Asymptotic behavior of flat surfaces
in hyperbolic 3-space

Dedicated to Professor Seiki Nishikawa on the occasion of his sixtieth birthday

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Abstract.   In this paper, we investigate the asymptotic behavior of regular ends of flat surfaces in the hyperbolic 3-space H^3. Gálvez, Martínez and Milán showed that when the singular set does not accumulate at an end, the end is asymptotic to a rotationally symmetric flat surface. As a refinement of their result, we show that the asymptotic order (called *pitch p*) of the end determines the limiting shape, even when the singular set does accumulate at the end. If the singular set is bounded away from the end, we have $-1 < p \le 0$. If the singular set accumulates at the end, the pitch p is a positive rational number not equal to 1. Choosing appropriate positive integers n and m so that $p = n/m$, suitable slices of the end by horospheres are asymptotic to d-coverings (d-times wrapped coverings) of epicycloids or d-coverings of hypocycloids with $2n_0$ cusps and whose normal directions have winding number m_0, where $n = n_0 d$, $m = m_0 d$ (n_0, m_0 are integers or half-integers) and d is the greatest common divisor of $m - n$ and $m + n$. Furthermore, it is known that the caustics of flat surfaces are also flat. So, as an application, we give a useful explicit formula for the pitch of ends of caustics of complete flat fronts.

Introduction.

Let $f : D^* \to H^3$ be an immersion of the unit punctured disc $D^* = \{z \in \mathbf{C} ; 0 < |z| < 1\}$ into the hyperbolic 3-space H^3. Then f is called *flat* if the Gaussian curvature vanishes everywhere, and assuming this is the case, we call f an *end* of a flat surface. Since any flat surface is orientable [**KRUY**], this is the general setup for "ends" of flat surfaces. Moreover, f is called a *complete* end if f is

complete at the origin $z = 0$ with respect to the Riemannian metric induced by f. Then the two hyperbolic Gauss maps $G, G_* : D^* \to \partial H^3 = \boldsymbol{C} \cup \{\infty\}$ are defined on D^* [**GMM**]. If both $G(z)$ and $G_*(z)$ can be extended smoothly across $z = 0$, f is called a *regular* end, and otherwise f is called an *irregular* end.

Let ν be the unit normal vector field to f, and set

$$f_t : D^* \ni z \longmapsto f_t(z) = \mathrm{Exp}_{f(z)}\big(t\nu(z)\big) \in H^3$$

for each real number t, where "Exp" denotes the exponential map of the hyperbolic 3-space H^3 (see (1.8) in the next section for a more explicit description of f_t). This surface f_t is called a *parallel surface* of f. A parallel surface f_t may have singular points, but it will be considered here as a (*wave*) *front*, i.e., a surface which admits certain kinds of singularities (see [**GMM**], [**KUY2**]). Moreover, any parallel surface f_t, away from singular points, is flat if f is flat. It is often reasonable to begin arguments under the assumption that the flat surface is a front. When we wish to emphasize that assumption, we speak of it as a *flat front* instead of a flat surface.

From now on, we assume $f : D^* \to H^3$ is a flat front such that f_{t_o} is complete for some $t_o \in \boldsymbol{R}$. Even if f is a complete end, f_t might not be complete at the origin in general, that is, it can happen that the singular points of f_t accumulate at the origin. However, each f_t, including $f = f_0$, is weakly complete and of finite type in the sense of [**KRUY**] (see Definition 1.5). Moreover, for each non-umbilic point $z \in D^*$, there is a unique $t(z) \in \boldsymbol{R}$ so that $f_{t(z)}$ is not an immersion at z, i.e., z is a singular point of $f_{t(z)}$. Then the singular locus (or equivalently, the set of focal points) is the image of the map

$$C_f : D^* \setminus \{\text{umbilic points}\} \ni z \longmapsto f_{t(z)} \in H^3,$$

which is called the *caustic* (or *focal surface*) of f. Note that caustics can be defined not only for ends but globally for non-totally umbilic flat fronts. Roitman [**R**] proved that C_f is flat (in fact, it is locally a flat front, see [**KRSUY**] and [**KRUY**]), and gave a holomorphic representation formula for such caustics.

A caustic can have more symmetry than the original surface: Figure 1 shows a symmetric four-noid and its caustic. The caustic of the four-noid as in Figure 1 (right) has octahedral symmetry, though the original surface has only dihedral symmetry. This shows that an end of a caustic coming from an umbilic point of the original surface can be congruent to another of the caustic's ends coming from an end of the original surface.
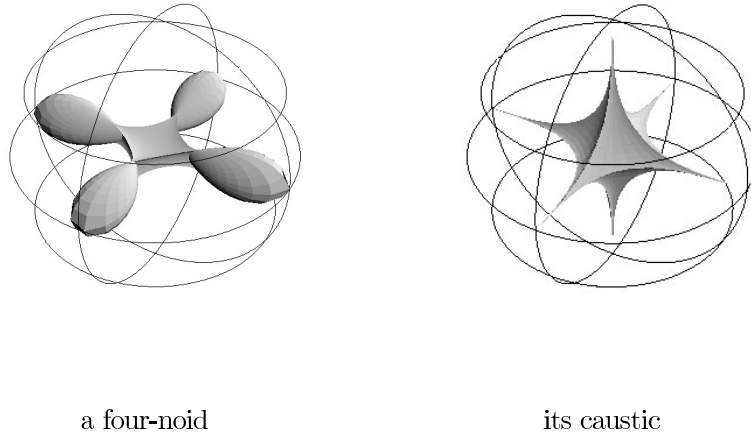
a four-noid   its caustic

Figure 1. A flat four-noid and its caustic.

As seen in Figure 1, the ends of caustics are typically highly acute, and the singular sets accumulate at the ends, even though non-cylindrical complete ends are tangent to the ideal boundary (see also Figure 6 in Section 4). Prompted by Roitman's work, the authors numerically examined such incomplete ends on several caustics and were surprised at their acuteness and at the additional symmetry as mentioned above, and so wished to analyze their behavior precisely. This is the central motivation of this paper, which is a sequel of the previous paper [**KRUY**].

As an analogue of a result in [**UY1**] for constant mean curvature one surfaces (CMC-1 surfaces) in H^3, [**GMM**] showed that a complete regular end is asymptotic to the m-fold cover of one of the rotationally symmetric flat surfaces, and that $m = 1$ implies proper embeddedness of the end. In order to state both this result and other new results, we fix the setting as follows: Let $f : D^* \to H^3$ be a weakly complete end of finite type, which we abbreviate as "*WCF-end*" (Weak completeness and finiteness were defined in [**KRUY**], and are also defined just before Fact 1.4).

Moreover, we assume the WCF-end f is regular. (The regularity for WCF-ends is defined in the same way as for complete ends.) Denoting by $\pi : H^3 \to \mathbf{R}^3_+$ the projection of H^3 to the Poincaré upper half-space model $\mathbf{R}^3_+ = \{(\zeta, h) \in \mathbf{C} \times \mathbf{R} \, ; \, h > 0\}$, we discuss the asymptotic behavior of regular WCF-ends in terms of $\pi \circ f$.

Note that WCF-ends are generalizations of complete ends. Moreover, *all ends of caustics of complete regular-ended flat fronts are regular WCF-ends* (*see* [**KRUY**, Theorems 7.4 and 7.6]).

Gálvez, Martínez and Milán [**GMM**] proved that each complete regular end is asymptotic to a finite covering of a rotationally symmetric end. The following proposition is essentially the same as their result, but now stated in terms of a geometric quantity we call the *pitch* (see Proposition A and Theorem B), and also in terms of the ratio of the Gauss maps (see [**KRUY**] or (1.26) for a definition):

PROPOSITION A.   *Suppose that the flat surface f is a complete regular end. Then for a sufficiently small $\varepsilon > 0$, the image $\pi \circ f(D_\varepsilon^*)$ $(D_\varepsilon^* = \{z \in \boldsymbol{C} \,|\, 0 < |z| < \varepsilon\})$ is congruent to a portion of the image of $[0, 2\pi) \times (0, h_0) \ni (t, h) \mapsto (\varphi_h(t), h) \in \boldsymbol{R}_+^3$ with*

$$\varphi_h(t) = ce^{imt}h^{1+p} + o(h^{1+p}), \tag{1}$$

for a non-zero constant c, a nonpositive constant p, and a positive integer m. Here, m is the multiplicity of the end as in (1.25), $o(h^{1+p})$ denotes terms of order higher than h^{1+p} as $h \to 0$, and the exponent p (called the pitch of f) is related to the ratio α of the Gauss maps by

$$p = -\frac{1+\alpha}{2} \in (-1, 0].$$

In particular, the pitch of each parallel surface f_t is also equal to p whenever f_t is complete.

Later, we shall give a refinement of this assertion, that is, we shall compute the second term of the expansion in (1) (Theorems 3.1 and 3.5).

A description of the asymptotic behavior of CMC-1 surfaces in H^3 was first given in [**UY1**], and refinements were given by Sa Earp and Toubiana [**ET**] and Daniel [**D**]. In particular, Daniel's refinement gives relationships between the flux and asymptotic behavior of complete regular ends of CMC-1 surfaces. To prove Proposition A and its refinements (Theorems 3.1 and 3.5), we define an analogue of the flux matrix as in [**RUY**]. In this sense, Theorems 3.1, 3.5 and Theorem B below are an analogue of Daniel's line of investigation.

On the other hand, the asymptotic behavior of an incomplete end, as in Theorem B below, has clearly not been analyzed in the case of CMC-1 surfaces, for the obvious reason that those surfaces do not have singularities. Analysis of the incomplete end case leads to a mysterious connection between flat surfaces and cycloid curves: A cycloid is the image of the map

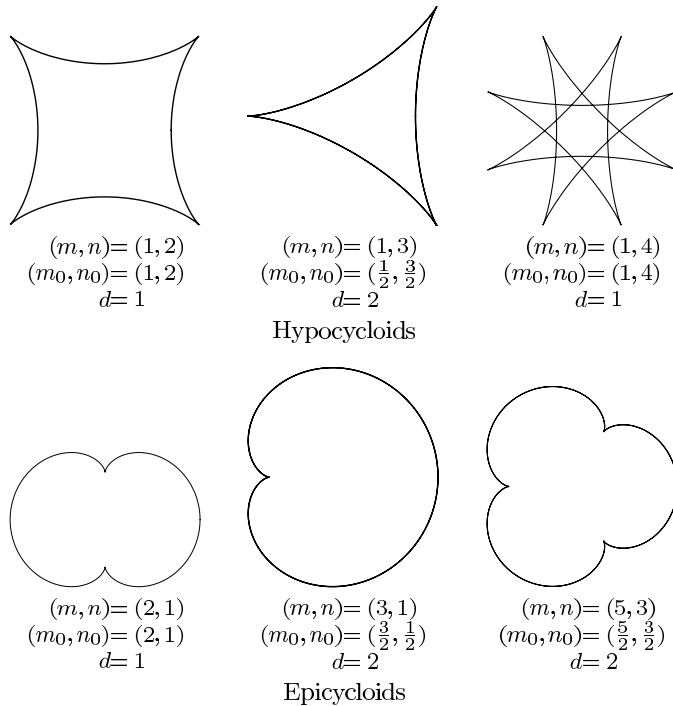$$\Gamma_{m,n}(t) = \frac{1}{m}\left[(m+n)e^{i(m-n)t} + (m-n)e^{i(m+n)t}\right].$$

$(m,n)= (1,2)$
$(m_0,n_0)= (1,2)$
$d= 1$

$(m,n)= (1,3)$
$(m_0,n_0)= (\frac{1}{2},\frac{3}{2})$
$d= 2$

$(m,n)= (1,4)$
$(m_0,n_0)= (1,4)$
$d= 1$

Hypocycloids

$(m,n)= (2,1)$
$(m_0,n_0)= (2,1)$
$d= 1$

$(m,n)= (3,1)$
$(m_0,n_0)= (\frac{3}{2},\frac{1}{2})$
$d= 2$

$(m,n)= (5,3)$
$(m_0,n_0)= (\frac{5}{2},\frac{3}{2})$
$d= 2$

Epicycloids

Figure 2. Cycloids.

Let d be the greatest common divisor of $m + n$ and $m - n$ and set $m = m_0 d$, $n = n_0 d$ $(m_0, n_0 \in (1/2)\boldsymbol{Z})$. Then the image of $\Gamma_{m,n}$ is determined by the pair (m_0, n_0) satisfying $m_0, n_0 \in (1/2)\boldsymbol{Z}$, $m_0 \pm n_0 \in \boldsymbol{Z}$ and $\mathrm{GCD}[m_0 + n_0, m_0 - n_0] = 1$. Since such a pair (m_0, n_0) corresponds bijectively to a rational number $n_0/m_0 \in \boldsymbol{Q}_+ \setminus \{1\}$, we denote the image of $\Gamma_{m,n}$ by $c_{n/m}(= c_{n_0/m_0})$, and is called an *epicycloid* if $n/m < 1$ and a *hypocycloid* if $n/m > 1$. It is well-known that $c_{n/m}$ is created by the trace of a point on a circle of signed radius $1 - p$ rolled along another circle of radius $2p = 2n/m = 2n_0/m_0$ without slippage, and has no self-intersections if and only if $|n_0 - m_0| = 1$. Cycloids admit $(3/2)$-cusps, at which their unit normal vectors are well-defined smooth vector fields. The number m_0 takes a value in $(1/2)\boldsymbol{Z}$, and is called the *winding number* of the cycloid c_{n_0/m_0}, which is the number of times that the unit normal vector to the cycloid winds around S^1 as the cycloid is traversed once. The map $\Gamma_{m,n}$ represents a d-covering of the cycloid $c_{n/m}$ (see Figure 2).

In order to describe our main results, we give an integer $2n$, which will correspond both to the number of singularities of a cycloid and to the number of

connected components of cuspidal edges appearing in an incomplete WCF-end. The canonical forms θ and ω associated to a flat front will be defined in Section 1. Note that incomplete regular WCF-ends are all cylindrical. If a regular WCF-end $f : D^* \to H^3$ is cylindrical, then $\rho(z) = \theta/\omega$ (given in (1.12)) is a nonvanishing holomorphic function when taking a suitable local coordinate z around the origin. Moreover if f is incomplete, then $|\rho(0)| = 1$ (see Lemma 1.20), and whenever ρ is non-constant, the ramification order n of ρ is computed as

$$n = 1 + \operatorname{ord}_0 \frac{d\rho}{\rho}, \qquad \frac{d\rho}{\rho} = \left(\frac{\hat{\theta}'}{\hat{\theta}} - \frac{\hat{\omega}'}{\hat{\omega}}\right) dz = \left(\frac{G_*''}{G_*'} - \frac{G''}{G'} + 2\frac{G' + G_*'}{G - G_*}\right) dz, \quad (2)$$

where $\omega = \hat{\omega} \, dz$, $\theta = \hat{\theta} \, dz$, $' = d/dz$ (see Section 1 and [**KRSUY**, (3-15)]), and $\operatorname{ord}_0 \varpi = k$ for a given meromorphic 1-form ϖ if it is written as $\varpi = z^k \varphi(z) \, dz$ $(\varphi(0) \neq 0)$.

THEOREM B.   *Suppose that the flat front f is an incomplete but weakly complete regular end of finite type (i.e., an incomplete regular WCF-end), which is not contained in a geodesic line in H^3. Then for a sufficiently small $\varepsilon > 0$, the image $\pi \circ f(D_\varepsilon^*)$ is congruent to a portion of the image of $[0, 2\pi) \times (0, h_0) \ni (t, h) \mapsto (\varphi_h(t), h) \in \mathbf{R}_+^3$, with*

$$\varphi_h(t) = h^{1+p} \Gamma_{m,n}(t) + o(h^{1+p}), \quad p = \frac{n}{m} \in (0, 1) \cup (1, \infty) . \tag{3}$$

Here, m is the ramification order of the hyperbolic Gauss map $G(z)$ at $z = 0$ (which coincides with that of the other hyperbolic Gauss map $G_*(z)$), and $n \, (\neq m)$ is the ramification order of the function $\rho = \theta/\omega$ at $z = 0$. The set of singular points of f consists only of cuspidal edges, corresponding to the cusps on the slices $\varphi_h(t)$ created by cutting with $h = $ constant. Here, the exponent p is again called the pitch of f. In particular, f has no self-intersections outside of a sufficiently large geodesic ball in H^3 if and only if $d = 1$ and $|m_0 - n_0| = 1$, where m_0, n_0 and d are the numbers determined by the map $\Gamma_{m,n}$.

It should be remarked that a surface given by

$$f_p(\vartheta, h) = (r(\vartheta)e^{i\vartheta} h^{1+p}, h) \in \mathbf{R}_+^3 \qquad (p > 0, \ p \neq 1)$$

is asymptotically flat with respect to the Poincaré metric of constant curvature -1 as $h \to +0$ if and only if

$$(p^2 - 1)\frac{du}{d\vartheta} = p^2 + (p^2 + 1)u^2 + u^4,$$

where $u = d\log r/d\vartheta$. The general solution of this ordinary differential equation for $p = n/m$ characterizes the cycloids $c_{n/m}$, which removes the mystery why cycloids appear in the asymptotic behavior of ends of flat surfaces (see Appendix B).

Theorem B implies that the pitch of any incomplete WCF-end is a positive rational number not equal to 1, so let us use the following terminology: an incomplete WCF-end is of *hypocycloid-type* if the pitch is greater than 1, or of *epicycloid-type* if the pitch is less than 1, respectively.

Theorem B is particularly useful for studying caustics of complete flat fronts in H^3, as those caustics have incomplete ends in general. The pitch of each end of the caustic can be computed using just the pair of hyperbolic Gauss maps for the original flat front as described in Theorem 4.2 in the final section, and it then tells us the asymptotic behavior of the end of the caustic.

Even if we do not know a priori whether the end is complete, any regular WCF-end is asymptotic to either (1) in Proposition A or (3) in Theorem B. Thus the *pitch p* is a single entity that encompasses both cases (1) and (3). As a corollary of Proposition A, Theorem B and [**KRUY**, Propositions 3.1 and 7.3], we have:

COROLLARY C. *The pitch p of a complete regular end takes its value in $(-1, 0]$, and the pitch p of an incomplete regular WCF-end takes its value in $\boldsymbol{Q}_+ \setminus \{1\}$, where $\boldsymbol{Q}_+$ is the set of positive rational numbers. Moreover, a regular WCF-end is*

- *a snowman-type end if and only if $-1 < p < -1/2$,*
- *a horospherical end if and only if $p = -1/2$,*
- *an hourglass-type end if and only if $-1/2 < p < 0$,*
- *a complete cylindrical end if and only if $p = 0$,*
- *an end of epicycloid-type with $2n$ cusps and winding number m if and only if $p = n/m \in (0, 1)$,*
- *an end of hypocycloid-type with $2n$ cusps and winding number m if and only if $p = n/m \in (1, \infty)$.*

Snowman-type ends, horospherical ends, hourglass-type ends and cylindrical ends were defined in [**KRUY**] using properties of the canonical 1-forms and the Hopf differential (see Definition 1.16), and asymptotic behavior was not established there. But as a consequence of Proposition A and Corollary C, snowman-type ends, horospherical ends, hourglass-type ends and complete cylindrical ends

are now known to be asymptotic to the finite covering of a snowman, the horosphere, an hourglass and a cylinder, respectively (see Figure 3).
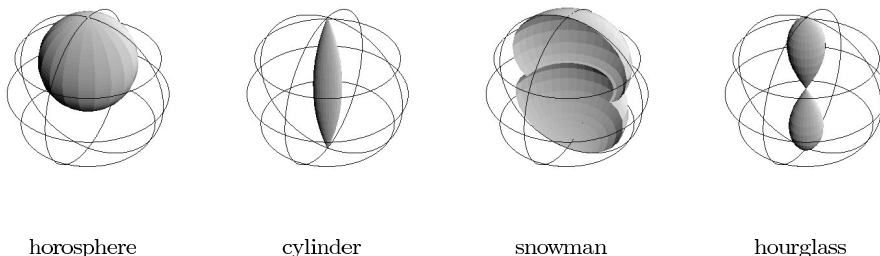

horosphere   cylinder   snowman   hourglass

Figure 3.   Flat fronts of revolution, shown in the Poincaré ball model for H^3.

We will see that any WCF-end of epicycloid-type or hypocycloid-type is necessarily a cylindrical end, but it is not complete (see Lemma 1.20).

Corollary C means that the shape of any end tells us what its pitch p is, and vice versa. For example, Figure 1 (right) indicates the caustic of a flat front of genus 0 with 4 ends (Figure 1 left). The central end (converging to the north pole in ∂H^3) shown there has four cuspidal edges. Since the winding number of slices of the end in this case is 1, we can conclude that the pitch of the end is $p = 2$ (cf. Example 4.4 in Section 4).

1.   Fundamental properties of regular ends of flat fronts.

In this section, we shall describe fundamental properties of flat fronts in the hyperbolic 3-space. See [**KUY1**], [**KUY2**], [**KRUY**] for precise arguments and proofs.

The hyperbolic space. The hyperbolic 3-space H^3 of constant sectional curvature -1 is realized as the upper half component of the hyperboloid of the Minkowski 4-space $\boldsymbol{L}^4$ with inner product $\langle\,,\,\rangle$ of signature $(-,+,+,+)$:

$$H^3 = \{x = (x_0, x_1, x_2, x_3) \in \boldsymbol{L}^4\,;\, \langle x, x \rangle = -1, x_0 > 0\}. \tag{1.1}$$

Identifying $\boldsymbol{L}^4$ with the set $\mathrm{Herm}(2)$ of 2×2 Hermitian matrices as

$$\boldsymbol{L}^4 \ni (x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \qquad \left(i = \sqrt{-1} \right), \qquad (1.2)$$

we can write

$$H^3 = \{X \in \mathrm{Herm}(2)\,;\, \det X = 1, \mathrm{trace}\, X > 0\} \qquad\qquad (1.3)$$
$$= \{uu^* \,|\, u \in SL(2, \boldsymbol{C})\} = SL(2, \boldsymbol{C})/SU(2) \qquad (u^* = {}^t\bar{u}).$$

The complex Lie group $SL(2, \boldsymbol{C})$ acts isometrically on H^3 by

$$\iota_u : H^3 \ni X \longmapsto uXu^* \in H^3 \qquad (u \in SL(2, \boldsymbol{C})). \qquad (1.4)$$

In fact, the identity component of the isometry group of H^3 is identified with $PSL(2, \boldsymbol{C}) = SL(2, \boldsymbol{C})/\{\pm 1\}$.

We consider the projection

$$\pi : H^3 \ni (x_0, x_1, x_2, x_3) \longmapsto \frac{1}{x_0 - x_3}(x_1 + ix_2, 1) \in \boldsymbol{R}_+^3, \qquad (1.5)$$

where $\boldsymbol{R}_+^3 := \{(\zeta, h) \in \boldsymbol{C} \times \boldsymbol{R}\,;\, h > 0\}$. The map π is an isometry from H^3 to the Poincaré upper half-space model

$$\left(\boldsymbol{R}_+^3, \frac{|d\zeta|^2 + dh^2}{h^2} \right). \qquad (1.6)$$

Under the parametrization as in (1.3), we can write

$$\pi(uu^*) = \frac{(u_{11}\overline{u_{21}} + u_{12}\overline{u_{22}}, 1)}{u_{21}\overline{u_{21}} + u_{22}\overline{u_{22}}}, \qquad \text{where} \quad u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in SL(2, \boldsymbol{C}). \quad (1.7)$$

The ideal boundary ∂H^3 is identified with $\boldsymbol{C} \cup \{\infty\}$ as in (A.1) in Appendix A.

Flat fronts. A smooth map $f : M^2 \to H^3$ from a 2-manifold into the hyperbolic 3-space is called a *front* if there exists a Legendrian immersion $L_f : M^2 \to T_1^* H^3$ into the unit cotangent bundle of H^3 whose projection is f. Identifying $T_1^* H^3$ with the unit tangent bundle $T_1 H^3$, L_f corresponds to the *unit normal vector field* $\nu : M^2 \to \boldsymbol{L}^4$ of f, that is, the immersion $(f, \nu) : M^2 \to T_1 H^3$ satisfies $\langle \nu, \nu \rangle = 1$ and

$\langle \nu, df \rangle = 0$. A point $x \in M^2$ where $\operatorname{rank}(df)_x < 2$ is called a *singularity* or *singular point*.

The *parallel front* f_t of a front f at distance t is given by $f_t(x) = \operatorname{Exp}_{f(x)}(t\nu(x))$, where "Exp" denotes the exponential map of H^3. In the model for H^3 as in (1.1), we can write

$$f_t = (\cosh t)f + (\sinh t)\nu, \quad \nu_t = (\cosh t)\nu + (\sinh t)f, \qquad (1.8)$$

where ν_t is the unit normal vector field of f_t.

Based on the fact that any parallel surface of a flat surface is also flat at regular points (i.e., non-singular points), we define flat fronts as follows: A front $f : M^2 \to H^3$ is called a *flat front* if, for each $x \in M^2$, there exists $t \in \mathbf{R}$ such that the parallel front f_t is a flat immersion at x. By definition, $\{f_t\}$ forms a family of flat fronts. We assume this is the case. As in (1.3), the hyperbolic 3-space H^3 can be considered as a subset of $SL(2, \mathbf{C})$, and there exist a complex structure on M^2 and a holomorphic Legendrian immersion

$$\mathscr{E}_f : \widetilde{M}^2 \longrightarrow SL(2, \mathbf{C})$$

$$\text{such that} \quad f = \mathscr{E}_f \mathscr{E}_f^* \quad \text{and} \quad \nu = \mathscr{E}_f e_3 \mathscr{E}_f^* \quad \left(e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), (1.9)$$

where $\widetilde{M}^2$ is the universal cover of M^2 (see [**GMM**] and [**KUY1**]). We call $\mathscr{E}_f$ the *holomorphic Legendrian lift* of the flat front f. Here, $\mathscr{E}_f$ being a holomorphic Legendrian map means that the $\mathfrak{sl}(2, \mathbf{C})$-valued 1-form $\mathscr{E}_f^{-1} d\mathscr{E}_f$ is off-diagonal, where $\mathfrak{sl}(2, \mathbf{C})$ is the Lie algebra of $SL(2, \mathbf{C})$ (see [**GMM**], [**KUY1**], [**KUY2**], [**KRUY**]). So, we can write

$$\mathscr{E}_f^{-1} d\mathscr{E}_f = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}, \qquad (1.10)$$

for holomorphic 1-forms ω and θ on $\widetilde{M}^2$. We call ω and θ the *canonical forms*.

The first and second fundamental forms $ds^2 = \langle df, df \rangle$ and $\mathbb{I} = -\langle df, d\nu \rangle$ are given by

$$ds^2 = |\omega + \bar{\theta}|^2 = Q + \overline{Q} + (|\omega|^2 + |\theta|^2), \qquad Q = \omega\theta,$$
$$\mathbb{I} = |\theta|^2 - |\omega|^2. \qquad (1.11)$$

Note that $|\omega|^2$ and $|\theta|^2$ are well-defined on M^2 itself, though ω and θ are generally

only defined on $\widetilde{M}^2$. The holomorphic 2-differential Q appearing in the $(2,0)$-part of ds^2 is defined on M^2, and is called the *Hopf differential* of f. By definition, the umbilic points of f equal the zeros of Q. Defining a meromorphic function on $\widetilde{M}^2$ by

$$\rho = \frac{\theta}{\omega}, \tag{1.12}$$

then $|\rho| : M^2 \to [0, +\infty]$ is well-defined on M^2, and $x \in M^2$ is a singular point of f if and only if $|\rho(x)| = 1$.

We note that the $(1,1)$-part of the first fundamental form

$$ds_{1,1}^2 = |\omega|^2 + |\theta|^2 \tag{1.13}$$

is positive definite on M^2 because it is the pull-back of the canonical Hermitian metric of $SL(2, \boldsymbol{C})$ by the immersion $\mathscr{E}_f$. Moreover, $2ds_{1,1}^2$ coincides with the pull-back of the canonical metric on the unit cotangent bundle $T_1^* H^3$ by the Legendrian lift L_f of f (which is the sum of the first and third fundamental forms, see [**KUY2**, Section 2] for details). The complex structure on M^2 is compatible with the conformal metric $ds_{1,1}^2$. Note that any flat front is orientable ([**KRUY**, Theorem B]). Throughout this paper, we always consider M^2 as a Riemann surface with this complex structure, for each flat front $f : M^2 \to H^3$.

The two *hyperbolic Gauss maps* are defined as

$$G = \frac{E_{11}}{E_{21}}, \quad G_* = \frac{E_{12}}{E_{22}}, \qquad \text{where} \quad \mathscr{E}_f = (E_{ij}).$$

It can be shown that the hyperbolic Gauss maps are well-defined as meromorphic functions on M^2. In fact, geometrically, G and G_* represent the intersection points in the ideal boundary $\partial H^3 = \boldsymbol{C} \cup \{\infty\}$ of H^3 of the two oppositely-oriented normal geodesics emanating from f in the ν and $-\nu$ directions, respectively (see Proposition A.2 in Appendix A). In particular, parallel fronts have the same hyperbolic Gauss maps. For $u \in SL(2, \boldsymbol{C})$, the change $\mathscr{E}_f \mapsto u\mathscr{E}_f$ corresponds to the rigid motion $f \mapsto \iota_u \circ f = ufu^*$ in H^3 as in (1.4). Under this change, the hyperbolic Gauss maps change by the Möbius transformation:

$$G \mapsto u \star G = \frac{u_{11}G + u_{12}}{u_{21}G + u_{22}}, \qquad G_* \mapsto u \star G_* = \frac{u_{11}G_* + u_{12}}{u_{21}G_* + u_{22}}, \tag{1.14}$$

where $u = (u_{ij})$, in contrast to the canonical forms ω, θ which are unchanged. The canonical forms, the hyperbolic Gauss maps and the Hopf differential are related as follows: Let g and g_* be holomorphic functions on the universal cover $\widetilde{M}^2$ of M^2 such that $dg = \omega$ and $dg_* = \theta$. Then it holds that

$$S(g) - S(G) = S(g_*) - S(G_*) = 2Q, \quad S(h) = \left\{ \left(\frac{h''}{h'} \right)' - \frac{1}{2} \left(\frac{h''}{h'} \right)^2 \right\} dz^2, \quad (1.15)$$

where z is a local complex coordinate and $' = d/dz$, that is, $S(\cdot)$ denotes the Schwarzian derivative with respect to z. The relations in (1.15) suggest us that G, G_*, ω and θ can be considered as pairs (G, ω) and (G_*, θ). In fact, as we shall see in (1.18), the front f can be represented via the pair (G, ω) or (G_*, θ).

DEFINITION 1.1. The canonical form ω (resp. θ) is said to be *associated with* G (resp. G_*).

The holomorphic Legendrian lift $\mathscr{E}_f$ has a $U(1)$-ambiguity, that is,

$$\mathscr{E}_f^\tau = \mathscr{E}_f \begin{pmatrix} e^{i\tau/2} & 0 \\ 0 & e^{-i\tau/2} \end{pmatrix} \qquad (\tau \in \boldsymbol{R})$$

is also a holomorphic Legendrian lift of f. Under this transformation, the canonical forms and the function ρ change as

$$\omega \mapsto e^{i\tau}\omega, \qquad \theta \mapsto e^{-i\tau}\theta, \qquad \rho \mapsto e^{-2i\tau}\rho, \qquad (1.16)$$

in contrast to the hyperbolic Gauss maps G, G_* which are unchanged. On the other hand, the projection of

$$\mathscr{E}_f^\natural = \mathscr{E}_f \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad (1.17)$$

is also the same front f, but the unit normal $\mathscr{E}_f^\natural e_3 (\mathscr{E}_f^\natural)^* = -\nu$ is reversed, where ν is the unit normal in (1.9). We call $\mathscr{E}_f^\natural$ the *dual* of $\mathscr{E}_f$ (see [**KRUY**, Remark 2.1]). The hyperbolic Gauss maps $G^\natural$, $G_*^\natural$, the canonical forms $\omega^\natural$, $\theta^\natural$ and the Hopf differential $Q^\natural$ are related to the original data by

$$(G^\natural, \omega^\natural) = (G_*, \theta), \qquad (G_*^\natural, \theta^\natural) = (G, \omega), \qquad Q^\natural = Q.$$

A holomorphic Legendrian lift $\mathscr{E}_f$ can be expressed by the pair (G, ω) of the hyperbolic Gauss map G and the canonical form ω, as in [**KUY1**]:

$$\mathscr{E}_f = \begin{pmatrix} GC & d(GC)/\omega \\ C & dC/\omega \end{pmatrix}, \qquad \text{where} \quad C = i\sqrt{\frac{\omega}{dG}}. \qquad (1.18)$$

We use the above formula in what follows, but, using the duality (1.17), we could express $\mathscr{E}_f$ in terms of the pair (G_*, θ) as well. The fact that we have these two different expressions for $\mathscr{E}_f$ will play a crucial role in our investigation of the asymptotic behavior of WCF-ends.

Another representation formula for $\mathscr{E}_f$ in terms of the hyperbolic Gauss maps is given in [**KUY1**]:

$$\mathscr{E}_f = \begin{pmatrix} G/\xi & \xi G_*/(G - G_*) \\ 1/\xi & \xi/(G - G_*) \end{pmatrix} \qquad \left(\xi = \delta \exp \int_{z_0}^{z} \frac{dG}{G - G_*} \right), \qquad (1.19)$$

where $z_0 \in M^2$ is a base point and $\delta \in \mathbf{C} \setminus \{0\}$ is a constant. Note that the choice on δ corresponds to the $U(1)$-ambiguity, as well as to the family of parallel fronts. The canonical form ω and the Hopf differential Q are expressed as

$$\omega = -\frac{dG}{\xi^2}, \qquad Q = -\frac{dG\,dG_*}{(G - G_*)^2}. \qquad (1.20)$$

REMARK 1.2 (Flat surfaces in de Sitter 3-space). We set

$$S_1^3 = \{ X \in \mathrm{Herm}(2) \,;\, \det X = -1 \}$$

$$= \{ ue_3 u^* \,;\, u \in SL(2, \mathbf{C}) \} = SL(2, \mathbf{C})/SU(1,1) \quad \left(e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right),$$

which gives a Lorentzian space form of positive curvature called *de Sitter 3-space*. A smooth map $f : M^2 \to S_1^3$ is called a *spacelike front* if its unit normal vector field ν is globally defined on M^2 and gives a front in H^3. The unit normal vector field ν of flat fronts in H^3 gives spacelike flat fronts in de Sitter 3-space S_1^3, and vice versa.

REMARK 1.3 (A characterization of the horosphere). If either G or G_* is constant, the Hopf differential Q vanishes everywhere because of (1.20). Then the surface lies in a horosphere.

Ends of flat fronts. Let $f : M^2 \to H^3$ be a flat front. If M^2 is homeomorphic to a compact Riemann surface $\overline{M}^2$ excluding a finite number of points $p_1, \ldots, p_n$, each point p_j represents an *end* of f. Moreover, if a neighborhood of p_j is *biholomorphic* to the punctured disc $D^* = \{z \in \boldsymbol{C} \,;\, 0 < |z| < 1\}$, then p_j is called a *puncture-type* end. *We often refer to the restriction of f to a neighborhood D^* as the end, as well.*

Puncture-type ends can appear in a flat front with some kinds of "completeness" properties: A flat front $f : M^2 \to H^3$ is called *complete* if there exists a symmetric 2-tensor T such that $T = 0$ outside a compact set $C \subset M^2$ and $ds^2 + T$ is a complete metric of M^2. In other words, the set of singular points of f is compact and each divergent path has infinite length (see Definition 1.5 below). On the other hand, f is called *weakly complete* (resp. *of finite type*) if the metric $ds^2_{1,1}$ as in (1.13) is complete (resp. of finite total curvature). The following fact is fundamental:

FACT 1.4 (**[KRUY**, Proposition 3.2]).   *If a flat front $f : M^2 \to H^3$ is weakly complete and of finite type, then there exist a compact Riemann surface $\overline{M}^2$ and a finite set of points $\{p_1, \ldots, p_n\}$ such that M^2 is biholomorphic to $\overline{M}^2 \setminus \{p_1, \ldots, p_n\}$.*

We can also define *completeness* of an end itself:

DEFINITION 1.5.   An end $f : D^* \to H^3$ is
- *complete* if f is complete at the origin, that is, the set of singular points does not accumulate at the origin and any path in D^* approaching the origin has infinite length, or
- *incomplete WCF* if $ds^2_{1,1}$ in (1.13) is complete at the origin, the total curvature of $ds^2_{1,1}$ on a neighborhood of the origin is finite, and f is incomplete at the origin.

Namely, an incomplete WCF-end is an "incomplete, *W*eakly *C*omplete end of *F*inite type".

FACT 1.6 (**[KUY2]**, **[KRUY**, Proposition 3.1]).   *A complete end is a WCF-end. Conversely, a WCF-end $f : D^* \to H^3$ is complete if the singular set does not accumulate at the origin.*

The weak completeness and the finite-type property of a complete end are shown in **[KUY2**, Corollary 3.4] and **[KRUY**, Proposition 3.1] respectively. The second assertion of Fact 1.6 follows from **[KRUY**, Theorem 3.3].

FACT 1.7 (**[GMM]**, **[KUY2]**, **[KRUY**, Proposition 3.2]).   *Let $f : D^* \to H^3$ be a WCF-end of a flat front. Then the canonical forms ω and θ are expressed as*

$$\omega = z^\mu \omega_1(z)\,dz, \qquad \theta = z^{\mu_*}\theta_1(z)\,dz \qquad (\mu, \mu_* \in \boldsymbol{R}, \mu + \mu_* \in \boldsymbol{Z}),$$

where ω_1 and θ_1 are holomorphic functions in z which do not vanish at the origin. In particular, the function $|\rho| : D^* \to [0, +\infty]$ as in (1.12) can be extended across the end 0.

Here $|\omega|^2$ and $|\theta|^2$ are considered as conformal flat metrics on D_ε^* for sufficiently small $\varepsilon > 0$. The real numbers μ and μ_* are the orders of the metrics $|\omega|^2$ and $|\theta|^2$ at the origin respectively, that is,

$$\mu = \operatorname{ord}_0 |\omega|^2, \qquad \mu_* = \operatorname{ord}_0 |\theta|^2. \tag{1.21}$$

Since $ds_{1,1}^2 = |\omega|^2 + |\theta|^2$ in (1.13) is complete at the origin, it holds that

$$\min\{\mu, \mu_*\} = \min\left\{\operatorname{ord}_0 |\omega|^2, \operatorname{ord}_0 |\theta|^2\right\} \leq -1 \tag{1.22}$$

for a WCF-end. By (1.11), the order of the Hopf differential is

$$\operatorname{ord}_0 Q = \mu + \mu_* = \operatorname{ord}_0 |\omega|^2 + \operatorname{ord}_0 |\theta|^2, \tag{1.23}$$

where $\operatorname{ord}_0 Q = m$ if $Q = z^m\big(a + o(1)\big)dz^2$ holds for some $a \neq 0$.

The following assertion is essentially shown in the proof of [**KRUY**, Theorem 3.4]. However, for the sake of convenience we give a proof here.

PROPOSITION 1.8. *Let $f : D^* \to H^3$ be a complete end of a flat front. Then the parallel front f_t as in (1.8) is a WCF-end. Conversely, for an incomplete WCF-end $f : D^* \to H^3$ of a flat front, f_t is a complete end for any $t \neq 0$.*

PROOF. The canonical forms of the parallel front f_t are expressed by those of f as

$$\omega_t = e^t \omega, \qquad \theta_t = e^{-t}\theta, \qquad \rho_t = e^{-2t}\rho.$$

By (1.22), the completeness of $ds_{1,1}^2$ is preserved by taking parallel fronts. On the other hand, it follows from [**KRUY**, (3.2)] that the finiteness of total curvature of $ds_{1,1}^2$ for f_t is equivalent to the finiteness of orders of $|\omega_t|^2$ and $|\theta_t|^2$ at the end $z = 0$. This implies that the finiteness of the total curvature of $ds_{1,1}^2$ is also preserved by taking parallel fronts. By Fact 1.6, if f is complete, it is a weakly complete end of finite type. Hence so is f_t, that is, f_t is WCF. Conversely, if f is incomplete WCF,

then $|\rho(0)| = 1$ (cf. Lemma 1.20) and $|\rho_t(0)| = e^{-2t}|\rho(0)| = e^{-2t} \neq 1$ for $t \neq 0$. Since the singular point $z \in D^*$ of f_t is characterized by $|\rho_t(z)| = 1$, the singular set of f_t $(t \neq 0)$ does not accumulate at the end. Hence f_t $(t \neq 0)$ is complete at the origin. □

Behavior of regular ends.

Fact 1.9 ([**GMM**], [**KUY2**]).   *The hyperbolic Gauss maps G, G_* of a weakly complete end $f : D^* \to H^3$ of a flat front satisfy either*

- *both G and G_* have at most pole singularities at the origin, and have the same value at the end, or*
- *both G and G_* have essential singularities at the origin.*

We call the end *regular* if both G and G_* have at most poles, and *irregular* otherwise. The following lemma is well known:

Lemma 1.10 ([**GMM**], [**KUY2**]).   *A WCF-end $f : D^* \to H^3$ of a flat front is regular if and only if the Hopf differential has a pole of order at most 2 at 0, that is, $\mathrm{ord}_0 Q \geq -2$ holds.*

Proposition 1.11.   *Let $f : D^* \to H^3$ be a regular weakly complete end, and denote its hyperbolic Gauss maps by G and G_*. Then*

$$\lim_{z \to 0} \pi \circ f(z) = \big(G(0), 0\big) = \big(G_*(0), 0\big)$$

holds if $G(0) \neq \infty$, where $\pi : H^3 \to \mathbf{R}_+^3$ is the projection as in (1.5).

Proof.   The proof of [**KUY2**, Lemma 3.10] applies to weakly complete ends as well. So we have $G(0) = G_*(0) (= a)$, where $a \in \mathbf{C} \cup \{\infty\}$. By a suitable rigid motion in H^3, we may assume $a \neq \infty$. In this case, we can write

$$G = a + \psi(z), \quad G_* = a + \psi_*(z), \qquad \big(\psi(0) = \psi_*(0) = 0\big),$$

where $\psi(z)$, $\psi_*(z)$ are holomorphic functions defined on a sufficiently small closed disc $\{|z| \leq \varepsilon\}$. We write $\pi \circ f = (\zeta, h)$. Then by (1.19), we have

$$\frac{1}{h} = |E_{21}|^2 + |E_{22}|^2 = \frac{1}{|\xi|^2} + \frac{|\xi|^2}{|G - G_*|^2} \geq \frac{2}{|G - G_*|} \longrightarrow +\infty \qquad (z \to 0).$$

In particular, $h \to 0$ as $z \to 0$. On the other hand, we have

$$\zeta = \frac{E_{11}\overline{E_{21}} + E_{12}\overline{E_{22}}}{|E_{21}|^2 + |E_{22}|^2} = \frac{G|E_{21}|^2 + G_*|E_{22}|^2}{|E_{21}|^2 + |E_{22}|^2}$$

$$= \frac{(a + \psi(z))|E_{21}|^2 + (a + \psi_*(z))|E_{22}|^2}{|E_{21}|^2 + |E_{22}|^2} = a + \frac{\psi(z)|E_{21}|^2 + \psi_*(z)|E_{22}|^2}{|E_{21}|^2 + |E_{22}|^2}.$$

Thus we have

$$|\zeta - a| \leq \max_{|z| \leq \varepsilon}\{|\psi(z)|, |\psi_*(z)|\}.$$

The right-hand side tends to zero as $\varepsilon \to 0$, hence the left-hand side $|\zeta - a|$ converges to zero as $z \to 0$. This completes the proof. $\square$

From now on, we consider a regular WCF-end $f : D^* \to H^3$ of a flat front. By a rigid motion, we may assume $G(0)(= G_*(0)) \neq \infty$. In the case where G and G_* are both non-constant (cf. Remark 1.3), we have the expressions

$$\begin{aligned} G(z) &= a + b_1 z^{m_1} + o(z^{m_1}), \\ G_*(z) &= a + b_2 z^{m_2} + o(z^{m_2}), \end{aligned} \qquad \big(a = G(0) = G_*(0), \ b_1, b_2 \in \mathbf{C} \setminus \{0\}\big) \qquad (1.24)$$

on a neighborhood of $z = 0$. We set

$$m = \min\{m_1, m_2\} = \min\{\mathrm{ord}_0\, G'(z), \ \mathrm{ord}_0\, G'_*(z)\} + 1, \qquad (1.25)$$

which is called the *multiplicity* of the end f. In the case where one of G, G_* is constant, the *multiplicity* of the end is defined to be the ramification order of whichever of G or G_* is nonconstant. The multiplicity m of the end has the following important property.

FACT 1.12 ([**GMM**] and [**KUY2**]). *Let $f : D^* \to H^3$ be a complete regular end. Then the multiplicity m of f is equal to 1 if and only if $f(D_\varepsilon^*)$ is properly embedded for a sufficiently small $\varepsilon > 0$.*

Recall that (see [**KRUY**, (7.1)]) the constant

$$\alpha = \begin{cases} (dG_*/dG)(0) & (\text{if } |(dG_*/dG)(0)| \leq 1), \\ (dG/dG_*)(0) & (\text{if } |(dG_*/dG)(0)| > 1) \end{cases} \qquad (1.26)$$

is called the *ratio of the Gauss maps*. As seen in [**KRUY**, Propositions 3.1 and

7.3], α is a real number which is not equal to 1, so $\alpha \in [-1, 1)$.

In particular, if $\alpha \neq 0$, the ramification orders of G and G_* coincide, and are equal to the multiplicity of the end.

To fix the expression of the ratio of Gauss maps uniquely, we wish to distinguish the pairs (G, ω) and (G_*, θ) of f (given just before Definition 1.1) as follows:

DEFINITION 1.13. The pair (G, ω) (resp. (G_*, θ)) is a *dominant pair* with respect to the regular end $z = 0$ if $|(dG_*/dG)(0)| \leq 1$ (resp. $|(dG_*/dG)(0)| \geq 1$). Moreover, (G, ω) (resp. (G_*, θ)) is called the *strictly dominant pair* if $|(dG_*/dG)(0)| < 1$ (resp. $|(dG_*/dG)(0)| > 1$).

REMARK 1.14. For a regular WCF-end, (G, ω) and (G_*, θ) are both dominant if and only if $\alpha = -1$, which corresponds to a regular cylindrical end (see Definition 1.16 and Proposition 1.17 below). If (G_*, θ) is strictly dominant, (G, ω) is not strictly dominant. In this case, by taking the dual as in (1.17), we can exchange the roles of (G, ω) and (G_*, θ). Thus, we may always assume that (G, ω) is a dominant pair. Then it holds that

$$\alpha = (dG_*/dG)(0), \qquad m = m_1. \tag{1.27}$$

In particular, we have the expressions

$$\begin{aligned} G(z) &= a + cz^m + o(z^m), \\ G_*(z) &= a + \alpha cz^m + o(z^m), \end{aligned} \qquad \left(c \neq 0\right). \tag{1.28}$$

We shall use frequently these expressions, or more normalized forms of them. The following assertion holds:

PROPOSITION 1.15. *Let (G, ω) be a dominant pair. Then the ratio of Gauss maps α and the multiplicity m of the end satisfy the following identity*

$$\mu = -\frac{1+\alpha}{1-\alpha} m - 1 (\leq -1), \qquad \text{that is,} \qquad \alpha = \frac{1+\mu+m}{1+\mu-m}, \tag{1.29}$$

where $\mu = \mathrm{ord}_0|\omega|^2$. In particular, $\mu_* = \mathrm{ord}_0|\theta|^2$ satisfies

$$\mu + \mu_* \geq -2, \quad \mu_* \geq -1, \quad (\mu \leq -1). \tag{1.30}$$

PROOF. Substituting (1.28) into (1.19) and (1.20), and noticing that $\alpha \neq 1$, we have (1.29). The second assertion follows from (1.23) and Lemma 1.10. □

Since the Hopf differential Q is written as in (1.20), it has the following expansion

$$Q = \frac{1}{z^2} \left(\frac{-m^2 \alpha}{(1-\alpha)^2} + o(1) \right) dz^2,$$

where $o(1)$ is a term having order higher than 1 as $z \to 0$. The term

$$q_{-2} = \frac{-m^2 \alpha}{(1-\alpha)^2} \tag{1.31}$$

is called the *top-term coefficient* of Q.

DEFINITION 1.16 (cf. [**KRUY**, Definition 7.1]). A regular WCF-end $f : D^* \to H^3$ of a flat front is called
  (1) *horospherical* if $q_{-2} = 0$, that is, $\mathrm{ord}_0 Q \geq -1$,
  (2) of *snowman-type* if $q_{-2} < 0$,
  (3) of *hourglass-type* if $q_{-2} > 0$ and $\mathrm{ord}_0 |\omega|^2 \neq \mathrm{ord}_0 |\theta|^2$, or
  (4) *cylindrical* if $\mathrm{ord}_0 |\omega|^2 = \mathrm{ord}_0 |\theta|^2$. (In this case, q_{-2} is positive. See Corollary 1.18 below.)

These types of ends are characterized as follows:

PROPOSITION 1.17. *Let $f : D^* \to H^3$ be a regular WCF-end of a flat front and (G, ω) a dominant pair. Then the end is*
  (1) *horospherical if and only if $\alpha = 0$, that is, $\mu = -m - 1$,*
  (2) *snowman-type if and only if $0 < \alpha < 1$, that is, $\mu < -m - 1$,*
  (3) *hourglass-type if and only if $-1 < \alpha < 0$, that is, $-m - 1 < \mu < -1$, and*
  (4) *cylindrical if and only if $\alpha = -1$, that is, $\mu = -1$.*

PROOF. If $\alpha \neq 0$, $\mathrm{ord}_0 Q = -2$ because of (1.31), and

$$\mathrm{ord}_0 |\theta|^2 = -2 - \mathrm{ord}_0 |\omega|^2 = -2 - \mu = -\frac{\alpha + 1}{\alpha - 1} m - 1, \tag{1.32}$$

because of (1.23) and (1.29). Then the conclusion follows. □

COROLLARY 1.18. *If a regular WCF-end of a flat front is cylindrical, then it*

holds that

$$q_{-2} > 0, \qquad \text{and} \qquad \text{ord}_0|\omega|^2 = \text{ord}_0|\theta|^2 = -1.$$

PROOF.   Substitute $\alpha = -1$ into (1.31) and (1.32).   □

EXAMPLE 1.19 (Flat fronts of revolution).   Take a positive integer m and $\alpha \in [-1, 1)$, and set $(G, G_*) = (z^m, \alpha z^m)$. Then by (1.19), we have a flat front $f : \boldsymbol{C} \setminus \{0\} \to H^3$ whose canonical forms are given by

$$\omega = -\frac{m}{\delta^2}\, z^\mu\, dz, \qquad \theta = \frac{m\alpha\delta^2}{(1-\alpha)^2}\, z^{-2-\mu}\, dz \qquad \left(\mu = \frac{\alpha+1}{\alpha-1}\, m - 1\right),$$

where δ is a constant as in (1.19). The front f is the m-fold cover of the *hourglass* (resp. *the snowman*) if $-1 < \alpha < 0$ (resp. $0 < \alpha < 1$). When $\alpha = 0$, f gives the horosphere (resp. the m-fold branched cover of the *horosphere* with branch point $z = \infty$) if $m = 1$ (resp. $m \geq 2$). In the case of $\alpha = -1$, f gives the m-fold cover of a *cylinder* if $|\delta|^2 \neq 2$. When $\alpha = -1$ and $|\delta|^2 = 2$, all points are singularities of f, and the image $f(\boldsymbol{C} \setminus \{0\})$ is the geodesic joining 0 and ∞. Here, we identify ∂H^3 with $\boldsymbol{C} \cup \{\infty\}$ as in (A.1) in Appendix A. In all cases, f is a flat front of revolution whose axis is the geodesic joining 0 and $\infty \in \partial H^3$, see Figure 3 in the introduction.

Conversely, any flat front of revolution whose axis is the geodesic joining 0 and $\infty \in \partial H^3$ is obtained in such a way. In particular, one can choose the complex coordinate z such that $G = z^m$, and the canonical form $\omega = cz^\mu\, dz$, where c is a non-zero constant.

Behavior of singular points on a regular WCF-end.

LEMMA 1.20.   *A WCF-end $f : D^* \to H^3$ of a flat front is cylindrical if and only if $\rho(z) = \theta/\omega$ as in (1.12) is a nonvanishing holomorphic function near $z = 0$. On the other hand, a regular WCF-end f is incomplete if and only if it is cylindrical and $|\rho(0)| = 1$.*

PROOF.   Note that this lemma holds not only for regular ends but for WCF-ends. The first assertion is obvious. In particular, if the end is not cylindrical, $\mu \neq \mu_*$ in (1.21) and then

$$\lim_{z \to 0} |\rho(z)| = 0 \quad \text{or} \quad +\infty.$$

This implies that the singular set $\{|\rho| = 1\}$ does not accumulate at the origin.

Thus incomplete ends are all cylindrical. Moreover, if the singular set accumulates at the origin, then $|\rho(0)| = 1$. Conversely, assume that f is cylindrical and $|\rho(0)| = 1$. By the $U(1)$-ambiguity as in (1.16), one can assume $\rho(0) = 1$ without loss of generality. If ρ is constant, all points are singular, and then the end is incomplete. Otherwise, ρ can be expanded as $\rho(z) = 1 + bz^n + o(z^n)$ $(b \neq 0)$, where n is the ramification order of ρ. Hence one can take a complex coordinate w $(w(0) = 0)$ such that

$$\log \rho = w^n. \tag{1.33}$$

Then the singular set

$$\{|\rho| = 1\} = \{w \,;\, \mathrm{Re}(w^n) = 0\} \tag{1.34}$$

accumulates at the origin. $\qquad\square$

PROPOSITION 1.21. *Let $f : D^* \to H^3$ be an incomplete regular WCF-end of a flat front, whose image is not contained in a geodesic line in H^3. Then, for a sufficiently small $\varepsilon > 0$, only cuspidal edge singularities appear in the image $f(D_\varepsilon^*)$, and the set of cuspidal edges has $2n$ components, where n is the ramification order of $\rho(z)$ at $z = 0$.*

PROOF. By Lemma 1.20, $|\rho|$ is a well-defined function on a neighborhood of the origin satisfying $|\rho(0)| = 1$. By the $U(1)$-ambiguity as in (1.16), we may assume $\rho(0) = 1$ without loss of generality. If ρ is identically 1, then [**KRSUY**, Proposition 4.7] yields that f is rotationally symmetric, and then, the image of f is a geodesic line. Thus ρ is not identically 1, and then we can take a complex local coordinate w around the origin as in (1.33). Hence the set of singularities is expressed as in (1.34), which consists of $2n$ rays starting at the origin in the w-plane.

By Proposition 1.17 and Lemma 1.20, we have $\alpha = -1$. Thus the Hopf differential Q expands as

$$Q = \frac{m^2}{4z^2} \big(1 + o(1)\big) \, dz^2, \tag{1.35}$$

where m is the multiplicity of the end. On the other hand, a singular point which is not a cuspidal edge point must be a zero of the imaginary part of the function

$$\sqrt{\zeta_c} = \frac{d\big(\log \rho\big)}{\sqrt{Q}},$$

(see [**KRSUY**, Proposition 4.7]), and we have the following expansion

$$\sqrt{\zeta_c(w)} = \frac{2n}{m}\,w^n\big(1 + o(1)\big),$$

where w is the local coordinate near the origin given in (1.33). Then the singular set is given by $\{\mathrm{Re}(w^n) = 0\}$, and the zeros of the imaginary part of $\sqrt{\zeta_c}$ are approximated by $\{\mathrm{Im}(w^n) = 0\}$. Since the two sets $\{\mathrm{Re}(w^n) = 0\}$ and $\{\mathrm{Im}(w^n) = 0\}$ are disjoint near $w = 0$, there are no singular points other than cuspidal edge points near $z = 0$. □

2.   Flux and axes of ends.

The flux matrix. Let $f : D^* \to H^3$ be an end of a flat front such that the complex structure of D^* is compatible with the metric (1.13). Regarding $H^3 \subset SL(2, \boldsymbol{C})$ as in (1.3), the *flux matrix* of f is defined by

$$\Phi_f = \frac{i}{2\pi} \int_\gamma (\partial f) f^{-1} \in \mathfrak{sl}(2, \boldsymbol{C}), \tag{2.1}$$

where $\mathfrak{sl}(2, \boldsymbol{C})$ is the Lie algebra of $SL(2, \boldsymbol{C})$ and γ is an arbitrary loop in D^* going around the origin in the counterclockwise direction. Here, ∂f is the $(1,0)$-part of df, that is, $\partial f = f_z\, dz$ for a complex coordinate z.

We first show the following:

PROPOSITION 2.1.   *Let* $\mathscr{E}_f : \widetilde{D}^* \to SL(2, \boldsymbol{C})$ *be a holomorphic Legendrian lift of* f, *where* $\widetilde{D}^*$ *is the universal cover of* D^*. *Then the following formula holds*:

$$(\partial f)f^{-1} = d\mathscr{E}_f \mathscr{E}_f^{-1} = \frac{1}{(G - G_*)^2} \begin{pmatrix} -G_* dG - G dG_* & G_*^2 dG + G^2 dG_* \\ -dG - dG_* & G_* dG + G dG_* \end{pmatrix}, \tag{2.2}$$

where G *and* G_* *are the hyperbolic Gauss maps. In particular, the* $\mathfrak{sl}(2, \boldsymbol{C})$-*valued 1-form* $(\partial f)f^{-1}$ *is holomorphic, and common to the parallel family* $\{f_t\}_{t\in\boldsymbol{R}}$, *that is,* $(\partial f)f^{-1} = (\partial f_t)f_t^{-1}$ *holds*.

PROOF.   Since $\mathscr{E}_f$ is holomorphic, $(\partial f)f^{-1} = d\mathscr{E}_f \mathscr{E}_f^{-1}$ follows from $f = \mathscr{E}_f \mathscr{E}_f^*$. Then by (1.19), we have the conclusion. □

As defined in [**KRUY**], a smooth map $f : M^2 \to H^3$ on a 2-manifold M^2 is called a flat *p-front* if for each $x \in M^2$, there exists a neighborhood U of x such that the restriction of f to U is a flat front. Roughly speaking, a p-front is locally a

front, but its unit normal vector field ν may not be globally single-valued. The caustics (i.e., focal surfaces) of flat fronts are also flat, but in general they are not fronts but only p-fronts. So if we wish to analyze the asymptotic behavior of ends of caustics, we must work in the category of p-fronts. A p-front is called *non-co-orientable* if it is not a front.

We now assume $f : M^2 \to H^3$ is a weakly complete flat p-front of finite type. Since Fact 1.4 holds also for flat p-fronts (see [**KRUY**, Proposition 5.4]), there exist a compact Riemann surface $\overline{M}^2$ and a finite set of points $\{p_1, \ldots, p_n\}$ such that M^2 is biholomorphic to $\overline{M}^2 \setminus \{p_1, \ldots, p_n\}$. Though G and G_* may have essential singularities at p_j, the holomorphic form $(\partial f)f^{-1}$ is a globally defined $\mathfrak{sl}(2, \boldsymbol{C})$-valued 1-form on M^2. Thus the total sum of the residues at $p_1, \ldots, p_n$ vanishes:

COROLLARY 2.2 (The balancing formula). *Let $f : M^2 \to H^3$ be a weakly complete flat p-front of finite type. Then the sum of flux matrices over its ends vanishes.*

This suggests that the flux matrices just defined might be useful for the global study of flat fronts, like as for the cases of CMC surfaces in $\boldsymbol{R}^3$ (cf. [**KKS**]) and CMC-1 surfaces in H^3 (cf. [**RUY**]).

By definition, the flux matrices are meaningful not only for regular ends but also irregular ends for which the hyperbolic Gauss maps have essentially singularities at the end. However we shall treat only regular ends in this paper.

On the other hand, if the end is not a front but a p-front, by taking the double cover, it becomes a front (see [**KRUY**, Corollary 5.2]). So, from now on, we shall usually work in the category of fronts.

Next, we shall define a projection:

$$\Pi : \boldsymbol{C}^2 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \ni \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \frac{x}{y} \in P^1(\boldsymbol{C}) = \boldsymbol{C} \cup \{\infty\}. \qquad (2.3)$$

Since, for the Poincaré upper half-space model $\boldsymbol{R}^3_+$, the ideal boundary ∂H^3 can be identified with $\boldsymbol{C} \cup \{\infty\}$ (see (A.1) in Appendix A), the image of Π is contained in ∂H^3.

THEOREM 2.3. *Let a flat front $f : D^* \to H^3$ be a regular WCF-end. Then there exists an eigenvector $\boldsymbol{v}$ of the flux matrix Φ_f such that the projection $\Pi(\boldsymbol{v}) \in \boldsymbol{C} \cup \{\infty\}$ equals the limiting value of the end, that is,*

$$\Pi(\boldsymbol{v}) = \lim_{z \to 0} \zeta(z) = G(0) = G_*(0) \,,$$

where $\pi \circ f(z) = \big(\zeta(z), h(z)\big)$. *Moreover, the flux matrix Φ_f is a lower triangular matrix if $G(0) = 0$.*

Proof.   An isometric action ι_u $(u \in SL(2, \boldsymbol{C}))$, as in (1.4), induces a flat front ufu^* congruent to f, and the flux matrix of ufu^* is given by $u\Phi_f u^{-1}$. This implies that an eigenvector of Φ_{ufu^*} must be $u\boldsymbol{v}$. On the other hand, the isometric action induces a transformation of the ideal boundary so that

$$\partial H^3 = \boldsymbol{C} \cup \{\infty\} \ni \zeta \mapsto u \star \zeta = \frac{u_{11}\zeta + u_{12}}{u_{21}\zeta + u_{22}} \in \boldsymbol{C} \cup \{\infty\} = \partial H^3 \quad \big(u = (u_{ij})\big).$$

Thus we have

$$\Pi(u\boldsymbol{v}) = u \star \Pi(\boldsymbol{v}) \qquad \left(\boldsymbol{v} \in \boldsymbol{C}^2 \bigg\backslash \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}\right),$$

which implies that the map Π is equivariant. So to prove the assertion, we may assume that $G(0) = G_*(0) = 0$, replacing f by ufu^* for a suitable isometry $u \in SL(2, \boldsymbol{C})$ if necessary. Without loss of generality, we may assume that (G, ω) is a dominant pair (see Remark 1.14). Then the hyperbolic Gauss maps are written as in (1.28) for $a = G(0) = 0$. Thus, we have

$$G_*^2 dG + G^2 dG_* = \big(mc^3\alpha(\alpha + 1)z^{3m-1} + o(z^{3m-1})\big)\, dz, \tag{2.4}$$

$$(G - G_*)^2 = c^2(1 - \alpha)^2 z^{2m} + o(z^{2m}). \tag{2.5}$$

Since the ratio α of the Gauss maps is not equal to 1, $(G_*^2 dG + G^2 dG_*)/(G - G_*)^2$ is a holomorphic 1-form at $z = 0$. In particular, it follows from (2.1), (2.2) that the flux matrix Φ_f is a lower triangular matrix, and $\boldsymbol{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is one of the eigenvectors. Then we have $\Pi(\boldsymbol{v}) = 0 = G(0) = G_*(0)$ which proves the assertion. $\square$

The eigenvalues of the flux matrix are related to the ratio of the Gauss maps:

Theorem 2.4.   *The eigenvalues of the flux matrix Φ_f of a regular WCF-end $f : D^* \to H^3$ of a flat front are*

$$\pm \frac{2m\alpha}{(1 - \alpha)^2} \left(= \mp \frac{2q_{-2}}{m} \right),$$

where m is the multiplicity of the end (cf. (1.28)), α is the ratio of the Gauss maps (1.26), and q_{-2} is the top-term coefficient of the Hopf differential (1.31). In particular, if $\alpha \neq 0$ (that is, if f is not horospherical), then Φ_f is diagonalizable.

PROOF.   Let us take the same notation as in the proof of Theorem 2.3. Then the diagonal components are just the eigenvalues of Φ_f since Φ_f is a triangular matrix. We have $G_* dG + G dG_* = (2mc^2 \alpha z^{2m-1} + o(z^{2m-1}))dz$, and by (2.5), we have

$$\frac{G_* dG + G dG_*}{(G - G_*)^2} = \frac{1}{z}\left(\frac{2m\alpha}{(1-\alpha)^2} + o(1)\right)dz.$$

It follows from (2.1), (2.2) that the eigenvalues of Φ_f are $\pm 2m\alpha/(1-\alpha)^2$, which are equal to $\mp 2q_{-2}/m$ by (1.31). $\qquad\square$

By Theorem 2.4, Φ_f is diagonalizable if $\alpha \neq 0$. In this case, the flux matrix has two linearly independent eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$. Then there exists a unique geodesic line in H^3 connecting $\Pi(\boldsymbol{v}_1)$ and $\Pi(\boldsymbol{v}_2) = G(0) \in \partial H^3$, which is called the *axis of the flux matrix* Φ_f. We denote this geodesic by $\overline{\boldsymbol{v}_1, \boldsymbol{v}_2}$. By Theorem 2.3, one endpoint of the axis is just the limit point of f.

We finish this subsection with some lemmas concerning the axis, which will be needed in the following sections.

LEMMA 2.5.   *Let $f : D^* \to H^3$ be a regular WCF-end of a flat front. Then the flux matrix Φ_f is diagonal if and only if the axis is the geodesic joining the origin and infinity, i.e., the h-coordinate axis $\{(\zeta, h); \zeta = 0\}$ in the upper half-space model $\boldsymbol{R}_+^3$.*

PROOF.   If Φ_f is a diagonal matrix, the eigenvectors are $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and vice versa. In this case, $\Pi(\boldsymbol{v}_1) = \infty$ and $\Pi(\boldsymbol{v}_2) = 0$. $\qquad\square$

LEMMA 2.6.   *Let $f : D^* \to H^3$ be a regular WCF-end of a flat front. Assume that Φ_f has the axis $\overline{\boldsymbol{v}_1, \boldsymbol{v}_2}$. Let $\tilde{f}$ be an end congruent to f, that is, $\tilde{f} = ufu^*$ for some $u \in SL(2, \boldsymbol{C})$. Then the axis of the flux matrix $\Phi_{\tilde{f}}$ is given by $\overline{u\boldsymbol{v}_1, u\boldsymbol{v}_2}$.*

PROOF.   As we have already noted in the proof of Theorem 2.3, if $\boldsymbol{v}$ is an eigenvector of Φ_f, then $u\boldsymbol{v}$ is an eigenvector of $\Phi_{\tilde{f}}$. $\qquad\square$

LEMMA 2.7.   *Let $f : D^* \to H^3$ be a regular WCF-end of a flat front. If the ratio α of the Gauss maps is not zero, then there exists an orientation preserving*

isometry ι of H^3 such that the axis of the flux matrix $\Phi_{\iota \circ f}$ coincides with the geodesic joining ∞ and 0, that is, the h-coordinate axis in $\boldsymbol{R}_+^3$.

PROOF.   As we have seen, Φ_f is diagonalizable if $\alpha \neq 0$. Hence this lemma is a direct consequence of the two lemmas above. $\square$

The indentation number. In this subsection, we introduce the *maximum indentation number n* of a regular WCF-end, which is a positive integer. Firstly, we will define a positive integer l_γ, called the *indentation number*, determined by the choice of geodesic γ asymptotic to the point $G(0) = G_*(0) \in \partial H^3$ of f. Then n is the maximum of l_γ for such geodesics:

Let $f : D^* \to H^3$ be a regular WCF-end of a flat front and take a (parametrized) geodesic $\gamma(s)$ $(s \in \boldsymbol{R})$ in H^3 whose endpoint

$$\gamma(+\infty) = \lim_{s \to +\infty} \gamma(s) \in \partial H^3 = \boldsymbol{C} \cup \{\infty\}$$

coincides with $G(0) = G_*(0) \in \partial H^3$. Here, we identify ∂H^3 with $\boldsymbol{C} \cup \{\infty\}$ as in (A.1) in Appendix A. Then there exists a rigid motion ι_u in H^3 $(u \in SL(2, \boldsymbol{C}))$ such that

$$\iota_u \circ \gamma(-\infty) = \infty, \qquad \iota_u \circ \gamma(+\infty) = u \star G(0) = 0. \tag{2.6}$$

Note that such a $u \in SL(2, \boldsymbol{C})$ is unique up to the change

$$u \longmapsto \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} u \qquad (\delta \in \boldsymbol{C} \setminus \{0\}). \tag{2.7}$$

We may assume that the pair (G, ω) is a dominant pair, and define a meromorphic function

$$A_\gamma = \frac{d(u \star G_*)}{d(u \star G)}, \tag{2.8}$$

which does not depend on the ambiguity as in (2.7), that is, depends only on γ. When $A_\gamma(z)$ is not a constant function, the ramification order $l(= l_\gamma)$ of the meromorphic function $A_\gamma(z)$ at $z = 0$ is called the *indentation number* with respect to the geodesic γ. On the other hand, if $A_\gamma(z)$ is constant, we set

$$l_\gamma = \infty. \tag{2.9}$$

This exceptional case corresponds exactly to the case that f is rotationally symmetric with respect to the axis γ (see Example 1.19).

Since (G, ω) is dominant, the ramification number of G at 0 is equal to the multiplicity m of the end. Then, replacing f by $\iota_u \circ f$ for a suitable $u \in SL(2, \boldsymbol{C})$, there exists a complex coordinate z on a neighborhood of the origin such that

$$G(z) = z^m. \tag{2.10}$$

Unless $A_\gamma(z)$ is constant, the other hyperbolic Gauss map $G_*(z)$ can be written as

$$G_*(z) = \alpha z^m + \alpha_1 z^{m+l} + o(z^{m+l}), \qquad (\alpha_1 \neq 0, \ l = l_\gamma), \tag{2.11}$$

where α is the ratio of the Gauss maps (see Remark 1.14).

We denote by $[\gamma]$ the image of the geodesic γ.

LEMMA 2.8. *Let $f : D^* \to H^3$ be a regular WCF-end of a flat front with multiplicity m. Then one of the following three cases occurs*:
  (1) *The end f is horospherical, and the indentation number l_γ does not depend on the choice of geodesic γ.*
  (2) *The end f is not horospherical, and the indentation number l_γ does not depend on the choice of geodesic γ. Moreover, $l_\gamma < m$ holds.*
  (3) *The end f is not horospherical, and there exists a unique geodesic σ satisfying $\sigma(+\infty) = G(0)$ such that*

$$l_\gamma \begin{cases} = m & ([\gamma] \neq [\sigma]), \\ > m & ([\gamma] = [\sigma]) \end{cases}$$

  holds for each geodesic γ satisfying $\gamma(+\infty) = G(0)$.

PROOF. Without loss of generality, we may assume that $G(0) = G_*(0) = 0$ by replacing (G, G_*) with $(u \star G, u \star G_*)$ $(u \in SL(2, \boldsymbol{C}))$ if necessary. We fix a geodesic γ_0 such that

$$\gamma_0(+\infty) = 0, \qquad \gamma_0(-\infty) = \infty$$

and let $l = l_{\gamma_0}$ be the indentation number with respect to γ_0.

Firstly, we consider the case $l_{\gamma_0} < \infty$. Then we may assume that $G(z)$ and $G_*(z)$ satisfy (2.10) and (2.11) respectively. We now take another geodesic γ such that

$$\gamma(+\infty) = 0, \qquad (a =)\gamma(-\infty) \in \boldsymbol{C} \setminus \{0\}, \tag{2.12}$$

and set

$$u\big(= (u_{ij})\big) = \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix}. \tag{2.13}$$

Then

$$\iota_u \circ \gamma(+\infty) = 0 \qquad \text{and} \qquad \iota_u \circ \gamma(-\infty) = \infty$$

hold. Here,

$$A_\gamma(z) = \frac{d(u \star G_*)}{d(u \star G)} = \left(\frac{u_{21}G + u_{22}}{u_{21}G_* + u_{22}}\right)^2 \frac{dG_*}{dG} = \left(\frac{-a^{-1}G + 1}{-a^{-1}G_* + 1}\right)^2 \frac{dG_*}{dG}. \tag{2.14}$$

By (2.10) and (2.11), we get

$$A_\gamma(z) = \left(1 - \frac{2}{a}(1-\alpha)z^m + o(z^m)\right)\left(\alpha + \frac{(m+l)\alpha_1}{m}z^l + o(z^l)\right).$$

If f is horospherical, that is, $\alpha = 0$, then the first non-vanishing term of $A_\gamma(z)$ is z^l, and we have $l_\gamma = l$. This implies that the indentation number l_γ does not depend on γ, that is, (1) holds.

Next, we consider the case that f is not horospherical, that is, $\alpha \neq 0$.

CASE A: Suppose that $l < m$, then

$$A_\gamma(z) = \alpha + \frac{(m+l)\alpha_1}{m}z^l + o(z^l),$$

which implies $l_\gamma = l$. Therefore (2) holds.

CASE B: Suppose that $l > m$, then we have

$$A_\gamma(z) = \alpha\left(1 - \frac{2}{a}(1-\alpha)z^m + o(z^m)\right). \tag{2.15}$$

Since $\alpha \neq 1$, we have $l_\gamma = m$. Thus we have

$$l_{\gamma_0} = l > m = l_\gamma \qquad ([\gamma] \neq [\gamma_0]).$$

This is case (3).

CASE C: Suppose that $l = m$. Then it holds that

$$A_\gamma(z) = \alpha + \left(2\alpha_1 - \frac{2}{a}(1-\alpha)\alpha \right) z^m + o(z^m).$$

Then $l_\gamma = m$ holds, unless

$$a = \gamma(-\infty) = \frac{(1-\alpha)\alpha}{\alpha_1}.$$

This exceptional value a determines the image $[\gamma]$ of the geodesic γ uniquely. Denoting this geodesic γ by σ, we fall into case (3).

Finally, we consider the case $l_{\gamma_0} = \infty$, which implies that dG_*/dG is constant. If f is horospherical, it is an m-fold cover of the end of the horosphere (see Example 1.19). Then G_* vanishes identically, and $d(u \star G_*)/d(u \star G)$ vanishes for all $u \in SL(2, \boldsymbol{C})$. This implies (1). On the other hand, if f is not horospherical, that is, $\alpha \neq 0$, then we have an expression

$$G(z) = z^m, \qquad G_*(z) = \alpha z^m,$$

which gives an m-fold cover of a flat front of revolution whose rotational axis is γ_0 (see Example 1.19). Take a geodesic γ ($[\gamma] \neq [\gamma_0]$) and a matrix $u \in SL(2, \boldsymbol{C})$ satisfying (2.12) and (2.13). Then by (2.14), we have (2.15), since $dG_*/dG = \alpha$. This implies $l_\gamma = m$. Therefore (3) holds. $\qquad \square$

DEFINITION 2.9.   Let f be as above. According to Lemma 2.8, we call f a *centerless end*, if it satisfies (1) or (2). On the other hand, f is called a *centered end* if it satisfies (3). When f is a centered end, the unique geodesic σ is called the *principal axis* of the end f. The maximum of the numbers l_γ, that is,

$$n = \max\{l_\gamma \,;\, \gamma \text{ is a geodesic such that } \gamma(+\infty) = G(0)\}$$

is called the *maximum indentation number* of the end f. If f is a centered end, then $n = l_\sigma$ holds for the principal axis σ.

REMARK 2.10.   Suppose that a regular WCF-end f is non-horospherical and

embedded. The embeddedness implies that f is complete (i.e., free of singularities) and has multiplicity $m = 1$. Hence only the case (3) in Lemma 2.8 occurs for f. In other words, an embedded, non-horospherical, regular WCF-end must be centered.

The flux axis and the principal axis. Definition 2.9 of the principal axis for a centered end looks rather technical. However, it is nothing but the axis of the flux matrix. In this subsection, we assume that *the end is not horospherical*. In fact, for a horospherical end, the only eigenvalue of the flux matrix Φ_f is 0 and hence the axis of the flux matrix cannot be defined.

THEOREM 2.11.   *Let $f : D^* \to H^3$ be a centered regular WCF-end of a flat front in the sense of Definition 2.9. Then the principal axis coincides with the axis of the flux matrix Φ_f.*

PROOF.   Without loss of generality, we may assume that (G, ω) is a dominant pair, $G(0) = 0$ and the principal axis σ is the geodesic joining ∞ and 0. Then by Lemma 2.5, the axis of the flux matrix Φ_f coincides with σ if and only if Φ_f is diagonal. Moreover, by Theorem 2.3, Φ_f is a lower triangular matrix because $G(0) = 0$. Then it is sufficient to show the lower-left component of the flux matrix vanishes.

If the end is rotationally symmetric, the assertion is obvious, since the axis of the flux matrix is the rotation axis. So we assume that the end is not rotationally symmetric. In this case, we may assume (2.10), (2.11), and

$$\alpha \neq 0, 1, \qquad m < l_\sigma = l < \infty,$$

where l is the maximum indentation number. (Here, $\alpha \neq 0$ because the end is not horospherical, $\alpha \neq 1$ because the end is of finite type, and $l_\sigma > m$ because the end is centered.)

Substituting these into (2.2), the lower-left component of $(\partial f)f^{-1}$ is computed as

$$-\frac{dG + dG_*}{(G - G_*)^2} = -\frac{m(1 + \alpha)z^{m-1} + (m + l)\alpha_1 z^{m+l-1} + o(z^{m+l-1})}{\left((1 - \alpha)z^m - \alpha_1 z^{m+l} + o(z^{m+l})\right)^2} \, dz$$

$$= -\frac{m(1 + \alpha)}{(1 - \alpha)^2} z^{-m-1} \left[1 + \alpha_1 \left(\frac{2}{1 - \alpha} + \frac{m + l}{m(1 + \alpha)}\right) z^l + o(z^l)\right] dz.$$

Since $l > m(\geq 1)$, the residue of this form at the origin vanishes. This completes the proof.   □

Normalization of dominant pairs. To prove the main theorems, we introduce the normalized form of the canonical form ω:

LEMMA 2.12. *Let $f : D^* \to H^3$ be a regular WCF-end of a flat front with $G(0) = 0$, and $l = l_\gamma$ the indentation number with respect to the geodesic γ joining ∞ and 0. Assume $l_\gamma < \infty$, that is, f is not rotationally symmetric with respect to γ (see (2.9)). Suppose that (G, ω) is a dominant pair (see Remark 1.14). Then there exist a (unique) local complex coordinate z around the origin and a diagonal matrix $u \in SL(2, \boldsymbol{C})$ such that*

(i) *if the end is not cylindrical,*

$$u \star G = z^m \quad \text{and} \quad \omega = -mz^\mu \big(1 + bz^l + o(z^l)\big)^2 dz \qquad (\mu < -1, \ b \in \boldsymbol{R}_+), \qquad (2.16)$$

(ii) *if the end is cylindrical,*

$$u \star G = z^m \quad \text{and} \quad \omega = -\lambda z^{-1} \big(1 + z^l + o(z^l)\big)^2 dz \qquad (\lambda \in \boldsymbol{R}_+), \qquad (2.17)$$

where m is the multiplicity of the end; moreover, the end is incomplete if and only if $\lambda = m/2$.

PROOF. We can take a coordinate z such that G and G_* are written as (2.10) and (2.11), respectively. Substituting these into (1.19) and (1.20), a direct calculation verifies that ω has the following expression:

$$\omega = c_0 z^\mu \big(1 + b_0 z^l + o(z^l)\big)^2 dz \qquad (2.18)$$

where c_0 and b_0 are non-zero constants which can be computed explicitly, and μ is given by the relation (1.29) in Proposition 1.15. Moreover, by Proposition 1.15, $\mu \le -1$ holds. Note that $b_0 \ne 0$ because f is not rotationally symmetric with respect to γ.

(i) First, we assume that the end is non-cylindrical, that is, $\mu < -1$ holds. Let k be a non-zero constant and take a new coordinate w as $z = kw$. Then ω is written as

$$\omega = c_0 k^{\mu+1} w^\mu \big(1 + k^l b_0 w^l + o(w^l)\big)^2 dw.$$

Here, choose k so that

$$k = \kappa e^{i\beta}, \qquad \text{where} \quad \kappa = \left|\frac{m}{c_0}\right|^{\frac{1}{\mu+1}}, \quad \beta = -\frac{\arg b_0}{l}.$$

Then we have

$$\omega = -me^{i\tau}w^{\mu}\big(1 + bw^l + o(w^l)\big)^2 dw,$$

$$\text{where} \quad b = \kappa^l|b_0| \in \boldsymbol{R}_+, \quad \tau = \arg c_0 + \beta(\mu + 1) + \pi.$$

Using the $U(1)$-ambiguity as in (1.16), we can write ω as

$$\omega = -mw^{\mu}\big(1 + bw^l + o(w^l)\big)^2 dw.$$

On the other hand, G is written as $G = z^m = k^m w^m$. Let

$$u = \begin{pmatrix} k^{-m/2} & 0 \\ 0 & k^{m/2} \end{pmatrix} \in SL(2, \boldsymbol{C}). \tag{2.19}$$

Then $u \star G = w^m$ and the isometry ι_u of H^3 preserves 0 and ∞ in ∂H^3. Finally, we need only to recall that ω is unchanged when f changes to $\iota_u \circ f$.

(ii) If the end is cylindrical, $\mu = -1$ holds. Take a coordinate w as $z = kw$ for a non-zero constant k. Then ω in (2.18) is written as

$$\omega = c_0 w^{-1}\big(1 + k^l b_0 w^l + o(w^l)\big)^2 dw.$$

In this case, if we set $k = b_0^{-1/l}$, we have

$$\omega = -\lambda e^{i\tau}w^{-1}\big(1 + w^l + o(w^l)\big)^2 dw \qquad \big(\lambda = |c_0|, \tau = \arg c_0 + \pi\big).$$

Then again using the $U(1)$-ambiguity and the matrix u in (2.19), we have (2.17).

Now, we shall prove the last assertion: By (1.31) and (2.17), ρ is expanded as

$$\rho = \frac{\theta}{\omega} = \frac{\theta\omega}{\omega^2} = \frac{Q}{\omega^2} = \frac{m^2}{4\lambda^2}\big(1 + o(1)\big).$$

Since $\lambda \in \boldsymbol{R}_+$, Lemma 1.20 implies that the end is incomplete if and only if $\lambda = m/2$. $\qquad\square$

3.   Asymptotic behavior of regular WCF-ends.

In this section, we shall state and prove refinements of Proposition A in the introduction. We also give a proof of Theorem B here.
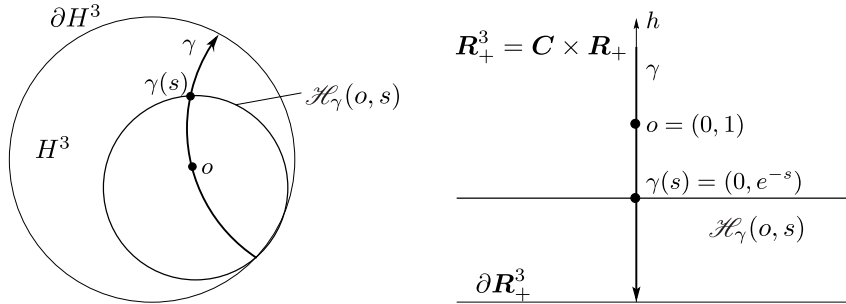
Figure 4. A family of horospheres.

Notations—A family of horospheres. To state the results, we prepare notations: Let γ be an oriented geodesic in H^3 and fix a point $o \in \gamma$. Parametrize γ by the arclength parameter s such that $\gamma(0) = o$. For each $s \in \boldsymbol{R}$, we denote by $\mathscr{H}_\gamma(o, s)$ the horosphere which meets ∂H^3 at $\gamma(-\infty)$ and intersects γ perpendicularly at the point $\gamma(s)$, see Figure 4, left. Since each $\mathscr{H}_\gamma(o, s)$ is isometric to the Euclidean plane, one can introduce a canonical coordinate system on $\mathscr{H}_\gamma(o, s)$ such that $\gamma(s)$ corresponds to the origin of the Euclidean plane as follows (see (3.2)): We wish to work in the upper half-space model $\boldsymbol{R}^3_+$ of H^3 as in (1.5) and (1.6). We always project H^3 to $\boldsymbol{R}^3_+$ so that γ is mapped to the downward oriented h-axis (vertical axis) and o is mapped to $(0, 1) \in \boldsymbol{C} \times \boldsymbol{R}_+ = \boldsymbol{R}^3_+$. Then

$$\mathscr{H}_\gamma(o, s) = \{(\zeta, e^{-s}) \,|\, \zeta \in \boldsymbol{C}\} \subset \boldsymbol{C} \times \boldsymbol{R}_+ = \boldsymbol{R}^3_+. \tag{3.1}$$

In this case, the isometry between $\mathscr{H}_\gamma(o, s) \subset H^3$ and the Euclidean plane is given by

$$\hat{\pi} : \mathscr{H}_\gamma(o, s) \ni (\zeta, h) = (\zeta, e^{-s}) \longmapsto \frac{\zeta}{h} = e^s \zeta \in \boldsymbol{C} = \boldsymbol{E}^2, \tag{3.2}$$

because of (1.6), where $\boldsymbol{E}^2$ denotes the Euclidean plane $(\boldsymbol{R}^2; x, y)$ with the canonical metric $dx^2 + dy^2$. See Figure 4, right.

Statements of the theorems. First, we consider non-cylindrical regular WCF-ends. In this case, they are complete regular ends by Lemma 1.20.

THEOREM 3.1 (Non-cylindrical case).   *Let $f : D^* \to H^3$ be a non-cylindrical complete regular end of a flat front with multiplicity m, and let γ be a geodesic in H^3 with $\gamma(+\infty) = G(0) \in \partial H^3$, where G is the hyperbolic Gauss map. Then there*

exists a unique point $o \in \gamma$ such that, for s large enough, $f(D^*) \cap \mathscr{H}_\gamma(o, s)$ is identified with a curve in $\boldsymbol{C} = \boldsymbol{E}^2$ parametrized by t as

$$e^{imt} h^p \big(1 + R_\gamma(h, t)\big) \qquad \left(p = -\frac{1 + \alpha}{2} \in (-1, 0)\right) \tag{3.3}$$

using $\hat{\pi}$ in (3.2), where α is the ratio of the Gauss maps (1.26), $h = e^{-s}$, and R_γ is a complex-valued function of two real variables (h, t) such that

$$\lim_{h \to +0} R_\gamma(h, t) = 0.$$

Moreover, assume the indentation number l_γ with respect to γ is finite (see (2.9)). Then, setting

$$N_{p,j,m}(h, t) = \{2(p+1)\cos jt\} h^\beta, \quad \text{where} \quad \beta = \beta_{p,j,m} = \frac{j(1+p)}{m} > 0,$$

$$S_p(h) = -\frac{1}{4(p+1)} h^{-2p} \tag{3.4}$$

for positive integers j, m and a real number $p \in (-1, 0)$, we have

   (i) *If f is a centerless end, then for any geodesic γ with $\gamma(+\infty) = G(0)$, there exists $b \in \boldsymbol{R}_+$ (when b is used) such that*

$$R_\gamma(h, t) = \begin{cases} bN_{p,n,m}(h, t) + o(h^\beta) & (if \ n(1+p) < -2pm), \\ bN_{p,n,m}(h, t) + S_p(h) + o(h^\beta) & (if \ n(1+p) = -2pm), \\ S_p(h) + o(h^{-2p}) & (if \ n(1+p) > -2pm), \end{cases} \tag{3.5}$$

     where n is the maximum indentation number, and $\beta = \beta_{p,n,m}$ as in (3.4).

  (ii) *If f is a centered end, then for the principal axis σ, there exists $b \in \boldsymbol{R}_+$ such that $R_\sigma(h, t)$ is written as in (3.5), where $n(> m)$ is the maximum indentation number.*

Figure 5 shows the end with $G = z$, $\omega = -z^{-5} \exp(2z^3)\,dz$ in the upper half-space model (left) and the view of it from the bottom (right). In each of two figures, the end point corresponds to the center of the figure.

REMARK 3.2.   For a centered end f as in Theorem 3.1, take a geodesic γ with $\gamma(+\infty) = G(0)$ which does not coincide with the principal axis. Then there exists a unique $b_\gamma \in \boldsymbol{R}_+$ (when b_γ is used) such that
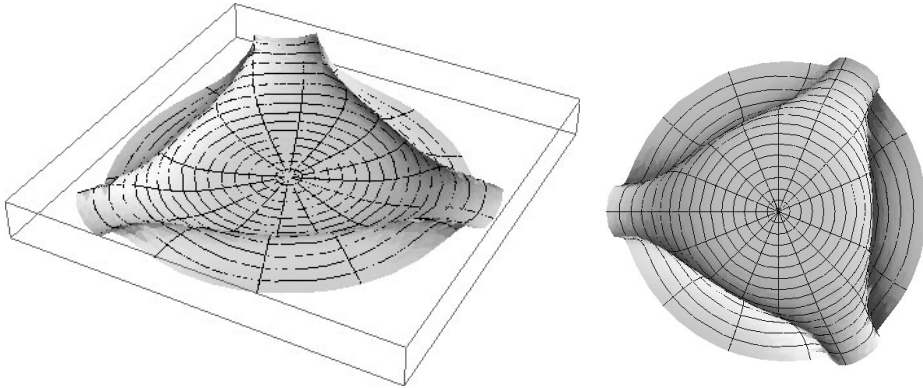
Figure 5. A centered end with $m = 1$, $n = 3$ and $p = -0.8$.

$$
R_\gamma(h,t) = \begin{cases} b_\gamma N_{p,m,m}(h,t) + o(h^{1+p}) & \left(\text{if } -1 < p < -\dfrac{1}{3}\right), \\[2ex] b_\gamma N_{-\frac{1}{3},m,m}(h,t) + S_{-\frac{1}{3}}(h) + o(h^{2/3}) & \left(\text{if } p = -\dfrac{1}{3}\right), \\[2ex] S_p(h) + o(h^{-2p}) & \left(\text{if } -\dfrac{1}{3} < p < 0\right). \end{cases}
\tag{3.6}
$$

This formula can be proved similarly to the proof of Theorem 3.1. Although a geometric interpretation of the bifurcation at $p = -1/3$ will not be provided here, we remark that this bifurcation again appears, also at $-1/3$, even if we instead were to use the height functions induced by slicing with the family of horospheres which meet the end at infinity.

REMARK 3.3. When $m = 1$, that is, the case of an embedded end, only the case (ii) occurs.

REMARK 3.4. If f is rotationally symmetric, $n = \infty$ by definition and the end is of centered type. In this case, $n = \infty$ implies $n(1 + p) > -2pm$ occurs, and then the third equation of (3.5) holds for the principal axis and (3.6) holds for non-principal axes.

THEOREM 3.5 (Cylindrical case). *Let $f : D^* \to H^3$ be a cylindrical regular WCF-end of a flat front with multiplicity m, and let n be its maximum indentation number. Take a geodesic γ in H^3 with $\gamma(+\infty) = G(0) \in \partial H^3$, where G is the hyperbolic Gauss map. We set*

$$V_{m,l,c}(t) = \left(\frac{4c^2 + m^2}{4cm}\right)^{l/m} \left[2\left(c + \frac{m^2}{4c}\right)\cos lt - i\frac{ml}{c}\sin lt\right],$$

$$\Gamma_{m,l}(t) = \frac{e^{imt}}{m} V_{m,l,\frac{m}{2}}(t) = \frac{1}{m}\left[(m+l)e^{i(m-l)t} + (m-l)e^{i(m+l)t}\right]. \tag{3.7}$$

We remark that $\Gamma_{m,l}(t)$ represents a cycloid as in Theorem B in the introduction. Then we have:

(i) *If f is a centerless end, that is, $n < m$, then for any geodesic γ with $\gamma(+\infty) = G(0)$, there exist $\lambda \in \mathbf{R}_+$ and a unique point $o \in \gamma$ such that, for s large enough, $\hat{\pi}(f(D^*) \cap \mathscr{H}_\gamma(o,s))$ is a curve in $\mathbf{C} = \mathbf{E}^2$ parametrized by t as*

$$\frac{1}{m}e^{imt}\left[\left(\lambda - \frac{m^2}{4\lambda}\right) + V_{m,n,\lambda}(t)h^\beta + o(h^\beta)\right],$$

$$\text{where} \quad \beta = \frac{n}{m}, \quad \text{and} \quad h = e^{-s}. \tag{3.8}$$

In particular, if f is incomplete, then $\lambda = m/2$ and $\hat{\pi}(f(D^*) \cap \mathscr{H}_\gamma(o,s))$ is parametrized as

$$\Gamma_{m,n}(t)h^\beta + o(h^\beta), \qquad \text{where} \quad \beta = \frac{n}{m}. \tag{3.9}$$

(ii) *If f is a centered end which is not rotationally symmetric, that is, $m < n < +\infty$, (3.8) or (3.9) holds when γ coincides with the principal axis σ.*

REMARK 3.6.   For a centered end f as in Theorem 3.5, take a geodesic γ with $\gamma(+\infty) = G(0)$ which does not coincide with the principal axis. Then there exists a unique point $o \in \gamma$ such that the intersection $\hat{\pi}(f(D^*) \cap \mathscr{H}_\gamma(o,s))$ is parametrized as

$$\frac{1}{m}e^{imt}\left[\left(\lambda - \frac{m^2}{4\lambda}\right) + V_{m,m,\lambda}(t)h + o(h)\right] \qquad (h = e^{-s}). \tag{3.10}$$

In particular, when f is incomplete, we have $\lambda = m/2$. Then (3.10) yields that $\hat{\pi}(f(D^*) \cap \mathscr{H}_\gamma(o,s))$ is parametrized as $2h + o(h)$.

REMARK 3.7.   If one chooses another point o' on the geodesic γ, the asymptotic behavior changes as follows: Under the situations of Theorems 3.1 or

3.5, take a point o' on the geodesic γ such that the signed distance between o (which is uniquely determined) and o' is τ, that is, $\mathscr{H}_\gamma(o', s) = \mathscr{H}_\gamma(o, s + \tau)$. We have the following:

- When the end is non-cylindrical, $\hat{\pi}(f(D^*) \cap \mathscr{H}_\gamma(o', s))$ is parametrized as

$$e^{-\tau p} e^{imt} h^p \big(1 + o(1)\big) \qquad (h = e^{-s}),$$

  where $o(1)$ denotes a higher order term in h.

- In the case of a complete cylindrical end, $\hat{\pi}(f(D^*) \cap \mathscr{H}_\gamma(o', s))$ is parametrized as

$$\frac{1}{m} e^{imt} \left(\left(\lambda - \frac{m^2}{4\lambda} \right) + e^{-\tau\beta} V_{m,n,\lambda}(t) h^\beta + o(h) \right) \qquad (h = e^{-s}),$$

  under the assumption that (3.8) holds.

- In the case of an incomplete cylindrical end, $\hat{\pi}(f(D^*) \cap \mathscr{H}_\gamma(o', s))$ is parametrized as

$$e^{-\tau\beta} \Gamma_{m,n}(t) h^\beta + o(h^\beta) \qquad (h = e^{-s})$$

  under the assumption that (3.9) holds.

First, we prove Proposition A and Theorem B in the introduction as corollaries of Theorems 3.1 and 3.5. After that their proofs are given.

PROOF OF PROPOSITION A. Take a geodesic γ with $\gamma(+\infty) = G(0)$. Then by taking the first terms of (3.3) and (3.8) with $\lambda \neq m/2$, $\hat{\pi}(f(D^*) \cap \mathscr{H}_\gamma(o, s))$ is parametrized as

$$
\begin{cases}
e^{imt} h^p \big(1 + o(1)\big) & \text{if the end is not cylindrical,} \\
\dfrac{1}{m} \left(\lambda - \dfrac{m^2}{4\lambda} \right) e^{imt} \big(1 + o(1)\big) & \text{if the end is cylindrical.}
\end{cases}
$$

Hence by the correspondence (3.2), we have Proposition A. $\qquad\square$

PROOF OF THEOREM B. Take a geodesic γ with $\gamma(+\infty) = G(0)$ arbitrarily when the end is centerless. If the end is centered, we take γ to be the principal axis. Then by (3.9) and (3.2), we get (3) of Theorem B, where n is the maximum indentation number. Finally, applying Proposition 1.21 we can conclude that n is the ramification order of ρ, since the map $\Gamma_{m,n}(t)$ has $2n$ cusps in $[0, 2\pi)$. $\qquad\square$

COROLLARY 3.8.   *Let $f : D^* \to H^3$ be an incomplete regular WCF-end.*
(1) *The maximum indentation number of f coincides with the ramification order of ρ at $z = 0$.*
(2) *f is epicycloid-type (or hypocycloid-type) if and only if it is centerless (or centered).*

PROOF.
(1) We have already seen this in the proof of Theorem B.
(2) Recall that the multiplicity m of the end and the ramification order n of ρ determine which type the end f is, in such a way that it is epicycloid-type if $n < m$ and is hypocycloid-type if $n > m$. Since n equals the indentation number, the condition $n < m$ or $n > m$ implies that the end f is centerless or centered, respectively. $\qquad\square$

REMARK 3.9.   Here we provide another proof of Corollary 3.8 (1): Take a geodesic γ with $\gamma(+\infty) = G(0)$ arbitrarily when the end is centerless, and to be principal when the end is centered. Then the indentation number l_γ equals the maximum indentation number n, which is not equal to the multiplicity m of the end (see Lemma 2.8 (2), (3));

$$l_\gamma = n \neq m. \tag{3.11}$$

On the other hand, the formula (2) in the introduction with expansions (2.10), (2.11) gives

$$\frac{d\rho}{\rho} = \left\{ \frac{\alpha_1}{m} (m+l)(m-l)z^{l-1} + o(z^{l-1}) \right\} dz, \quad (l = l_\gamma). \tag{3.12}$$

It follows from (3.11), (3.12) that the ramification order of ρ equals n, the maximum indentation number.

Proofs of Theorems 3.1 and 3.5.

PROOF OF THEOREM 3.1.   By Lemma 2.12, there exist a unique isometry $u \in SL(2, \boldsymbol{C})$ and a complex coordinate z such that (2.16) holds. Then $\iota_u \circ \gamma$ is the geodesic joining ∞ and 0, that is, the h-axis of the upper half-space model $\boldsymbol{R}^3_+$ in (1.5), and we write G instead of $u \star G$.

Replacing f by $\iota_u \circ f$, we assume γ itself is the geodesic joining ∞ and 0.

We set $o = (0, 1)$ in $\boldsymbol{R}^3_+$, which is a point on the geodesic γ. In this case, the family of horospheres is written as in (3.1). So, by (3.2), if the image of $\pi \circ f$ is parametrized as $\big(\zeta(h, t), h\big)$, the image under $\hat{\pi}$ of the intersection $f(D^*) \cap \mathscr{H}_\gamma(o, s)$ is parametrized as $t \mapsto \zeta(h, t)/h \in \boldsymbol{C} = \boldsymbol{E}^2$, where $h = e^{-s}$. Thus, to prove the theorem, it is sufficient to compute ζ/h in terms of h.

According to (1.7), we have $\zeta/h = E_{11}\overline{E_{21}} + E_{12}\overline{E_{22}}$ and $1/h = E_{21}\overline{E_{21}} + E_{22}\overline{E_{22}}$. Hence we start with calculations of E_{ij}. Substituting (2.16) into (1.18), we obtain

$$E_{11} = -z^{(1+\mu+m)/2}\big(1 + bz^l + o(z^l)\big),$$

$$E_{12} = \frac{1}{2m}\,z^{-(1+\mu-m)/2}\big[(1+\mu+m) + b\big(2l - (1+\mu+m)\big)z^l + o(z^l)\big],$$

$$E_{21} = -z^{(1+\mu-m)/2}\big(1 + bz^l + o(z^l)\big),$$

$$E_{22} = \frac{1}{2m}\,z^{-(1+\mu+m)/2}\big[(1+\mu-m) + b\big(2l - (1+\mu-m)\big)z^l + o(z^l)\big].$$

Thus,

$$
\begin{aligned}
E_{21}\overline{E_{21}} &= r^{\frac{-m}{1+p}}\big(1 + 2br^l \cos lt + o(r^l)\big),\\[4pt]
E_{22}\overline{E_{22}} &= \frac{1}{4}\,r^{\frac{-m}{1+p}}\left(\frac{1}{(1+p)^2}\,r^{\frac{-2mp}{1+p}} + o\big(r^{\frac{-2mp}{1+p}}\big)\right),\\[4pt]
E_{11}\overline{E_{21}} &= e^{imt}r^{\frac{mp}{1+p}}\big(1 + 2br^l \cos lt + o(r^l)\big),\\[4pt]
E_{12}\overline{E_{22}} &= \frac{1}{4}\,e^{imt}r^{\frac{mp}{1+p}}\left(\frac{-(2p+1)}{(1+p)^2}\,r^{\frac{-2mp}{1+p}} + o\big(r^{\frac{-2mp}{1+p}}\big)\right),
\end{aligned}
\tag{3.13}
$$

where $z = re^{it}$. Here, we used the relation

$$p = -\frac{1+\alpha}{2} = -\frac{1+\mu}{1+\mu-m} \in (-1,0),$$

see Proposition 1.15. Change the coordinate re^{it} to ηe^{it} so that $\eta = r^{m/(1+p)}$. Then

$$
\begin{aligned}
E_{21}\overline{E_{21}} &= \eta^{-1}\big(1 + (2b\cos lt)\eta^\beta + o(\eta^\beta)\big),\\[4pt]
E_{22}\overline{E_{22}} &= \frac{1}{4}\,\eta^{-1}\left(\frac{1}{(1+p)^2}\,\eta^{-2p} + o(\eta^{-2p})\right),\\[4pt]
E_{11}\overline{E_{21}} &= e^{imt}\eta^p\big(1 + (2b\cos lt)\eta^\beta + o(\eta^\beta)\big),\\[4pt]
E_{12}\overline{E_{22}} &= \frac{1}{4}\,e^{imt}\eta^p\left(\frac{-(2p+1)}{(1+p)^2}\,\eta^{-2p} + o(\eta^{-2p})\right),
\end{aligned}
\tag{3.14}
$$

where $\beta = \beta_{p,l,m}$ as in (3.4).

The case $(1+p)l < -2pm$: In this case, $\beta < -2p$ holds. Then by (3.14), we have

$$\frac{1}{h} = E_{21}\overline{E_{21}} + E_{22}\overline{E_{22}} = \frac{1}{\eta}\left(1 + (2b\cos lt)\eta^\beta + o(\eta^\beta)\right),$$
$$h = \eta\left(1 - (2b\cos lt)\eta^\beta + o(\eta^\beta)\right),$$
$$\eta = h\left(1 + (2b\cos lt)h^\beta + o(h^\beta)\right).$$

Thus, $\hat{\pi}\left(f(D^*) \cap \mathscr{H}_\gamma(o,s)\right)$ is parametrized as

$$\frac{\zeta}{h} = E_{11}\overline{E_{21}} + E_{12}\overline{E_{22}} = e^{imt}\eta^p\left(1 + (2b\cos lt)\eta^\beta + o(\eta^\beta)\right) \qquad (3.15)$$
$$= e^{imt}h^p\left(1 + bN_{p,l,m}(h,t) + o(h^\beta)\right) \qquad (h = e^{-s}).$$

The case $(1+p)l > -2pm$: In this case, $\beta > -2p$ holds. Then by (3.14), we have

$$\frac{1}{h} = E_{21}\overline{E_{21}} + E_{22}\overline{E_{22}} = \frac{1}{\eta}\left(1 + \frac{1}{4(1+p)^2}\eta^{-2p} + o(\eta^{-2p})\right),$$
$$h = \eta\left(1 - \frac{1}{4(1+p)^2}\eta^{-2p} + o(\eta^{-2p})\right),$$
$$\eta = h\left(1 + \frac{1}{4(1+p)^2}h^{-2p} + o(h^{-2p})\right).$$

Thus, $\hat{\pi}\left(f(D^*) \cap \mathscr{H}_\gamma(o,s)\right)$ is parametrized as

$$\frac{\zeta}{h} = E_{11}\overline{E_{21}} + E_{12}\overline{E_{22}} = e^{imt}\eta^p\left(1 - \frac{2p+1}{4(1+p)^2}\eta^{-2p} + o(\eta^{-2p})\right) \qquad (3.16)$$
$$= e^{imt}h^p\left(1 - \frac{1}{4(p+1)}h^{-2p} + o(h^{-2p})\right)$$
$$= e^{imt}h^p\left(1 + S_p(h) + o(h^{-2p})\right).$$

The case $(1+p)l = -2pm$: In this case, $\beta = -2p$ holds, and by a similar

calculation, the intersection $\hat{\pi}(f(D^*) \cap \mathscr{H}_\gamma(o, s))$ is parametrized as

$$\frac{\zeta}{h} = e^{imt} h^p \left(1 + b N_{p,l,m}(h, t) + S_p(h) + o(h^{-2p})\right). \tag{3.17}$$

The case of centerless end: In this case, l in (2.16) is always equal to the maximum indentation number n because of (1) and (2) in Lemma 2.8. Substituting $l = n$ into (3.15), (3.16) and (3.17), we have (3.5).

The case of centered end: If the geodesic γ coincides with the principal axis σ, l in (2.16) equals the maximum indentation number n. Then (ii) holds. We also explain Remark 3.2 here. If the geodesic γ is not a principal axis, l equals m. Substituting $l = m$ into (3.15), (3.16) and (3.17), we have (3.6). □

PROOF OF THEOREM 3.5. By Lemma 2.12, we normalize as in (2.17). Then by (1.18), we have

$$E_{11} = -\sqrt{\frac{\lambda}{m}} z^{\frac{m}{2}} \left(1 + z^l + o(z^l)\right),$$

$$E_{12} = \frac{1}{2\sqrt{\lambda m}} z^{\frac{m}{2}} \left(m + (2l - m) z^l + o(z^l)\right),$$

$$E_{21} = -\sqrt{\frac{\lambda}{m}} z^{-\frac{m}{2}} \left(1 + z^l + o(z^l)\right),$$

$$E_{22} = \frac{1}{2\sqrt{\lambda m}} z^{-\frac{m}{2}} \left(-m + (2l + m) z^l + o(z^l)\right).$$

Hence, we successively have

$$\frac{1}{h} = E_{21} \overline{E_{21}} + E_{22} \overline{E_{22}} = \frac{1}{m} \left(\lambda + \frac{m^2}{4\lambda}\right) r^{-m} \left(1 + o(1)\right),$$

$$h = \left(\frac{4\lambda m}{4\lambda^2 + m^2}\right) r^m \left(1 + o(1)\right),$$

$$\frac{\zeta}{h} = \frac{e^{imt}}{m} \left[\left(\lambda - \frac{m^2}{4\lambda}\right) + \left(2\left(\lambda + \frac{m^2}{4\lambda}\right) \cos lt - i\frac{ml}{\lambda} \sin lt\right) r^l + o(r^l)\right],$$

where $z = re^{it}$. Setting $r^m = \eta$, we have

$$h = \left(\frac{4\lambda m}{4\lambda^2 + m^2}\right)\eta\big(1 + o(1)\big),$$

$$\frac{\zeta}{h} = \frac{e^{imt}}{m}\left[\left(\lambda - \frac{m^2}{4\lambda}\right) + \right.$$

$$\left. \left(\frac{4\lambda m}{4\lambda^2 + m^2}\right)^{-\beta}\left(2\left(\lambda + \frac{m^2}{4\lambda}\right)\cos lt - i\frac{ml}{\lambda}\sin lt\right)h^\beta + o(h^\beta)\right],$$

where $\beta = l/m$.

If the end is centered and γ is not principal, then $l = m$ and we have (3.10). In all other cases, l equals the maximum indentation number n, and (3.8) holds.

When the end is incomplete, $\lambda = m/2$ holds because of Lemma 2.12. Then the first term of (3.8) vanishes, and we have (3.9). $\qquad\square$

REMARK 3.10 (Behavior of the singular curvature). In [**SUY**], the notion of singular curvature for cuspidal edges of fronts was introduced. It was seen there that the singular curvature of a cuspidal edge is negative, (resp. positive) if and only if the cuspidal edge curves outward, (resp. curves inward) with respect to the location of the surface. (For representative figures, see [**SUY**].) Let $f : D^* \to H^3$ be a regular incomplete WCF-end. Since f is flat in H^3, the extrinsic curvature K_{ext} is identically 1. Then the singular curvature κ_s of cuspidal edges on f is negative, by [**SUY**, Theorem 3.1]. Let (ω, θ) be the canonical forms associated with f, and set $\rho = \theta/\omega$. Here, we take a complex coordinate $z = x + iy$ so that the image of the x-axis is a cuspidal edge and $z = 0$ corresponds to the incomplete end. Then we can write

$$\rho(z) = 1 + i\rho_1 z^n + o(z^n) \qquad (\rho_1 \in \mathbf{R} \setminus \{0\}).$$

A somewhat lengthy but straightforward calculation gives the following explicit formula for the singular curvature on the x-axis:

$$\kappa_s = -\frac{n|\rho_1|}{2m^2}x^n + o(x^n),$$

where m is the multiplicity of the end. This implies that the singular curvature is always negative, and hence that the cuspidal edges always curve outward with respect to the surface. Moreover, the limiting value of the singular curvature is zero at the end, in accordance with the fact that the cuspidal edges become asymptotically straight as they extend out to the end.

4. The pitch of caustics.

As pointed out in the introduction, the caustic C_f of a flat front gives locally a flat front. In this section, we give a useful formula for the pitch of an end of C_f. We suppose that the flat front $f : M^2 \to H^3$ is weakly complete and of finite type. Then there exist a compact Riemann surface $\overline{M}^2$ and finitely many points $p_1, \ldots, p_n \in \overline{M}^2$ so that $M^2 = \overline{M}^2 \setminus \{p_1, \ldots, p_n\}$. Moreover, we assume the restriction of f to a neighborhood of any p_j corresponds to a regular WCF-end. So we call each p_j a regular WCF-end of f. On the other hand, let

$$q_1, \ldots, q_m \in M^2$$

be all of the umbilics of f. Then it is known that the caustic

$$C_f : \overline{M}^2 \setminus \{p_1, \ldots, p_n, q_1, \ldots, q_m\} \longrightarrow H^3$$

is a weakly complete flat p-front of finite type, and $\{p_1, \ldots, p_n, q_1, \ldots, q_m\}$ are all regular ends. In particular, the number of ends of C_f is $m + n$. The ends $p_1, \ldots, p_n$ of C_f come from the ends of f, so they are called *E-ends*. The ends $q_1, \ldots, q_m$ of C_f come from the umbilics of f, so they are called *U-ends*. (See [**KRUY**].) Even when C_f is not a front but only a p-front, by taking its double cover, we may consider it as a front.

From now on, we fix a point

$$e = p_j \quad \text{or} \quad q_k \qquad (j = 1, \ldots, n, \ k = 1, \ldots, m)$$

and denote by

$$C : D^* = D \setminus \{e\} \longrightarrow H^3$$

a restriction of C_f around a neighborhood $D(\subset \overline{M}^2)$ of the end e of C_f, which is a regular WCF-end of a p-front. If the unit normal vector field of C is globally defined on D^* (namely C is a front), the end C is called *co-orientable*. (In this case, C itself is a regular WCF-end of a front.) Otherwise, C is called *non-co-orientable*. If C is non-co-orientable, it is not a front, but taking the double cover $\pi : \hat{D}^* \to D^*$, then $C \circ \pi$ is a regular WCF-end of a front.

PROPOSITION 4.1 ([**KRUY**, Theorems 7.4 and 7.6]). *All of the U-ends $q_1, \ldots, q_m$ are incomplete regular WCF-ends of the p-front C_f. Each E-end p_j is an incomplete regular WCF-end of C_f, unless p_j is a snowman-type end of f.*

If p_j is a snowman-type end of f, then C_f has a complete cylindrical end at p_j, and the pitch is equal to 0. So to give a formula for the pitch of the ends of C_f, we may assume that p_j is not a snowman-type end of f. We can prove the following:

THEOREM 4.2. *Let $C : D^* \to H^3$ be a regular WCF p-front end, which is the restriction of the caustic C_f around an E-end $z = p_j$ or a U-end $z = q_k$. The end is incomplete if and only if $z = q_k$, or $z = p_j$ and p_j is not a snowman-type end of f. In this case, for a sufficiently small $\varepsilon > 0$, the image of C in $\mathbf{R}_+^3$ is congruent to a portion of the image of $[0, 2\pi) \times (0, h_0) \ni (t, h) \mapsto (\varphi_h(t), h) \in \mathbf{R}_+^3$, with*

$$\varphi_h(t) = h^{1+p}\Gamma_{m_c,n_c}(t) + o(h^{1+p}), \quad p = \frac{n_c}{m_c} \in (0,1) \cup (1, \infty) , \qquad (4.1)$$

for the map Γ_{m_c,n_c} of a cycloid, and for

$$n_c = \frac{1}{2}\operatorname{ord} Q + \operatorname{ord}\big(S(G_*) - S(G)\big) + 3,$$

$$m_c = \begin{cases} \dfrac{1}{2}\operatorname{ord} Q + m_j + 1 & (\text{if } z = p_j) \\[2mm] \dfrac{1}{2}\operatorname{ord} Q & (\text{if } z = q_k), \end{cases}$$

where m_j is the multiplicity of the end p_j of f. In particular, the pitch $p = n_c/m_c$ of C is a rational number. (In fact, if p_j is a snowman-type end of f, it is a complete cylindrical end of C_f, and its pitch vanishes.)

PROOF. The multiplicity m_c of the end p_j or q_k of C_f has been computed in [**KRUY**, Theorems 7.4 and 7.6]. So we consider only the formula for n_c. The canonical forms (ω_c, θ_c) of C_f are given by (see [**KRUY**, (6.5)])

$$\omega_c = i\sqrt{Q} + \frac{d(\log \rho)}{4}, \quad \theta_c = -i\sqrt{Q} + \frac{d(\log \rho)}{4},$$

where $\rho = \theta/\omega$, $Q = \omega\theta$ and $(\omega, \theta) = (\hat{\omega}dz, \hat{\theta}dz)$ is a pair of canonical forms of f. (The meaning of the square root $\sqrt{Q}$ is explained in [**KRUY**].) Then the Hopf differential Q_c of C_f is given by

$$Q_c = \omega_c\theta_c = Q + \left(\frac{d(\log \rho)}{4}\right)^2.$$

We set $\rho_c = \theta_c/\omega_c$. By a straightforward calculation, we have

$$d(\log \rho_c) = \frac{1}{Q_c} \left(\hat{\omega}_c \hat{\theta}'_c - \hat{\omega}'_c \hat{\theta}_c \right) dz^3$$

$$= \frac{i\sqrt{Q}}{2Q_c} \left\{ \left(\frac{\hat{\theta}'}{\hat{\theta}} \right)' - \frac{1}{2} \left(\frac{\hat{\theta}'}{\hat{\theta}} \right)^2 - \left(\frac{\hat{\omega}'}{\hat{\omega}} \right)' + \frac{1}{2} \left(\frac{\hat{\omega}'}{\hat{\omega}} \right)^2 \right\} dz^2$$

$$= \frac{i\sqrt{Q}}{2Q_c} \left\{ \left(2Q + S(G_*) \right) - \left(2Q + S(G) \right) \right\} = \frac{i\sqrt{Q}}{2Q_c} \left(S(G_*) - S(G) \right),$$

where $' = d/dz$ and $S(\cdot)$ denotes the Schwarzian derivative as in (1.15). Since $z = p_j$ or $z = q_k$ is incomplete, it must be cylindrical and the order of Q_c at the end equals -2 (i.e., a pole of order 2). The singular set of C_f is represented as $\{|\rho_c| = 1\}$. Then

$$n_c = \operatorname{ord} \left(d(\log \rho_c) \right) + 1$$

$$= \frac{1}{2} \operatorname{ord}(Q) - \operatorname{ord}(Q_c) + \operatorname{ord} \left(S(G_*) - S(G) \right) + 1$$

$$= \frac{1}{2} \operatorname{ord}(Q) + \operatorname{ord} \left(S(G_*) - S(G) \right) + 3.$$

If $\operatorname{ord} Q$ at the end is even, C is a front, and the assertion follows directly from Theorem B. On the other hand, if $\operatorname{ord} Q$ is odd, then C is non-co-orientable. In this case, we get the assertion by applying Theorem B to the double cover of C. □

We shall now compute the pitches of ends of some complete flat fronts and their caustics, showing that a variety of cases do indeed occur on global examples.

EXAMPLE 4.3 (Flat fronts of revolution). Recall the flat fronts of revolution as in Example 1.19. The caustic of the horosphere is the empty set because the horosphere is totally umbilic, and the caustic of a hyperbolic cylinder is a geodesic line. The caustic of an hourglass is a geodesic, which can be considered as a parallel surface of the hyperbolic cylinder, regardable as a regular incomplete cylindrical end with pitch $p = \infty$. The caustic of the snowman is congruent to a hyperbolic cylinder.

EXAMPLE 4.4. The third and the fourth authors [**UY2**] constructed constant mean curvature one surfaces in H^3 from a given hyperbolic Gauss map and a polyhedron of constant Gaussian curvature 1. Here, we explain the

canonical symmetric flat k-noid via a construction similar to that one. We consider a domain D_k in $\mathbf{C}$ bounded by a regular k-polygon P_k. Consider a pair of $\overline{D_k}$ and glue them along P_k. Then we get an abstract flat surface with k conical singularities, up to a homothety, which gives a flat symmetric conformal metric $|\omega|^2$ on $S^2 = \mathbf{C} \cup \{\infty\}$, that is,

$$\omega = c(z^k - 1)^{-2/k}\, dz \qquad (k \geq 3).$$

We set $G = z$. By [**KRUY**, Theorem 4.1], the pair (G, ω) gives a flat symmetric k-noid

$$f : \mathbf{C} \cup \{\infty\} \setminus \{z^k = 1\} \longrightarrow H^3,$$

whose Hopf differential Q and other hyperbolic Gauss map G_* are given by

$$Q = \frac{(k-1)z^{k-2}}{(z^k - 1)^2}\, dz^2, \qquad G_* = z^{1-k}.$$

In particular, $z = 0, \infty$ are umbilics. Since $\lim_{z \to 1}(z-1)^2 Q/dz^2 = (k-1)/k^2 > 0$ and the ends of f are congruent to each other, the ends of f are all of hourglass type, and they are embedded, as G does not branch at the ends. Moreover, their pitch is given by

$$p = -\frac{k-2}{2k-2} < 0.$$

Next, we consider the caustic C_f. The points $z = 0, \infty$ are ends coming from the umbilics of f. Since Q has order $k - 2$ at those two points, we have $m_c = (k-2)/2$. Since $G_*(z) = z^{1-k}$, $S(G_*)$ has order -2 at $z = 0, \infty$. Thus

$$n_c = \frac{1}{2}\operatorname{ord} Q + \operatorname{ord} S(G_*) + 3 = \frac{k}{2} \ ,$$

and $n_c/m_c = k/(k-2) > 1$ gives the pitch of the end $z = 0, \infty$. They are hypocycloid-type ends.

On the other hand, the other remaining ends of C_f are all congruent. By a similar computation, we have $m_c = 1$ and $n_c = 2$, and thus the pitch is equal to 2, namely, these ends are of hypocycloid-type. In particular, the case of $k = 4$ (Figure 1 in the introduction) is very interesting. As pointed out in the

introduction, the caustic C_f has octahedral symmetry. In this case, one can easily get $n_c = 2$ from the picture since the four cuspidal edges accumulate at each end.

EXAMPLE 4.5. We consider a cone $C_d(\subset \boldsymbol{R}^3)$ over the domain D_d in $\boldsymbol{C}$ bounded by a regular d-polygon P_d, which gives a polyhedron whose sides consist of d regular triangles. Consider a pair of $\overline{C}_d$ and glue them along $\overline{D}_d$. Then we get an abstract flat surface with $(d+2)$-conical singularities, which gives a flat symmetric conformal metric $|\omega|^2$ on $S^2 = \boldsymbol{C} \cup \{\infty\}$. We set $G = z^k$. Then the pair (G, ω) gives a flat front with $d + 2$ ends and dihedral symmetry

$$f : \boldsymbol{C} \setminus \big(\{0\} \cup \{z^d = 1\}\big) \longrightarrow H^3,$$

whose Hopf differential Q and other hyperbolic Gauss map G_* are given by

$$Q = -\frac{k(k+d)z^{d-2}}{(z^d - 1)^2}\, dz^2, \qquad G_* = z^{k+d}.$$

In particular, f has no umbilics. The ends $z = 0, \infty$ are ends of multiplicity k (embedded if and only if $k = 1$) and with pitch $p = -1/2$, that is, they are horospherical. On the other hand, the other d ends are all mutually congruent, and they are embedded snowman-type ends with pitch $p = -(2k+d)/(2k+2d) < -1/2$.

Next, we consider the caustic C_f. Since f has no umbilics, the ends of C_f are the same as those of f. The ends $z = 0, \infty$ of C_f satisfy $m_c = k + d/2$ and $n_c = d/2$. So the pitch is equal to $p_c = d/(2k+d)$, and thus they are epicycloid-type. The other ends of C_f are mutually congruent. Since they are caustics of snowman-type ends, they are regular complete cylindrical ends (i.e., $p_c = 0$). See Figure 6.

Appendix A. The hyperbolic Gauss maps

In this appendix, we show that the hyperbolic Gauss maps defined as limits of the normal geodesic of the flat front coincide with G and G_* defined in Section 1.

Let $\boldsymbol{L}^4$ be the Minkowski 4-space with the Lorentzian metric $\langle\,,\,\rangle$ and consider the hyperbolic 3-space H^3 as the hyperboloid as in (1.1). We denote the cone of future pointing light-like vectors by L_+:

$$L_+ = \{n = (n_0, n_1, n_2, n_3) \in \boldsymbol{L}^4 \,;\, \langle n, n \rangle = 0, n_0 > 0\}.$$

The multiplicative group $\boldsymbol{R}_+$ acts on L_+ by scalar multiplication. The *ideal boundary* of H^3 is defined as
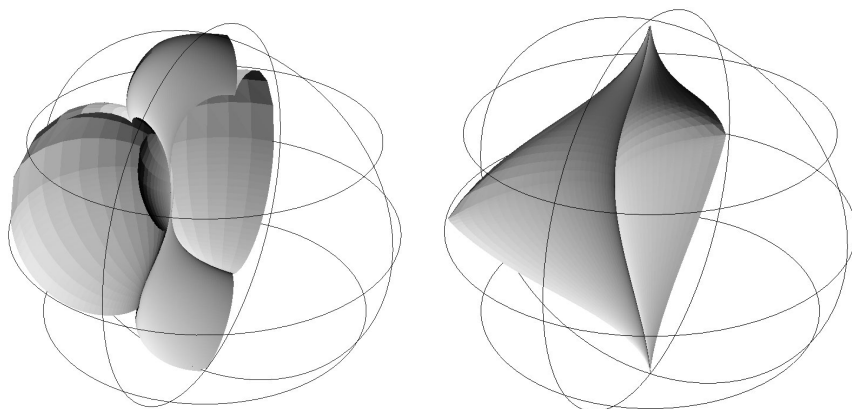
Figure 6.   One of three congruent portions of the front with data $G = z^k$ and $G_* = z^{k+d}$ and one of three congruent portions of its caustic, for $(k, d) = (1, 3)$.

$$\partial H^3 = L_+/\boldsymbol{R}_+.$$

This is also considered as the *asymptotic classes of geodesics*. In fact, if we denote by $\gamma_{x,\boldsymbol{v}}$ the geodesic starting at x with velocity $\boldsymbol{v}$ ($|\boldsymbol{v}| = 1$),

$$\text{the asymptotic class of } \gamma_{x,\boldsymbol{v}}(s) = (\cosh s)x + (\sinh s)\boldsymbol{v}$$
$$\leftrightarrow [x + \boldsymbol{v}] \in \partial H^3 = L_+/\boldsymbol{R}_+ \tag{A.1}$$
$$\leftrightarrow \frac{1}{(x_0 + v_0) - (x_3 + v_3)}\left((x_1 + v_1) + i(x_2 + v_2)\right) \in \boldsymbol{C} \cup \{\infty\}$$

are one-to-one correspondences, where $x = (x_0, x_1, x_2, x_3)$, $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$.

Now, identify $\boldsymbol{L}^4$ with the set of 2×2 hermitian matrices Herm(2) as in (1.2):

$$\boldsymbol{L}^4 \ni (x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \quad \left(i = \sqrt{-1}\right).$$

Then the Lorentzian inner product $\langle\ ,\ \rangle$ is represented by

$$\langle X, Y \rangle = -\frac{1}{2}\operatorname{trace} X\widetilde{Y},$$

where $\widetilde{Y}$ is the cofactor matrix of Y, that is, $\widetilde{Y}Y = Y\widetilde{Y} = (\det Y)\mathrm{id}$ holds. In

particular, $\langle X, X \rangle = -\det X$. Here, we can write

$$H^3 = \{x \in \operatorname{Herm}(2)\,;\, \det x = 1, \operatorname{trace} x > 0\} = \{aa^* \,|\, a \in SL(2, \boldsymbol{C})\}. \qquad (\text{A.2})$$

We write

$$e_0 = \operatorname{id}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (\text{A.3})$$

Let $x \in H^3$ and $X, Y \in T_x H^3$. The tangent space $T_x H^3$ is identified with the orthogonal compliment of the position vector x in $\boldsymbol{L}^4 = \operatorname{Herm}(2)$. Then we define a skew-symmetric bilinear form

$$T_x H^3 \times T_x H^3 \ni X, Y \mapsto X \times Y = \frac{i}{2}\left(Xx^{-1}Y - Yx^{-1}X\right) \in T_x H^3, \qquad (\text{A.4})$$

called the *exterior product*, where $x \in H^3$ is considered as a matrix in $SL(2, \boldsymbol{C})$, and products of the right-hand side are matrix multiplications. Then one can show the following:

- $X \times Y$ is perpendicular to both X and Y.
- If ι is an orientation preserving isometry of H^3, $\iota_* X \times \iota_* Y = \iota_*(X \times Y)$. In particular, $e_1 \times e_2 = e_3$ holds for matrices as in (A.3), where e_j $(j = 1, 2, 3)$ are considered as vectors in $T_{e_0} H^3$.
- $(x, X, Y, X \times Y)$ is a positively oriented basis of $\boldsymbol{L}^4$.

Let $f : D^* \to H^3$ be a complete regular end as in Section 1, ν the unit normal vector field and $\mathscr{E}_f$ its holomorphic Legendrian lift. Then we have:

PROPOSITION A.1. *At each regular point of f,*
$$\nu = \operatorname{sign}\big(|\hat{\theta}|^2 - |\hat{\omega}|^2\big)\frac{f_u \times f_v}{|f_u \times f_v|}$$

holds. That is, ν is compatible to the orientation of D^* if and only if $|\hat{\theta}|^2 - |\hat{\omega}|^2 > 0$, where $z = u + iv$ is the complex coordinate and $\omega = \hat{\omega}\,dz$, $\theta = \hat{\theta}\,dz$.

PROOF. By (1.10), we have

$$f_z = \mathscr{E}_f \begin{pmatrix} 0 & \hat{\theta} \\ \hat{\omega} & 0 \end{pmatrix} \mathscr{E}_f^*, \qquad f_{\bar{z}} = \mathscr{E}_f \begin{pmatrix} 0 & \overline{\hat{\omega}} \\ \overline{\hat{\theta}} & 0 \end{pmatrix} \mathscr{E}_f^*.$$

Thus, using (A.4), we have

$$f_u \times f_v = -2i f_z \times f_{\bar{z}} = \left(|\hat{\theta}|^2 - |\hat{\omega}|^2 \right) \mathscr{E}_f \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathscr{E}_f^*.$$

Then we have the conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now, we define

$$G_\pm = [f \pm \nu] : D^* \longrightarrow \partial H^3,$$

where $[\ ]$ denotes the equivalence class in $\partial H^3 = L_+ / \boldsymbol{R}_+$.

PROPOSITION A.2.   Let $\mathscr{E}_f = (E_{ij})$ be the holomorphic Legendrian lift of f. Then under the identification as in (A.1), it holds that

$$G_+ = G = \frac{E_{11}}{E_{21}}, \qquad G_- = G_* = \frac{E_{12}}{E_{22}}.$$

PROOF.   We denote $f + \nu = (n_0, n_1, n_2, n_3)$, and consider $\boldsymbol{L}^4$ as Herm(2). Then

$$\begin{pmatrix} n_0 + n_3 & n_1 + in_2 \\ n_1 - in_2 & n_0 - n_3 \end{pmatrix} = f + \nu = \mathscr{E}_f \mathscr{E}_f^* + \mathscr{E}_f e_3 \mathscr{E}_f^* = 2\mathscr{E}_f \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathscr{E}_f^*$$

$$= 2 \begin{pmatrix} E_{11}\overline{E_{11}} & E_{11}\overline{E_{21}} \\ E_{21}\overline{E_{11}} & E_{21}\overline{E_{21}} \end{pmatrix}.$$

By (A.1), the map $[f + \nu]$ is identified with

$$\frac{n_1 + in_2}{n_0 - n_3} = \frac{E_{11}\overline{E_{21}}}{E_{21}\overline{E_{21}}} = \frac{E_{11}}{E_{21}} = G.$$

On the other hand, since

$$f - \nu = 2\mathscr{E}_f \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \mathscr{E}_f^* = -2 \begin{pmatrix} E_{12}\overline{E_{12}} & E_{12}\overline{E_{22}} \\ E_{22}\overline{E_{12}} & E_{22}\overline{E_{22}} \end{pmatrix},$$

$[f - \nu]$ is identified with

$$\frac{E_{12}\overline{E_{22}}}{E_{22}\overline{E_{22}}} = \frac{E_{12}}{E_{22}} = G_*.$$

This concludes the proof. □

Appendix B. A differential equation of cycloids

We will show that the plane curve $\Gamma : \vartheta \mapsto r(\vartheta)e^{i\vartheta}$ determined by a general solution r of

$$\begin{cases} \dfrac{d\log r}{d\vartheta} = u \\[2mm] (p^2 - 1)\dfrac{du}{d\vartheta} = p^2 + (p^2 + 1)u^2 + u^4 \qquad (p > 0, \; p \neq 1) \end{cases} \tag{B.1}$$

is a hypo-/epi-cycloid if $p = n/m$.

Integrating the second equation of (B.1), we have

$$\arctan u - \frac{1}{p}\arctan\frac{u}{p} = \vartheta + C_1. \tag{B.2}$$

Without loss of generality, we take the arbitrary constant C_1 in (B.2) to be zero, because the constant C_1 can be cancelled by a change of the parameter of the curve Γ. From now on, we let $u = u(\vartheta)$ be the implicit function determined by (B.2) with $C_1 = 0$.

We introduce a new parameter $s(= s(\vartheta))$ by

$$s = \vartheta - \arctan u(\vartheta). \tag{B.3}$$

Note that s is monotone in ϑ because $ds/d\vartheta = (1 + u^2)/(1 - p^2) \neq 0$.

It follows from (B.2) and (B.3) that

$$-\frac{1}{p}\arctan\frac{u(\vartheta)}{p} = s, \tag{B.4}$$

that is,

$$u(\vartheta) = -p\tan(ps). \tag{B.4'}$$

It follows from (B.3) and (B.4′) that

$$\vartheta = s - \arctan\big(p\tan(ps)\big). \tag{B.5}$$

Next, we rewrite the first equation of (B.1) in terms of s instead of ϑ. Since the first equation of (B.1) is equivalent to

$$\frac{d\log r}{ds}\frac{ds}{d\vartheta} = u, \qquad \text{i.e.,} \quad \frac{d\log r}{ds} = u\frac{d\vartheta}{ds}, \tag{B.6}$$

we first calculate $d\vartheta/ds$. In fact, differentiating (B.5), we have

$$\frac{d\vartheta}{ds} = \frac{(1-p^2)\cos^2(ps)}{\cos^2(ps) + p^2\sin^2(ps)}. \tag{B.7}$$

Substituting (B.4′) and (B.7) into (B.6), we have

$$\frac{d\log r}{ds} = -\frac{p(1-p^2)\sin(2ps)}{(1+p^2) + (1-p^2)\cos(2ps)}. \tag{B.8}$$

It implies that

$$r^2 = C_2\{(1+p^2) + (1-p^2)\cos(2ps)\}, \tag{B.9}$$

where C_2 is an arbitrary non-zero constant.

On the other hand, let us also describe $e^{2i\vartheta}$ in terms of s. In fact, it follows from (B.5) that

$$\begin{aligned}
e^{i\vartheta} &= e^{is}\exp\big(-i\arctan(p\tan(ps))\big) \\
&= e^{is}\{\cos(\arctan(p\tan(ps))) - i\sin(\arctan(p\tan(ps)))\} \\
&= e^{is}\left\{\frac{1}{\sqrt{1+(p\tan(ps))^2}} - i\frac{p\tan(ps)}{\sqrt{1+(p\tan(ps))^2}}\right\}.
\end{aligned}$$

Hence, we have

$$e^{2i\vartheta} = e^{2is} \frac{1 - 2ip\tan(ps) - p^2\tan^2(ps)}{1 + p^2\tan^2(ps)} \tag{B.10}$$

$$= e^{2is} \frac{(1 - p^2) + (1 + p^2)\cos 2ps - 2ip\sin 2ps}{(1 + p^2) + (1 - p^2)\cos 2ps}.$$

It follows from (B.9) and (B.10) that

$$(re^{i\vartheta})^2 = C_2 e^{2is}\{(1 - p^2) + (1 + p^2)\cos 2ps - 2ip\sin 2ps\}$$

$$= \frac{C_2}{2} e^{2is}\{(1 - p)e^{ips} + (1 + p)e^{-ips}\}^2.$$

Hence, we can conclude that

$$re^{i\vartheta} = C_3\{(1 - p)e^{i(1+p)s} + (1 + p)e^{i(1-p)s}\}$$

for arbitrary non-zero constant C_3.

In the case of $p = n/m$, using a new parameter $t = s/m$, we have

$$\Gamma : re^{i\vartheta} = C\{(m - n)e^{i(m+n)t} + (m + n)e^{i(m-n)t}\}\big(= C \cdot \Gamma_{m,n}(t)\big) \tag{B.11}$$

for arbitrary non-zero constant C. The equation (B.11) proves that the solutions of (B.1) give hypo-/epi-cycloids.

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