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Compression theorems for surfaces and their applications

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Abstract. Let M be a complete glued surface whose sectional curvature is greater than or equal to k and $\triangle pqr$ a geodesic triangle domain with vertices p, q, r in M. We prove a compression theorem that there exists a distance nonincreasing map from $\triangle pqr$ onto the comparison triangle domain $\widetilde{\triangle} pqr$ in the two-dimensional space form with sectional curvature k. Using the theorem, we also have some compression theorems and an application to a circular billiard ball problem on a surface.

1. Introduction.

The compression theorem of Rubinstein and Weng ([**9**]) is stated as follows.

Theorem 1 (Compression theorem). *Suppose $\triangle p_i q_i r_i (i = 1, 2)$ are two triangles on spheres S_i with radii r_1, r_2 $(r_1 < r_2)$ respectively. Suppose the circular measures of the sides of $\triangle p_i q_i r_i$ $(i = 1, 2)$ are less than π. If $d(p_1, q_1) = d(p_2, q_2)$, $d(q_1, r_1) = d(q_2, r_2)$, $d(r_1, p_1) = d(r_2, p_2)$, then there exists a map h of $\triangle p_1 q_1 r_1$ onto $\triangle p_2 q_2 r_2$ so that $d(x_1, y_1) \geq d(h(x_1), h(y_1))$ for any points x_1, y_1 in $\triangle p_1 q_1 r_1$ where $d(\cdot, \cdot)$ denotes the distance function. Moreover, if x_1, y_1 are not on the same side, then the inequality strictly holds.*

They claim that the radius of the inscribed circle and the circumscribed circle of a triangle can be compared to the one of a comparison triangle, and introduce other applications ([**9**]). Moreover, they have stated that the Steiner ratio of a sphere in the minimal network problem is $\sqrt{3}/2$ as an application of this theorem. Weng and Rubinstein ([**13**]) have stated the compression theorems for convex surfaces. We will study these theorems in a wider class of surfaces.

Let M be a glued surface. Here we say that a two-dimensional topological manifold M without boundary is by definition a *glued surface* if a surface M has a decomposition $M = \cup_{\alpha \in \Lambda} M_\alpha$ such that

(1) M_α is a two-dimensional smooth complete Riemannian manifold with piecewise smooth boundary for any $\alpha \in \Lambda$.
(2) $\mathrm{Int} M_\alpha \cap M_\beta = \varnothing$ if $\alpha \neq \beta \in \Lambda$, where $\mathrm{Int} M_\alpha$ is the interior of M_α.
(3) If S is the set of points $p \in M$ such that p belongs to the boundary of some component M_α and it is not smooth at p, then $\inf\{d(p, q) \mid p, q \in S, p \neq q\}$ is positive, where $d(\cdot, \cdot)$ is the natural distance function associated with the Riemannian metric. Furthermore, for a point $p \in S$ there exist finitely many $\alpha \in \Lambda$ with $p \in M_\alpha$.

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The surfaces of objects are sometimes like glued surfaces. The boundary of a domain in the Euclidean space $\boldsymbol{E}^3$ which is given by some inequalities $f_j(x,y,z) \leq c_j$ is also sometimes a glued surface. A glued surface is a two-dimensional Riemannian manifold with piecewise smooth metric such that the set of non-differentiable points is a graph consisting of differentiable curves as its edges. Jacobi vector fields along geodesics in glued Riemannian manifolds have been studied in [**6**], [**8**], [**10**], [**11**] and [**12**]. We call a point p in the interior of a component M_α a *regular* point, a point in the boundaries of just two components M_α and M_β a *smooth gluing* point if the boundaries ∂M_α and ∂M_β are differentiable at the point p, and other points *singular* points.

We say that a glued surface M is *with curvature* $\geq k$ if the Gaussian curvature is greater than or equal to k at any regular point, the sum of geodesic curvature of gluing boundaries is nonnegative at any smooth gluing point and the sum of angles at any singular point is less than 2π. Here the geodesic curvature $\kappa(p)$ at a point p in the boundary B_α of M_α is given by $\nabla_X N_\alpha = -\kappa(p)X$ where N_α is the inward unit normal vector field to B_α of M_α and X is a tangent vector to B_α at p. Let $M(k)$ denote a complete simply connected surface with constant Gaussian curvature k. If k is positive, zero and negative, then $M(k)$ is isometric to a sphere with radius $1/\sqrt{k}$, a Euclidean plane and a hyperbolic plane with curvature k, respectively. Let $T(p,q)$ denote a minimal geodesic segment connecting p and q for points $p,q \in M$. We say that $\triangle pqr$ is a *geodesic triangle domain* for points $p,q,r \in M$ if $\triangle pqr$ is a simply connected domain bounded by $T(p,q) \cup T(q,r) \cup T(r,p)$, and that a triangle $\widetilde{\triangle} pqr$ in $M(k)$ is a *comparison triangle domain* to $\triangle pqr$ if the lengths of its sides are the same as the ones of $\triangle pqr$. The points $\tilde{p},\tilde{q},\tilde{r}$ denote the corresponding vertices of $\widetilde{\triangle} pqr$ to p,q, and r, respectively. Namely, $\widetilde{\triangle} pqr = \triangle \tilde{p}\tilde{q}\tilde{r}$.

Let D be a domain in a glued surface M. We say that D is *convex* if there exists a minimal geodesic segment $T(p,q)$ contained in D for any points $p,q \in D$. Let D and $\widetilde{D}$ be closed convex domains in a glued surface M and $M(k)$, respectively, such that the boundaries ∂D and $\partial \widetilde{D}$ are rectifiable curves and their lengths are equal. We say that a surjective map $h : D \longrightarrow \widetilde{D}$ is a *compression map* from D onto $\widetilde{D}$ if $d(x,y) \geq d(h(x),h(y))$ for any points $x,y \in D$ and the restriction map $h : \partial D \longrightarrow \partial \widetilde{D}$ preserves the length of any subarc of ∂D.

In the present paper we will prove some compression theorems for glued surfaces with curvature $\geq k$.

THEOREM 2. *Let M be a glued surface with curvature $\geq k$ and $\triangle pqr$ an arbitrary convex geodesic triangle domain in M. Let $\widetilde{\triangle} pqr$ be a comparison triangle domain in $M(k)$. Then there exists a compression map from $\triangle pqr$ onto $\widetilde{\triangle} pqr$.*

Let p be a point in a glued surface M and let a positive number a be less than the diameter of M. Let $C'(p,a) = \{x \in M \mid d(p,x) = a\}$. The set $C'(p,a)$ divides M into at least two parts. In general, $C'(p,a)$ is not connected. We say that a connected component $C(p,a)$ of $C'(p,a)$ is a *circle* with center p and radius a. If the domain bounded with $C'(p,a)$ is convex, then $C'(p,a)$ is connected and at least of class C^1.

Let n be an integer greater than 2. Let $p_1,\dots,p_n$ be points in $C(p,a)$ which are in this order and $p_{n+1} = p_1$. We say that $\cup_{i=1}^n T(p_i,p_{i+1})$ is a *regular n-gon* if $d(p_i,p_{i+1}) = d(p_{i+1},p_{i+2})$ for all $i = 1,\dots,n-1$. A regular n-gon $\cup_{i=1}^n T(p_i{}',p_{i+1}{}')$ in $M(k)$ with vertices $p_1{}', \ldots, p_n{}'$ and $d(p_1{}', p_2{}') = d(p_1, p_2)$ is called a *comparison regular n-gon* to $\cup_{i=1}^{n} T(p_i, p_{i+1})$. In general, the radius of the circle in which the vertices $p_1{}', \ldots, p_n{}'$ lie is not equal to a.

THEOREM 3. *Let M be a glued surface with curvature $\geq k$ and let C be a circle in M. Assume that a regular n-gon P with vertices in C bounds a simply connected convex domain D containing the center of C. If $\widetilde{D}$ is the domain bounded by a comparison regular n-gon to P in $M(k)$, then there exists a compression map from D onto $\widetilde{D}$.*

Let C be a circle in a glued surface M with length L. A circle $\widetilde{C}$ in $M(k)$ with length L is called a *comparison circle* to C. If a circle C in M is convex, then the domain D bounded by C is simply connected and has at most one singular point which is the center of C.

THEOREM 4. *Let M be a glued surface with curvature $\geq k$ and let C be a circle in M. Assume that C bounds a convex domain D containing the center of C. If $\widetilde{D}$ is the domain in $M(k)$ bounded by a comparison circle to C, then there exists a compression map from D onto $\widetilde{D}$.*

We will apply compression theorems to a Steiner minimum tree problem and a circular billiard ball problem. Let M be a glued surface. Let P be a finite set of points in M. A shortest network interconnecting P is called a *Steiner minimum tree* which is denoted as $\mathrm{SMT}(P)$. An $\mathrm{SMT}(P)$ may have vertices which are not in P. Such vertices are called *Steiner points*. The Steiner minimum tree problem is an interesting subject to study (cf. [**5**], [**9**]). The following is a direct application of compression theorems which was used in computing the Steiner ratio of spheres by Rubinstein and Weng ([**9**]).

THEOREM 5. *Let M be a glued surface with curvature $\geq k$ and D a convex domain in M. Assume that there exist a comparison domain $\widetilde{D}$ in $M(k)$ and a compression map from D onto $\widetilde{D}$. Let P be a finite set of points $\{p_i\}$ in the boundary of D and $\widetilde{P}$ the set $\{\tilde{p}_i\}$ with $\tilde{p}_i = h(p_i)$. Then, $L(\mathrm{SMT}(P))$ is greater than or equal to $L(\mathrm{SMT}(\widetilde{P}))$ where $L(\mathrm{SMT}(P))$ is the length of $\mathrm{SMT}(P)$.*

We are going to show a theorem concerning a circular billiard ball problem on surfaces. Let C be a simple closed curve of class C^1 with length L in a glued surface M which bounds a domain D. We require that a geodesic line in D is reflected on the boundary C under the law that the angle of reflection with C is equal to the angle of incidence. Then such a geodesic line is called a *reflecting geodesic line*. The reflecting geodesic lines may be considered to be billiard ball trajectories. Let $\gamma : (-\infty, \infty) \longrightarrow D$ be a reflecting geodesic line with unit speed such that it hits C at $\cdots < t_{i-1} < t_i < t_{i+1} < \cdots$. Let $L(\gamma(t_i), \gamma(t_{i+1}))$ be the arclength of C from $\gamma(t_i)$ to $\gamma(t_{i+1})$ measured with anticlockwise rotation for all i. We say that

$$\alpha = \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} L\big(\gamma(t_{i-1}), \gamma(t_i)\big)$$

is the *slope* (or α/L the *rotation number*) of γ. Then, the inequality $0 \leq \alpha \leq L$ holds.

THEOREM 6. *Let M be a glued surface with curvature ≥ 0 and let C be a convex circle with length L. Then given α with $0 < \alpha < L$ there exists a reflecting geodesic line $\gamma : (-\infty, \infty) \longrightarrow D$ with slope α such that*

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d\big(\gamma(t_{i-1}), \gamma(t_i)\big) \geq \frac{L}{\pi} \sin \frac{\pi \alpha}{L}.$$

The right hand side of the inequality above is the average such as in the left hand side for a reflecting line with slope α in a comparison circle in a Euclidean plane $M(0)$. We will see how to find γ in Section 8.

2. Preliminaries.

Alexandrov spaces: Le M be a glued surface with curvature $\geq k$. Then, M is an Alexandrov space with curvature $\geq k$. Hence, M satisfies the following properties ([**3**], [**4**], [**8**]).

Any minimal geodesic segment does not pass through a singular point in M, except for its endpoints. Two minimal geodesic segments with the same endpoints do not intersect at any other point.

Let $\triangle pqr$ be a geodesic triangle in M and $\widetilde{\triangle} pqr$ a comparison triangle to $\triangle pqr$ in $M(k)$. If k is positive we assume that the perimeter of $\triangle pqr$ is less than $2\pi/\sqrt{k}$.

(1) If $x \in T(q, r)$, $\tilde{x} \in T(\tilde{q}, \tilde{r})$ with $d(q, x) = d(\tilde{q}, \tilde{x})$, then $d(p, x) \geq d(\tilde{p}, \tilde{x})$.
(2) If $x \in T(p, q)$, $y \in T(p, r)$, $\tilde{x} \in T(\tilde{p}, \tilde{q})$ and $\tilde{y} \in T(\tilde{p}, \tilde{r})$ with $d(p, x) = d(\tilde{p}, \tilde{x})$ and $d(p, y) = d(\tilde{p}, \tilde{y})$, then $d(x, y) \geq d(\tilde{x}, \tilde{y})$.
(3) $\angle pqr \geq \angle \tilde{p}\tilde{q}\tilde{r}$, $\angle qrp \geq \angle \tilde{q}\tilde{r}\tilde{p}$ and $\angle rpq \geq \angle \tilde{r}\tilde{p}\tilde{q}$.

Convex circles: Let C be a circle in a glued surface with curvature $\geq k$ and let p be the center of C.

LEMMA 7. *If the domain D bounded by C is convex, then C is of class C^1, the domain D is simply connected and any point in D other than the center of C cannot be singular.*

PROOF. Suppose there exists a point $q \in C$ such that there exist two minimal geodesic segments from p to q. Then, C is not differentiable at q and the outer angle of C at q is less than π. Hence, there exist points q_1 and q_2 in C near q such that the unique minimal geodesic segment $T(q_1, q_2)$ is not contained in D, contradicting that D is convex. Since $T(p, x)$ depend continuously on $x \in C$, we see that $D = \cup_{x \in C} T(p, x)$, and, hence, D is homeomorphic to a disk in an Euclidean plane. This completes the proof. □

Circular billiards: Let $c : (-\infty, \infty) \longrightarrow M$ be a parametrization of a convex circle C with length L by arclength and let $\gamma : (-\infty, \infty) \longrightarrow D$ be a reflecting geodesic line where D is the domain bounded by C. Let $s = (s_i)_{i \in \boldsymbol{Z}}$ be a sequence such that $\gamma(t_i) = c(s_i)$, $0 < s_{i+1} - s_i < L$ for all $i \in \boldsymbol{Z}$ where $\boldsymbol{Z}$ is the set of all integers. Then, $\sum_{i=1}^{n} L(\gamma(t_{i-1}), \gamma(t_i)) = s_n - s_0$ for all positive integers n. Let $a_0 < a_n < a_0 + nL$ be given. Let $H(u_1, \ldots, u_{n-1})$ be a function given by

$H(u_1, \ldots, u_{n-1}) = -\sum_{i=1}^{n} d(c(u_{i-1}), c(u_i))$ for $0 < u_i - u_{i-1} < L$ where $u_0 = a_0$ and $u_n = a_n$. Then, there exists a sequence $a_1, \ldots, a_{n-1}$ where H assumes the minimum and a broken geodesic $\cup_{i=1}^{n} T(c(a_{i-1}), c(a_i))$ makes a reflecting geodesic. We say that a reflecting geodesic line γ is *minimal* if any subsegment $\gamma \mid [t_i, t_j]$ of γ is given as above for all $i < j$. Given α with $0 < \alpha < L$ there exists a minimal reflecting geodesic line $\gamma : (-\infty, \infty) \longrightarrow D$ with slope α (see [**1**], [**7**]). For a circle $\widetilde{C}$ with length L in the Euclidean plane $\boldsymbol{E}^2$ a reflecting geodesic line $\cup_{i=-\infty}^{\infty} T(\tilde{c}(\alpha(i-1)), \tilde{c}(\alpha i))$ is always minimal where $\tilde{c} : (-\infty, \infty) \longrightarrow \boldsymbol{E}^2$ is a parametrization of $\widetilde{C}$ by arclength.

3. Contraction map.

Throughout this section let M be a glued surface. Let p, q, r, s be points in M such that $d(p, q) = d(s, r)$. Let A (or B, resp.) be a simple curve connecting q and r (or s and p, resp.). Assume that $d(p, A) = d(p, q)$, $d(s, A) = d(s, r)$ and that $K = T(p, q) \cup A \cup T(r, s) \cup B$ is a simple curve which divides M into two parts. Let D be a domain bounded by K.

Lemma 8. *There exists a map $g : D \longrightarrow T(p, q)$ such that*

(1) *$d(x, y) \geq d(g(x), g(y))$ for any $x, y \in D$,*
(2) *$d(x, y) = d(g(x), g(y))$ for any $x, y \in T(s, r)$,*
(3) *$g(x) = x$ for any $x \in T(p, q)$.*

Proof. Let $g : D \longrightarrow T(p, q)$ be a map given as follows. If $d(x, A) \geq d(p, q)$ for $x \in D$, then $g(x) = p$. If $d(x, A) < d(p, q)$ for $x \in D$, then $g(x)$ is the point in $T(p, q)$ such that $d(g(x), q) = d(x, A)$. We prove that the map g satisfies the condition. Let $x, y \in D$. We can assume without loss of generality that $d(x, A) \geq d(y, A)$. Let w, z be the feet of x, y on A, respectively, i.e., $d(x, w) = d(x, A)$ and $d(y, z) = d(y, A)$ hold with $w, z \in A$. Then we have the inequality

$$\begin{aligned} d(x, y) &\geq d(x, z) - d(y, z) \geq d(x, w) - d(y, z) \\ &\geq d(g(x), q) - d(g(y), q) = d(g(x), g(y)). \end{aligned}$$

If $x, y \in T(s, r)$, then $w = z = r$ and the equalities hold. This completes the proof. □

We call a map satisfying the properties in this lemma a *contraction map* of D to $T(p, q)$. We will need 2 special cases to make a compression map.

Lemma 9. *Let D be a sector with vertex p whose boundary is $T(p, q) \cup C(p) \cup T(p, r)$ where $C(p)$ is a subarc of a circle connecting q and r with center p. Then there exists a contraction map g of the sector D to the radius $T(p, q)$.*

Any geodesic biangle is a kind of a sector.

Lemma 10. *Let D be a geodesic biangle domain whose boundary consists of 2 minimal geodesic segments connecting p and q. Then, there exists a contraction map of D to a minimal geodesic segment $T(p, q)$.*

4. Basic partition and map for triangles.

In this section we show how to divide a triangle domain into two thinner triangle domains and other domains. The method will be used for all triangle domains which will appear in the proof of Theorem 2 in Section 5.

Let $\triangle pqr$ be a convex geodesic triangle domain in a glued surface with curvature $\geq k$ and $\widetilde{\triangle} pqr$ a comparison triangle domain in $M(k)$. Let $\tilde{s}_i{}^n$ be the point in $T(\tilde{q}, \tilde{r})$ with $d(\tilde{q}, \tilde{s}_i{}^n) = nd(\tilde{q}, \tilde{r})/2^i$ in $M(k)$ for any $n = 0, 1, \ldots, 2^i$ and let $S_i = \{T(\tilde{p}, \tilde{s}_i{}^n) \mid n = 0, 1, \ldots, 2^i\} \cup T(\tilde{q}, \tilde{r})$ be sets in $M(k)$ for all positive integer i.

Let m be the midpoint between q and r in $T(q, r)$. Then, by (2) in Section 2, we have that $d(p, m) \geq d(\tilde{p}, \tilde{m})$. If $d(p, m) = d(\tilde{p}, \tilde{m})$, then we set $D = T(p, q) \cup T(p, m) \cup T(p, r) \cup T(q, r)$ and define a map $g : D \longrightarrow S_1$ as the union of isometric maps of corresponding sides. Then, the map g is distance nonincreasing as was seen in Section 2.

We assume that $d(p, m) > d(\tilde{p}, \tilde{m})$. Since $\triangle pqr$ is simply connected, the connected component $C(q)$ (and $C(r)$, resp.) of the circle passing through m with center q (and r, resp.) and radius $d(m, q)$ divides the triangle $\triangle pqr$ into two parts. We have two possibilities. Both $C(q)$ and $C(r)$ intersect the same side of the triangle $\triangle pqr$, say $T(p, r)$. Or $C(q)$ intersects $T(p, q)$ and $C(r)$ intersects $T(p, r)$. In the former case, the intersection point $s \in T(p, r)$ with $C(q)$ satisfies the inequality

$$\begin{aligned} d(p, s) &= d(p, r) - d(r, s) < d(p, r) - d(r, m) \\ &= d(\tilde{p}, \tilde{r}) - d(\tilde{r}, \tilde{m}) < d(\tilde{p}, \tilde{m}), \end{aligned}$$

since $C(q)$ is in the same side as q with respect to $C(r)$. We can take a point $r' \in C(q)$ such that $d(p, r') = d(\tilde{p}, \tilde{m})$ and $d(p, x) > d(\tilde{p}, \tilde{m})$ for any $x \in C(q)$ where x is between m and r'. In the same way, we can take a point q' such that $d(p, q') = d(\tilde{p}, \tilde{m})$ and $d(p, x) > d(\tilde{p}, \tilde{m})$ for any $x \in C(r)$ where x is between m and q'. In the latter case, the intersection point $s \in C(q)$ satisfies the inequality

$$\begin{aligned} d(p, s) &= d(p, q) - d(q, s) \\ &= d(\tilde{p}, \tilde{q}) - d(\tilde{q}, \tilde{m}) < d(\tilde{p}, \tilde{m}). \end{aligned}$$

We can take a point $r' \in C(q)$ such that $d(p, r') = d(\tilde{p}, \tilde{m})$ and $d(p, x) > d(\tilde{p}, \tilde{m})$ for any $x \in C(q)$ where x is between m and r'. In the same way, we can take a point q' such that $d(p, q') = d(\tilde{p}, \tilde{m})$ and $d(p, x) > d(\tilde{p}, \tilde{m})$ for any $x \in C(r)$ where x is between m and q'. Thus we have two convex geodesic triangle domains $\triangle pqr'$ and $\triangle prq'$ in $\triangle pqr$ in both cases. In fact, the convexity of $\triangle prq'$ is proved as follows. Let $T(x, y)$ be a minimal geodesic segment connecting $x \in \triangle prq'$ and $y \in \triangle prq'$. If $T(x, y)$ is contained in $\triangle prq'$, then we have nothing to prove. Suppose $T(x, y)$ is not contained in $\triangle prq'$. We may assume without loss of generality that $x \in T(p, q')$ and $y \in T(q', r)$ and $T(x, y)$ intersect $C(r)$ at some point z which is between m and q'. Then, we have

$$\begin{aligned} d(x, y) &= d(x, z) + d(z, y) \\ &\geq d(x, q') + d(q', y), \end{aligned}$$

since

$$d(x,z) \geq d(p,z) - d(p,x)$$
$$\geq d(p,q') - d(p,x) = d(x,q')$$

and

$$d(z,y) \geq d(r,z) - d(r,y)$$
$$= d(r,q') - d(r,y) = d(q',y).$$

Since the inner angle of $\triangle prq'$ at q' is less than or equal to π, there exists a curve in $\triangle prq'$ which connects x and y and whose length is less than or equal to $d(x,q')+d(q',y)$. Thus, we can find a minimal geodesic segment connecting x and y in $\triangle prq'$, contradicting that $T(x,y)$ is a minimal geodesic segment. This shows that $\triangle prq'$ is convex.

Let $D = \triangle pqr - (\mathrm{Int}\triangle pqr' \cup \mathrm{Int}\triangle prq')$. We will make a map $g : D \longrightarrow S_1$. Let $x \in D$. If $x \in T(p,q)$ (and $x \in T(p,r)$, resp.), then $g(x)$ is the point in $T(\tilde{p},\tilde{q})$ (and $T(\tilde{p},\tilde{r})$, resp.) such that $d(p,x) = d(\tilde{p},g(x))$. Assume that $x \notin T(p,q) \cup T(p,r)$. Set $p_0 = p$, $p_1 = q$, $p_2 = r$, $\tilde{p}_0 = \tilde{p}$, $\tilde{p}_1 = \tilde{q}$, $\tilde{p}_2 = \tilde{r}$ for convenience. Assume that the nearest vertex of $\triangle pqr$ to x is p_j. If $d(p_j,x) \leq d(\tilde{p}_j,\tilde{m})$, then $g(x)$ is the point in $T(\tilde{p}_j,\tilde{m})$ with $d(p_j,x) = d(\tilde{p}_j,g(x))$. Otherwise, $g(x) = \tilde{m}$.

We have to prove that the map g satisfies the condition: $d(x,y) \geq d(g(x),g(y))$ for any points $x,y \in D$ and the equality holds if x,y are in the same sides of $\triangle pqr$. We divide $\triangle pqr$ into 6 parts in such a way that $\triangle pqr = \triangle pqr' \cup \triangle prq' \cup B(q) \cup B(r) \cup B(p) \cup E$, where $B(q)$ is the domain bounded by $T(q,r') \cup C(q) \cup T(q,m)$, $B(r)$ is the domain bounded by $T(r,q') \cup C(r) \cup T(r,m)$, $B(p)$ is the domain bounded by $T(p,r') \cup C(p) \cup T(p,q')$ and E is the remainder part. Here $C(p)$ is the subarc of the circle connecting q' and r' with center p and others. Then, $D = \partial\triangle pqr' \cup \partial\triangle prq' \cup B(q) \cup B(r) \cup B(p) \cup E$ where ∂K is the boundary of any set K. It follows from the properties in Sections 2 and 3 that the restrictions of g to those sets, $g \mid \partial\triangle pqr'$, $g \mid \partial\triangle prq'$, $g \mid B(q)$, $g \mid B(r)$, $g \mid B(p)$, $g \mid E$, satisfy the distance non-increasing condition which we are going to prove. Let $x,y \in D$. Let $T(x_0,x_1), T(x_1,x_2), \ldots, T(x_{n-1},x_n)$ be the subsegments of $T(x,y)$ each of which is contained in a single component of the decomposition of $\triangle pqr$ where $x_0 = x$ and $x_n = y$. Thus we have

$$d(x,y) = d(x_0,x_1) + \cdots + d(x_{n-1},x_n)$$
$$\geq d(g(x_0),g(x_1)) + \cdots + d(g(x_{n-1}),g(x_n))$$
$$\geq d(g(x),g(y))$$

In the next section we will divide these triangle domains into thinner triangle domains successively in the same way as above and make a compression map h of $\triangle pqr$ by using this partition. Then we use the notation D and g given as above.

5. Proof of Theorem 2.

We inductively make the subsets D_i of $\triangle pqr$ such that $D_1 \subset D_2 \subset \cdots \subset D_i \subset \cdots$ and the compression maps $h_i : D_i \longrightarrow S_i$ with $h_i \mid D_j = h_j$ for all $j < i$.

Let $D_1 = D$ and $h_1 = g$ where D and g were made for $\triangle pqr$ in the previous section. Assume that D_i and h_i are constructed so that $\triangle pqr - D_i$ consists of 2^i open triangle domains. Each triangle domain is divided into two triangle domains and one set D as in the basic partition of the previous section. Let D_{i+1} be the union of D_i and 2^i D's. As was seen in the previous section, each D is mapped onto the set $T(\tilde{p}, {\tilde{s}_i}^k) \cup T(\tilde{p}, {\tilde{s}_i}^{k+1}) \cup T(\tilde{p}, {\tilde{s}_{i+1}}^\ell) \cup T({\tilde{s}_i}^k, {\tilde{s}_i}^{k+1})$ for some k by a compression map where ${\tilde{s}_{i+1}}^\ell$ is the midpoint between ${\tilde{s}_i}^k$ and ${\tilde{s}_i}^{k+1}$. We define a map $h_{i+1} : D_{i+1} \longrightarrow S_{i+1}$ as follows. If $x \in D_i$, then $h_{i+1}(x) = h_i(x)$. If $x \in D_{i+1} - D_i$ and $x \in D$ where D is one of 2^i D's as in the previous section, then $h_{i+1}(x)$ is by definition the point sent by the compression map g defined on D.

Let $X = \cup_{i=1}^\infty D_i$ and $W = \triangle pqr - X$. The length of opposite sides to p for triangles in $\triangle pqr - \mathrm{Int} D_i$ is $d(\tilde{q}, \tilde{r})/2^i$. Therefore, W consists of segments one of whose endpoints is p, and geodesic biangle domains one of whose vertices is p. There exists the isometric map from each of these segments $T(p, s)$ to the segment connecting $\tilde{p}$ and the point $\tilde{s}$ in $T(\tilde{q}, \tilde{r})$ corresponding to s. There exists also a contraction map from each of these geodesic biangle domains to the segment connecting $\tilde{p}$ and the point $\tilde{s}$ in $T(\tilde{q}, \tilde{r})$ corresponding to the other vertex than p. Now we define a map $h : \triangle pqr \longrightarrow \widetilde{\triangle} pqr$ by combining these maps.

We show that the map h is a compression map of $\triangle pqr$ onto $\widetilde{\triangle} pqr$. Let W_1 be the set of all points x in W such that x is a limit point of a sequence of points in some sides of triangles and let W_2 be the set of all points x in W such that x is an interior point of some geodesic biangle domain. Let $x, y \in \triangle pqr$. *Case* (1): If $x, y \in D_i$ for some i, then $d(x, y) \geq d(h_i(x), h_i(y)) = d(h(x), h(y))$. *Case* (2): If $x, y \in W_1$, then $d(x, y) \geq d(h(x), h(y))$, since x, y are limit points of sequences of points in some sides of triangles. *Case* (3): Assume that $x, y \in W_2$. Let x_1, y_1 be points in the boundaries of geodesic biangle domains which contain x and y such that $x_1, y_1 \in T(x, y)$, $T(x, x_1) - \{x_1\} \subset W_2$, $T(y_1, y) - \{y_1\} \subset W_2$. Then, we have the inequality

$$\begin{aligned} d(x, y) &= d(x, x_1) + d(x_1, y_1) + d(y_1, y) \\ &\geq d(h(x), h(x_1)) + d(h(x_1), h(y_1)) + d(h(y_1), h(y)) \\ &\geq d(h(x), h(y)), \end{aligned}$$

by using *Case* (2). For other cases the inequalities required are proved in the same way. This completes the proof of Theorem 2.

6. Proof of Theorem 3.

Let p_0 be the center of a circle C and $D = \cup_{i=1}^n \triangle p_0 p_i p_{i+1}$ a regular n-gon whose vertices are in C. Let $\widetilde{D} = \cup_{i=1}^n \triangle \tilde{p}_0 {p_i}' {p_{i+1}}'$. Notice that $\triangle \tilde{p}_0 {p_i}' {p_{i+1}}'$ may not be comparison triangle domains to $\triangle p_0 p_i p_{i+1}$ for $i = 1, \ldots, n$. In general, $\triangle p_0 p_i p_{i+1}$ may not be convex. In such a case the distance $d_i(\cdot, \cdot)$ is defined as the infimum of the lengths of curves which are contained in $\triangle p_0p_ip_{i+1}$ and used instead of the distance $d(\cdot,\cdot)$, and, then, the comparison theorems as in Section 2 and compression theorem are true also. It should be noted that $d_i(x,y) = d(x,y)$ for any points $x, y \in \triangle p_0p_ip_{i+1}$ such that a minimal geodesic segment connecting x and y is contained in $\triangle p_0p_ip_{i+1}$. The triangle domain $\triangle p_0p_ip_{i+1}$ is mapped to an isosceles comparison triangle domain $\widetilde{\triangle} p_0p_1p_2$ by a compression map h_i for every $i = 1,\dots,n$. Since $\sum_{i=1}^{n} \angle p_ip_0p_{i+1} \le 2\pi$ and $\angle p_ip_0p_{i+1} \ge \angle\tilde{p}_1\tilde{p}_0\tilde{p}_2$ for all $i = 1,\dots,n$, we have $\angle\tilde{p}_1\tilde{p}_0\tilde{p}_2 \le 2\pi/n = \angle p_1{}'\tilde{p}_0p_2{}'$. It follows from Lemma 1 in [**9**] that for every $i = 1,\dots,n$ there exists a map $h_i{}' : \triangle\tilde{p}_0\tilde{p}_1\tilde{p}_2 \longrightarrow \triangle\tilde{p}_0p_i{}'p_{i+1}{}'$ such that $d(x,y) \ge d(h_i{}'(x), h_i{}'(y))$ for $x, y \in \triangle\tilde{p}_0\tilde{p}_1\tilde{p}_2$ and $d(x,y) = d(h_i{}'(x), h_i{}'(y))$ for $x, y \in T(\tilde{p}_1,\tilde{p}_2)$. Let $h : D \longrightarrow \widetilde{D}$ be a map given by sending $x \in \triangle p_0p_ip_{i+1}$ to $h_i{}'h_i(x)$ for all $i = 1,\dots,n$. Then, h is a compression map from D onto $\widetilde{D}$. This completes the proof.

7. Proof of Theorem 4.

In order that a convex circle C will be approximated by regular n-gons we first prove the following lemma.

LEMMA 11. *Let n be any integer greater than 2. There exists a regular n-gon $\cup_{i=1}^{n} T(p_i.p_{i+1})$ whose vertices are in C where $p_{n+1} = p_1$. It satisfies that $d(p_i,p_{i+1}) \le L/n$ for all $i = 1,\dots,n$ where L is the length of C.*

PROOF. Let $c : (-\infty,\infty) \longrightarrow M$ be a parametrization of a circle C with length L by arclength such that $c(0) = p_1$. For a point $q = c(t)$ let $q' = c(a) = c(a+L)$ be the antipodal point of q in C with $a < t < a+L$. Namely, the antipodal point q' satisfies that $d(q,q') = \max\{d(q,x) \mid x \in C\}$. Let p be the center of convex circle C and let x, y, z be distinct points in C such that $T(x,z)$ intersects $T(p,y)$ at a point w. Then, it follows that $d(x,y) < d(x,z)$, since $d(x,y) < d(x,w) + d(w,y) < d(x,w) + d(w,z) = d(x,z)$. This means that there exists only one antipodal point q', and, moreover, that either there exists a unique minimal geodesic $T(q,q')$ connecting q and q', or some biangle domain with vertices q and q' contains the center p of C. Hence, if $f : [a, a+L] \longrightarrow \boldsymbol{R}$ is a function given by $f(s) = d(q,c(s))$ for $s > t$ and $f(s) = -d(q,c(s))$ for $s < t$, then the function $f(s)$ is monotone increasing. In particular, it follows that for any b with $0 < b < d(q,q')$ there exist just two points $q_1 = c(s_1)$ and $q_2 = c(s_2)$ in C such that $d(q,q_1) = d(q,q_2) = b$ and $a < s_2 < t < s_1 < a+L$.

Let $s_0 \in [0,L]$ be such that for any s with $0 < s < s_0$ there exists a broken geodesic $\cup_{i=1}^{n} T(p_i,p_{i+1})$ satisfying that the points $p_1,\dots,p_{n+1}$ are in this order in C, $p_1 = c(0)$, $p_{n+1} = c(s)$ and $d(p_1,p_2) = \cdots = d(p_n,p_{n+1})$. Let u_0 be the maximum of these s_0. We have to prove that $u_0 = L$. Obviously, it follows that $u_0 > 0$. Suppose that $u_0 < L$. If p_{i+1} is not the antipodal point of p_i in C for every $i = 1,\dots,n$, then we can find a number s_0 with $s_0 > u_0$ such that it satisfies the condition. Hence, we suppose that there exists at least one p_{i+1} which is the antipodal point of p_i in C. Then, p_{i-1} is not the antipodal point of p_i in C or p_{i+2} is not the antipodal point of p_{i+1} in C. This implies that $d(p_{i-1},p_i) < d(p_i,p_{i+1})$ or $d(p_i,p_{i+1}) > d(p_{i+1},p_{i+2})$, contradicting the choice of u_0. This completes the proof. □

We prove Theorem 4. Let C be a convex circle as in Theorem 4. Let P_n be a regular n-gon with vertices in C and $h_n : D_n \longrightarrow \widetilde{D}_n$ a compression map where D_n and $\widetilde{D}_n$ are the domains bounded by P_n and a comparison regular n-gon $\widetilde{P}_n$ in $M(k)$ to P_n with center $\widetilde{p}_0$, respectively.

Let N be a countable dense set in $\mathrm{Int}D$. Since $M(k)$ is finitely compact and D_n converges to D as $n \longrightarrow \infty$, there exists a subsequence $\{m\}$ of $\{n\}$ such that $h_m(q)$ is defined for sufficiently large m and converges to a point $h(q)$ as $m \longrightarrow \infty$ for any $q \in N$. We make a compression map $h : D \longrightarrow \widetilde{D}$ as follows. Let p be a point in $\mathrm{Int}D$. For any positive ϵ there exist a point $q \in N$ with $d(p,q) < \epsilon/3$ and an m_0 such that both $h_m(p)$ and $h_m(q)$ are contained in D_m for any $m \geq m_0$ and $d(h_m(q), h_k(q)) < \epsilon/3$ for any $k, m \geq m_0$. Then we have the inequality

$$d(h_m(p), h_k(p)) \leq d(h_m(p), h_m(q)) + d(h_m(q), h_k(q)) + d(h_k(q), h_k(p)) < \epsilon.$$

Since $M(k)$ is complete, we see that $h_m(p)$ converges to a point $h(p)$ as $m \longrightarrow \infty$. Let p be a point in $\partial D = C$. Suppose a sequence $\{q_\ell\}$ with $q_\ell \in \mathrm{Int}D$ converges to p as $\ell \longrightarrow \infty$. Since $d(h_m(q_\ell), h_m(q_k)) \leq d(q_\ell, q_k)$ for sufficiently large m, we have $d(h(q_\ell), h(q_k)) \leq d(q_\ell, q_k)$. Hence, the sequence $\{h(q_\ell)\}$ is a Cauchy sequence, and, therefore, converges to a point $h(p)$. The map h is obviously a compression map. This completes the proof of Theorem 4.

8. Proof of Theorem 6.

Let C be a convex circle as in Theorem 6 and let $\widetilde{C}$ be a comparison circle in the Euclidean plane. Let α be as in Theorem 6 and $r_j = m/n$ a sequence of rational numbers converging to α/L. Then there exists a periodic and minimal reflecting geodesic line $\tilde{\gamma}_j$ in $\widetilde{D}$ with slope r_jL, namely $s_{i+n} = s_i + mL$ hold for all integers i where $\tilde{c}(s_i) = \tilde{\gamma}_j(t_i)$ as in Section 2 (see [**1**], [**7**]). Suppose $\tilde{\gamma}_j(t_0) = \tilde{c}(0)$. Let $h : D \longrightarrow \widetilde{D}$ be a compression map given in Theorem 4. Then, the length of a broken segment $\cup_{i=1}^n T(h^{-1}(c(s_i)), h^{-1}(c(s_{i+1})))$ is greater than or equal to $(nL/\pi)\sin \pi r_j$. Thus, if $\gamma_j : (-\infty, \infty) \longrightarrow D$ is a periodic and minimal reflecting geodesic with slope r_jL, then the average of lengths of γ_j is greater than or equal to $(L/\pi)\sin \pi r_j$, since γ_j is periodic. The slope is continuous for minimal reflecting geodesic lines. We can find a reflecting geodesic line with slope α satisfying the condition in Theorem 6. This completes the proof of Theorem 6.

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