 0$. So $(z_1 z_2)^p \in M$.

Case 2: $p \neq q$.

By Case 1, $\sum_{j_1 = j_2} a_{j_1,j_2}(z_1 z_2)^{j_1} \in M$. Let $f_0 = f - \sum_{j_1 = j_2} a_{j_1,j_2}(z_1 z_2)^{j_1}$, it is easy to see that $f_0 \in M$.

By Lemma 2.3,

$$\begin{aligned} f_0 &= P_M(f_0) \\ &= \sum_{j_1 > j_2} a_{j_1,j_2}\big(c_{j_1,j_2} z_1^{j_1} z_2^{j_2} + d_{j_1,j_2} z_1^{j_2} z_2^{j_1}\big) + \sum_{j_1 > j_2} a_{j_2,j_1}\big(c_{j_1,j_2} z_1^{j_2} z_2^{j_1} + d_{j_1,j_2} z_1^{j_1} z_2^{j_2}\big) \\ &= \sum_{j_1 > j_2}\big(a_{j_1,j_2} c_{j_1,j_2} + a_{j_2,j_1} d_{j_1,j_2}\big) z_1^{j_1} z_2^{j_2} + \sum_{j_1 > j_2}\big(a_{j_1,j_2} d_{j_1,j_2} + a_{j_2,j_1} c_{j_1,j_2}\big) z_1^{j_2} z_2^{j_1}, \end{aligned}$$

then

$$\begin{cases} a_{j_1,j_2} = a_{j_1,j_2} c_{j_1,j_2} + a_{j_2,j_1} d_{j_1,j_2} \\ a_{j_2,j_1} = a_{j_1,j_2} d_{j_1,j_2} + a_{j_2,j_1} c_{j_1,j_2}\,, \end{cases}$$

i.e.

$$\begin{cases} (1 - c_{j_1,j_2}) a_{j_1,j_2} = d_{j_1,j_2} a_{j_2,j_1} & (1) \\ (1 - c_{j_1,j_2}) a_{j_2,j_1} = d_{j_1,j_2} a_{j_1,j_2}\,. & (2) \end{cases}$$

Put (1) in (2) $\times\, a_{j_1,j_2}$, and put (2) in (1) $\times\, a_{j_1,j_2}$, then

$$\begin{cases} d_{j_1,j_2} a^2_{j_2,j_1} = d_{j_1,j_2} a^2_{j_1,j_2} \\ (1-c_{j_1,j_2}) a^2_{j_1,j_2} = (1-c_{j_1,j_2}) a^2_{j_2,j_1}\,. \end{cases}$$

If $d_{j_1,j_2} = 1 - c_{j_1,j_2} = 0$, then $P_M(z_1^{j_1} z_2^{j_2}) = z_1^{j_1} z_2^{j_2} \in M$. So $z_1^p z_2^q \in M$, since $a_{p,q} \neq 0$.

If either d_{j_1,j_2} or $1 - c_{j_1,j_2}$ is not zero, then $a_{j_1,j_2} = \pm a_{j_2,j_1}$.

If $a_{j_1,j_2} = -a_{j_2,j_1}$, put it in (1), we have $c_{j_1,j_2} = 1 + d_{j_1,j_2}$, then $c^2_{j_1,j_2} = |1 + d_{j_1,j_2}|^2 > 1$, it is a contradiction with Lemma 2.3. So $a_{j_1,j_2} = a_{j_2,j_1}$. Then

$$f_0 = \sum_{j_1 > j_2} a_{j_1,j_2} \big(z_1^{j_1-j_2} + z_2^{j_1-j_2}\big)(z_1 z_2)^{j_2}.$$

We might as well let $p > q$, if $(z_1^{p-q} + z_2^{p-q})(z_1 z_2)^q = z_1^p z_2^q + z_1^q z_2^p \in M^\perp$, then

$$\big\langle f_0, z_1^p z_2^q + z_1^q z_2^p \big\rangle = 0,$$

i.e.

$$\big\| z_1^p z_2^q + z_1^q z_2^p \big\|^2 a_{p,q} = 0,$$

so $a_{p,q} = 0$, it is a contradiction. By Lemma 2.3, $(z_1^{p-q} + z_2^{p-q})(z_1 z_2)^q \in M$.

Similarly, if $q > p$, $(z_1^{q-p} + z_2^{q-p})(z_1 z_2)^p \in M$. □

THEOREM 2.5. *Suppose M is a reducing subspace of T_{z^N} in $A^2_\alpha(D^2)$, then there exist nonnegative integers a, b, k, m with $0 \le m \le N-1$ and $a, b \in \{0,1\}$ such that*

$$Span\big\{(a z_1^k + b z_2^k)(z_1 z_2)^{m+lN} : l = 0, 1, 2, \dots\big\} \subseteq M.$$

In particular, M is minimal, if and only if,

$$M = Span\big\{(a z_1^k + b z_2^k)(z_1 z_2)^{m+lN} : l = 0, 1, 2, \dots\big\}.$$

PROOF. If M is a reducing subspace of T_{z^N} in $A^2_\alpha(D^2)$, then by Lemma 2.4, for some nonnegative integers a, b, k, m with $0 \le m \le N-1$, and $a, b \in \{0,1\}$,

$$\big(a z_1^k + b z_2^k\big)(z_1 z_2)^m \in M.$$

We know that M is invariant under the operators T_{z^N} and $T^*_{z^N}$, so

$$Span\{(az_1^k + bz_2^k)(z_1z_2)^{m+lN} : l = 0, 1, 2, \dots\} \subseteq M.$$

It is easy to see that $Span\{(az_1^k+bz_2^k)(z_1z_2)^{m+lN} : l = 0, 1, 2, \dots\}$ is a reducing subspace of T_{z^N} in $A^2_\alpha(D^2)$. If M is minimal, then

$$M = Span\{(az_1^k + bz_2^k)(z_1z_2)^{m+lN} : l = 0, 1, 2, \dots\}.$$

It is obvious that $Span\{(az_1^k + bz_2^k)(z_1z_2)^{m+lN} : l = 0, 1, 2, \dots\}$ is a minimal reducing subspace. So it is the only minimal reducing subspace of T_{z^N} in $A^2_\alpha(D^2)$. □

3. Beurling-type theorem of the weighted Bergman space over bidisk.

In this section we will show the Beurling-type theorem about invariant subspaces of the weighted Bergman space over bidisk. The basis of our proof is the following result which was obtained in the context of general Hilbert space; see Theorem 6.14 in [**3**] for details.

Let H be a separable Hilbert space and let $T : H \to H$ be a bounded linear operator satisfying:
(a) $\|Tx + y\|^2 \le 2(\|x\|^2 + \|Ty\|^2)$, $x, y \in H$;
(b) $\bigcap\{T^nH : n \ge 0\} = 0$,
then we have
(i) T is one to one and has closed range, so that the operator T^*T is invertible,
(ii) $H = [\mathscr{E}] = \bigvee\{T^nx : x \in \mathscr{E}, n \ge 0\}$, where $\mathscr{E} = ker(T^*) = H \ominus TH$.

In fact, under some assumption, we have that (i) implies (a).

Lemma 3.1. *Let H be a separable Hilbert space and let $T : H \to H$ be a bounded linear operator. If the operator T is one to one and has closed range, and satisfies*

$$TT^* + (T^*T)^{-1} \le 2I,$$

then the operator T satisfies condition (a)*, that is*

$$\|Tf + g\|^2 \le 2(\|f\|^2 + \|Tg\|^2), \quad f, g \in H.$$

PROOF. Let $g = (T^*T)^{-\frac{1}{2}}h$ in condition (a), then the condition (a) is equivalent to the inequality

$$\left\|Tf + (T^*T)^{-\frac{1}{2}}h\right\|^2 \leq 2(\|f\|^2 + \|h\|^2).$$

Consider the operator $R : H \oplus H \to H$ defined by

$$R(f,h) = Tf + (T^*T)^{-\frac{1}{2}}h, \quad (f,h) \in H \oplus H,$$

then we have

$$R^*(h) = \left(T^*h, (T^*T)^{-\frac{1}{2}}h\right), \quad h \in H,$$

it follows that

$$RR^* = TT^* + (T^*T)^{-1}.$$

Since

$$TT^* + (T^*T)^{-1} \leq 2I,$$

where I is the identity operator on H, then $RR^* \leq 2I$, thus $\|R\| \leq \sqrt{2}$.

For $(f,h) \in H \oplus H$, $\|R(f,h)\|^2 \leq (\sqrt{2})^2\|I(f,h)\|^2$, that is

$$\left\|Tf + (T^*T)^{-\frac{1}{2}}h\right\|^2 \leq 2(\|f\|^2 + \|h\|^2).$$

Thus the result is proved. □

Through describing the corresponding matrix of $T_{z_i} (i = 1, 2)$, we have the following theorem.

THEOREM 3.1. *Given $-1 < \alpha \leq 0$ and T_{z_1} is a bounded linear operator on $A^2_\alpha(D^2)$, then $T_{z_1}T^*_{z_1} + (T^*_{z_1}T_{z_1})^{-1} \leq 2I$, where I is the identity operator on $A^2_\alpha(D^2)$.*

PROOF. For any $n = 0, 1, 2, \ldots$, in the closed subspace $X^{(1)}_n$, the operator S_n can be represented as a $\aleph_0 \times \aleph_0$ matrix:

$$S_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \sqrt{w_1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{\frac{w_2}{w_1}} & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \sqrt{\frac{w_m}{w_{m-1}}} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{\frac{w_{m+1}}{w_m}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then the operator S_n^* has the matrix form as

$$S_n^* = \begin{pmatrix} 0 & \sqrt{w_1} & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{\frac{w_2}{w_1}} & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{\frac{w_m}{w_{m-1}}} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \sqrt{\frac{w_{m+1}}{w_m}} & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus the operators $S_nS_n^*$, $(S_n^*S_n)^{-1}$ respectively have the matrix forms as

$$S_nS_n^* = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & w_1 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \frac{w_2}{w_1} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{w_m}{w_{m-1}} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix},$$

$$(S_n^*S_n)^{-1} = \begin{pmatrix} \frac{1}{w_1} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \frac{w_1}{w_2} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \frac{w_2}{w_3} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{w_m}{w_{m+1}} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}.$$

Therefore,

$$S_nS_n^* + (S_n^*S_n)^{-1} = \begin{pmatrix} \frac{1}{w_1} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & w_1 + \frac{w_1}{w_2} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \frac{w_2}{w_1} + \frac{w_2}{w_3} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{w_m}{w_{m-1}} + \frac{w_m}{w_{m+1}} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}.$$

It is easily concluded that $\frac{w_m}{w_{m+1}} + \frac{w_m}{w_{m-1}} \leq 2$, $m = 0, 1, 2, \ldots$, for $-1 < \alpha \leq 0$. So

$$S_nS_n^* + \left(S_n^*S_n\right)^{-1} \leq 2\tilde{I},$$

where $\tilde{I}$ is the restriction of the identity operator I to the closed subspace $X_n^{(1)}$. Since

$$T_{z_1} = \sum_{n=0}^{\infty} \oplus T_{z_1}|X_n^{(1)} = \sum_{n=0}^{\infty} \oplus S_n,$$

so $T_{z_1}T_{z_1}^* + (T_{z_1}^*T_{z_1})^{-1} \leq 2I$, the result is proved. □

THEOREM 3.2. *Suppose $-1 < \alpha \leq 0$ and M is an invariant subspace of T_{z_1} in $A_\alpha^2(D^2)$. Then M is generated by $M \ominus T_{z_1}M$, that is*

$$M = [M \ominus T_{z_1}M].$$

PROOF. According to Theorem 3.1 that

$$T_{z_1}T_{z_1}^* + \left(T_{z_1}^*T_{z_1}\right)^{-1} \le 2I,$$

then by Lemma 3.1, condition (a) holds for T_{z_1}. Let T be the restriction of T_{z_1} to the invariant subspace M, then T satisfies condition (a) and (b), and therefore the result is now immediate from Theorem 6.14 in [**3**]. □

THEOREM 3.3. *Suppose* $-1 < \alpha \le 0$ *and* M *is an invariant subspace of* T_{z_2} *in* $A_\alpha^2(D^2)$*. Then* M *is generated by* $M \ominus T_{z_2}M$*, that is*

$$M = [M \ominus T_{z_2}M].$$

PROOF. The proof follows from the symmetry of z_1, z_2 and Theorem 3.2. □

COROLLARY 3.1. *Suppose* $-1 < \alpha \le 0$*. If*

$$M = [M \ominus T_{z_1}M] \bigcap [M \ominus T_{z_2}M],$$

then M *is an invariant subspace of* T_z *in* $A_\alpha^2(D^2)$*.*

PROOF. It obviously follows from $T_z = T_{z_1}T_{z_2}$ and Theorem 3.2, Theorem 3.3. □

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