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C^∞-vectors of irreducible representations of exponential solvable Lie groups

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Abstract. Let G be an exponential solvable Lie group, and π be an irreducible unitary representation of G. Then by induction from a unitary character of a connected subgroup, π is realized in an L^2-space of functions on a homogeneous space. We are concerned with C^∞vectors of π from a viewpoint of rapidly decreasing properties. We show that the subspace $\mathscr{S}\mathscr{E}$ consisting of vectors with a certain property of rapidly decreasing at infinity can be embedded as the space of the C^∞vectors in an extension of π to an exponential group including G. Using the space $\mathscr{S}\mathscr{E}$, we also give a description of the space $\mathscr{A}\mathscr{S}\mathscr{E}$ related to Fourier transforms of L^1-functions on G. We next obtain an explicit description of C^∞vectors for a special case. Furthermore, we consider a space of functions on G with a similar rapidly decreasing property and show that it is the space of the C^∞vectors of an irreducible representation of a certain exponential solvable Lie group acting on $L^2(G)$.

1. Introduction.

Let G be an exponential solvable group with Lie algebra $\mathfrak{g}$, and π be an irreducible unitary representation of G. According to the orbit method, there exist a linear form $l \in \mathfrak{g}^*$ and a real polarization $\mathfrak{h}$ at l such that the representation π is realized as the induced representation $\mathrm{ind}_H^G \chi_l$ from χ_l of H, where $H = \exp \mathfrak{h}$ is the connected and simply connected subgroup with Lie algebra $\mathfrak{h}$ and χ_l is the unitary character of H defined by $\chi_l(\exp X) = e^{il(X)}$ for $X \in \mathfrak{h}$.

Suppose that G is nilpotent, and realize π on $L^2(\boldsymbol{R}^m)$ by taking a supplementary Malcev basis to $\mathfrak{h}$ and identifying G/H with $\boldsymbol{R}^m$. Then by results of Kirillov [**5**] and Corwin-Greenleaf-Penney [**4**], it is well known that the action of the enveloping algebra $\mathscr{U}(\mathfrak{g})$ forms the algebra of differential operators with polynomial coefficients, and the space of the C^∞vectors is precisely the Schwartz space $\mathscr{S}(\boldsymbol{R}^m)$.

However, when G is a general exponential solvable Lie group, the space of the C^∞vectors does not have such simple characterizations. For example, the action of $\mathscr{U}(\mathfrak{g})$ may involve multiplications of exponential functions which require C^∞vectors to have a property of rapidly decreasing at infinity in one direction but do not necessarily require such property in another direction.

In this paper, we investigate structures of the C^∞vectors from a viewpoint of some

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rapidly decreasing properties. In section 2, under a standard realization of π, we are concerned with the subspace $\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$ consisting of functions with a rapidly decreasing property defined in Definition 2.3. We shall show that it can be embedded as the space of the C^∞vectors in a space of irreducible representation π_{l_0} of an exponential solvable group $F \supset G$ such that the restriction of π_{l_0} to G is equivalent to π. By using this space $\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$, we also describe the space $\mathscr{A}\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$ introduced by Ludwig [**7**], which is included in the image of Fourier transforms of L^1-functions on G of finite ranks. In section 3, we shall give an explicit characterization of C^∞vectors when G can be described as $G = N^l N$, where N and N^l are the subgroups corresponding to the nilradical of $\mathfrak{g}$ and its stabilizer for l, respectively. In section 4, we are also concerned with the space $\mathscr{S}\mathscr{E}(G)$, a space of functions on G with a similar property of rapidly decreasing at infinity, and we shall show that it is the space of C^∞vectors of an irreducible representation of a certain exponential solvable Lie group acting on $L^2(G)$.

2. The space $\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$.

Let G be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$ (for details on the theory of exponential solvable Lie groups see [**6**] and [**3**]). Let $\mathfrak{n}$ be a nilpotent ideal including $[\mathfrak{g}, \mathfrak{g}]$. (For instance we can take the nilradical of $\mathfrak{g}$.) Let $\pi \in \hat{G}$ be an irreducible unitary representation of G, and $l \in \mathfrak{g}^*$ be a real linear form such that the coadjoint orbit $G \cdot l$ corresponds to π. We denote by $\mathfrak{g}^l = \mathfrak{g}(l)$ and $\mathfrak{n}^l$ the stabilizers defined as follows:

$$\mathfrak{g}^l = \mathfrak{g}(l) := \{X \in \mathfrak{g};\ l([X, \mathfrak{g}]) = \{0\}\},$$
$$\mathfrak{n}^l := \{X \in \mathfrak{g};\ l([X, \mathfrak{n}]) = \{0\}\}.$$

DEFINITION 2.1 (see [**9**]). We say that a polarization $\mathfrak{h}$ at $l \in \mathfrak{g}^*$ is adapted to $\mathfrak{n}$, if

1. $\mathfrak{h} \cap \mathfrak{n}$ is a polarization at $l_{|\mathfrak{n}}$
2. $[\mathfrak{n}^l, \mathfrak{h} \cap \mathfrak{n}] \subset \mathfrak{h} \cap \mathfrak{n}$.

Then $\mathfrak{h}$ is a Pukanszky polarization and there exists a polarization $\mathfrak{h}_0 \subset \mathfrak{n}^l$ at $l_{|\mathfrak{n}^l}$ such that $\mathfrak{h} = \mathfrak{h}_0 + (\mathfrak{h} \cap \mathfrak{n})$ and $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{n}^l$.

REMARK 2.2. (1) For any l and $\mathfrak{n}$, there exists a polarization adapted to $\mathfrak{n}$. For example, a Vergne polarization associated with a refinement of the series of ideals $\{0\} \subset \mathfrak{n} \subset \mathfrak{g}$ is adapted to $\mathfrak{n}$.

(2) Let $\mathfrak{h}_\mathfrak{n}$ be a polarization at $l_{|\mathfrak{n}}$, such that

$$[\mathfrak{n}^l, \mathfrak{h}_\mathfrak{n}] \subset \mathfrak{h}_\mathfrak{n}.$$

If $\mathfrak{h}_0 \subset \mathfrak{n}^l$ denotes any polarization at $l_{|\mathfrak{n}^l}$, then

$$\mathfrak{h} := \mathfrak{h}_0 + \mathfrak{h}_\mathfrak{n}$$

is a Pukanszky polarization at l. Let $\mathfrak{m} := \mathfrak{n}^l \cap \mathfrak{n} \cap \ker(l)$. Then $\mathfrak{m}$ is an ideal of $\mathfrak{n}^l$

and $\mathfrak{n}^l/\mathfrak{m}$ is either abelian or a direct sum of a central ideal and a Heisenberg algebra. In particular any polarization $\mathfrak{h}_0 \subset \mathfrak{n}^l$ at $l_{|\mathfrak{n}^l}$ is a Pukanszky polarization, since $\mathfrak{n}^l/\mathfrak{m}$ is at most nilpotent of step 2.

Let $\mathfrak{h}$ be a polarization at l adapted to $\mathfrak{n}$, $H = \exp \mathfrak{h}$, χ_l a unitary character of H such that $d\chi_l = il$. Let $\mathscr{D}(G/H)$ be the space of all continuous functions $f : G \to \boldsymbol{C}$ with compact support modulo H, such that $f(gh) = \frac{\Delta_H(h)}{\Delta_G(h)} f(g)$ for all $h \in H$ and $g \in G$. On this space there exists a unique positive left invariant linear functional

$$f \mapsto \oint_{G/H} f(g) d\mu_{G/H}(g) \tag{2.1}$$

(see [**3**]). Then we realize π as $\pi = \pi_{l,H} = \mathrm{ind}_H^G \chi_l$ in $\mathscr{H}_\pi$, where $\mathscr{H}_\pi = L^2(G/H, \chi_l)$ is the completion with respect to the norm $\| \ \|_\pi$ of the space $\mathscr{D}(G/H, \chi_l)$ of the continuous functions ϕ with compact support modulo H on G such that

1. $\phi(gh) = \chi_l(h)^{-1} \Delta_{H,G}^{1/2}(h) \phi(g)$ for all $h \in H$, $g \in G$.
2. $\|\phi\|_\pi^2 := \oint_{G/H} |\phi(g)|^2 d\mu_{G/H}$,

where Δ_G and Δ_H are the modular functions of G and H, respectively, and $\Delta_{H,G}^{1/2} = (\Delta_H/\Delta_G)^{1/2}$.

Taking coexponential bases $\{T_1, \cdots, T_\nu\}$ for $\mathfrak{n}^l + \mathfrak{n}$ in $\mathfrak{g}$, $\{T_{\nu+1}, \cdots, T_m\}$ for $\mathfrak{n} + \mathfrak{h}$ in $\mathfrak{n}^l + \mathfrak{n}$, $\{R_1, \cdots, R_v\}$ for $\mathfrak{h}$ in $\mathfrak{n} + \mathfrak{h}$, we identify G/NH with $\boldsymbol{R}^m$, NH/H with $\boldsymbol{R}^v$ by $t = (t_1, \cdots, t_m) \mapsto E(t) := \exp t_1 T_1 \cdots \exp t_m T_m$ modulo HN, $r = (r_1, \cdots, r_v) \mapsto E(r) := \exp r_1 R_1 \cdots \exp r_v R_v$ modulo H, respectively, and G/H with $\boldsymbol{R}^{m+v}$ by $(t, r) \mapsto E(t, r) : = E(t)E(r)$ modulo H.
We can now express the integral (2.1) as an integral on $\boldsymbol{R}^{m+v}$:

$$\oint_{G/H} f(g) d\mu_{G/H}(g) = \int_{\boldsymbol{R}^{m+v}} f(E(t,r)) dt dr, \ f \in \mathscr{D}(G/H),$$

(see [**6**]).

Definition 2.3. Let $\mathfrak{D}_{t,r}$ be the space of all differential operators on $\boldsymbol{R}^{m+v}$ with polynomial coefficients and let $\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$ be the space of all functions $\phi \in \mathscr{H}_{\pi_{l,H}}$ such that

1. ϕ is smooth,
2. $$\|\phi\|_{a,D}^2 := \int_{\boldsymbol{R}^{m+v}} e^{a\|t\|} |D(\phi \circ E)(t,r)|^2 dt dr < \infty, \quad \forall a \in \boldsymbol{R}_+, \forall D \in \mathfrak{D}_{t,r}.$$

(Here $\|t\|$ denotes a norm on $\boldsymbol{R}^{m+v}$.)

Remark that this space is independent of the choice of coexponential bases (see [**6**]).

2.1. $\mathscr{S}\mathscr{E}$-space and C^∞ vectors.

We shall define an exponential solvable group $F \supset G$ such that its Lie algebra $\mathfrak{f}$ is of the form $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}$, where $\mathfrak{a}$ is an abelian ideal and $[\mathfrak{n}+\mathfrak{h}, \mathfrak{a}] = \{0\}$. We also show that any linear functional l_0 of $\mathfrak{f}$ whose restriction to $\mathfrak{g}$ equals l satisfies the condition $\dim(\mathfrak{f}(l_0)) = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a})$, where $\mathfrak{f}(l_0) = \{X \in \mathfrak{f};\ l_0([X,\mathfrak{f}]) = \{0\}\}$, which implies that $G \cdot l_0 = F \cdot l_0$, and show that $\mathfrak{p} := \mathfrak{h} + \mathfrak{a}$ is a polarization at l_0 with the Pukanszky condition.

For every l_0, we have that the restriction $\pi_{l_0,P|G}$ of $\pi_{l_0,P}$ to G and $\pi_{l,H}$ are equivalent; the G-equivariant unitary mapping $R_{l_0} : \mathscr{H}_{\pi_{l_0,P}} \to \mathscr{H}_{\pi_{l,H}}$

$$R_{l_0}\phi = \phi_{|G}$$

is a unitary intertwining operator and its inverse S_{l_0} is given by

$$S_{l_0} : \mathscr{H}_{\pi_{l,H}} \to \mathscr{H}_{\pi_{l_0,P}}, \quad S_{l_0}\phi(g \exp A) := e^{-il_0(A)}\phi(g),\ g \in G, A \in \mathfrak{a}.$$

We obtain a new set of norms on the space $\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})$ by letting for every element $U \in \mathscr{U}(\mathfrak{f})$,

$$\|\phi\|_{l_0,U} := \|d\pi_{l_0,P}(U)S_{l_0}\phi\|_{\pi_{l_0}}.$$

It is easy to see that for every $U \in \mathscr{U}(\mathfrak{f})$, we have $a \in \boldsymbol{R}_+$ and an element $D \in \mathfrak{D}_{t,r}$ such that

$$\|\phi\|_{l_0,U} \le \|\phi\|_{a,D}, \quad \text{for all } \phi \in \mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}).$$

Indeed, if we use the coordinates (t,r) for G/H, then for any $X \in \mathscr{U}(\mathfrak{g})$ we have that $d\pi_{l,H}(X)$ is a differential operator with coefficients which are bounded by $e^{a\|t\|}(1+\|r\|)^k$ for some $a, k \in \boldsymbol{R}_+$. This shows that

$$S_{l_0}(\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})) \subset \mathscr{H}^\infty_{\pi_{l_0,P}}. \tag{2.2}$$

THEOREM 2.4. *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group, $\mathfrak{n}$ be a nilpotent ideal such that $\mathfrak{n} \supset [\mathfrak{g},\mathfrak{g}]$, $l \in \mathfrak{g}^*$, and $\mathfrak{h}$ be a polarization at l adapted to $\mathfrak{n}$. Then there exists an exponential solvable Lie group F with Lie algebra $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}$ which satisfies the following*:

(1) *$\mathfrak{a}$ is an abelian ideal of dimension $2m = 2\dim(\mathfrak{g}/(\mathfrak{n}+\mathfrak{h}))$ and $[\mathfrak{n}+\mathfrak{h},\mathfrak{a}] = \{0\}$, and there exists a coexponential basis $\{X_j\}_{1\le j\le m}$ for $\mathfrak{n}+\mathfrak{h}$ in $\mathfrak{g}$ and a basis $\{A_1,\cdots,A_m,B_1,\cdots,B_m\}$ of $\mathfrak{a}$ such that*

$$[X_j, A_k] = \delta_{j,k}A_k, \quad [X_j, B_k] = -\delta_{j,k}B_k, \quad 1 \le j,k \le m.$$

(2) *For all extension $l_1 \in \mathfrak{f}^*$ of l, we have $\dim(\mathfrak{f}(l_1)) = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a})$, and the subalgebra $\mathfrak{p} = \mathfrak{h} + \mathfrak{a}$ is a Pukanszky polarization at l_1 adapted to $\mathfrak{n}+\mathfrak{a}$.*

(3) *There exists an extension* $l_0 \in \mathfrak{f}^*$ *of* l *such that the family of norms* $\{\| \ \|_{a,D}, a \in \boldsymbol{R}_+, D \in \mathfrak{D}_{t,r}\}$ *is equivalent to the family of norms* $\{\| \ \|_{l_0,U},\ U \in \mathscr{U}(\mathfrak{f})\}$ *and we have that*

$$\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}) = R_{l_0}(\mathscr{H}^\infty_{\pi_{l_0},P}),$$

where $P = \exp \mathfrak{p}$.

PROOF. By (2.2), we have only to show that $\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}) \supset R_{l_0}(\mathscr{H}^\infty_{\pi_{l_0},P})$. We make an induction on the dimension of G. If $\mathfrak{g}$ is abelian or $\mathfrak{n} = \mathfrak{g}$, the statement is trivial. Suppose that $l = 0$ on an abelian ideal $\mathfrak{i} \neq \{0\}$. Then $\mathfrak{h} \supset \mathfrak{i}$. Let $\dot{\mathfrak{g}} = \mathfrak{g}/\mathfrak{i}$, $\dot{\mathfrak{n}} = (\mathfrak{n}+\mathfrak{i})/\mathfrak{i}$, $\dot{\mathfrak{h}} = \mathfrak{h}/\mathfrak{i}$, $\dot{G} = \exp \dot{\mathfrak{g}} = G/I$, $I = \exp \mathfrak{i}$. Then, denoting quotient maps by $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$, $Q : G \to G/I$, we have $\dot{\pi} \in \widehat{\dot{G}}$ such that $\dot{\pi} \circ Q = \pi$, and we have $\dot{\pi} = \mathrm{ind}_{\dot{H}}^{\dot{G}} \chi_{\dot{l}}$, where $\dot{l} \circ q = l$. By the induction hypothesis for $(\dot{G},\dot{\mathfrak{n}},\dot{l},\dot{\mathfrak{h}})$, there exist an exponential solvable Lie group $\dot{F} = \exp \dot{\mathfrak{f}}$, $\dot{\mathfrak{f}} = \dot{\mathfrak{g}} \ltimes \dot{\mathfrak{a}}$ and an extension $\dot{l}_0 \in \dot{\mathfrak{f}}^*$ of $\dot{l}$ with the required properties.

Let $\mathfrak{f} = \mathfrak{g} \ltimes \dot{\mathfrak{a}}$ defined by $[X, \dot{A}] := [q(X), \dot{A}]$ for $X \in \mathfrak{g}$, $\dot{A} \in \dot{\mathfrak{a}}$, and an extension $l_0 \in \mathfrak{f}^*$ of l be defined by $l_0|_{\dot{\mathfrak{a}}} = \dot{l}_0$. Then we have that $\mathfrak{f}$ and l_0 has the required properties for $(G,\mathfrak{n},l,\mathfrak{h})$.

Suppose $l \neq 0$ on any non-zero abelian ideal. Let $\mathfrak{g}_1$ be a minimal ideal contained in $\mathfrak{n}$.

Then there are following possibilities (see [**6**]):

(1) $\mathfrak{g}_1$ is non-central. Then $\dim(\mathfrak{g}_1) = 1$ or 2:

a) There exist $Y \in \mathfrak{g}_1$, $\lambda \in \mathfrak{g}^*$, and $X \in \mathfrak{g}^*$ such that $\mathfrak{g}_1 = \boldsymbol{R}Y$, $l(Y) = 1$,

$$[U,Y] = \lambda(U)Y \text{ for all } U \in \mathfrak{g},$$
$$\lambda(X) = 1.$$

b) There exist $Y_1, Y_2 \in \mathfrak{g}_1$, $\lambda \in \mathfrak{g}^*$, $\omega \in \boldsymbol{R} \setminus \{0\}$ and $X \in \mathfrak{g}^*$ such that $l(Y_1) \neq 0$, $\mathfrak{g}_1 = \boldsymbol{R}Y_1 \oplus \boldsymbol{R}Y_2$, and

$$[U,Y_1] = \lambda(U)(Y_1 - \omega Y_2), \quad [U,Y_2] = \lambda(U)(\omega Y_1 + Y_2) \quad \text{for all } U \in \mathfrak{g}^*,$$
$$\lambda(X) = 1.$$

(2) $\mathfrak{g}_1$ is the center of $\mathfrak{g}$. Then $\mathfrak{g}_1$ is one dimensional because of the assumption of l. Let $Z \in \mathfrak{g}_1$ such that $l(Z) = 1$.

(2-1) Suppose first that $\mathfrak{g}_1$ is properly contained in $\mathfrak{n}$. Let $\mathfrak{g}_2$ be a minimal ideal modulo $\mathfrak{g}_1$ such that $\mathfrak{g}_2 \subset \mathfrak{n}$. Then

a) $\mathfrak{g}_2$ is two-dimensional and there exist $Y \in \ker(l) \cap \mathfrak{g}_2$, $X \in [\mathfrak{g},\mathfrak{g}] \cap \ker(l)$, $T \in \ker(l)$, $\lambda, \gamma \in \mathfrak{g}^*$ such that

$$[U,Y] = \lambda(U)Y + \gamma(U)Z \quad \text{for all } U \in \mathfrak{g}, \tag{2.3}$$
$$\lambda(T) = 1,\ \lambda(X) = 0,\ \gamma(T) = 0,\ \gamma(X) = 1.$$

Then we have $[T,X]\in -X+(\ker(\lambda)\cap\ker(\gamma))$, and $\ker(\gamma)+\mathfrak{n}=\mathfrak{g}$.

b) $\mathfrak{g}_2$ is two-dimensional and there exist $Y\in\ker(l)\cap\mathfrak{g}_2,\ X\in\ker(l),\gamma\in\mathfrak{g}^*$ such that

$$[U,Y]=\gamma(U)Z \quad \text{for all } U\in\mathfrak{g}, \qquad (2.4)$$
$$\gamma(X)=1.$$

Then we have two subcases:

b-1) $\mathfrak{n}+\ker(\gamma)=\mathfrak{g}$.

b-2) $\mathfrak{n}+\ker(\gamma)\neq\mathfrak{g}$ (here λ is necessarily 0).

c) $\mathfrak{g}_2$ is 3-dimensional and there exist $Y_1,Y_2\in\mathfrak{g}_2\cap\ker(l),\ X_1,X_2\in[\mathfrak{g},\mathfrak{g}]\cap\ker(l), T\in\ker(l),\ \lambda,\gamma_1,\gamma_2\in\mathfrak{g}^*,\omega\in\boldsymbol{R}^*$, such that for all $U\in\mathfrak{g}$

$$\begin{aligned}[U,Y_1]&=\lambda(U)(Y_1-\omega Y_2)+\gamma_1(U)Z,\\ [U,Y_2]&=\lambda(U)(\omega Y_1+Y_2)+\gamma_2(U)Z,\end{aligned} \qquad (2.5)$$

$$\gamma_j(X_i)=\delta_{i,j},\ i,j=1,2,\quad \gamma_1(T)=\gamma_2(T)=0,$$
$$0=\lambda(X_1)=\lambda(X_2),\ \lambda(T)=1.$$

Then we have $\mathfrak{n}+\ker(\gamma_1)\cap\ker(\gamma_2)=\mathfrak{g}$.

(2-2) Suppose now that $\boldsymbol{R}Z=\mathfrak{g}_1=\mathfrak{n}$. Then $\mathfrak{n}^l=\mathfrak{g}$. Since the center is one dimensional, our $\mathfrak{g}$ is the Heisenberg algebra, if $\mathfrak{g}$ is not abelian, which we assume. We can take a basis $\{X_1,\cdots,X_n,Y_1,\cdots,Y_n,Z\}$ such that $[X_i,Y_j]=\delta_{i,j}Z$ $(i,j=1,\cdots,n)$ and so that $\mathfrak{h}$ is spanned by $\{Y_1,\cdots,Y_n,Z\}$.

CASE (1): Let $\mathfrak{k}:=\ker(\lambda)$. Then $\mathfrak{k}$ is an ideal, $\mathfrak{k}\supset\mathfrak{n}+\mathfrak{n}^l$, and $\mathfrak{g}_1\subset\mathfrak{h}\subset\mathfrak{k}$. We have $\pi=\mathrm{ind}_K^G\pi_1$, where $\pi_1=\mathrm{ind}_H^K\chi_l$, $K=\exp\mathfrak{k}$.

By the induction hypothesis for $(K,\mathfrak{n},l|_{\mathfrak{k}},\mathfrak{h})$, there exists $\tilde{F}=\exp\tilde{\mathfrak{f}}$ such that $\mathfrak{k}\ltimes\tilde{\mathfrak{a}}=\tilde{\mathfrak{f}}$, where $\tilde{\mathfrak{a}}$ is an abelian ideal such that $[\mathfrak{n}+\mathfrak{h},\tilde{\mathfrak{a}}]=\{0\}$, with the required properties: For all extension $l_1\in\tilde{\mathfrak{f}}^*$ of $l|_{\mathfrak{k}}$, we have $\dim(\tilde{\mathfrak{f}}(l_1))=\dim(\mathfrak{k}(l|_{\mathfrak{k}}))+\dim(\tilde{\mathfrak{a}})$, and the subalgebra $\tilde{\mathfrak{p}}=\mathfrak{h}+\tilde{\mathfrak{a}}$ is a Pukanszky polarization at l_1. And there exists an extension $\tilde{l}\in\tilde{\mathfrak{f}}^*$ of $l|_{\mathfrak{k}}$ such that the corresponding family of norms are equivalent and such that

$$\mathscr{S}\mathscr{E}(K,\mathfrak{n},l|_{\mathfrak{k}},\mathfrak{h})=R_{\tilde{l}}(\mathscr{H}^{\infty}_{\pi_{\tilde{l},\tilde{P}}}).$$

We have in case (1) a) that $[X,Y]=Y$, and in case (1) b) that $[X,Y_1]=Y_1-\omega Y_2$, $[X,Y_2]=\omega Y_1+Y_2$, and in both cases that $[\mathfrak{g}_1,\mathfrak{k}]=\{0\}$.

Let $\mathfrak{a}=\tilde{\mathfrak{a}}\oplus\boldsymbol{R}A\oplus\boldsymbol{R}B$ be an abelian Lie algebra, and $\mathfrak{f}=\mathfrak{g}\ltimes\mathfrak{a}$ defined by

$$\mathfrak{f}=\mathfrak{g}\oplus\mathfrak{a}=\boldsymbol{R}X\oplus\mathfrak{k}\oplus\tilde{\mathfrak{a}}\oplus\boldsymbol{R}A\oplus\boldsymbol{R}B,$$
$$[A,\tilde{\mathfrak{f}}]=[B,\tilde{\mathfrak{f}}]=\{0\},\ [X,A]=A,\ [X,B]=-B,\ [\tilde{\mathfrak{a}},X]=\{0\}.$$

Let $\mathfrak{e} = \tilde{\mathfrak{f}} \oplus \boldsymbol{R}A \oplus \boldsymbol{R}B = \mathfrak{k} \oplus \tilde{\mathfrak{a}} \oplus \boldsymbol{R}A \oplus \boldsymbol{R}B$, $E = \exp \mathfrak{e}$, $F = \exp \mathfrak{f}$.

By the assumption of l and $[\mathfrak{g}_1, \mathfrak{k}] = \{0\}$, we have that $\dim(\mathfrak{g}_1/(\mathfrak{g}(l) \cap \mathfrak{g}_1)) = 1$ and $\dim(\mathfrak{g}(l)) + 1 = \dim(\mathfrak{k}(l|_{\mathfrak{k}}))$. Let $l_0 \in \mathfrak{f}^*$ be an extension of l and $l_1 = l_0|_{\tilde{\mathfrak{f}}}$. We also have that $\dim(\mathfrak{g}_1/(\mathfrak{f}(l_0) \cap \mathfrak{g}_1)) = 1$ and $\dim(\mathfrak{f}(l_0)) = \dim(\tilde{\mathfrak{f}}(l_1)) + 1$. In fact, suppose first that $l_0|_{\boldsymbol{R}A + \boldsymbol{R}B} \neq 0$. If $l_0(A) l_0(B) \neq 0$ and $C = \alpha A + \beta B \in \ker(l_0) \setminus \{0\}$, where $\alpha, \beta \in \boldsymbol{R}$, then $C' := \alpha A - \beta B \notin \ker(l_0)$, $[X, C'] = C$, $[X, C] = C'$, and the mapping $\tilde{\mathfrak{f}}(l_1) \oplus \boldsymbol{R}C' \ni V \mapsto V - ((l_0([X, V]))/(l_0(C')))C \in \mathfrak{f}(l_0)$ gives a linear isomorphism of $\tilde{\mathfrak{f}}(l_1) \oplus \boldsymbol{R}C'$ and $\mathfrak{f}(l_0)$. If $l_0(A) = 0$ and $l_0(B) \neq 0$, then taking $Y_0 \in \mathfrak{g}_1 \setminus \mathfrak{f}(l_0)$, we have $\tilde{\mathfrak{f}}(l_1) \oplus \boldsymbol{R}A \oplus \boldsymbol{R}B = \mathfrak{f}(l_0) \oplus \boldsymbol{R}Y_0$. Similarly, if $l_0|_{\boldsymbol{R}A + \boldsymbol{R}B} = 0$, then $\mathfrak{f}(l_0) \supset \boldsymbol{R}A \oplus \boldsymbol{R}B$ and taking $Y_0 \in \mathfrak{g}_1 \setminus \mathfrak{f}(l_0)$, we have $\tilde{\mathfrak{f}}(l_1) \oplus \boldsymbol{R}A \oplus \boldsymbol{R}B = \mathfrak{f}(l_0) \oplus \boldsymbol{R}Y_0$. Since $\dim(\tilde{\mathfrak{f}}(l_1)) = \dim(\mathfrak{k}(l|_k)) + \dim(\tilde{\mathfrak{a}})$, we have

$$\begin{aligned} \dim(\mathfrak{f}(l_0)) &= \dim(\tilde{\mathfrak{f}}(l_1)) + 1 = \dim(\mathfrak{k}(l|_{\mathfrak{k}})) + \dim(\tilde{\mathfrak{a}}) + 1 \\ &= \dim(\mathfrak{k}(l|_{\mathfrak{k}})) + \dim(\mathfrak{a}) - 1 = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a}). \end{aligned}$$

Since $\tilde{\mathfrak{p}} = \mathfrak{h} + \tilde{\mathfrak{a}}$ is a Pukanszky polarization at $l_0|_{\tilde{\mathfrak{f}}}$ adapted to $\mathfrak{n} + \tilde{\mathfrak{a}}$ and $\boldsymbol{R}A \oplus \boldsymbol{R}B$ is central in $\mathfrak{e}$, we also have that $\mathfrak{p} = \tilde{\mathfrak{p}} \oplus \boldsymbol{R}A \oplus \boldsymbol{R}B$ is a Pukanszky polarization at l_0 adapted to $\mathfrak{n} + \mathfrak{a}$. Letting l_0 be an extension of $\tilde{l}$ such that $l_0(B) \neq 0$, and $\tilde{\pi} = \pi_{\tilde{l}, \tilde{P}} = \operatorname{ind}_{\tilde{P}}^{\tilde{F}} \chi_{\tilde{l}}$, we realize $\tau = \tau_{l_0} = \operatorname{ind}_P^E \chi_{l_0}$ in $\mathscr{H}_{\tilde{\pi}}$ by $\tau(\tilde{x}a)v = \chi_{l_0}(a)\tilde{\pi}(\tilde{x})v$ for $v \in \mathscr{H}_{\tilde{\pi}}$, $\tilde{x} \in \tilde{F}$, $a \in \exp(\boldsymbol{R}A + \boldsymbol{R}B)$.

Now, we realize $\pi_{l_0,P} = \operatorname{ind}_P^F \chi_{l_0}$ as $\operatorname{ind}_E^F \tau_{l_0}$ on $L^2(\boldsymbol{R}, \mathscr{H}_{\tilde{\pi}})$. Then for $\phi = \phi(x) \in L^2(\boldsymbol{R}, \mathscr{H}_{\tilde{\pi}})$, we have in case (1) a)

$$\begin{aligned} & d\pi_{l_0,P}(X)\phi(x) = -\frac{d}{dx}\phi(x), \\ & d\pi_{l_0,P}(A)\phi(x) = il_0(A)e^{-x}\phi(x), \\ & d\pi_{l_0,P}(B)\phi(x) = il_0(B)e^{x}\phi(x), \\ & d\pi_{l_0,P}(Y)\phi(x) = ie^{-x}\phi(x), \\ & d\pi_{l_0,P}(V)\phi(x) = d\tilde{\pi}(\operatorname{Ad}(\exp(-xX))V)(\phi(x)), \quad V \in \tilde{\mathfrak{f}} \end{aligned} \tag{2.6}$$

and in case (1) b)

$$\begin{aligned} & d\pi_{l_0,P}(X)\phi(x) = -\frac{d}{dx}\phi(x), \\ & d\pi_{l_0,P}(A)\phi(x) = il_0(A)e^{-x}\phi(x), \\ & d\pi_{l_0,P}(B)\phi(x) = il_0(B)e^{x}\phi(x), \\ & d\pi_{l_0,P}(Y_1)\phi(x) = ie^{-x}(l(Y_1)\cos(\omega x) + l(Y_2)\sin(\omega x))\phi(x), \\ & d\pi_{l_0,P}(Y_2)\phi(x) = ie^{-x}(-l(Y_1)\sin(\omega x) + l(Y_2)\cos(\omega x))\phi(x), \\ & d\pi_{l_0,P}(V)\phi(x) = d\tilde{\pi}(\operatorname{Ad}(\exp(-xX))V)(\phi(x)), \quad V \in \tilde{\mathfrak{f}}. \end{aligned} \tag{2.7}$$

Since we can regard

$$G/H = (G/K)(K/H) = (G/K)(K/NH)(NH/H),$$

we can use the coordinates t, r for K/H, and for G/K we use the coordinate x. We show that for $a \in \boldsymbol{R}$ and $D = \frac{\partial^n}{\partial x^n} \otimes D_{t,r},\ D_{t,r} \in \mathfrak{D}_{t,r}$ there exists a finite family $\{U_1, \cdots, U_N\}$ in $\mathscr{U}(\mathfrak{f})$, such that $\| \ \|_{a,D} \leq \sum_{j=1}^{N} \| \ \|_{l_0,U_j}$. Indeed, by the induction hypothesis, there exists a finite family $\{\tilde{U}_1, \cdots, \tilde{U}_{\tilde{N}}\}$ in $\mathscr{U}(\tilde{\mathfrak{f}})$, such that

$$\begin{aligned}
&\int_{K/H} e^{a\|t\|} |D_{t,r}\phi(\exp(xX)E(t,r))|^2 dtdr \\
&\leq \left(\sum_{j=1}^{\tilde{N}} \left(\int_{\tilde{F}/\tilde{P}} |d\tilde{\pi}(\tilde{U}_j) S_{\tilde{l}}\phi(\exp{(xX)}k)|^2 d\mu_{\tilde{F}/\tilde{P}}(k) \right)^{1/2} \right)^2 \\
&\leq \tilde{N}^2 \sup_{j=1,\cdots,\tilde{N}} \int_{\tilde{F}/\tilde{P}} |d\tilde{\pi}(\tilde{U}_j) S_{\tilde{l}}\phi(\exp{(xX)}k)|^2 d\mu_{\tilde{F}/\tilde{P}}(k)
\end{aligned}$$

for all $\phi \in \mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$.

Let d_i be the degree of $\tilde{U}_i$ in $\mathscr{U}(\tilde{\mathfrak{f}})$ and let $\{V_1^i, \cdots, V_{M_i}^i\}$ be a basis of $\mathscr{U}(\tilde{\mathfrak{f}})_{d_i}$, the subspace of $\mathscr{U}(\tilde{\mathfrak{f}})$ consisting of the elements of degree $\leq d_i$. Then $\mathrm{Ad}(\exp(xX))\tilde{U}_i = \sum_{j=1}^{M_i} \psi_j^i(x) V_j^i,\ x \in \boldsymbol{R}$, where the functions ψ_j^i are C^∞ and are bounded by exponential functions. Therefore

$$\tilde{U}_i = \mathrm{Ad}(\exp(-xX))(\mathrm{Ad}(\exp(xX))\tilde{U}_i) = \sum_{j=1}^{M_i} \psi_j^i(x) \mathrm{Ad}(\exp(-xX)) V_j^i \tag{2.8}$$

and so

$$\begin{aligned}
&\|\phi\|_{a,c,D}^2 \\
&\quad = \int_{\boldsymbol{R}} \int_{K/H} e^{c|x|} e^{a\|t\|} \left| \frac{\partial^n}{\partial x^n} D_{t,r}\phi(\exp(xX)E(t,r)) \right|^2 dxdtdr \\
&\quad \leq \int_{\boldsymbol{R}} e^{c|x|} \tilde{N}^2 \sup_{i=1,\cdots,\tilde{N}} \int_{\tilde{F}/\tilde{P}} \left| d\tilde{\pi}(\tilde{U}_i) S_{\tilde{l}} \frac{\partial^n}{\partial x^n} \phi(\exp(xX)k) \right|^2 d\mu_{\tilde{F}/\tilde{P}}(k) dx \\
&\quad \leq \int_{\boldsymbol{R}} \tilde{N}^2 e^{c|x|} \\
&\sum_{i=1}^{\tilde{N}} M_i^2 \int_{\tilde{F}/\tilde{P}} |\psi_j^i(x)|^2 \sum_{j=1}^{M_i} \left| d\tilde{\pi}(\mathrm{Ad}(\exp(-xX))V_j^i) S_{\tilde{l}} \frac{\partial^n}{\partial x^n} \phi(\exp(xX)k) \right|^2 d\mu_{\tilde{F}/\tilde{P}}(k) dx \\
&\quad \leq \tilde{N}^2 \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{M_i} M_i^2 \\
&\int_{\boldsymbol{R}} \int_{\tilde{F}/\tilde{P}} e^{c|x|} |\psi_j^i(x)|^2 \left| d\tilde{\pi}(\mathrm{Ad}(\exp(-xX))V_j^i) S_{\tilde{l}} \frac{\partial^n}{\partial x^n} \phi(\exp{(xX)}k) \right|^2 d\mu_{\tilde{F}/\tilde{P}}(k) dx \\
&\quad \leq \tilde{N}^2 \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{M_i} M_i^2 \int_{\boldsymbol{R}} \int_{\tilde{F}/\tilde{P}} C_{i,j} e^{\alpha_{i,j}|x|} \left| d\pi_{l_0,P}(V_j^i) \frac{\partial^n}{\partial x^n} S_{l_0}\phi(\exp{(xX)}k) \right|^2 d\mu_{\tilde{F}/\tilde{P}}(k) dx
\end{aligned}$$

with some constant $C_{i,j}, \alpha_{i,j} \in \boldsymbol{R}_+$. It follows now from the formulas in (2.6) and (2.7), that there exists a finite family $U_1, \cdots, U_N$ in $\mathscr{U}(\mathfrak{f})$, such that

$$\| \ \|_{a,c,D} \leq \sum_{j=1}^{N} \| \ \|_{l_0,U_j}$$

and so

$$\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}) = R_{l_0}(\mathscr{H}^\infty_{\pi_{l_0,P}}).$$

CASE (2-1) a), b), c): $\mathfrak{g}_1 \neq \mathfrak{n}$. Let $\mathfrak{k} = \ker(\gamma)$ in case a) and b), resp. $\mathfrak{k} = \ker(\gamma_1) \cap \ker(\gamma_2)$ in case c), and $\mathfrak{k}_0 = \{U \in \mathfrak{g}; [U, \mathfrak{g}_2] = \{0\}\}$. We remark that $\mathfrak{g}_2 \cap \mathfrak{g}(l) = \mathfrak{g}_1$ and $\mathfrak{k}(l|_\mathfrak{k}) = \mathfrak{g}(l) + \mathfrak{g}_2$ because of our assumption. Thus we have $\dim(\mathfrak{k}(l|_\mathfrak{k})) = \dim(\mathfrak{g}(l)) + 1$ in cases a) and b), resp. $\dim(\mathfrak{k}(l|_\mathfrak{k})) = \dim(\mathfrak{g}(l)) + 2$ in case c).

We have two possibilities: either i): $\mathfrak{g}_2 \subset \mathfrak{h}$, or ii): $\mathfrak{g}_2 \not\subset \mathfrak{h}$.

CASE (2-1) a), b), c); i): We begin with case i). Then $\mathfrak{h}$ must be contained in $\mathfrak{k}$. If not, there exists $X' \in \mathfrak{h} \setminus \mathfrak{k}$. But then $X' = \alpha T + \beta X + X_0$ where $\beta \in \boldsymbol{R}$, $\beta \neq 0$, $X_0 \in \mathfrak{k}_0$ (in case a), $X' = \alpha X + X_0$ with $X_0 \in \mathfrak{k}$, $\alpha \neq 0$ (in case b), $X' = \alpha T + \beta X_1 + \delta X_2 + X_0$ where $\alpha, \beta, \delta \in \boldsymbol{R}$, $\beta^2 + \delta^2 \neq 0$, $X_0 \in \mathfrak{k}_0$ (in case c) and so by (2.3), (2.4) and (2.5),

$$\{0\} = l([X', \mathfrak{g}_2]) = l([X', \boldsymbol{R}Y_1 + \boldsymbol{R}Y_2]) = l((\beta \boldsymbol{R} + \delta \boldsymbol{R})Z) \neq \{0\},$$

since $\mathfrak{g}_2 \subset \mathfrak{h}$ in case c) and similarly in the other two cases. This contradiction tells us that $\mathfrak{h} \subset \mathfrak{k}$.

Hence we have $\pi = \mathrm{ind}_K^G(\mathrm{ind}_H^K \chi_l)$, and by the induction hypothesis for $(K, \mathfrak{k} \cap \mathfrak{n}, l|_\mathfrak{k}, \mathfrak{h})$, there exists $\tilde{\mathfrak{f}} = \mathfrak{k} \ltimes \tilde{\mathfrak{a}}$ such that $[\tilde{\mathfrak{a}}, \tilde{\mathfrak{a}}] = \{0\}$, $[\tilde{\mathfrak{a}}, (\mathfrak{k} \cap \mathfrak{n}) + \mathfrak{h}] = \{0\}$, and having the required properties: $\dim(\tilde{\mathfrak{f}}(l_1)) = \dim(\tilde{\mathfrak{a}}) + \dim(\mathfrak{k}(l|_\mathfrak{k}))$ holds for any extension $l_1 \in \tilde{\mathfrak{f}}^*$ of $l|_\mathfrak{k}$, the subalgebra $\tilde{\mathfrak{p}} = \mathfrak{h} + \tilde{\mathfrak{a}}$ is a polarization at $\tilde{l}$, and there exists an extension $\tilde{l}$ such that

$$\mathscr{S}\mathscr{E}(K, \mathfrak{k} \cap \mathfrak{n}, l|_\mathfrak{k}, \mathfrak{h}) = R_{\tilde{l}}(\mathscr{H}^\infty_{\pi_{\tilde{l},\tilde{P}}}).$$

We first treat case a), b-1), and c). Recalling $\mathfrak{g} = \boldsymbol{R}X \oplus \mathfrak{k}$ in case a) and b-1), resp. $\mathfrak{g} = \boldsymbol{R}X_1 \oplus \boldsymbol{R}X_2 \oplus \mathfrak{k}$, in case c), we define $\mathfrak{a} = \tilde{\mathfrak{a}}$, and $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}$ by $[\boldsymbol{R}X, \mathfrak{a}] = \{0\}$, resp. $[\boldsymbol{R}X_1 + \boldsymbol{R}X_2, \mathfrak{a}] = \{0\}$. Let $\mathfrak{p} = \mathfrak{h} + \mathfrak{a}$ $(= \tilde{\mathfrak{p}})$. Then, for an extension $l_0 \in \mathfrak{f}^*$ of l and $l_1 = l_0|_{\tilde{\mathfrak{f}}}$, we have $\dim(\tilde{\mathfrak{f}}(l_1)) = \dim(\mathfrak{f}(l_0)) + 1$ in case a) and b-1) and $\dim(\tilde{\mathfrak{f}}(l_1)) = \dim(\mathfrak{f}(l_0)) + 2$ in case c), and, by the induction hypothesis $\dim(\tilde{\mathfrak{f}}(l_1)) = \dim(\mathfrak{a}) + \dim(\mathfrak{k}(l|_\mathfrak{k}))$, we have

$$\dim(\mathfrak{f}(l_0)) = \dim(\tilde{\mathfrak{f}}(l_1)) - 1 = \dim(\mathfrak{a}) + \dim(\mathfrak{k}(l|_\mathfrak{k})) - 1 = \dim(\mathfrak{a}) + \dim(\mathfrak{g}(l))$$

in a) and b),

$$\dim(\mathfrak{f}(l_0)) = \dim(\tilde{\mathfrak{f}}(l_1)) - 2 = \dim(\mathfrak{a}) + \dim(\mathfrak{k}(l|_\mathfrak{k})) - 2 = \dim(\mathfrak{a}) + \dim(\mathfrak{g}(l))$$

in case c). Thus $\mathfrak{p}$ is a polarization at l_0, and $\mathfrak{p}$ is adapted to $\mathfrak{n}+\mathfrak{a}$.

Let l_0 be an extension of $\tilde{l}$, and we realize in case a) and b-1), $\pi_{l_0,P}=\mathrm{ind}_P^F\chi_{l_0}$ as $\mathrm{ind}_{\tilde{F}}^F\tilde{\pi}$, where $\tilde{\pi}=\mathrm{ind}_P^{\tilde{F}}\chi_{l_0}$, on $L^2(\boldsymbol{R},\mathscr{H}_{\tilde{\pi}})$. For $\phi=\phi(x)\in C^\infty(\boldsymbol{R},\mathscr{H}_{\tilde{\pi}})$, we have

$$d\pi_{l_0,P}(X)\phi(x)=-\frac{d}{dx}\phi(x), \tag{2.9}$$

$$d\pi_{l_0,P}(T)\phi(x)=\left(\frac{1}{2}+x\frac{d}{dx}+d\tilde{\pi}(T+P(x))\right)\phi(x), \tag{2.10}$$

$$d\pi_{l_0,P}(V)\phi(x)=d\tilde{\pi}(\mathrm{Ad}(\exp(-xX))V)(\phi(x)),\quad V\in\mathfrak{k}_0+\mathfrak{a}, \tag{2.11}$$

$$d\pi_{l_0,P}(Y)\phi(x)=-ix(\phi(x)), \tag{2.12}$$

where $P(x)$ is a $\mathfrak{k}_0$-valued polynomial in x, in case a). In case b-1) we have (2.9), (2.11) and (2.12). In case c) we have $G=\exp\boldsymbol{R}X_1\exp\boldsymbol{R}X_2K$ and we realize $\pi_{l_0,P}=\mathrm{ind}_P^F\chi_{l_0}$ as $\mathrm{ind}_{\tilde{F}}^F\tilde{\pi}$, where $\tilde{\pi}=\mathrm{ind}_P^{\tilde{F}}\chi_{l_0}$, on $L^2(\boldsymbol{R}^2,\mathscr{H}_{\tilde{\pi}})$. For $\phi=\phi(x_1,x_2)\in C^\infty(\boldsymbol{R}^2,\mathscr{H}_{\tilde{\pi}})$, we have

$$d\pi_{l_0,P}(X_1)\phi(x_1,x_2)=-\frac{\partial}{\partial x_1}\phi(x_1,x_2),$$
$$d\pi_{l_0,P}(X_2)\phi(x_1,x_2)=-\frac{\partial}{\partial x_2}\phi(x_1,x_2)+d\tilde{\pi}(Q(x_1,x_2))\phi(x_1,x_2),$$
$$d\pi_{l_0,P}(T)\phi(x_1,x_2)=\left(1+(x_1-\omega x_2)\frac{\partial}{\partial x_1}+(x_1\omega+x_2)\frac{\partial}{\partial x_2}+d\tilde{\pi}(T+R(x_1,x_2))\right)\phi(x_1,x_2),$$
$$d\pi_{l_0,P}(V)\phi(x_1,x_2)=d\tilde{\pi}(\mathrm{Ad}(\exp(-x_2X_2)\exp(-x_1X_1))V)(\phi(x_1,x_2)),\quad V\in\mathfrak{k}_0+\mathfrak{a},$$
$$d\pi_{l_0,P}(Y_1)\phi(x_1,x_2)=-ix_1(\phi(x_1,x_2)),$$
$$d\pi_{l_0,P}(Y_2)\phi(x_1,x_2)=-ix_2(\phi(x_1,x_2)),$$

where $Q(x_1,x_2),R(x_1,x_2)$ are $\mathfrak{k}_0$-valued polynomial in x_1,x_2. Remarking that $\mathrm{Ad}(\exp(-x_2X_2)\exp(-x_1X_1))V$ is also a polynomial in x_1,x_2 since $X_1,X_2\in[\mathfrak{g},\mathfrak{g}]$, we can show similarly as in case (1), that the family of norms $\{\|\ \|_{a,D},a\in\boldsymbol{R}_+,D\in\mathfrak{D}_{t,r}\}$ is equivalent to the family of norms $\{\|\ \|_{l_0,U},U\in\mathscr{U}(\mathfrak{f})\}$ and so we have that in case a), case b-1) and c)

$$\phi\in R_{l_0}(\mathscr{H}^\infty_{\pi_{l_0,P}})\Longleftrightarrow\phi\in\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}).$$

We next treat case b-2). We define $\mathfrak{a}=\tilde{\mathfrak{a}}+\boldsymbol{R}A+\boldsymbol{R}B$ and $\mathfrak{f}=\mathfrak{g}\ltimes\mathfrak{a}$ with

$$[X,A]=A,[X,B]=-B,[A,\tilde{\mathfrak{f}}]=[B,\tilde{\mathfrak{f}}]=\{0\},$$

and $\mathfrak{p}=\mathfrak{h}+\mathfrak{a}=\tilde{\mathfrak{p}}+\boldsymbol{R}A+\boldsymbol{R}B$. Let $l_0\in\mathfrak{f}^*$ be an extension of l. Then it can be deduced as in case (1) that $\dim(\mathfrak{f}(l_0))=\dim(\mathfrak{g}(l))+\dim(\mathfrak{a})$ and $\mathfrak{p}$ is a polarization at l_0 adapted to $\mathfrak{n}+\mathfrak{a}$. Let $\mathfrak{e}=\tilde{\mathfrak{f}}\oplus\boldsymbol{R}A\oplus\boldsymbol{R}B$, and $E=\exp\mathfrak{e}$. We take an extension l_0 of $\tilde{l}$ such that $l_0(A)\neq 0$ and $l_0(B)\neq 0$, and realize $\tau=\mathrm{ind}_P^E\chi_{l_0}$ in $\mathscr{H}_{\tilde{\pi}}$ by $\tau(ka)\phi=\chi_{l_0}(a)\tilde{\pi}(k)\phi$ for $\phi\in\mathscr{H}_{\tilde{\pi}}$, $k\in K$, $a\in\exp(\boldsymbol{R}A+\boldsymbol{R}B)$. As in case (1), we realize $\pi_{l_0,P}=\mathrm{ind}_P^F\chi_{l_0}=\mathrm{ind}_E^F\tau$ in

$L^2(\boldsymbol{R}, \mathscr{H}_{\tilde\pi})$, and we have

$$\begin{aligned} d\pi_{l_0,P}(X)\phi(x) &= -\frac{d}{dx}\phi(x), \\ d\pi_{l_0,P}(A)\phi(x) &= il_0(A)e^{-x}\phi(x), \\ d\pi_{l_0,P}(B)\phi(x) &= il_0(B)e^{x}\phi(x), \\ d\pi_{l_0,P}(V)\phi(x) &= d\tilde\pi(\mathrm{Ad}(\exp(-xX))V)(\phi(x)), \quad V \in \tilde{\mathfrak{f}}, \\ d\pi_{l_0,P}(Y)\phi(x) &= -ix\phi(x). \end{aligned}$$

We can also show similarly as in case (1) that

$$\phi \in R_{l_0}(\mathscr{H}^\infty_{\pi_{l_0,P}}) \Longleftrightarrow \phi \in \mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h}).$$

Case (2-1) a), b), c); ii): We come now to case ii). Now $\mathfrak{h} \not\subset \mathfrak{k}$. We take $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{k} + \mathfrak{g}_2$. Since $\mathfrak{n}^l \subset \mathfrak{g}_2^l = \mathfrak{k}$, and $\mathfrak{h}$ is adapted to $\mathfrak{n}$, which implies $\mathfrak{h} = (\mathfrak{h}\cap\mathfrak{n}) + (\mathfrak{h}\cap\mathfrak{n}^l)$, we may choose subspaces $\mathscr{X} \subset \mathfrak{n}\cap\ker(l)$ and $\mathscr{Y} \subset \mathfrak{g}_2 \cap \ker(l)$ so that $\mathfrak{h} = \mathscr{X} \oplus (\mathfrak{h}\cap\mathfrak{k})$ and $\mathfrak{h}' = \mathscr{Y} \oplus (\mathfrak{h}\cap\mathfrak{k})$. We remark that $\mathfrak{h}'$ is a polarization at l adapted to $\mathfrak{n}$ and $\dim(\mathscr{X}) = \dim(\mathscr{Y}) (\leq 2)$. Applying the result i) above to $(G, l, \mathfrak{n}, \mathfrak{h}')$, we have $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}$ with an abelian ideal $\mathfrak{a}$ with the required properties; $[\mathfrak{n}+\mathfrak{h}', \mathfrak{a}] = \{0\}$, $\dim(\mathfrak{f}(l_0)) = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a})$ for any extension $l_0 \in \mathfrak{f}^*$ of l, and there exists an extension l_0 such that

$$\phi \in R_{l_0}(\mathscr{H}^\infty_{\pi_{l_0,P'}}) \Longleftrightarrow \phi \in \mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h}'),$$

where $\mathfrak{p}' = \mathfrak{h}' + \mathfrak{a}$, $P' = \exp\mathfrak{p}'$.

Let $\mathfrak{p} = \mathfrak{h} + \mathfrak{a} = \mathscr{X} \oplus (\mathfrak{h}\cap\mathfrak{k}) \oplus \mathfrak{a}$ and $P = \exp\mathfrak{p}$. Since $\mathscr{X} \subset \mathfrak{n}$, we have $[\mathfrak{n}+\mathfrak{h}, \mathfrak{a}] = \{0\}$, and the subalgebra $\mathfrak{p}$ is also a Pukanszky polarization at l_0 adapted to $\mathfrak{n}+\mathfrak{a}$. We take a subspace $\mathscr{M} \subset \mathfrak{n}$ such that $\mathfrak{n} = \mathscr{M} \oplus \mathscr{X} \oplus (\mathfrak{n}\cap\mathfrak{k})$. (We regard $\mathscr{M} = \{0\}$ if $\mathfrak{h}+\mathfrak{k} = \mathfrak{g}$.) Then we have

$$\begin{aligned} NH/H &= (\exp\mathscr{M}\exp\mathscr{X}(N\cap K)H/G_2H)(G_2H/H) \\ &= (\exp\mathscr{M}(N\cap K)/(N\cap K\cap H)G_2)(\exp\mathscr{Y}), \\ NH'/H' &= \exp\mathscr{X}\exp\mathscr{M}(N\cap K)H'/H' \\ &= \exp\mathscr{X}(\exp\mathscr{M}(N\cap K)/(N\cap K\cap H)G_2). \end{aligned}$$

Remarking that $\mathfrak{n}\cap\mathfrak{k} = \mathfrak{n}\cap\mathfrak{k}_0$, we can take coexponential bases $\{Y_i, R_j\}$ for $\mathfrak{n}\cap\mathfrak{h}$ in $\mathfrak{n}$ and $\{X_i, R_j\}$ for $\mathfrak{n}\cap\mathfrak{h}'$ in $\mathfrak{n}$ so that $\{X_i\}_{i=1,\dim(\mathscr{X})}$ is a basis of $\mathscr{X}$, $\{Y_i\}_{i=1,\dim(\mathscr{Y})}$ is a basis of $\mathscr{Y}$, and $\{R_j\}_{j=1,\cdots,w}$ is a coexponential basis for $\mathfrak{n}\cap\mathfrak{h}'$ $(= (\mathfrak{n}\cap\mathfrak{k}\cap\mathfrak{h}) + \mathfrak{g}_2 = (\mathfrak{n}\cap\mathfrak{k}_0\cap\mathfrak{h}) + \mathfrak{g}_2)$ in $\mathscr{M}\oplus(\mathfrak{n}\cap\mathfrak{k})$ $(= \mathscr{M}\oplus(\mathfrak{n}\cap\mathfrak{k}_0))$, where $w := \dim((\mathscr{M}\oplus(\mathfrak{n}\cap\mathfrak{k}))/(\mathfrak{h}'\cap\mathfrak{n}))$. We identify $NH/H = \boldsymbol{R}^w \oplus \mathscr{Y}$ by $(r, y) \mapsto E(r)E(y)$, and $NH'/H' = \mathscr{X} \oplus \boldsymbol{R}^w$ by $(x, r) \mapsto E(x)E(r)$, where $E(r) := \exp r_1R_1 \cdots \exp r_wR_w$, $E(x) := \exp(x_1X_1)$, $E(y) := \exp(y_1Y_1)$ (for the case of $\dim(\mathscr{X}) = 1$), $E(x) := \exp(x_1X_1)\exp(x_2X_2)$, $E(y) := \exp(y_1Y_1)\exp(y_2Y_2)$ (for the case of $\dim(\mathscr{X}) = 2$.)

The intertwining operator u between the space of $\mathrm{ind}_H^G\chi_l$ and $\mathrm{ind}_{H'}^G\chi_l$ is given by

$$u\phi(g) = \oint_{H'/H'\cap H} \phi(gy)\chi_l(y)\Delta_{G,H'}^{-1/2}(y)dy, \quad \phi \in \mathscr{H}_{\pi_{l,H}}$$

(see [**2**]), which is in our coordinates (x,r) for NH'/H', (r,y) for NH/H, and t for $G/NH' = G/NH$ given by

$$u\phi(t,x,r) = \int_{\boldsymbol{R}^{\dim(\mathscr{Y})}} \phi(t,r(x),y)e^{-il([x,y]-[x,g(x)]+h(x))}dy,$$

where $r(x)$ is a $\boldsymbol{R}^w$-valued polynomial, $h(x)$ is an $\mathfrak{h}\cap\mathfrak{n}\cap\mathfrak{k}_0$-valued polynomial and $g(x)$ is a $\mathscr{Y}$-valued polynomial in x such that $E(x)E(r)E(x)^{-1} = E(r(x))\exp g(x)\exp h(x)$. Hence the operator u maps $\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})$ onto $\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}')$. Therefore,

$$\begin{aligned}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}) &= u^{-1}(u(\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}))) \\ &= u^{-1}(R_{l_0}(\mathscr{H}^\infty_{\pi_{l_0,P'}})) \\ &= R_{l_0}(\mathscr{H}^\infty_{\pi_{l_0,P}}).\end{aligned}$$

CASE (2-2): $\mathfrak{n} = \mathfrak{g}_1$. We define $\mathfrak{f} = \mathfrak{g}\ltimes\mathfrak{a}$, where $\mathfrak{a}$ is an abelian ideal spanned by $\{A_1,\cdots,A_n,B_1,\cdots,B_n\}$, by $[X_j,A_k] = \delta_{j,k}A_j$, $[X_j,B_k] = -\delta_{j,k}B_j$, $[Y_j,A_k] = [Y_j,B_k] = 0$ $(j,k = 1,\cdots,n)$. Let $\mathfrak{p} := \mathfrak{h}+\mathfrak{a}$. Then for all extension $l_0 \in \mathfrak{f}^*$ of l we have that $\mathfrak{f}(l_0)$ is spanned by $\{Z,\ A_j - l_0(A_j)Y_j,\ B_j + l_0(B_j)Y_j,\ 1\le j\le n\}$ and $\dim(\mathfrak{f}(l_0)) = \dim(\mathfrak{g}(l)) + \dim(\mathfrak{a})$ since $\mathfrak{g}(l) = \boldsymbol{R}Z$. Thus $\mathfrak{p}$ is a polarization at l_0 adapted to $\mathfrak{n}+\mathfrak{a}$. We choose an extension l_0 such that $l_0(A_j)\neq 0$, $l_0(B_j)\neq 0$ for all $j = 1,\cdots,n$, and realize $\pi_{l_0,P}$ as $\mathrm{ind}_P^F\chi_{l_0}$ on $L^2(\boldsymbol{R}^n)$. Then for smooth functions $\phi = \phi(x_1,\cdots,x_n) \in L^2(\boldsymbol{R}^n)$, we have

$$\begin{aligned}&d\pi_{l_0,P}(X_j)\phi(x_1,\cdots,x_n) = -\frac{\partial}{\partial x_j}\phi(x_1,\cdots,x_n),\\ &d\pi_{l_0,P}(A_j)\phi(x_1,\cdots,x_n) = il_0(A_j)e^{-x_j}\phi(x_1,\cdots,x_n),\\ &d\pi_{l_0,P}(B_j)\phi(x_1,\cdots,x_n) = il_0(B_j)e^{x_j}\phi(x_1,\cdots,x_n),\\ &d\pi_{l_0,P}(Y_j)\phi(x_1,\cdots,x_n) = i(l_0(Y_j)-x_j)\phi(x_1,\cdots,x_n), \quad j = 1,\cdots,n.\end{aligned}$$

Noting that $G/NH = G/H$, we get

$$\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}) = R_{l_0}(\mathscr{H}^\infty_{\pi_{l_0,P}}).$$

□

2.2. $\mathscr{S}\mathscr{E}^\infty$-space and $\mathscr{A}\mathscr{S}\mathscr{E}$-space.

Using our $\mathscr{S}\mathscr{E}$-space, we shall describe the $\mathscr{A}\mathscr{S}\mathscr{E}$-space introduced in [**7**], where it is denoted by $\mathscr{E}\mathscr{S}$ (see Remark 2.7). Let $G = \exp\mathfrak{g}$, $\mathfrak{n}$, $l\in\mathfrak{g}^*$ be as above, $\mathfrak{h}$ be a polarization at l adapted to $\mathfrak{n}$, and $\mathfrak{h}_\mathfrak{n} = \mathfrak{h}\cap\mathfrak{n}$. Let $\mathscr{P}(\mathfrak{h})$ be the set of all the polarizations

$\check{\mathfrak{h}}$ at l such that $\check{\mathfrak{h}} \cap \mathfrak{n} = \mathfrak{h}_{\mathfrak{n}}$ and $\check{\mathfrak{h}}$ is adapted to $\mathfrak{n}$. By Remark 2.2, a polarization $\check{\mathfrak{h}}$ belongs to $\mathscr{P}(\mathfrak{h})$ if and only if $\check{\mathfrak{h}} = \mathfrak{h}_0 + \mathfrak{h}_{\mathfrak{n}}$, where $\mathfrak{h}_0 \subset \mathfrak{n}^l$ is a polarization at $l_{|\mathfrak{n}^l}$. Let $\check{H} = \exp \check{\mathfrak{h}}$, and we denote by $\mathscr{T}_{\mathfrak{h}\check{\mathfrak{h}}} : \mathscr{H}_{\pi_{l,\check{H}}} \to \mathscr{H}_{\pi_{l,H}}$ the intertwining operator of $\mathrm{ind}_{\check{H}}^G \chi_l$ and $\mathrm{ind}_H^G \chi_l$;

$$\mathscr{T}_{\mathfrak{h}\check{\mathfrak{h}}}\phi(g) = \oint_{H/(H\cap\check{H})} \phi(gy)\chi_l(y)\Delta_{G,H}^{-1/2}(y)d\mu_{H/(H\cap\check{H})}(y), \quad \phi \in \mathscr{H}_{\pi_{l,\check{H}}}$$

(see [**2**]).

Definition 2.5. We define

$$\mathscr{S}\mathscr{E}^\infty(G, \mathfrak{n}, l, \mathfrak{h}) := \cap_{\check{\mathfrak{h}}\in\mathscr{P}(\mathfrak{h})} \mathscr{T}_{\mathfrak{h}\check{\mathfrak{h}}}(\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \check{\mathfrak{h}})).$$

We also define the space $\mathscr{A}\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$ ([**6**], [**7**]). Recall that for defining $\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$, we regard $G/H = (G/NH) \times (NH/H) = \boldsymbol{R}^{m+v}$; now we decompose G/H as

$$G/H = (G/N^l N) \times (N^l N/NH) \times (NH/H) = \boldsymbol{R}^{\nu+u+v},$$

where $m = \nu + u$, taking coexponential bases $\{T_1, \cdots, T_\nu\}$ for $\mathfrak{n}^l + \mathfrak{n}$ in $\mathfrak{g}$, $\{S_1 := T_{\nu+1}, \cdots, S_u := T_{\nu+u}\}$ for $\mathfrak{n} + \mathfrak{h}$ in $\mathfrak{n}^l + \mathfrak{n}$, $\{R_1, \cdots, R_v\}$ for $\mathfrak{h}$ in $\mathfrak{n} + \mathfrak{h}$, and letting $\boldsymbol{R}^\nu \ni t = (t_1, \cdots, t_\nu) \mapsto E(t) = \exp t_1 T_1 \cdots \exp t_\nu T_\nu$, $\boldsymbol{R}^u \ni s = (s_1, \cdots, s_u) \mapsto E(s) = \exp s_1 S_1 \cdots \exp s_u S_u$, $\boldsymbol{R}^v \ni r = (r_1, \cdots, r_v) \mapsto E(r) = \exp r_1 R_1 \cdots \exp r_v R_v$, and $\boldsymbol{R}^{\nu+u+v} \ni (t, s, r) \mapsto E(t, s, r) = E(t)E(s)E(r)$.

Definition 2.6. Let $\mathfrak{D}_{t,s,r}$ be the space of all differential operators on $\boldsymbol{R}^{\nu+u+v}$ with polynomial coefficients and let $\mathscr{A}\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$ be the space of all functions $\phi \in \mathscr{H}_{\pi_{l,H}}$ such that

1. ϕ is smooth,
2. $$\|\phi\|_{a,b,D}^2 := \int_{\boldsymbol{R}^{\nu+u+v}} e^{a\|t\|} e^{b\|s\|} |D(\phi \circ E)(t, s, r)|^2 dt ds dr < \infty, \forall\ (a, b) \in \boldsymbol{R}_+^2, D \in \mathfrak{D}_{t,s,r}.$$
3. The same conditions 1 and 2 hold for the partial Fourier transform $\hat{\phi}_s$ of ϕ in s, where
$$\hat{\phi}_s(t, s, r) = \int_{\boldsymbol{R}^u} \phi \circ E(t, x, r) e^{i\langle x,s\rangle} dx.$$

Remark 2.7. The space $\mathscr{A}\mathscr{S}\mathscr{E}$ is also independent of the choice of coexponential bases. We have $\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h}) \supset \mathscr{A}\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$; A function $\phi \in \mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$ belongs to $\mathscr{A}\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$ if and only if ϕ satisfies the condition 3 above. In the paper [**7**] this space has been denoted by $\mathscr{E}\mathscr{S}(G, \mathfrak{n}, l, \mathfrak{h})$. We write here the letter $\mathscr{A}$ in front to indicate that the functions ϕ contained in $\mathscr{A}\mathscr{S}\mathscr{E}(G, \mathfrak{n}, l, \mathfrak{h})$ are analytic in the direction x. It has been shown in [**7**] and [**1**] that for ϕ and ψ in $\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})$ there exists a function $f \in L^1(G)$ and more precisely in the subalgebra $\mathscr{S}\mathscr{E}(G)$ (see section 4), such that

$$\pi_{l,H} f(\xi) = \langle \xi, \psi \rangle \phi, \ \xi \in \mathscr{H}_{\pi_{l,H}}.$$

Theorem 2.8. *Let $G = \exp\mathfrak{g}$, $\mathfrak{n}$, l, $\mathfrak{h}$ be as above. Then we have*

$$\mathscr{S}\mathscr{E}^\infty(G,\mathfrak{n},l,\mathfrak{h}) = \mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}).$$

Proof. The proof is by induction on $\dim(\mathfrak{g})$. We shall use the framework of induction in the proof of Theorem 2.4.

If $\mathfrak{n} = \{0\}$, the statement is trivial. Suppose that $l = 0$ on an abelian ideal $\mathfrak{i} \neq \{0\}$, and let $\dot{G}, \dot{\mathfrak{n}}, \dot{l}, \dot{\mathfrak{h}}, \dot{\pi}$ be as in the proof of Theorem 2.4. Then we get the conclusion by the induction hypothesis for $(\dot{G}, \dot{\mathfrak{n}}, \dot{l}, \dot{\mathfrak{h}})$ and $\dot{\pi}$ because we can naturally identify $\mathscr{S}\mathscr{E}^\infty(\dot{G}, \dot{\mathfrak{n}}, \dot{l}, \dot{\mathfrak{h}})$ with $\mathscr{S}\mathscr{E}^\infty(G,\mathfrak{n},l,\mathfrak{h})$ and $\mathscr{A}\mathscr{S}\mathscr{E}(\dot{G}, \dot{\mathfrak{n}}, \dot{l}, \dot{\mathfrak{h}})$ with $\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})$.

Suppose $l \neq 0$ on any non-zero abelian ideal. Taking a minimal ideal $\mathfrak{g}_1$ contained in $\mathfrak{n}$, we use the same notations as those in the proof of Theorem 2.4.

Case (1): Letting $\mathfrak{k} := \ker(\lambda)$, $K = \exp\mathfrak{k}$, we have $\check{\mathfrak{h}} \subset \mathfrak{k}$ for all $\check{\mathfrak{h}} \in \mathscr{P}(\mathfrak{h})$ and

$$\mathscr{S}\mathscr{E}^\infty(K,\mathfrak{n},l_{|\mathfrak{k}},\mathfrak{h}) = \mathscr{A}\mathscr{S}\mathscr{E}(K,\mathfrak{n},l_{|\mathfrak{k}},\mathfrak{h})$$

by the induction hypothesis. Since $\mathfrak{k}$ is an ideal including $\mathfrak{n}^l + \mathfrak{n}$, we have

$$G/N^l N = (G/K)(K/N^l N),$$

and obtain the conclusion $\mathscr{S}\mathscr{E}^\infty(G,\mathfrak{n},l,\mathfrak{h}) = \mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})$.

Case (2-1) a),b),c): $\mathfrak{g}_1 \neq \mathfrak{n}$. Let $\mathfrak{k} = \ker(\gamma)$ in case a) and b), resp. $\mathfrak{k} = \ker(\gamma_1) \cap \ker(\gamma_2)$ in case c).

Case (2-1) a),b),c); i): $\mathfrak{g}_2 \subset \mathfrak{h}$. Then any polarization $\check{\mathfrak{h}} \in \mathscr{P}(\mathfrak{h})$ is contained in $\mathfrak{k}$. We have $\mathfrak{n} + \mathfrak{k} = \mathfrak{g}$ in cases a),b-1),c), $\mathfrak{n} \subset \mathfrak{k}$ in case b-2). Since

$$(\mathfrak{k} \cap \mathfrak{n})^{l_{|\mathfrak{k}}} = \mathfrak{n}^l + \mathfrak{g}_2,$$

and

$$G/N^l N = \begin{cases} K/(K\cap N)^{l_{|\mathfrak{k}}}(K \cap N) & \text{cases a),b-1),c)} \\ (G/K)(K/N^l N) = (G/K)(K/(K\cap N)^{l_{|\mathfrak{k}}}(K\cap N)) & \text{case b-2),} \end{cases}$$

$$N^l N/NH = N^l(K\cap N)/(K\cap N)H = (K\cap N)^{l_{|\mathfrak{k}}}(K\cap N)/(K\cap N)H$$

we can deduce the conclusion from the induction hypothesis

$$\mathscr{S}\mathscr{E}^\infty(K,\mathfrak{k}\cap\mathfrak{n},l_{|\mathfrak{k}},\mathfrak{h})=\mathscr{A}\mathscr{S}\mathscr{E}(K,\mathfrak{k}\cap\mathfrak{n},l_{|\mathfrak{k}},\mathfrak{h}).$$

Case (2-1) a),b),c); ii): $\mathfrak{g}_2\not\subset\mathfrak{h}$. Then $\check{\mathfrak{h}}\not\subset\mathfrak{k}$ for any polarization $\check{\mathfrak{h}}\in\mathscr{P}(\mathfrak{h})$. Let $\mathfrak{h}'=(\mathfrak{h}\cap\mathfrak{k})+\mathfrak{g}_2$, and according to the notations in case (2-1) a),b),c) ii) of the proof of Theorem 2.4, we have $\mathfrak{h}=\mathscr{X}\oplus(\mathfrak{h}\cap\mathfrak{k})$, $\mathscr{X}\subset\mathfrak{n}\cap\ker(l)$ and $\mathfrak{h}'=(\mathfrak{h}\cap\mathfrak{k})\oplus\mathscr{Y}$, $\mathscr{Y}\subset\mathfrak{g}_2\cap\ker(l)$, and we identify $NH/H=\boldsymbol{R}^w\oplus\mathscr{Y}$ by $(r,y)\mapsto E(r)E(y)$, and $NH'/H'=\mathscr{X}\oplus\boldsymbol{R}^w$ by $(x,r)\mapsto E(x)E(r)$. Then the intertwining operator $\mathscr{T}_{\mathfrak{h}'\mathfrak{h}}$ is given by

$$\mathscr{T}_{\mathfrak{h}'\mathfrak{h}}\phi(t,s,x,r)=\int_{\boldsymbol{R}^{\dim(\mathscr{Y})}}\phi(t,s,r(x),y)e^{-il([x,y]-[x,g(x)]+h(x))}dy, \tag{2.13}$$

where $r(x)$, $h(x)$ and $g(x)$ are polynomials whose values are in $\boldsymbol{R}^w$, $\mathfrak{h}\cap\mathfrak{n}\cap\mathfrak{k}_0$ and $\mathscr{Y}$, respectively. Thus we have

$$\mathscr{T}_{\mathfrak{h}'\mathfrak{h}}(\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}))=\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}').$$

For any polarization $\check{\mathfrak{h}}=\mathfrak{h}_0+\mathfrak{h}_{\mathfrak{n}}\in\mathscr{P}(\mathfrak{h})$, letting $\check{\mathfrak{h}}'=(\check{\mathfrak{h}}\cap\mathfrak{k})+\mathfrak{g}_2$, we also get the expression of $\mathscr{T}_{\check{\mathfrak{h}}'\check{\mathfrak{h}}}$ as (2.13), and we have

$$\mathscr{T}_{\check{\mathfrak{h}}'\check{\mathfrak{h}}}(\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\check{\mathfrak{h}}))=\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\check{\mathfrak{h}}').$$

Applying the result i) above, we have

$$\mathscr{S}\mathscr{E}^\infty(G,\mathfrak{n},l,\mathfrak{h}')=\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}').$$

The set $\mathscr{P}(\mathfrak{h}')$ consists of polarizations $\mathfrak{h}_1=\mathfrak{h}_0+(\mathfrak{h}'\cap\mathfrak{n})=\mathfrak{h}_0+(\mathfrak{h}\cap\mathfrak{k}\cap\mathfrak{n})+\mathfrak{g}_2$, with some polarization $\mathfrak{h}_0\subset\mathfrak{n}^l\subset\mathfrak{k}$ at $l_{|\mathfrak{n}^l}$. Thus we have $\mathscr{P}(\mathfrak{h}')=\{(\check{\mathfrak{h}}\cap\mathfrak{k})+\mathfrak{g}_2;\check{\mathfrak{h}}\in\mathscr{P}(\mathfrak{h})\}$. Writing $\check{\mathfrak{h}}'=(\check{\mathfrak{h}}\cap\mathfrak{k})+\mathfrak{g}_2$ for each $\check{\mathfrak{h}}\in\mathscr{P}(\mathfrak{h})$, we have

$$\begin{aligned}
\mathscr{T}_{\mathfrak{h}'\mathfrak{h}}(\mathscr{S}\mathscr{E}^\infty(G,\mathfrak{n},l,\mathfrak{h}))&=\cap_{\check{\mathfrak{h}}\in\mathscr{P}(\mathfrak{h})}\mathscr{T}_{\mathfrak{h}'\mathfrak{h}}\circ\mathscr{T}_{\mathfrak{h}\check{\mathfrak{h}}}(\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\check{\mathfrak{h}}))\\
&=\cap_{\check{\mathfrak{h}}\in\mathscr{P}(\mathfrak{h})}\mathscr{T}_{\mathfrak{h}'\mathfrak{h}}\circ\mathscr{T}_{\mathfrak{h}\check{\mathfrak{h}}'}\circ\mathscr{T}_{\check{\mathfrak{h}}'\check{\mathfrak{h}}}(\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\check{\mathfrak{h}}))=\cap_{\check{\mathfrak{h}}\in\mathscr{P}(\mathfrak{h})}\mathscr{T}_{\mathfrak{h}'\mathfrak{h}}\circ\mathscr{T}_{\mathfrak{h}\check{\mathfrak{h}}'}(\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\check{\mathfrak{h}}'))\\
&=\cap_{\check{\mathfrak{h}}'\in\mathscr{P}(\mathfrak{h}')}\mathscr{T}_{\mathfrak{h}'\check{\mathfrak{h}}'}(\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\check{\mathfrak{h}}'))=\mathscr{S}\mathscr{E}^\infty(G,\mathfrak{n},l,\mathfrak{h}')\\
&=\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}')=\mathscr{T}_{\mathfrak{h}'\mathfrak{h}}(\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})).
\end{aligned}$$

Thus we have $\mathscr{S}\mathscr{E}^\infty(G,\mathfrak{n},l,\mathfrak{h})=\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})$.

Case (2-2): $\mathfrak{n}=\mathfrak{g}_1$ whence $\mathfrak{n}^l=\mathfrak{g}$. We can take a basis $\{X_1,\cdots,X_n,Y_1,\cdots,Y_n,Z\}$ of $\mathfrak{g}$ such that $[X_i,Y_j]=\delta_{i,j}Z$, $l(Z)=1$, $l(X_i)=l(Y_i)=0$ $(i,j=1,\cdots,n)$, and $\mathfrak{h}$ is spanned by $\{Y_1,\cdots,Y_n,Z\}$. Let $\mathfrak{h}_2$ be a subalgebra spanned by $\{X_1,\cdots,X_n,Z\}$. Then $\mathfrak{h}_2\in\mathscr{P}(\mathfrak{h})$. We identify G/H with $\mathscr{X}:=\boldsymbol{R}X_1\oplus\cdots\oplus\boldsymbol{R}X_n$ and G/H_2 with $\mathscr{Y}:=\boldsymbol{R}Y_1\oplus\cdots\oplus\boldsymbol{R}Y_n$, and realize $\mathrm{ind}_H^G\chi_l$ and $\mathrm{ind}_{H_2}^G\chi_l$, respectively. Then the intertwining operator $\mathscr{T}_{\mathfrak{h}\mathfrak{h}_2}$ is described by

$$\mathscr{T}_{\mathfrak{h}\mathfrak{h}_2}\phi(x_1,\cdots,x_n)=\int_{\mathscr{Y}}\phi(y_1,\cdots,y_n)e^{-i(x_1y_1+\cdots+x_ny_n)}dy_1\cdots dy_n,\quad \phi\in\mathscr{H}_{\pi_{l,H_2}}.$$

Let $\phi_0\in\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})\cap\mathscr{T}_{\mathfrak{h}\mathfrak{h}_2}(\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}_2))$. Then $\phi_0=\mathscr{T}_{\mathfrak{h}\mathfrak{h}_2}\phi$ with $\phi\in\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}_2)$, and we have that ϕ is obtained by Fourier transform of ϕ_0, $\phi=c\hat{\phi}_0$ with some constant c, and ϕ satisfies the conditions 1 and 2 of Definition 2.6, which implies that $\phi_0\in\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})$. Conversely, let $\mathfrak{h}'$ be a polarization at l. Taking subspaces $\mathscr{Y}$, $\mathscr{W}$, $\mathscr{V}$ such that $\mathfrak{h}=\mathscr{Y}\oplus(\mathfrak{h}\cap\mathfrak{h}')$, $\mathfrak{h}'=\mathscr{W}\oplus(\mathfrak{h}\cap\mathfrak{h}')$, $\mathscr{W}\subset\ker(l)$, and $\mathfrak{g}=\mathscr{V}\oplus\mathscr{W}\oplus\mathscr{Y}\oplus(\mathfrak{h}\cap\mathfrak{h}')$, we identify G/H with $\mathscr{V}\oplus\mathscr{W}$ by $(V,W)\mapsto\exp V\exp W$, G/H' with $\mathscr{V}\oplus\mathscr{Y}$ by $(V,Y)\mapsto\exp V\exp Y$. Then we have

$$\mathscr{T}_{\mathfrak{h}\mathfrak{h}'}\phi(V,W)=\int_{\mathscr{Y}}\phi(V,Y)e^{-il([W,Y])}dY,\quad \phi\in\mathscr{H}_{\pi_{l,H'}}.$$

If $\phi_0\in\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})$, then the function $\psi:=\phi_0\circ E$ has the property that all its partial Fourier transforms are exponentially decreasing. Hence $\phi_0=\mathscr{T}_{\mathfrak{h},\mathfrak{h}'}\mathscr{T}^{-1}_{\mathfrak{h},\mathfrak{h}'}\phi_0$ with $\phi:=\mathscr{T}^{-1}_{\mathfrak{h},\mathfrak{h}'}\phi_0\in\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}')$. Thus we have $\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})\subset\mathscr{T}_{\mathfrak{h}\mathfrak{h}'}(\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h}'))$, and therefore, we have the conclusion $\mathscr{S}\mathscr{E}^{\infty}(G,\mathfrak{n},l,\mathfrak{h})=\mathscr{A}\mathscr{S}\mathscr{E}(G,\mathfrak{n},l,\mathfrak{h})$. □

3. The case $G=N^lN$.

Let $\mathfrak{n}$ again be the nilradical of $\mathfrak{g}$ and suppose that $\mathfrak{g}=\mathfrak{n}^l+\mathfrak{n}$. Let $\mathfrak{h}_{\mathfrak{n}}$ be a polarization at $l_{|\mathfrak{n}}$, such that $[\mathfrak{n}^l,\mathfrak{h}_{\mathfrak{n}}]\subset\mathfrak{h}_{\mathfrak{n}}$. and let $\mathfrak{h}_0\subset\mathfrak{n}^l$ be any polarization at $l_{|\mathfrak{n}^l}$. As we have seen in section 2, the subalgebra $\mathfrak{h}:=\mathfrak{h}_0+\mathfrak{h}_{\mathfrak{n}}$ is a Pukanszky-polarisation at l. Taking a subspace $\mathfrak{x}\subset\mathfrak{n}^l$ such that $\mathfrak{g}=\mathfrak{x}\oplus(\mathfrak{n}+\mathfrak{h})$, and a coexponential basis $\{T_1,\cdots,T_m\}$ for $\mathfrak{h}_{\mathfrak{n}}$ in $\mathfrak{n}$, we identify G/H with $\boldsymbol{R}^m\times\mathfrak{x}$ through the mapping

$$(t_1,\cdots,t_m,X)\mapsto E(t,X)=\exp t_1T_1\cdots\exp t_mT_m\exp X.$$

Let $H=\exp\mathfrak{h}, H_{\mathfrak{n}}:=\exp(\mathfrak{h}_{\mathfrak{n}}), H_0=\exp\mathfrak{h}_0$. The invariant linear functional $\oint_{G/H}d\mu_{G/H}$ is given in these coordinates by

$$\oint_{G/H}f(g)d\mu_{G/H}(g)=\int_{\boldsymbol{R}^m\times\mathfrak{x}}f(E(t,X))\Delta_{G/H}(\exp X)dtdX,\tag{3.14}$$

where $\Delta_{G/H}(\exp X):=e^{-\mathrm{tr}_{\mathfrak{n}/\mathfrak{h}_{\mathfrak{n}}}(\mathrm{ad}X)}$, $X\in\mathfrak{x}$.

In order to see this, let us denote by $\nu(f)$ the concrete expression on the right of equation (3.14). Since $\mathfrak{h}_{\mathfrak{n}}$ is $\mathfrak{n}^l$-invariant, we see that the positive functional ν is N-invariant. If we take $Y\in\mathfrak{n}^l$, denoting by $\lambda(g)$, $g\in G$, left translation by g, then

$$\begin{aligned}\nu(\lambda(\exp Y)f) &= \int_{\boldsymbol{R}^m\times\mathfrak{x}} f(\exp(-Y)E(t)\exp(X))\Delta_{G/H}(\exp X)dtdX\\ &= \int_{\boldsymbol{R}^m\times\mathfrak{x}} \Delta_{G/H}(\exp(-Y))f(E(t)\exp(-Y)\exp(X))\Delta_{G/H}(\exp X)dtdX\\ &= \int_{\boldsymbol{R}^m\times\mathfrak{x}} f(E(t)\exp(-Y)\exp(X))\Delta_{G/H}(\exp(-Y+X))dtdX\\ &= \nu(f).\end{aligned}$$

The uniqueness of $\oint_{G/H} d\mu(g)$ tells us that equation (3.14) is valid. In particular, for every ξ in the Hilbert space $\mathscr{H}_{\pi_{l,H}} = L^2(G/H,\chi_l)$, the L^2-norm of ξ is given by

$$\|\xi\|_2^2 = \oint_{\boldsymbol{R}^m\times\mathfrak{x}} |\xi(E(t,X))|^2\Delta_{G/H}(\exp X)dtdX.$$

Let χ_l be the unitary character of H whose differential is the linear functional $il_{|\mathfrak{h}}$ and let $\pi = \pi_{l,H} = \mathrm{ind}_H^G\chi_l$.

DEFINITION 3.1. Let $\mathfrak{D}_{t,\mathfrak{x}}$ be the space of all differential operators on $\boldsymbol{R}^m\times\mathfrak{x}$ with polynomial coefficients and let $\mathscr{S}_{t,\mathfrak{x}} = \mathscr{S}(G,\mathfrak{n},l,\mathfrak{h})$ be the space of all functions $\phi\in\mathscr{H}_{\pi_{l,H}}$ such that

1. ϕ is smooth,
2. $$\|\phi\|_D^2 := \int_{\boldsymbol{R}^m\times\mathfrak{x}} |D(\phi\circ E)(t,X)|^2\Delta_{G/H}(\exp X)dtdX < \infty, \quad \forall D\in\mathfrak{D}_{t,\mathfrak{x}}.$$

Denote by $\mathscr{S}(V)$ the Schwartz space of rapidly decreasing smooth functions on the real finite dimensional vector space V. With this notation, we see that

$$\mathscr{S}_{t,\mathfrak{x}} = \{\phi: G\to\boldsymbol{C},\ (\Delta_{G/H}\cdot\phi)\circ E\in\mathscr{S}(\boldsymbol{R}^m\times\mathfrak{x})\},$$

since the mapping $D\mapsto M_\Delta\circ D\circ M_\Delta^{-1}$, $D\in\mathfrak{D}_{t,\mathfrak{x}}$, where M_Δ denotes multiplication with the function $\Delta_{G/H}$, is a bijection of $\mathfrak{D}_{t,\mathfrak{x}}$.

THEOREM 3.2. *Let $G=\exp\mathfrak{g}$ be an exponential solvable Lie group, let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$ and let $l\in\mathfrak{g}^*$. Suppose that $\mathfrak{g}=\mathfrak{n}^l+\mathfrak{n}$. Choose a polarization $\mathfrak{h}_\mathfrak{n}$ at $l_{|\mathfrak{n}}$, such that $[\mathfrak{n}^l,\mathfrak{h}_\mathfrak{n}]\subset\mathfrak{h}_\mathfrak{n}$. Then the space $\mathscr{H}^\infty := \mathscr{H}^\infty_{\pi_{l,H}}$ of the C^∞ vectors of the representation $\pi:=\pi_{l,H}$ and the Fréchet space $\mathscr{S}_{t,\mathfrak{x}}$ coincide.*

PROOF. By a theorem of [**8**], the C^∞ vectors of the representation π are smooth functions. For fixed $X\in\mathfrak{x}$ and $\xi\in\mathscr{H}^\infty$, we see that the function

$$N\ni n\mapsto \xi_X(n) = \xi(n\exp(X))$$

satisfies the covariance condition

$$\xi_X(nh) = \chi_l(h^{-1})\xi_X(n),\ \ h \in H_{\mathfrak{n}}, n \in N,$$

since $\chi_l(\exp(X)h\exp(-X)) = \chi_l(h)$ for all $X \in \mathfrak{n}^l$ and all $h \in H_n$. Therefore, multiplying ξ with a smooth function $\varphi \in C_c^\infty(G/H)$, we obtain an element $\eta := (\varphi\xi)_X \in \mathscr{H}_{\pi_{\mathfrak{n}}}^\infty$ (where $\pi_{\mathfrak{n}} := \mathrm{ind}_{H_{\mathfrak{n}}}^N \chi_{l_{|\mathfrak{n}}}$) and hence by Kirillov's theorem, for any element D in $\mathfrak{D}_t$, there exists a u_D in the enveloping algebra $\mathscr{U}(\mathfrak{n})$ such that $D = d\pi_{\mathfrak{n}}(u_D)$ on $\mathscr{H}_{\pi_{\mathfrak{n}}}^\infty$. Now, if we let run φ through an approximate unit, we see that

$$D(\xi) = d\pi(u_D)\xi,\ \ \xi \in \mathscr{H}^\infty. \tag{3.15}$$

Hence $\mathfrak{D}_t \subset d\pi(\mathscr{U}(\mathfrak{g}))$. For $Y \in \mathfrak{n}^l$, we have that

$$\begin{aligned}&\pi(\exp(Y))\xi(E(t,X))\\ &\quad = \xi(\exp(t_1\mathrm{Ad}(\exp(-Y)T_1))\cdots\exp(t_m\mathrm{Ad}(\exp(-Y)T_m))\exp(-Y)\exp(X)),\end{aligned}$$

for $t \in \boldsymbol{R}^m, X, Y \in \mathfrak{n}^l$. This shows that

$$(d\pi(Y)(\varphi\xi)) \circ E(t,X) = D_Y(\varphi\xi \circ E)(t,X) + d\pi_0(Y)((\varphi\xi)_t)(\exp(X)),$$

where D_Y is some element in $\mathfrak{D}_t$ (acting only on the variable t), where $(\varphi\xi)_t$ is the function $(\varphi\xi)_t(\exp X) := \varphi\xi(E(t)\exp(X))$, which is contained in the Hilbert space $\mathscr{H}_0$ of the representation $\pi_0 := \mathrm{ind}_{H_0}^{N^l}\chi_{l_{|\mathfrak{n}^l}}$. Together with the relation (3.15) and the fact that $\mathfrak{D}_{\mathfrak{r}} = d\pi_0(\mathscr{U}(\mathfrak{n}^l))$ this shows that $\mathfrak{D}_{\mathfrak{r}}$ is contained in $d\pi(\mathscr{U}(\mathfrak{g}))$ and finally that

$$\mathfrak{D}_{t,\mathfrak{r}} = d\pi(\mathscr{U}(\mathfrak{g})). \tag{3.16}$$

In particular, the function $(\Delta_{G/H}\xi) \circ E$ is contained in $\mathscr{S}(\boldsymbol{R}^m \times \mathfrak{r})$. Conversely, because of (3.16) every smooth function η defined on G satisfying the covariance condition for H and χ_l, such that $\Delta_{G/H}\eta \circ E$ is in $\mathscr{S}(\boldsymbol{R}^m \times \mathfrak{r})$ is also contained in $\mathscr{H}^\infty$. □

4. The space $\mathscr{S}\mathscr{E}(G)$.

Let again $G = \exp\mathfrak{g}$ be an exponential solvable Lie group. We shall introduce special coordinates on G, which will allow us to write the product in G in a particularly simple way. Let $\mathfrak{n}$ again be the nilradical of $\mathfrak{g}$. Take an element T of $\mathfrak{g}$ which is in general position with respect to the roots of $\mathfrak{g}$. This means that for any two distinct roots λ, λ' of $\mathfrak{g}$ we have that $\lambda(T) - \lambda'(T) \neq 0$. This means that the mapping $\lambda \to \lambda(T)$ is an injection. For a root λ let

$$\mathfrak{g}_{\lambda,\boldsymbol{C}} = \{X \in \mathfrak{g}_{\boldsymbol{C}};\ (\lambda(T)\boldsymbol{I}_{\mathfrak{g}_{\boldsymbol{C}}} - \mathrm{ad}(T))^d(X) = 0,\ \text{for some } d \in \boldsymbol{N}^*\}.$$

By the usual rules we have that

$$[\mathfrak{g}_{\lambda,\boldsymbol{C}}, \mathfrak{g}_{\lambda',\boldsymbol{C}}] \subset \mathfrak{g}_{\lambda+\lambda',\boldsymbol{C}}$$

for two roots λ, λ'. Then $\mathfrak{g}_0 := \mathfrak{g}_{0,\boldsymbol{C}} \cap \mathfrak{g}$ is thus a nilpotent Lie subalgebra of $\mathfrak{g}$. Since by Jordan's theorem

$$\mathfrak{g}_{\boldsymbol{C}} = \mathfrak{g}_{0,\boldsymbol{C}} + \sum_{\lambda \neq 0} \mathfrak{g}_{\lambda,\boldsymbol{C}}$$

and since $\mathfrak{g}_{\lambda,\boldsymbol{C}}$, $\lambda \neq 0$, is contained in $[\mathfrak{g}_{\boldsymbol{C}}, \mathfrak{g}_{\boldsymbol{C}}] \subset \mathfrak{n}_{\boldsymbol{C}}$, we see that

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{n}.$$

Let us choose a subspace $\mathfrak{t} \subset \mathfrak{g}_0$, such that

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}.$$

We can now define a Lie group structure on the Lie algebra $\mathfrak{k} := \mathfrak{t} \oplus \mathfrak{n}_{\boldsymbol{C}}$. We use on the complexification $\mathfrak{n}_{\boldsymbol{C}}$ of $\mathfrak{n}$ the Campbell-Baker-Hausdorff multiplication $\cdot_C$ and we can write for $S, S' \in \mathfrak{t}$

$$S \cdot_C S' = S + S' + \frac{1}{2}[S, S'] + \cdots = (S + S') \cdot_C m(S, S'),$$

where $m : \mathfrak{t} \times \mathfrak{t} \to \mathfrak{n} \cap \mathfrak{g}_0$ is a polynomial mapping.

We define now on $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{n}_{\boldsymbol{C}}$ a multiplication $\cdot$ in the following way:

$$(S + U) \cdot (S' + U') := S + S' + m(S, S') \cdot_C (e^{\mathrm{ad}(-S')} U) \cdot_C U', \; U, U' \in \mathfrak{n}_{\boldsymbol{C}}, S, S' \in \mathfrak{t}. \quad (4.17)$$

In particular we have the relations

$$S \cdot U = S + U, \; U \cdot S = S + e^{-\mathrm{ad}(S)} U, \; S \in \mathfrak{t}, \; U \in \mathfrak{n}_{\boldsymbol{C}}.$$

It is easy to check that we obtain in this fashion a simply connected exponential solvable Lie group $K = (\mathfrak{k}, \cdot)$ and that this new Lie group contains a closed subgroup $(\mathfrak{g}, \cdot)$, which is isomorphic to G, since $\mathfrak{g} \subset \mathfrak{k}$. Denote also by $N_{\boldsymbol{C}}$ the subgroup $(\mathfrak{n}_{\boldsymbol{C}}, \cdot_C)$ of the Lie group $(\mathfrak{k}, \cdot)$.

The Haar measure on the group $(\mathfrak{g}, \cdot)$ is given by Lebesgue measure dx on the vector space $\mathfrak{g}$. Indeed, for a continuous function δ with compact support on $\mathfrak{g}$, we have that

$$\int_{\mathfrak{g}} \delta(x) dx = \int_{\mathfrak{t} \times \mathfrak{n}} \delta(T \cdot U) dU dT$$

and the left-invariance of this measure follows from the multiplication rule (4.17).

We define now a space of smooth functions on G, which will replace the well known Schwartz space of nilpotent Lie groups.

DEFINITION 4.1. Let $\mathfrak{D}_{\mathfrak{t},\mathfrak{n}}$ be the space of all differential operators on $\mathfrak{t}+\mathfrak{n}$ with polynomial coefficients and let $\mathscr{S}\mathscr{E}(G)$ be the space of all functions $\phi : G \to \boldsymbol{C}$ such that

1. ϕ is smooth,
2. $$\|\phi\|_{a,D}^2 := \int_{\mathfrak{t}+\mathfrak{n}} e^{a\|t\|}|D(\phi)(t+U)|^2 dtdU < \infty,\ \forall\ a \in \boldsymbol{R}_+, D \in \mathfrak{D}_{\mathfrak{t},\mathfrak{n}}.$$

The space $\mathscr{S}\mathscr{E}(G)$ is in fact independent of the choice of the subspace $\mathfrak{t}$. Indeed, for any subspace $\mathfrak{s}$ of $\mathfrak{g}_0$ such that $\mathfrak{s}\oplus\mathfrak{n}=\mathfrak{g}$, the mapping $E:\mathfrak{s}\times\mathfrak{n}\to\mathfrak{g}, E(S,U):=S\cdot U$ is a diffeomorphism, whose coordinate functions are polynomials in $U\in\mathfrak{n}$ and all the partial derivatives of them are exponentially bounded in S. This allows us to write

$$\mathscr{S}\mathscr{E}(G) = \Big\{\phi : G \to \boldsymbol{C}; \phi \text{ smooth}, \int_{\mathfrak{s}\times\mathfrak{n}} e^{a\|S\|}|D(\phi)(S\cdot U)|^2 dSdU < \infty, \forall\ a \in \boldsymbol{R}_+, D \in \mathfrak{D}_{\mathfrak{s},\mathfrak{n}}\Big\}.$$

We shall show in this section that the space $\mathscr{S}\mathscr{E}(G)$ is the space of the C^∞ vectors of an irreducible representation of a certain exponential solvable Lie group $\mathscr{G}$ acting on $L^2(G)$.

Let $\mathscr{T} := \{T_1,\cdots,T_m\}$ be a basis of $\mathfrak{t}$. Choose a Jordan-Hölder basis $\mathscr{B} = \{T_1,\cdots,T_m,U_1,\cdots,U_p\} =: \{X_1,\cdots,X_n\}$ of $\mathfrak{k}$ and for every $i=1,\cdots,n$, we choose a Jordan-Hölder basis $\mathscr{B}_i=\{U_1^i,\cdots,U_p^i\}$ for the endomorphism $\mathrm{ad}X_i$ of $\mathfrak{n}_{\boldsymbol{C}}$. Then the coefficients $a_{k,l}^i(t_i), t_i\in\boldsymbol{R}$, of the matrix of the endomorphism $\mathrm{Ad}(\exp(t_iX_i))$ with respect to the basis $\mathscr{B}_i$ are polynomials in t_i for $i>m$, are 0 for $k>l$ and for $k=l$ they are exponential functions of the form $e^{t_i\alpha(T_i)}, i\le m$, where α denotes a root of $\mathfrak{g}$. Hence, by replacing the basis $\mathscr{B}_i$ by the basis $\mathscr{B}$, for $T=\sum_{i=1}^m t_iT_i$ in $\mathfrak{t}$ and $U\in\mathfrak{n}$, the coefficients $a_{k,l}(T\cdot U)$ of the matrix of $\mathrm{Ad}(T\cdot U)$ with respect to the basis $\mathscr{B}$ are polynomials in $(t_1,\cdots,t_d,U)$ multiplied by exponential functions χ_α of the form $\chi_\alpha(T\cdot U)=e^{a_1t_1+\cdots+a_mt_m}$. We denote by $\mathscr{R}'$ the family of all these complex valued linear functionals α which appear in this way. Let also $\mathscr{R}''$ be the family of complex linear functionals of $\mathfrak{k}$ obtained as sums of j elements of $\mathscr{R}'$, with $j\le 2p$ and let

$$\mathscr{R} = \{\pm\beta, \beta\in\mathscr{R}''\}$$

and let $E_{\mathscr{R}}$ be the (finite) family of exponential functions of the form $e^\alpha, \alpha\in\mathscr{R}$.

DEFINITION 4.2. For a function f defined on a group K, let the left and right translates of f be defined by

$$\lambda(t)f(g) := f(t^{-1}g),\ \ \rho(t)f(g) := f(gt),\ \ g,t \in K.$$

Let now W be the span of all the left and right translates by elements of $N_{\boldsymbol{C}}$ of the complex polynomial functions of degree 1 defined on $\mathfrak{n}_{\boldsymbol{C}}$. Then every element of W is of total degree $\leq 2\dim(\mathfrak{n}) = 2p$ and so W is finite dimensional and left and right $N_{\boldsymbol{C}}$-invariant. Let $(P_j)_{j=1}^d$ be a basis of W. For $g \in N$ we have that the matrix coefficients $a_{i,j}, b_{i,j}$ defined by

$$\lambda(g)P_j = \sum_{i=1}^d a_{i,j}(g)P_i,\ \ \rho(g)P_j = \sum_{i=1}^d b_{i,j}(g)P_i$$

are also elements of W, hence they are polynomial functions of total degree $\leq 2p$. It follows that for every $P \in W$, there exist two finite families of elements of W, $(P_i)_i, (Q_i)_i$, such that

$$P(U \cdot U') = \sum_i P_i(U)Q_i(U'), U, U' \in \mathfrak{n}_{\boldsymbol{C}}. \tag{4.18}$$

We consider now the linear span V of the left translates of all linear functionals $l : \mathfrak{k} \to \boldsymbol{C}$. Since for every couple $(T,U),(T',U')$ the multiplication of these 2 elements is given by

$$(T' + U') \cdot (T + U) = T + T' + m(T,T') \cdot_C (e^{-\mathrm{ad}(T)}(U')) \cdot_C U,$$

it follows from (4.18) that the left translate of $l \in \mathfrak{k}_{\boldsymbol{C}}^*$ is given by

$$\begin{aligned}\lambda((T' + U')^{-1})l(T + U) &= l(T) + l(T') \\ &\quad + \sum_{i,j} P_i(m(T,T'))Q_{i,j}((e^{-\mathrm{ad}(T)}(U'))R_{i,j,}(U),\end{aligned}$$

where the different polynomial functions $P_i, Q_{i,j}$ and $R_{i,j}$ are contained in W. Hence $\lambda((T' + U')^{-1})l$ is a finite linear combination of polynomial functions of degree $\leq 2p$ in U, of degree $\leq 4p^2$ in T multiplied by exponential functions in T, which are all contained in $E_{\mathscr{R}}$. Hence V is a finite dimensional left invariant space of functions on $\mathfrak{k}$ and so is the vector space $\mathscr{V}$ of real valued functions on $\mathfrak{g}$, which is generated as a vector space by the restrictions to $\mathfrak{g}$ of the real parts of the elements of V and by the exponential functions $e^{\pm \mathrm{Re}\,\alpha}$, $\alpha \in \mathscr{R}$.

We obtain the group $\boldsymbol{G}$ as the semi-direct product of G with $\mathscr{V}$, i.e $\boldsymbol{G} = G \times \mathscr{V}$ with the multiplication defined by

$$(g', \varphi') \cdot (g, \varphi) := (g'g, \lambda(g^{-1})\varphi' + \varphi).$$

This group acts on $L^2(G)$ by left translations with the elements of G and by multiplication with the functions $\chi_\varphi = e^{-i\varphi}$, i.e. for $(g,\varphi) \in \boldsymbol{G},\ f \in L^2(G), s \in G$ we let

$$\Pi(g,\varphi)f(s) := e^{-i\varphi(g^{-1}s)} f(g^{-1}s).$$

It is easy to check that $(\Pi, L^2(G))$ is a unitary representation of $\boldsymbol{G}$ in the Hilbert space $L^2(G)$.

Theorem 4.3. *The representation* $(\Pi, L^2(G))$ *of* $\boldsymbol{G}$ *is irreducible and the* C^∞ *vectors of* Π *are the elements of* $\mathscr{S}\mathscr{E}(G)$.

Proof. Since every real valued linear functional l is contained in $\mathscr{V}$, it follows that for $\phi = l$,

$$\Pi(\phi)\xi = e^{-il}\xi,\ d\Pi(\phi)\xi = -il\xi,\ \xi \in L^2(G)^\infty.$$

Furthermore, for any $\alpha \in \mathscr{R}$ and $\phi = e^{\pm \operatorname{Re}\alpha} \in \mathscr{V}$, we have that

$$\Pi(\phi)\xi = e^{-ie^{\pm \operatorname{Re}\alpha}}\xi,\ d\Pi(\phi)\xi = -ie^{\pm \operatorname{Re}\alpha}\xi,\ \xi \in L^2(G)^\infty.$$

This shows that any C^∞-vector of Π is contained in our space $\mathscr{S}\mathscr{E}(G)$. Conversely, every function $f \in \mathscr{S}\mathscr{E}(G)$ will be mapped by $\mathfrak{g}$ into $\mathscr{S}\mathscr{E}(G) \subset L^2(G)$ and therefore $\mathscr{S}\mathscr{E}(G) \subset L^2(G)^\infty$.

In order to prove that Π is irreducible, let $(0) \neq \mathscr{H}_0$ be a closed Π-invariant subspace of $L^2(G)$ and let $\xi' \in \mathscr{H}_0^\perp$ and $0 \neq \eta' \in \mathscr{H}_0$. We replace ξ' and η' by $\xi = \Pi(\delta)\xi'$ resp. by $\eta = \Pi(\delta)\eta'$, where δ is a continuous function on G with a small compact support. Then ξ and η are themselves continuous functions and we have that

$$\langle \Pi(\varphi)\Pi(g)\eta, \Pi(g')\xi\rangle_2 = 0, \quad \text{for all } g, g' \in G, \varphi \in \mathscr{V}.$$

In particular for $\varphi = l \in \mathfrak{g}^*$ we get

$$\int_{\mathfrak{g}} e^{-il(x)} \lambda(g)\eta(x)\overline{\lambda(g')\xi(x)}dx = 0.$$

Hence for every $g, g' \in G$, we have that

$$\lambda(g)\eta(x)\overline{\lambda(g')\xi(x)} = 0 \quad \text{for all } x \in G.$$

This shows that $\xi = 0$ whenever $\eta \neq 0$. Finally $\xi' = 0$ and Π is irreducible.

□

References

[1] J. Andele, Noyaux d'opérateurs sur les groupes de Lie exponentiels, Thèse, Université de Metz, 1997.

[2] D. Arnal, H. Fujiwara and J. Ludwig, Opérateurs d'entrelacement pour les groupes de Lie exponentiels, Amer. J. Math., **118** (1996), 839–878.

[3] P. Bernat, N. Conze, M. Duflo, M. Lévy-Nahas, M. Rais, P. Renouard and M. Vergne, Représentations des groupes de Lie résolubles, Monographies de la Société Mathématique de France, Dunod, Paris, 1972.

[4] L. Corwin, F. P. Greenleaf and R. Penney, A general character formula for irreducible projections on L^2 of a nilmanifold, Math. Ann., **225** (1977), 21–32.

[5] A. A. Kirillov, Unitary representations of nilpotent Lie groups, Uspehi Mat. Nauk, **17** (1962), 57–110, translated to: Russ. Math. Surv., **17** (1962), 53–104.

[6] H. Leptin and J. Ludwig, Unitary representation theory of exponential Lie groups, de Gruyter, Berlin, 1994.

[7] J. Ludwig, Irreducible representations of exponential solvable Lie groups and operators with smooth kernels, J. Reine Angew. Math., **339** (1983), 1–26.

[8] N. S. Poulsen, On C^∞-vectors and intertwining bilinear forms for representations of Lie groups, J. Functional Analysis, **9** (1972), 87–120.

[9] L. Pukanszky, On a property of the quantization map for the coadjoint orbits of connected Lie groups, The orbit method in representation theory, Copenhagen, 1988, Progr. Math., **82**, Birkhäuser Boston, Boston, MA, 1990, pp. 187–211.

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