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Normality for an inclusion of ergodic discrete measured equivalence relations in the von Neumann algebraic framework

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Abstract. It is shown that for the inclusion of factors $(B \subseteq A) := (W^*(\mathscr{S}, \omega) \subseteq W^*(\mathscr{R}, \omega))$ corresponding to an inclusion of ergodic discrete measured equivalence relations $\mathscr{S} \subseteq \mathscr{R}$, $\mathscr{S}$ is normal in $\mathscr{R}$ in the sense of Feldman-Sutherland-Zimmer ([**9**]) if and only if A is generated by the normalizing groupoid of B. Though this fact has been already obtained in [**3**], we reprove it here by a quite different method.

1. Introduction.

Feldman and Moore obtained a beautiful result in [**8**] which states that, if a (separable) von Neumann algebra A contains a so-called Cartan subalgebra D, then there exist a discrete measured equivalence relation $\mathscr{R}$ on a standard Borel probability measure space (X, μ) and a normalized 2-cocycle ω on $\mathscr{R}$ in such a way that the given inclusion $(D \subseteq A)$ is identified with $(L^\infty(X, \mu) \subseteq W^*(\mathscr{R}, \omega))$, where $W^*(\mathscr{R}, \omega)$ is roughly the matrix algebra over $\mathscr{R}$ twisted by ω, and $L^\infty(X, \mu)$ is regarded as the algebra of diagonal matrices. Hence this result completely characterizes von Neumann algebras admitting Cartan subalgebras as those that arise from discrete equivalence relations.

Moreover, Aoi showed in [**1**] that, for such an inclusion $(D \subseteq A)$ as above, every intermediate von Neumann subalgebra B between D and A has the form $B = W^*(\mathscr{S}, \omega)$ for a (unique) Borel subrelation $\mathscr{S}$ of $\mathscr{R}$. This adds yet another evidence of the close connection between von Neumann algebras with Cartan subalgebras and discrete equivalence relations.

In the meantime, Feldman, Sutherland and Zimmer introduced in [**9**] a notion of normality for an inclusion of (ergodic) discrete equivalence relations, which is regarded as a groupoid analogue of normal subgroups in group theory.

Given the results of Feldman-Moore and Aoi mentioned above, we might expect that every phenomenon that occurs in equivalence relations can be in principle "translated" into the one in von Neumann algebras with Cartan subalgebras, and vice versa. Thus we might ask ourselves the following question: what kind of notion does "normality" correspond to in the framework of operator algebras? The purpose of this

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paper is to give a satisfactory answer to this question in the case where an intermediate subalgebra is a factor. Namely, we will characterize the normality in the sense of [**9**] in a purely operator-algebraic term.

We should remark that an answer to the above question has already been obtained in [**3**, Theorem 6.5]. The proof given there has an "ergodic theory" nature (partial Borel transformations, full groups, etc.) and was obtained by "going back and forth" between a von Neumann algebra and its associated equivalence relation. We will reprove this theorem in this paper, but our strategy here is quite different from the one adopted in [**3**]. We will try to avoid measure-theoretic complicated arguments and to remain inside of the operator-algebraic framework as much as we can. In addition, it seems of independent interest. Therefore, we believe that it is worth presenting our proof here.

The organization of the paper is as follows. Section 2 is for preparations. We recall the definitions of von Neumann algebras associated with discrete equivalence relations, group coactions and the Jones basic extension. In Section 3, we characterize the normality in terms of minimal coactions of discrete groups. This is the first operator-algebraic characterization of normality. In Section 4, we first introduce a notion of the normalizing groupoid for an inclusion of von Neumann algebras. Then we characterize the normality by using this normalizing groupoid. In Appendix, we discuss some properties of the assignment established in Section 4.

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2. Notation and terminology.

Throughout this paper, we assume that all von Neumann algebras have separable preduals.

For a faithful normal semifinite weight ϕ on a von Neumann algebra M, we set

$$\mathfrak{n}_\phi := \{x \in M : \phi(x^*x) < \infty\}, \qquad \mathfrak{m}_\phi := \mathfrak{n}_\phi^* \mathfrak{n}_\phi, \qquad \mathfrak{m}_\phi^+ := \mathfrak{m}_\phi \cap M_+.$$

More generally, for an operator valued weight T ([**10**], [**11**], [**17**]) from a von Neumann algebra M to a von Neumann subalgebra N, we set

$$\mathfrak{n}_T := \{x \in M : T(x^*x) \in N_+\}, \qquad \mathfrak{m}_T := \mathfrak{n}_T^* \mathfrak{n}_T, \qquad \mathfrak{m}_T^+ := \mathfrak{m}_T \cap M_+.$$

The Hilbert space obtained from ϕ by the GNS-construction will be denoted by H_ϕ, and we let $\Lambda_\phi : \mathfrak{n}_\phi \to H_\phi$ stand for the natural injection. As usual, we use the symbols J_ϕ, Δ_ϕ to denote the modular conjugation and the modular operator associated with ϕ. The automorphism group of M is denoted by $\mathrm{Aut}(M)$.

2.1. Discrete measured equivalence relations.

Throughout this paper, we fix a discrete measured equivalence relation $\mathscr{R}$ on a standard Borel probability space $(X, \mathfrak{B}, \mu)$ in which μ is quasi-invariant for $\mathscr{R}$. For a general theory for discrete measured equivalence relations, refer to [**7**] and [**8**]. We denote by ν the (σ-finite) measure on $\mathscr{R}$ given by

$$\nu(E) := \int_X |r^{-1}(x) \cap E| \, d\mu(x) \qquad (E\text{: Borel subset of } \mathscr{R}),$$

where $r : \mathscr{R} \to X$ is the projection onto the first coordinate, and $|S|$ in general stands for the cardinality of a (countable) set S. The Radon-Nikodym derivative associated with this measured equivalence relation will be denoted by δ.

We also fix a (normalized) Borel 2-cocycle ω from $\mathscr{R}$ into the one-dimensional torus $\mathbf{T}$ in what follows. We then write $W^*(\mathscr{R}, \omega)$ for the von Neumann algebra on the Hilbert space $L^2(\mathscr{R}, \nu)$ obtained by the Feldman-Moore construction from $\mathscr{R}$ and ω ([**7**]). Briefly, the construction is as follows. We first define the subspace $\mathfrak{A}_I$ of $L^2(\mathscr{R}, \nu)$ by

$$\mathfrak{A}_I := \{\xi \in L^2(\mathscr{R}, \nu) : \xi \text{ is } \delta\text{-bounded and } \|\xi\|_I < \infty\}.$$

See [**12**] and [**23**] for the definition and properties of $\mathfrak{A}_I$ and for the terminology used above. We then introduce a product and an involution on $\mathfrak{A}_I$ as follows:

$$(f * g)(x, z) := \sum_{y \sim x} f(x, y) g(y, z) \omega(x, y, z), \qquad f^\sharp(x, z) := \delta(x, z)^{-1} \overline{f(z, x)},$$

where $\sum_{y \sim x}$ stands for the sum taken over all y equivalent to x. By the same argument as in [**12**] and [**23**], one can show that $\mathfrak{A}_I$ is a left Hilbert algebra (in fact, a Tomita algebra) in $L^2(\mathscr{R}, \nu)$. The left von Neumann algebra of $\mathfrak{A}_I$ is denoted by $W^*(\mathscr{R}, \omega)$. The modular operator Δ, the modular conjugation J are given by

$$\Delta\xi := \delta\xi, \qquad \{J\xi\}(x, y) = \delta(x, y)^{-1/2} \overline{\xi(y, x)}, \qquad (\xi \in \mathfrak{A}_I).$$

The left multiplication of $f \in \mathfrak{A}_I$ will be denoted by $L^\omega(f)$: $L^\omega(f)\xi := f * \xi$. Remark that every element $a \in W^*(\mathscr{R}, \omega)$ can be in fact written as $a = L^\omega(f)$ for some $f \in L^2(\mathscr{R}, \nu)$ ([**8**]). The abelian von Neumann algebra $L^\infty(X, \mu)$ is embedded into $W^*(\mathscr{R}, \omega)$ through the representation $f \in L^\infty(X, \mu) \longmapsto f \circ r$. We will always identity $L^\infty(X, \mu)$ with its image D under this representation. This algebra D is called a Cartan subalgebra of $W^*(\mathscr{R}, \omega)$.

We define $[\mathscr{R}]_*$ to be the set of all bimeasurable nonsingular transformations ρ from a Borel subset $\mathrm{Dom}(\rho)$ of X onto a Borel subset $\mathrm{Im}(\rho)$ of X satisfying $(x, \rho(x)) \in \mathscr{R}$ for μ-a.e. $x \in \mathrm{Dom}(\rho)$. For any $\rho \in [\mathscr{R}]_*$, set $\Gamma(\rho) := \{(x, \rho(x)) : x \in \mathrm{Dom}(\rho)\}$. Then, for each measurable function g on X of absolute value one, $L^\omega(\delta^{-1/2}(g \circ r)\chi_{\Gamma(\rho^{-1})})$ is a partial isometry in $W^*(\mathscr{R}, \omega)$ whose initial and final projections are respectively $\chi_{\mathrm{Dom}(\rho)}$ and $\chi_{\mathrm{Im}(\rho)}$. Here χ_E in general stands for the characteristic function of a set E. We denote by $\mathscr{GN}(D)$ the set all partial isometries in $W^*(\mathscr{R}, \omega)$ obtained in this way from $\rho \in [\mathscr{R}]_*$ and call it the normalizing groupoid of D in $W^*(\mathscr{R}, \omega)$. It is known that $\mathscr{GN}(D)$ coincides with the set of all partial isometries $v \in W^*(\mathscr{R}, \omega)$ satisfying $v^*v, vv^* \in D$ and $vDv^* = Dvv^*$.

If $\mathscr{S}$ is a Borel equivalence subrelation of $\mathscr{R}$, then we may construct $W^*(\mathscr{S}, \omega)$ which is a von Neumann subalgebra of $W^*(\mathscr{R}, \omega)$.

A Borel map c from $\mathscr{R}$ into a (second countable) locally compact group K is said to be a (Borel) 1-cocycle if it satisfies

$$c(x,y)c(y,z) = c(x,z) \qquad \text{for all } x \sim y \sim z.$$

For a Borel 1-cocycle $c : \mathscr{R} \to K$, the essential range of c is the smallest closed subset $\sigma(c)$ of K such that $c^{-1}(\sigma(c))$ has complement of measure zero. The asymptotic range $r^*(c)$ of c is by definition $\bigcap\{\sigma(c_B) : B(\subseteq X)$: Borel and $\mu(B) > 0\}$, where c_B stands for the restriction of c to the reduction $\mathscr{R}_B := \{(x,y) \in \mathscr{R} : x, y \in B\}$.

From this point on, assume that $\mathscr{R}$ is ergodic. Let $\mathscr{S}$ be a Borel subrelation of $\mathscr{R}$. By [**9**], we may choose a countable family $\{\varphi_i\}_{i\in I}$ of Borel maps from X into itself such that (i) $(x, \varphi_i(x)) \in \mathscr{R}$ for all $i \in I$ and μ-a.e. $x \in X$; (ii) for μ-a.e. $x \in X$, $\{\mathscr{S}(\varphi_i(x))\}_{i\in I}$ is a partition of $\mathscr{R}(x)$, where $\mathscr{R}(x) := \{y \in X : (x,y) \in \mathscr{R}\}$. The family $\{\varphi_i\}_{i\in I}$ is called choice functions for $\mathscr{S} \subseteq \mathscr{R}$. Once choice functions $\{\varphi_i\}_{i\in I}$ are fixed, we can define the index cocycle $\sigma : \mathscr{R} \to \Sigma(I)$ of the pair $\mathscr{S} \subseteq \mathscr{R}$, where $\Sigma(I)$ denotes the full permutation group on I, by the following rule:

$$\sigma(x,y)(i) = j \iff (\varphi_i(y), \varphi_j(x)) \in \mathscr{S}.$$

We say (see [**9**, Theorem 2.2]) that $\mathscr{S}$ is normal in $\mathscr{R}$ if there are choice functions $\{\varphi_i\}_{i\in I}$ for $\mathscr{S} \subseteq \mathscr{R}$ such that $(\varphi_i(x), \varphi_i(y)) \in \mathscr{S}$ for all $i \in I$ and a.e. $(x,y) \in \mathscr{S}$. According to [**9**, Theorem 2.2], there are several equivalent definitions for normality. If $\mathscr{S}$ is ergodic, then one of them is phrased in terms of a 1-cocycle as follows: $\mathscr{S}$ is normal in $\mathscr{R}$ if there exist a countable discrete group Γ, often denoted by $\mathscr{R}/\mathscr{S}$, and a Borel 1-cocycle $c : \mathscr{R} \to \Gamma$ such that (i) the subrelation $\operatorname{Ker}(c) := \{(x,y) \in \mathscr{R} : c(x,y) = e\}$ coincides with $\mathscr{S}$; (ii) the asymptotic range $r^*(c)$, which turns out to be the same as the essential range $\sigma(c)$, equals Γ.

2.2. Group coactions on von Neumann algebras.

Let K be a (second countable) locally compact group. We denote by $W^*(K)$ the group von Neumann algebra of K, i.e., the von Neumann algebra generated by the left regular representation λ_K of K on $L^2(K)$. Remark that $W^*(K)$ is the left von Neumann algebra of the left Hilbert algebra $C_c(K)$ of all continuous functions on K with compact support, where we consider on $C_c(K)$ the usual convolution and involution. The faithful semifinite normal weight on $W^*(K)$ associated with the left Hilbert algebra $C_c(K)$ is denoted by φ_K, the Plancherel weight on $W^*(K)$. It is well-known that the predual $A(K)$ of $W^*(K)$ has a structure of a commutative involutive Banach algebra. It is called the Fourier algebra of K (cf. [**5**], [**16**]).

There is a special unital normal $*$-isomorphism Δ_K from $W^*(K)$ into $W^*(K) \otimes W^*(K)$, called the coproduct of $W^*(K)$, defined by

$$\Delta_K(x) := W_K(1 \otimes x)W_K^* \qquad (x \in W^*(K)),$$

where W_K is a unitary on $L^2(K) \otimes L^2(K) = L^2(K \times K)$ given by $\{W_K\xi\}(g,h) := \xi(hg, h)$ $(\xi \in L^2(K \times K))$.

A coaction of K on a von Neumann algebra A is a unital normal $*$-isomorphism α from A into $W^*(K) \otimes A$ satisfying $(\Delta_K \otimes id_A) \circ \alpha = (id_{W^*(K)} \otimes \alpha) \circ \alpha$.

Suppose that α is a coaction of K on a von Neumann algebra A.

(1) For each $k \in K$, we define the subspace $A^\alpha(k)$ to be the set of elements $a \in A$ that satisfies $\alpha(a) = \lambda_K(k) \otimes a$. We call $A^\alpha(k)$ the spectral subspace of α belonging to k. Note that $A^\alpha := A^\alpha(e)$, where e is the identity of K, is a von Neumann subalgebra of A. It is called the fixed-point algebra of α.
(2) The map T_α defined by $T_\alpha(a) := (\varphi_K \otimes id_A)(\alpha(a))$ is an operator valued weight from A to A^α. The coaction α is said to be integrable if T_α is semifinite.
(3) The crossed product of A by α is the von Neumann algebra $\widehat{K}\,_\alpha\!\ltimes A := (\alpha(A) \cup L^\infty(K) \otimes \mathbf{C})''$.
(4) We say that α is faithful if $\{(id_{W^*(K)} \otimes \phi)(\alpha(a)) : a \in A,\ \phi \in A_*\}'' = W^*(K)$.
(5) We say that α is minimal (see [**13**], [**20**]) if it is faithful and satisfies $A \cap (A^\alpha)' = \mathbf{C}$. We say that α is strictly outer ([**13**], [**21**]) if $\widehat{K}\,_\alpha\!\ltimes A \cap \alpha(A)' = \mathbf{C}$. In [**20**], the term "outer" is used for "strictly outer". Due to [**20**, Proposition 6.2], minimality is equivalent to strict outerness when the action is integrable.

For the spectral theory for coactions such as the (Arveson) spectrum, the Connes spectrum and so on, we refer the readers to [**16**].

Let c be a Borel 1-cocycle from $\mathscr{R}$ into a (second countable) locally compact group K. Then one can construct, without the assumption that $\mathscr{R}$ is ergodic, a coaction α_c of K on $W^*(\mathscr{R}, \omega)$ whose fixed-point algebra $W^*(\mathscr{R}, \omega)^{\alpha_c}$ is exactly $W^*(\mathrm{Ker}(c), \omega)$ (see [**2**, Section 4]) for the details). In particular, $W^*(\mathscr{R}, \omega)^{\alpha_c}$ contains the Cartan subalgebra D. We succeeded in proving in [**2**] that the converse of this statement is also true:

THEOREM 2.1 ([**2**]). *Let α be a coaction of a (second countable) locally compact group K on $W^*(\mathscr{R}, \omega)$, where $\mathscr{R}$ is not necessarily ergodic. If the fixed-point algebra $W^*(\mathscr{R}, \omega)^\alpha$ contains the Cartan subalgebra D, then there exists a Borel 1-cocycle $c : \mathscr{R} \to K$ such that $\alpha = \alpha_c$.*

2.3. Basic extension.

Let $B \subseteq A$ be an inclusion of factors with a faithful normal conditional expectation E_B. (In our situation considered in the following sections, such an expectation always exists uniquely.) Fix a faithful normal state ϕ on B and set $\theta := \phi \circ E_B$. Then the equation $e_B \Lambda_\theta(a) := \Lambda_\theta(E_B(a))$ defines a projection $e_B \in B(H_\theta)$ onto $[\Lambda_\theta(B)]$, where $[S]$ is in general the closed subspace spanned by a set S. We call e_B the Jones projection of the inclusion $B \subseteq A$. The basic extension of this inclusion (by E_B) is the factor, denoted by A_1, acting on H_θ generated by A and e_B. It is known that $A_1 = J_\theta B' J_\theta$.

According to [**14**] (see also [**13**, Section 2]), there exists a faithful normal semifinite operator valued weight $\hat{E}_B$, called the operator valued weight dual to E_B, from A_1 to A. It satisfies $\hat{E}_B(e_B) = 1$ [**14**, Lemma 3.1], so that $A e_B A \subseteq \mathfrak{m}_{\hat{E}_B}$.

3. Characterization of normality in terms of coactions.

Throughout this section, we fix an ergodic Borel subrelation $\mathscr{S}$ of $\mathscr{R}$. We then set $A := W^*(\mathscr{R}, \omega)$ and $B := W^*(\mathscr{S}, \omega)$ for some ω.

Theorem 3.1. *The subrelation $\mathscr{S}$ is normal in $\mathscr{R}$ if and only if there exist a countable discrete group Γ and a minimal coaction α of Γ on A such that $A^\alpha = B$.*

Proof. Suppose that $\mathscr{S}$ is normal. Hence there exist a countable discrete group Γ and a Borel 1-cocycle $c : \mathscr{R} \to \Gamma$ such that (i) $\mathrm{Ker}(c) = \mathscr{S}$; (ii) $r^*(c) = \Gamma$. Let α be the coaction of Γ on A obtained from c by the construction mentioned just before Theorem 2.1. Since $\mathrm{Ker}(c) = \mathscr{S}$, it follows that $A^\alpha = B$. By the ergodicity of $\mathscr{S}$, we have $A \cap B' = \mathbf{C}$. In the meantime, by (the proof of) [**2**, Theorem 6.3], we know that the essential range $\sigma(c)$ of c equals the (Arveson) spectrum $\mathrm{Sp}(\alpha)$ of α, and that $r^*(c)$ coincides with the Connes spectrum $\Gamma(\alpha)$ of α. Since $A^\alpha = B$ is a factor, we deduce that $\mathrm{Sp}(\alpha) = \Gamma(\alpha) = r^*(c) = \Gamma$. From [**2**, Lemma 6.1], we now see that α is faithful. Consequently, α is minimal.

Conversely, suppose that there exist a countable discrete group Γ and a minimal coaction α of Γ on A such that $A^\alpha = B$. By Theorem 2.1, there is a Borel 1-cocycle $c : \mathscr{R} \to \Gamma$ such that $\alpha = \alpha_c$. Since $A^\alpha = B$, it follows that $\mathrm{Ker}(c)$ equals $\mathscr{S}$. The minimality of α implies its strict outerness by [**20**, Proposition 6.2]. In particular, the crossed product $\widehat{\Gamma}_\alpha \ltimes A$ is a factor. From [**16**], the Connes spectrum $\Gamma(\alpha)$ is Γ. By the fact mentioned in the previous paragraph, we have $r^*(c) = \Gamma$. Therefore, $\mathscr{S}$ is normal in $\mathscr{R}$. □

4. Characterization of normality using normalizing groupoids.

As in the preceding section, let us fix an ergodic Borel subrelation $\mathscr{S}$ of $\mathscr{R}$. Put $A := W^*(\mathscr{R}, \omega)$ and $B := W^*(\mathscr{S}, \omega)$ for some ω. Let A_1 be the Jones extension of the inclusion $B \subseteq A$. We denote by E_B the unique faithful normal conditional expectation from A onto B and by e_B the Jones projection determined by E_B.

Before we proceed, we think it proper to mention the result of [**15**] in relation to the subject treated in this section (see also [**18**], [**6**], [**4**] and the papers therein). In [**15**], Kosaki studied an irreducible subfactor N of a properly infinite factor M with the condition that (i) N is of finite Jones index and of depth 2; (ii) $N' \cap \langle M, e_N \rangle$ is abelian. His main result states that there exist a finite group G and an outer action of α of G on N such that M is the crossed product of N by α. Particularly, N is the fixed-point algebra of an outer coaction of G on M. In this case, G can be obtained as the so-called Weyl group of $M \supseteq N$, that is, the normalizer group $\mathscr{N}(N)$ of N in M modulo the unitary group of N. Then a key observation is that G is in bijective correspondence with the minimal projections in $N' \cap \langle M, e_N \rangle$. This observation seems suggestive even in our case, since $B' \cap A_1$ is also abelian. We however note that, because $B \subseteq A$ is in general of infinite index, the Weyl group of $A \supseteq B$ is no longer useful to our situation, as Example (1) of [**3**] shows (namely, $\mathscr{S} \subseteq \mathscr{R}$ can be normal even if the Weyl group is trivial). Therefore, we need to find a good substitute for a normalizer group, which is the following.

Definition 4.1. Consider a von Neumann algebra P and a von Neumann subalgebra Q of P. Define

$$\mathscr{G}\mathscr{N}(Q) := \{v \in P : v : \text{partial isometry}, \ vQv^* \subseteq Q, \, v^*Qv \subseteq Q\}.$$

We call this set the normalizing groupoid of Q in P.

Proposition 4.2. *Let $B \subseteq A$ be as above. For any $v \in \mathscr{GN}(B) \setminus \{0\}$, there exists a unique minimal projection z_v in $A_1 \cap B'$ such that $ve_Bv^* = z_vvv^*$. The projection z_v equals e_B if and only if v belongs to B. Moreover, we have $\widehat{E}_B(z_v) = 1$.*

Proof. Let $v \in \mathscr{GN}(B) \setminus \{0\}$ and $p := vv^*$. For any $b \in B$, we have

$$\begin{aligned} ve_Bv^*pbp &= ve_B \cdot v^*v \cdot v^*bv \cdot v^* = v \cdot v^*v \cdot v^*bv \cdot e_Bv^* \\ &= pbve_Bv^* = pbpve_Bv^*. \end{aligned}$$

Hence we obtain

$$ve_Bv^* \in pA_1p \cap (pBp)' = pA_1p \cap (B')p = (A_1 \cap B')p.$$

It follows that there exists an element $z_v \in A_1 \cap B'$ such that $ve_Bv^* = z_vp$. This z_v is unique, since the induction $y \in B' \longmapsto yp \in (B')p$ is an isomorphism due to the factoriality of B. Because of this, we also see that z_v is a projection.

Suppose that e is a projection in $A_1 \cap B'$ such that $e \le z_v$. Then we get

$$v^*ev \le v^*z_vv = v^*z_vpv = v^*ve_Bv^*v = v^*ve_B. \tag{4.1}$$

In the meantime, by the same argument as in the previous paragraph, we find that there is a unique projection $q \in A_1 \cap B'$ such that $v^*ev = qv^*v$. By (4.1), we have $qv^*v \le e_Bv^*v$. Since the induction $y \in B' \longmapsto yp \in (B')p$ is an isomorphism, it follows that $q \le e_B$. Because e_B is minimal in $A_1 \cap B'$, we must have either $q = 0$ or $q = e_B$. If $q = 0$, then $v^*ev = 0$, which implies that $ep = 0$. By the induction $B' \to (B')p$ being an isomorphism, we get $e = 0$. If $q = e_B$, then $v^*ev = e_Bv^*v$. So we have

$$ep = v(v^*ev)v^* = v(e_Bv^*v)v^* = z_vp.$$

Hence $e = z_v$. Therefore, z_v is minimal in $A_1 \cap B'$.

If v belongs to B, then, by the uniqueness of z_v, we have $z_v = e_B$. Conversely, if $z_v = e_B$, then we have $ve_B = z_vv = e_Bv$. Namely, v commutes with e_B. Hence v belongs to B.

Since $A \cap B' = \mathbf{C}$, $\widehat{E}_B(z_v)$ is in $(0, \infty]$. But, since $ve_Bv^* = z_vvv^*$ and $\widehat{E}_B(e_B) = 1$, we have $\widehat{E}_B(z_v)vv^* = \widehat{E}_B(ve_Bv^*) = v\widehat{E}_B(e_B)v^* = vv^*$. Hence $\widehat{E}_B(z_v) = 1$. □

Corollary 4.3. *The set $\{z_v : v \in \mathscr{GN}(B) \setminus \{0\}\}$ coincides with $\{e_B\}$ if and only if $\mathscr{GN}(B)'' = B$.*

Lemma 4.4. *Let $v_1, v_2 \in \mathscr{GN}(B) \setminus \{0\}$. Then $z_{v_1}z_{v_2} = 0$ if and only if $E_B(v_1^*Bv_2) = \{0\}$. Moreover, $z_{v_1} = z_{v_2}$ if and only if $v_1^*Bv_2 \subseteq B$.*

Proof. Suppose first that $z_{v_1}z_{v_2} = 0$. Then, for any $b \in B$, we have

$$E_B(v_1^*bv_2)e_B = e_Bv_1^*bv_2e_B = v_1^*z_{v_1}bz_{v_2}v_2 = 0.$$

Hence $E_B(v_1^*bv_2) = 0$.

Conversely, suppose that $E_B(v_1^*Bv_2) = \{0\}$. By the computation in the previous paragraph, one has $v_1^*bz_{v_1}z_{v_2}v_2 = 0$ for any $b \in B$. If $z_{v_1} = z_{v_2}$ ($=: z$), we have $v_1^*bzv_2 = 0$, that is, $zv_1v_1^*bv_2v_2^* = 0$ for all $b \in B$. Since the map $x \in B \mapsto xz \in Bz$ is an isomorphism as noted before, it follows that $v_1v_1^*Bv_2v_2^* = \{0\}$. But this cannot happen, because B is a factor. Therefore, we conclude that $z_{v_1}z_{v_2} = 0$.

If $z_{v_1} = z_{v_2}$, then we have, for any $b \in B$:

$$v_1^*bv_2e_B = v_1^*bz_{v_2}v_2 = v_1^*z_{v_1}bv_2 = e_Bv_1^*bv_2.$$

Hence $v_1^*bv_2$ belongs to B.

Conversely, assume that $v_1^*Bv_2 \subseteq B$. Since B is factor, $v_1v_1^*Bv_2v_2^*$ never equals $\{0\}$. In particular, $v_1^*Bv_2$ is nonzero. So, by assumption, we have $E_B(v_1^*Bv_2) = v_1^*Bv_2 \neq \{0\}$. Therefore, we obtain $z_{v_1} = z_{v_2}$. □

In what follows, let ξ_0 stand for the cyclic and separating unit vector $\chi_{\mathscr{D}} \in L^2(\mathscr{R})$ for A, where $\mathscr{D} := \{(x,x) : x \in X\}$. We denote by θ the faithful normal state on A determined by the unit vector ξ_0.

LEMMA 4.5. *For any $v \in \mathscr{GN}(B) \setminus \{0\}$, z_v is the projection onto the closure of the subspace $BvB\xi_0$.*

PROOF. Let z be the projection onto the closed subspace $K := [BvB\xi_0]$. Since K is B-invariant, z is in B'. If $b \in B$, then we have $JbJ\xi_0 = \sigma^\theta_{-i/2}(b^*)\xi_0$. From this, we see that K is JBJ-invariant. Hence z belongs to $(JBJ)' = A_1$. Consequently, we obtain $z \in A_1 \cap B'$. If $b_1, b_2 \in B$, then we have

$$\begin{aligned} vv^*(b_1vb_2\xi_0) &= v(v^*b_1vb_2)\xi_0 = vE_B(v^*b_1vb_2)e_B\xi_0 \\ &= ve_Bv^*b_1vb_2e_B\xi_0 = vv^*z_v(b_1vb_2\xi_0). \end{aligned}$$

This shows that $vv^*|_K = vv^*z_v|_K$, i.e., $vv^*z = vv^*z_vz$. Since $x \in B \mapsto xz \in B'z$ is an isomorphism, it follows that $z = z_vz$. By the minimality of z_v, z_vz is either 0 or z_v. If $z_vz = 0$, then $vv^*z = 0$. Because $x \in B \mapsto xz \in B'z$ is an isomorphism again, we would get $vv^* = 0$, a contradiction. So we must have $z_vz = z_v$. In this case, we easily see that $z = z_v$. □

LEMMA 4.6. *Let $x \in A$. Then the following are equivalent*:

(1) *There are projections z_1, z_2 in B' such that $xe_B = z_1x$ and $x^*e_B = z_2x^*$.*
(2) *Both x^*Bx and xBx^* are contained in B.*

If one of the above conditions holds true, then we can take z_1 (resp. z_2) to be the proejction onto $[BxB\xi_0]$ (resp. $[Bx^*B\xi_0]$).

PROOF. (1) ⇒ (2): For any $b \in B$, we have, by assumption:

$$x^*bxe_B = x^*bz_1x = x^*z_1bx = e_Bx^*bx.$$

This shows that x^*bx belongs to B. Similarly, we obtain $xbx^* \in B$.

(2) ⇒ (1): Let z_1 (resp. z_2) be the projection onto $[BxB\xi_0]$ (resp. $[Bx^*B\xi_0]$). As we have seen in the proof of Lemma 4.5, both z_1 and z_2 belong to $A_1 \cap B'$.

By definition, we have $z_1xb\xi_0 = xb\xi_0$ for all $b \in B$. It follows that $z_1xe_B = xe_B$. Let $\eta \in [B\xi_0]^\perp$ and $\xi \in L^2(\mathscr{R})$. Then we may choose sequences $\{b(n,i) : n \in \boldsymbol{N},\ 1 \le i \le k_n\}$ and $\{c_n : n \in \boldsymbol{N},\ 1 \le i \le k_n\}$ in B such that $z_1\xi = \lim_{n\to\infty}\sum_{i=1}^{k_n} b(n,i)xc(n,i)\xi_0$. Since $x^*b(n,i)xc(n,i)$ is in B by assumption, we have

$$(z_1x\eta \mid \xi) = \lim_{n\to\infty}\left(\eta \,\middle|\, \sum_{i=1}^{k_n} x^*b(n,i)xc(n,i)\xi_0\right) = 0.$$

So we obtain $z_1x\eta = 0$. Thus $z_1x(1-e_B) = 0$. Consequently, $z_1x = xe_B$. A similar argument yields $x^*e_B = z_2x^*$. □

LEMMA 4.7. *Let $x \in A$ be a nonzero element satisfying $xBx^* \subseteq B$ and $x^*Bx \subseteq B$. If $x = w|x|$ be the polar decomposition of x, then w belongs to $\mathscr{GN}(B)$. Moreover, z_w equals the projection onto $[BxB\xi_0]$.*

PROOF. Since x^*x and xx^* are in B, their support projections w^*w and ww^* both belong to B. Moreover, due to Lemma 4.6, we have $xe_B = z_1x$ and $x^*e_B = z_2x^*$, where z_1 (resp. z_2) is the projection onto $[BxB\xi_0]$ (resp. $[Bx^*B\xi_0]$).

Since $|x|$ is in B, it is easy to see that $xe_B = (we_B)\cdot(|x|e_B)$ is the polar decomposition of xe_B. From this, we have

$$\begin{aligned}(we_B)(we_B)^* &= \text{the range projection of } xe_B\\ &= \text{the range projection of } z_1x\\ &= z_1\cdot(\text{the range projection of } x) = z_1ww^*.\end{aligned}$$

Thus $we_Bw^* = z_1ww^*$. In particular, $we_B = z_1w$.

Next note that $x^* = w^*\cdot w|x|w^*$ is the polar decomposition of x^*. Now, by applying the arguments made in the preceding paragraphs to x^* and z_2 this time, we obtain $w^*e_B = z_2w^*$. From Lemma 4.6, it follows that w^*Bw and wBw^* are contained in B. Therefore, w belongs to $\mathscr{GN}(B)$, and it satisfies $z_w = z_1$. □

Suggested by Lemma 4.5, we define, for any $a \in A$, z_a to be the projection onto the closed subspace $[BaB\xi_0]$. Since $[BaB\xi_0]$ is both B-invariant and JBJ-invariant, z_a belongs to $A_1 \cap B'$.

PROPOSITION 4.8. *We have $Jz_aJ = z_{a^*}$ for any $a \in A$.*

PROOF. Let $a \in A$. Since $z_a \in A_1 \cap B' \subseteq L^\infty(\mathscr{R})$, there exists a Borel subset $\mathscr{E}$ of

$\mathscr{R}$ such that $z_a = \chi_{\mathscr{E}}$. Then we have $Jz_aJ = \chi_{\mathscr{E}^{-1}}$, where $\mathscr{E}^{-1} := \{(x,y) : (y,x) \in \mathscr{E}\}$.

Note that Jz_aJ is the projection onto $T := J[BaB\xi_0]$. Since $Jx\xi_0 = \sigma^\theta_{-i/2}(x^*)\xi_0$ for all $x \in A$, it follows that $T = [B\sigma^\theta_{-i/2}(a^*)B\xi_0]$. By the previous paragraph, we particularly have $\chi_{\mathscr{E}^{-1}}\sigma^\theta_{-i/2}(a^*)\xi_0 = \sigma^\theta_{-i/2}(a^*)\xi_0$. Since $\sigma^\theta_{-i/2}(a^*)\xi_0 = \delta^{1/2}a^*\xi_0$, we get $\chi_{\mathscr{E}^{-1}}\delta^{1/2}a^*\xi_0 = \delta^{1/2}a^*\xi_0$. Because the Radon-Nikodym derivative δ is nonzero everywhere, it follows that $\chi_{\mathscr{E}^{-1}}a^*\xi_0 = a^*\xi_0$. Namely, $Jz_aJa^*\xi_0 = a^*\xi_0$. From this, we easily deduce that $Jz_aJ \geq z_{a^*}$. Replacing a by a^* in this inequality, we get $Jz_{a^*}J \geq z_a$. Thus $Jz_aJ = z_{a^*}$. □

Corollary 4.9. *Let $v_1, v_2 \in \mathscr{GN}(B) \setminus \{0\}$. Then the following are equivalent:*

(1) $z_{v_1} = z_{v_2}$.
(2) $v_1^*Bv_2 \subseteq B$.
(3) $v_1Bv_2^* \subseteq B$.

Proof. We have already proven the equivalence of (1) and (2) in Lemma 4.4. By Proposition 4.8, (1) is equivalent to the condition $z_{v_1^*} = z_{v_2^*}$. But this condition is equivalent to (3) due to Lemma 4.4. □

We denote by $\{z_\gamma\}_{\gamma\in\Gamma}$ the set of all distinct projections z in $A_1 \cap B'$ obtained as $z = z_v$ for some $v \in \mathscr{GN}(B) \setminus \{0\}$. Let us denote by γ_0 the element $\gamma \in \Gamma$ satisfying $z_\gamma = e_B$.

Thanks to Proposition 4.8, for any $\gamma \in \Gamma$, the equation

$$Jz_\gamma J = z_{\gamma^{-1}}$$

defines a unique element γ^{-1} of Γ. It is obvious that the mapping $\gamma \in \Gamma \longmapsto \gamma^{-1} \in \Gamma$ is a period two bijection on Γ.

Next we will define a product on Γ which turns Γ into a (countable) group.

Fix any $\gamma_1, \gamma_2 \in \Gamma$. Then choose $v_1, v_2 \in \mathscr{GN}(B)$ satisfying $z_{\gamma_i} = z_{v_i}$ $(i = 1, 2)$. Since B is a factor, there exists a nonzero partial isometry $u \in B$ such that $u^*u \leq v_2v_2^*$ and $uu^* \leq v_1^*v_1$. Then $v := v_1uv_2$ is a nonzero partial isometry in A. It is easy to check that v in fact belongs to $\mathscr{GN}(B)$. So there is a $\gamma_3 \in \Gamma$ such that $z_{\gamma_3} = z_v$. We will call γ_3 the product of γ_1 and γ_2, and write $\gamma_3 = \gamma_1\gamma_2$. Now we have one thing to prove in order to ensure that this product is in fact well-defined.

Lemma 4.10. *The product γ_3 obtained above is independent of the choices of v_1, $v_2 \in \mathscr{GN}(B)$ and $u \in B$ satisfying $z_{\gamma_1} = z_{v_1}$, $z_{\gamma_2} = z_{v_2}$, $u^*u \leq v_2v_2^*$ and $uu^* \leq v_1^*v_1$.*

Proof. Let (w_1, w_2, s) be another triple enjoying the same properties as (v_1, v_2, u) does. Thanks to Corollary 4.4, it suffices to show $w_2^*s^*w_1^*Bv_1uv_2 \subseteq B$. But, from Corollary 4.9, we have $w_2^*s^*w_1^*Bv_1uv_2 \subseteq w_2^*s^*Buv_2 \subseteq w_2^*Bv_2 \subseteq B$, as desired. □

Theorem 4.11. *The index set Γ, equipped with the map $\gamma \in \Gamma \mapsto \gamma^{-1} \in \Gamma$ and the product defined above, is a (countable) group, where γ_0 is the identity.*

Proof. Thanks to Lemma 4.10, the product on Γ introduced above is in fact well-defined.

For each $\gamma \in \Gamma$, choose a nonzero $v_\gamma \in \mathscr{G}\mathscr{N}(B)$ with $z_{v_\gamma} = z_\gamma$. We agree that $v_{\gamma_0} = 1$.

[Associativity] Let γ_1, γ_2 and γ_3 be in Γ. First, choose a nonzero partial isometry $u \in B$ such that $uu^* \leq v_{\gamma_1}^* v_{\gamma_1}$ and $u^*u \leq v_{\gamma_2} v_{\gamma_2}^*$. Next we choose a nonzero partial isometry $v \in B$ such that $vv^* \leq (uv_{\gamma_2})^*(uv_{\gamma_2})$ and $v^*v \leq v_{\gamma_3} v_{\gamma_3}^*$. Since u and v belong to B, it follows that $z_{uv_{\gamma_2}} = z_{\gamma_2}$ and $z_{\gamma_3} = z_{vv_{\gamma_3}}$. By definition, we have $z_{\gamma_1\gamma_2} = z_{v_{\gamma_1} uv_{\gamma_2}}$ and $z_{\gamma_2\gamma_3} = z_{uv_{\gamma_2} vv_{\gamma_3}}$. So we have

$$z_{(\gamma_1\gamma_2)\gamma_3} = z_{(v_{\gamma_1} uv_{\gamma_2}) vv_{\gamma_3}} = z_{(v_{\gamma_1})(uv_{\gamma_2} vv_{\gamma_3})} = z_{\gamma_1(\gamma_2\gamma_3)}.$$

Hence the product is associative.

[Identity] Since $v_{\gamma_0} = 1$, it immediately follows that $z_{\gamma\gamma_0} = z_{\gamma_0\gamma} = z_\gamma$ for all $\gamma \in \Gamma$. Namely, $\gamma\gamma_0 = \gamma_0\gamma = \gamma$. Hence γ_0 is the identity of Γ.

[Inverse] Let $\gamma \in \Gamma$. By definition and Proposition 4.8, we have $z_{\gamma^{-1}} = z_{v_\gamma^*}$. So we have

$$z_{\gamma\gamma^{-1}} = z_{v_\gamma v_\gamma^*} = z_{\gamma_0}.$$

This means that $\gamma\gamma^{-1} = \gamma_0$. Similarly, one can prove that $\gamma^{-1}\gamma = \gamma_0$. This completes the proof. □

Recall that $A_1 \cap B'$ is contained in $L^\infty(\mathscr{R})$. So, for each $\gamma \in \Gamma$, there is a Borel subset $\mathscr{E}_\gamma$ of $\mathscr{R}$ such that $z_\gamma = \chi_{\mathscr{E}_\gamma}$. We agree that $\{\mathscr{E}_\gamma : \gamma \in \Gamma\}$ is a disjoint family, and that $\mathscr{E}_{\gamma_0} = \mathscr{S}$.

LEMMA 4.12. *We have $A = \mathscr{G}\mathscr{N}(B)''$ if and only if $\sum_{\gamma\in\Gamma} z_\gamma = 1$.*

PROOF. Assume that $A = \mathscr{G}\mathscr{N}(B)''$. Suppose then that $1 - \sum_{\gamma\in\Gamma} z_\gamma$ is nonzero. In this case, we have $\nu(\mathscr{R} \setminus \bigcup_{\gamma\in\Gamma} \mathscr{E}_\gamma) > 0$. So there exists a map $\rho \in [\mathscr{R}]_*$ such that $\mu(\mathrm{Dom}(\rho)) > 0$ and $\Gamma(\rho^{-1}) \subseteq \mathscr{R} \setminus \bigcup_{\gamma\in\Gamma} \mathscr{E}_\gamma$, where $\Gamma(\rho^{-1})$ stands for the graph of ρ^{-1}. Put $v := L^\omega(\delta^{-1/2}\chi_{\Gamma(\rho^{-1})})$. Since $z_\gamma v\xi_0 = 0$ for all $\gamma \in \Gamma$, we have, for any $a, b \in B$;

$$z_\gamma a v \sigma^\theta_{-i/2}(b^*)\xi_0 = a z_\gamma v JbJ\xi_0 = aJbJ z_\gamma v\xi_0 = 0.$$

This shows that $z_\gamma z_v = 0$ for all $\gamma \in \Gamma$.

Since the linear span of nonzero monomials in elements of $\mathscr{G}\mathscr{N}(B)$ is σ-weakly dense in A, there exists a nonzero $x \in A$, expressed as a product of finite number of elements in $\mathscr{G}\mathscr{N}(B)$, such that $E_B(x^*v) \neq 0$. It is easy to see that xBx^* and x^*Bx are both contained in B. If $x = w|x|$ is the polar decomposition of x, then, by Lemma 4.7, w belongs to $\mathscr{G}\mathscr{N}(B)$. Hence $z_w = z_{\gamma_1}$, that is, $z_{\gamma_1} L^2(\mathscr{R}) = [BwB\xi_0]$ for some $\gamma_1 \in \Gamma$. Since $E_B(w^*v) \neq 0$, there are vectors $\xi, \eta \in L^2(\mathscr{R})$ such that $(E_B(w^*v)e_B\xi \mid \eta) \neq 0$. So we may choose $b_0 \in B$ such that $(E_B(w^*v)b_0\xi_0 \mid \eta) \neq 0$. We have

$$0 \neq (e_B w^* v b_0 \xi_0 \mid \eta) = (v b_0 \xi_0 \mid w e_B \eta).$$

Again, we may choose $b_1 \in B$ such that $(vb_0\xi_0 \mid wb_1\xi_0) \neq 0$. This means that z_v is not orthogonal to z_{γ_1}, a contradiction.

Conversely, assume that $\sum_{\gamma\in\Gamma} z_\gamma = 1$. Set $C := \mathscr{GN}(B)''$ and consider the Jones projection e_C of the inclusion $C \subseteq A$. For any $\gamma \in \Gamma$, choose a $v_\gamma \in \mathscr{GN}(B)$ such that $z_\gamma = z_{v_\gamma}$. Then we clearly have $[Bv_\gamma B\xi_0] \subseteq [C\xi_0]$. This implies that $z_\gamma \leq e_C$. So $1 = \sum_{\gamma\in\Gamma} z_\gamma \leq e_C$, i.e., $e_C = 1$. Therefore, C coincides with A. □

In what follows, we assume that $A = \mathscr{GN}(B)''$. From Lemma 4.12 and the fact that $A_1 \cap B'$ is abelian, it follows that $A_1 \cap B'$ is generated by the minimal projections $\{z_\gamma\}_{\gamma\in\Gamma}$. So $A_1 \cap B'$ is isomorphic to $\ell^\infty(\Gamma)$.

Let us fix a $\gamma \in \Gamma$. Put $T := E_B \circ \hat{E}_B$. As we saw in Proposition 4.2, one has $\hat{E}_B(z_\gamma) = 1$. So $T(z_\gamma) = 1$. In the meantime, by [**14**, Lemma 1.3], we have $T^{-1} = (\hat{E}_B)^{-1} \circ E_B^{-1} = E_B(J\,E_B^{-1}(\,\cdot\,)\,J)$. From this and $z_{\gamma^{-1}} = Jz_\gamma J$, we obtain

$$T^{-1}(z_\gamma) = E_B(JE_B^{-1}(z_\gamma)J) = E_B(\hat{E}_B(z_{\gamma^{-1}})) = 1.$$

From these results and [**13**, Lemma 2.7], it follows that the index $\operatorname{Ind} T_{z_\gamma}$ of the conditional expectation T_{z_γ} from $z_\gamma A_1 z_\gamma$ onto Bz_γ is 1. In other words, we have $z_\gamma A_1 z_\gamma = Bz_\gamma$. Keeping in mind that $b \in B \longmapsto bz_\gamma$ is an isomorphism, we find from this result that, for any $a \in A$, there exists a unique element $S_\gamma(a) \in B$ such that $z_\gamma a z_\gamma = S_\gamma(a) z_\gamma$. Since $z_\gamma a z_\gamma = az_\gamma$ for all $a \in B$, it follows that S_γ is a normal projection of norm one from A onto B. By [**17**, Proposition 10.17], we obtain $S_\gamma = E_B$. Thus we have proven

Lemma 4.13. *For any $\gamma \in \Gamma$, we have $z_\gamma a z_\gamma = E_B(a) z_\gamma$ for all $a \in A$.*

Lemma 4.14. *For any $\gamma \in \Gamma$, the subfactor Q_γ of A_1 generated by A and z_γ coincides with A_1.*

Proof. It suffices to show that e_B belongs to Q_γ. By Lemma 4.13, the σ-weak closure of $A + Az_\gamma A$ coincides with Q_γ. From this, it results that the σ-weak closure of $Az_\gamma A$ is a σ-weakly closed two-sided ideal of Q_γ, and hence coincides with Q_γ. So $Az_\gamma A$ is σ-weakly dense in Q_γ. From this, we see that, if T_γ is the restriction of $\hat{E}_B$ to Q_γ, then T_γ is still semifinite, because $T_\gamma(z_\gamma) = 1$. In particular, the restriction of $\hat{\theta} := \theta \circ \hat{E}_B$ to Q_γ is semifinite. Moreover, since $\sigma_t^{\hat{\theta}}(z_\gamma) = z_\gamma$ for all $t \in \boldsymbol{R}$, we have $\sigma_t^{\hat{\theta}}(Q_\gamma) = Q_\gamma$. It follows from [**19**] that there exists a unique faithful normal conditional expectation F from A_1 onto Q_γ such that $\hat{\theta} \circ F = \hat{\theta}$.

Take a $v \in \mathscr{GN}(B)$ such that $z_v = z_\gamma$, and denote by $\eta : B' \to (B')v^*v$ the induction $\eta(y) := yv^*v\ (y \in B')$. Note that $F(e_B)$ belongs to B' as well. Then we have

$$\eta(F(e_B)) = F(e_B)v^*v = F(v^*ve_B) = F(v^*z_\gamma v) = v^*z_\gamma v.$$

By a similar calculation, we have $\eta(e_B) = v^*z_\gamma v$. Since η is an isomorphism, we get $e_B = F(e_B) \in Q_\gamma$. □

PROPOSITION 4.15. *For each $\gamma \in \Gamma$, there exists a unique $*$-automorphism β_γ of A_1 such that $\beta_\gamma|_A = id_A$ and $\beta_\gamma(z_{\gamma_1}) = z_{\gamma_1\gamma^{-1}}$. Moreover, β is an action of Γ on A_1.*

PROOF. Let us fix a $\gamma \in \Gamma$. By Lemma 4.14, A_1 is generated by A and z_γ. We also have $z_\gamma \in (A_1 \cap B')_{E_B \circ \hat{E}_B}$. By Lemma 4.13 and [**13**, Lemma 2.4], there exists a $*$-automorphism τ_γ of A_1 satisfying $\tau_\gamma|_A = id_A$ and $\tau_\gamma(e_B) = z_\gamma$ and $\hat{E}_B \circ \tau_\gamma = \tau_\gamma \circ \hat{E}_B$.

Since τ_γ is an automorphism and $\tau_\gamma|_A = id_A$, we have $\tau_\gamma(A_1 \cap B') = A_1 \cap B'$. From this and the fact that $\{z_\gamma\}_{\gamma\in\Gamma}$ is the all minimal projections in $A_1 \cap B'$, there exists a bijection π_γ of Γ such that $\pi_\gamma(\gamma_0) = \gamma$ and $\tau_\gamma(z_{\gamma_1}) = z_{\pi_\gamma(\gamma_1)}$ for any $\gamma_1 \in \Gamma$. Fix any $\gamma_1 \in \Gamma$. Choose $v, w \in \mathscr{GN}(B)$ satisfying $z_v = z_{\gamma_1}$ and $z_w = z_\gamma$. Also take a suitable partial isometry $u \in B$ such that $z_{\gamma_1\gamma} = z_{vuw}$. We have $ve_B = z_{\gamma_1}v$. By applying τ_γ to this identity, we get $vz_\gamma = z_{\pi_\gamma(\gamma_1)}v$. From this, we have, for any $b_1, b_2 \in B$:

$$b_1 v z_\gamma u w b_2 \xi_0 = b_1 z_{\pi_\gamma(\gamma_1)} v u w b_2 \xi_0.$$

Since $uwb_2\xi_0$ belongs to the range of z_γ, the above identity can be rewritten as

$$b_1 v u w b_2 \xi_0 = z_{\pi_\gamma(\gamma_1)} b_1 v u w b_2 \xi_0.$$

Because the linear span of elements of the form $b_1vuwb_2\xi_0$ $(b_1, b_2 \in B)$ forms a dense subspace of the range of $z_{\gamma_1\gamma}$, we deduce that $z_{\gamma_1\gamma} \le z_{\pi_\gamma(\gamma_1)}$. Hence $z_{\gamma_1\gamma} = z_{\pi_\gamma(\gamma_1)}$. This means that $\pi_\gamma(\gamma_1) = \gamma_1\gamma$. Consequently, $\tau_\gamma(z_{\gamma_1}) = z_{\gamma_1\gamma}$ for all $\gamma_1 \in \Gamma$. By setting $\beta_\gamma := \tau_\gamma^{-1}$, we completes the proof. $\square$

We use the same notation β for the map $A_1 \to \ell^\infty(\Gamma) \otimes A_1$ defined by

$$\{\beta(x)\xi\}(\gamma) := \beta_\gamma(x)\xi(\gamma) \qquad (x \in A_1, \xi \in \ell^2(\Gamma) \otimes L^2(\mathscr{R})).$$

Note that this β is the action of $\underline{\ell^\infty(\Gamma)^{op}}$ on A_1 induced by the original action $\gamma \in \Gamma \mapsto \beta_\gamma \in \mathrm{Aut}(A_1)$. Here "the action of $\ell^\infty(\Gamma)^{op}$" means the one in the framework of locally compact quantum groups (see [**20**]). One can easily check that

$$\beta(z_\gamma) = \sum_{\gamma_1\in\Gamma} \delta_{\gamma_1} \otimes z_{\gamma\gamma_1^{-1}} \tag{4.2}$$

for any $\gamma \in \Gamma$, where $\delta_\gamma := \chi_{\{\gamma\}}$.

LEMMA 4.16. *The fixed-point algebra $(A_1)^\beta$ of the action β defined above coincides with A.*

PROOF. By Proposition 4.15, $(A_1)^\beta$ contains A. Let $P := (J(A_1)^\beta J)'$, which is an intermediate subfactor of $B \subseteq A$. Since $(A_1)^\beta$ is the basic extension of $P \subseteq A$, the Jones projection e_P induced by the (unique) faithful normal conditional expectation from A onto P is in $(A_1)^\beta \cap P' \subseteq A_1 \cap B'$. So there is a subset Γ_0 of Γ such that $e_P = \sum_{\gamma\in\Gamma_0} z_\gamma$. Since e_P belongs to $(A_1)^\beta$, it follows from Proposition 4.15 that we have, for any $\gamma_1 \in \Gamma$:

$$e_P = \beta_{\gamma_1}(e_P) = \sum_{\gamma \in \Gamma_0} z_{\gamma\gamma_1^{-1}} = \sum_{\gamma \in \Gamma_0 \gamma_1^{-1}} z_\gamma.$$

Hence $\Gamma_0 = \Gamma_0 \gamma_1^{-1}$ for any $\gamma_1 \in \Gamma$, which yields $\Gamma_0 = \Gamma$. Thus $e_P = 1$. Namely, $P = A$. Therefore, $(A_1)^\beta$ equals A. □

Theorem 4.17. *Let* $X := \sum_{\gamma \in \Gamma} \lambda(\gamma)^* \otimes z_\gamma \in W^*(\Gamma) \otimes A_1$. *Then the equation*

$$\alpha(a) := X^*(1 \otimes a)X \qquad (a \in A)$$

defines a strictly outer coaction α *of* Γ *on* A. *Moreover, if we set*

$$\pi(x) := X^* \beta(x) X \qquad (x \in A_1)$$

then the map π *gives a* $*$*-isomorphism from* A_1 *onto the crossed product* $\widehat{\Gamma}_\alpha \ltimes A$ *satisfying* $(id \otimes \pi)(X) = (W_\Gamma)_{12}$, $\pi(a) = \alpha(a)$ $(\forall a \in A)$ *and* $\hat{\alpha} \circ \pi = (id \otimes \pi) \circ \beta$.

Proof. Note that the unitary W_Γ introduced in Subsection 2.2 is given by

$$W_\Gamma = \sum_{\gamma \in \Gamma} \lambda_\Gamma(\gamma)^* \otimes \delta_\gamma.$$

By using this and Eq (4.2), one can easily check that

$$(id \otimes \beta)(X) = (W_\Gamma)_{12} X_{13}.$$

We also have $(\Delta_\Gamma \otimes id)(X^*) = (X^*)_{23}(X^*)_{13}$. Now, in order to obtain the assertion of this theorem, we have only to apply [**22**, Proposition 1.22] to our situation above. □

Lemma 4.18. *The fixed-point algebra* A^α *of the coaction* α *obtained in Theorem* 4.17 *equals* B.

Proof. According to Theorem 4.17, the coaction α is given by

$$\alpha(a) = \sum_{\gamma_1, \gamma_2 \in \Gamma} \lambda_\Gamma(\gamma_1 \gamma_2^{-1}) \otimes z_{\gamma_1} a z_{\gamma_2} \qquad (a \in A). \tag{4.3}$$

If $b \in B$, then, by (4.3),

$$\begin{aligned}\alpha(b) &= \sum_{\gamma_1, \gamma_2 \in \Gamma} \lambda_\Gamma(\gamma_1 \gamma_2^{-1}) \otimes z_{\gamma_1} b z_{\gamma_2} = \sum_{\gamma_1, \gamma_2 \in \Gamma} \lambda_\Gamma(\gamma_1 \gamma_2^{-1}) \otimes z_{\gamma_1} z_{\gamma_2} b \\ &= \sum_{\gamma_1 \in \Gamma} 1 \otimes z_{\gamma_1} b = 1 \otimes b.\end{aligned}$$

Hence b belongs to A^α. Conversely, if $a \in A^\alpha$, then we have

$$1 \otimes a = \sum_{\gamma_1,\gamma_2\in\Gamma} \lambda_\Gamma(\gamma_1\gamma_2^{-1}) \otimes z_{\gamma_1} a z_{\gamma_2} \qquad (a \in A). \tag{4.4}$$

Applying $\varphi_\Gamma \otimes id$ to both sides of (4.4), where φ_Γ is the Plancherel state on $W^*(\Gamma)$, we obtain $a = \sum_{\gamma_1\in\Gamma} z_{\gamma_1} a z_{\gamma_1}$. By Lemma 4.13, we have

$$\sum_{\gamma_1\in\Gamma} z_{\gamma_1} a z_{\gamma_1} = \sum_{\gamma_1\in\Gamma} E_B(a) z_{\gamma_1} = E_B(a).$$

Hence $a = E_B(a) \in B$. Therefore, we completes the proof. □

THEOREM 4.19. *The subrelation $\mathscr{S}$ is normal in $\mathscr{R}$ if and only if the normalizing groupoid $\mathscr{GN}(B)$ of B in A generates A.*

PROOF. Suppose that $\mathscr{S}$ is normal in $\mathscr{R}$. By Theorem 3.1, there exist a minimal coaction α of a countable discrete group Γ on A satisfying $A^\alpha = B$. Since $\Gamma(\alpha) = \mathrm{Sp}(\alpha) = \Gamma$ and $A^\alpha (= B)$ is a factor, we find that the spectral subspace $A^\alpha(\gamma)$ is nonzero for all $\gamma \in \Gamma$, and that the linear span of $\bigcup_{\gamma\in\Gamma} A^\alpha(\gamma)$ is σ-strongly* dense in A. Let $x \in A^\alpha(\gamma)$ for some $\gamma \in \Gamma$. If $x = u|x|$ be the polar decomposition of x, then $|x|$ is in $A^\alpha = B$ and u is in $A^\alpha(\gamma)$. Note that both u^*u and uu^* belong to B. Moreover, uBu^* and u^*Bu are contained in B. Hence u belongs to $\mathscr{GN}(B)$. In particular, x is in $\mathscr{GN}(B)''$. Therefore, $\mathscr{GN}(B)''$ equals A.

Conversely, suppose that $\mathscr{GN}(B)'' = A$. By Theorem 4.17 and Lemma 4.18, we now know that there exists a strictly outer (hence minimal) coaction of a discrete group Γ on A whose fixed-point algebra is B. From Theorem 3.1, $\mathscr{S}$ is normal in $\mathscr{R}$. □

Appendix

A. The range of the mapping $v \in \mathscr{GN}(B) \longmapsto z_v \in A_1 \cap B'$

We saw that every nonzero element $v \in \mathscr{GN}(B)$ gives rise to a minimal projection z_v in $A_1 \cap B'$ satisfying $\hat{E}_B(z_v) = 1$. In fact, by the arguments preceding Lemma 4.13, we know that $\hat{E}_B(z_v) = E_B^{-1}(z_v) = 1$. In particular, it follows (see [**13**]) that the index $\mathrm{Ind}\, T_{z_v}$ of the conditional expectation T_{z_v} from $z_v A_1 z_1$ onto $B z_v$ given by $T_{z_v} := z_v T(z_v)^{-1} T|_{z_v A_1 z_v}$ is 1, where $T :\, = E_B \circ \hat{E}_B$. In this Appendix, we will show that every minimal projection $z \in A_1 \cap B'$ with $\hat{E}_B(z) = E_B^{-1}(z) = 1$ arises in this way.

Let us fix a minimal projection $z \in A_1 \cap B'$ satisfying $\hat{E}_B(z) = E_B^{-1}(z) = 1$. Since we particularly have $\mathrm{Ind}\, T_z = T(z)T^{-1}(z) = 1$, it follows that $zA_1 z = Bz$.

Meanwhile, choose a Borel subset $\mathscr{E}$ of $\mathscr{R}$ such that $z = \chi_{\mathscr{E}}$. Then there is a $\rho \in [\mathscr{R}]_*$ such that $\Gamma(\rho^{-1}) \subseteq \mathscr{E}$ and $\nu(\Gamma(\rho^{-1})) > 0$. Put $v :\, = L^\omega(\delta^{-1/2}\chi_{\Gamma(\rho^{-1})}) \in \mathscr{GN}(D)$. Denote by z_0 the projection onto the closed subspace $[BvB\xi_0]$. As we saw before, z_0 is a projection in $A_1 \cap B'$. Since $zv\xi_0 = v\xi_0$ by the definition of v, we have, for any $b, c \in B$:

$$zbvc\xi_0 = zJ\sigma^\theta_{-i/2}(c^*)Jbv\xi_0 = J\sigma^\theta_{-i/2}(c^*)Jzbv\xi_0 = J\sigma^\theta_{-i/2}(c^*)Jvc\xi_0 = bvc\xi_0.$$

From this, we see that z_0 is majorized by z. By the minimality of z, we obtain $z_0 = z$. In particular, we have

$$zve_B = ve_B. \tag{A.1}$$

Lemma A.1. *The partial isometry $v \in \mathcal{GN}(D)$ defined above belongs to $\mathcal{GN}(B)$.*

Proof. By (A.1), we have $zve_Bv^* = ve_Bv^*$. Thus ve_Bv^* is a projection in A_1 majorized by z. Since $zA_1z = Bz$, there exists a unique projection p in B such that $ve_Bv^*(= z(ve_Bv^*)z) = pz$. From this, we have $ve_B = (ve_Bv^*)v = pzv = zpv$. So $v\xi_0 = ve_B\xi_0 = zpv\xi_0 = pv\xi_0$. Since ξ_0 is separating for A, we get $v = pv$. Hence $ve_B = zv$. By (the proof of) Lemma 4.6, we obtain $v^*Bv \subseteq B$.

Let $z' := JzJ \in A_1 \cap B'$. Then $z' = \chi_{\mathcal{E}^{-1}}$, where $\mathcal{F}^{-1} := \{(x, y) : (y, x) \in \mathcal{F}\}$ for a subset $\mathcal{F}$ of $\mathcal{R}$. Since $\Gamma(\rho) = \Gamma(\rho^{-1})^{-1} \subseteq \mathcal{E}^{-1}$, we have $zv^*\xi_0 = v^*\xi_0$. From this, it follows that z' is the projection onto the closed subspace $[Bv^*B\xi_0]$. So we get $z'v^*e_B = v^*e_B$. Meanwhile, we have $z'A_1z' = Bz'$. By the same arguments as in the case of the projection z, we can deduce the inclusion $vBv^* \subseteq B$. By Lemma 4.7, v belongs to $\mathcal{GN}(B)$. □

From the discussion made above, we now have the following.

Theorem A.2. *The mapping $v \in \mathcal{GN}(B) \setminus \{0\} \longmapsto z_v \in A_1$ has the image which consists exactly of the minimal projections z in $A_1 \cap B'$ satisfying $\hat{E}_B(z) = E_B^{-1}(z) = 1$.*

Remark. If z is a minimal projection in $A_1 \cap B'$ satisfying $\hat{E}_B(z) = E_B^{-1}(z) = 1$, as above, we obviously have $\operatorname{Ind} T_z = 1$. Conversely, if $z \in A_1 \cap B'$ satisfies $\operatorname{Ind} T_z = 1$, then $zA_1z = Bz$. Consider the partial isometry v constructed in the discussion preceding Lemma A.1. By proceeding as in the proof of Lemma A.1, it can be verified that we have $ve_B = zv$ and $v^*e_B = z'v^*$, where $z' = JzJ$. In particular, $ve_Bv^* = zvv^*$ and $v^*e_Bv = z'v^*v$. Since $\hat{E}_B(e_B) = 1$, it follows that $\hat{E}_B(z) = \hat{E}_B(z') = 1$. Therefore, we have proven that $\operatorname{Ind} T_z = 1$ if and only if $\hat{E}_B(z) = E_B^{-1}(z) = 1$.

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