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Gap modules for direct product groups

Dedicated to Professor Masayoshi Kamata on his 60th birthday

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Abstract. Let G be a finite group. A gap G-module V is a finite dimensional real G-representation space satisfying the following two conditions:

(1) The following strong gap condition holds: $\dim V^P > 2\dim V^H$ for all $P < H \le G$ such that P is of prime power order, which is a sufficient condition to define a G-surgery obstruction group and a G-surgery obstruction.

(2) V has only one H-fixed point 0 for all large subgroups H, namely $H \in \mathscr{L}(G)$.
A finite group G not of prime power order is called a gap group if there exists a gap G-module. We discuss the question when the direct product $K \times L$ is a gap group for two finite groups K and L. According to [**5**], if K and $K \times C_2$ are gap groups, so is $K \times L$. In this paper, we prove that if K is a gap group, so is $K \times C_2$. Using [**5**], this allows us to show that if a finite group G has a quotient group which is a gap group, then G itself is a gap group. Also, we prove the converse: if K is not a gap group, then $K \times D_{2n}$ is not a gap group. To show this we define a condition, called NGC, which is equivalent to the non-existence of gap modules.

1. Introduction.

Let G be a finite group and p a prime. In this paper we assume that the trivial group is also called a p-group. We denote by $\mathscr{P}_p(G)$ a set of p-subgroups of G, define the Dress subgroup $G^{\{p\}}$ as the smallest normal subgroup of G whose index is a power of p, possibly 1, and let denote by $\mathscr{L}_p(G)$ the family of subgroups L of G which contains $G^{\{p\}}$. Set

$$\mathscr{P}(G) = \bigcup_p \mathscr{P}_p(G) \quad \text{and} \quad \mathscr{L}(G) = \bigcup_p \mathscr{L}_p(G).$$

Let V be a G-module V. We say that V is $\mathscr{L}(G)$-*free*, if $V^{G^{\{p\}}} = 0$ holds for any prime p. Set $\mathscr{D}(G)$ as a set of pairs (P, H) of subgroups of G such that $P < H \le G$ and $P \in \mathscr{P}(G)$. We denote by $\underline{\mathscr{D}}(G)$ be a set of all elements (P, H) of $\mathscr{D}(G)$ with $P \notin \mathscr{L}(G)$. Clearly note that this set equals to $\mathscr{D}(G)$ if $\mathscr{P}(G) \cap \mathscr{L}(G) = \varnothing$ holds. We define a function $d_V : \mathscr{D}(G) \to \boldsymbol{Z}$ by

$$d_V(P, H) = \dim V^P - 2\dim V^H.$$

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We say that V is positive (resp. nonnegative, resp. zero) at (P, H), if $d_V(P, H)$ is positive (resp. nonnegative, resp. zero). For a finite group G not of prime power order, a real G-module V is called an almost gap G-module, if V is an $\mathscr{L}(G)$-free real G-module such that $d_V(P, H) > 0$ for all $(P, H) \in \underline{\mathscr{D}}(G)$. If $\mathscr{L}(G) \cap \mathscr{P}(G) = \emptyset$ holds, an almost gap G-module is called a gap G-module. We can stably apply the equivariant surgery theory to gap G-modules. We say that G is a/an (almost) gap group if there is a/an (almost) gap G-module. A finite group G is called an Oliver group if there does not exist a normal series $P \triangleleft H \triangleleft G$ such that P and G/H are of prime power order and H/P is cyclic. A finite group G has a smooth action on a disk without fixed points if and only if G is an Oliver group, and G has a smooth action on a sphere with exactly one fixed point if and only if G is an Oliver group (cf. Oliver [**7**] and Laitinen-Morimoto [**3**]).

It is an important task to decide whether a given group G is a gap group. In fact, if a finite Oliver group G is a gap group, then one can apply equivariant surgery to convert an appropriate smooth action of G on a disk D into a smooth action of G on a sphere S with $S^G = M = D^G$, where $\dim M > 0$ (cf. Morimoto [**4**, Corollary 0.3]).

Laitinen and Morimoto [**3**] defined the G-module

$$V(G) = (\boldsymbol{R}[G] - \boldsymbol{R}) - \bigoplus_p (\boldsymbol{R}[G/G^{\{p\}}] - \boldsymbol{R}),$$

which is useful to construct a gap G-module, and proved that a finite group G has a smooth action on a sphere with any number of fixed points if and only if G is an Oliver group. This G-module also plays an important role in this paper. The purpose of this paper is to study the question when a direct product group is a gap group. The main theorem of this paper concerns a direct product $K \times D_{2n}$, where D_{2n} is the dihedral group of order $2n$ for $n \geq 1$ ($D_2 = C_2$ and $D_4 = C_2 \times C_2$).

THEOREM 1.1. *Let n be a positive integer and let K be a finite group. Then K is a gap group if and only if $G = K \times D_{2n}$ is a gap group.*

This paper is a continuation of our joint work with M. Morimoto and M. Yanagihara [**5**]. The key idea of the proof can be found in [**6**]. In [**5**, Theorem 3.5], we have shown that if $\mathscr{P}(K) \cap \mathscr{L}(K) = \emptyset$ and $K \times C_2$ is a gap group, so is $K \times F$ for any finite group F. In Lemma 5.1, we show that if K is a gap group, so is $K \times C_2$, which is the case where $n = 1$ in the main theorem. Using Lemma 5.1 and [**5**, Theorem 3.5], we obtain the following theorem.

THEOREM 1.2. *If a finite group G has a quotient group which is a gap group, then G itself is a gap group.*

Recall that G is an Oliver group if it has a quotient group which is an Oliver group.

The organization of the paper is as follows. In Section 2, we estimate $d_V(P,H)$ for a $(K \times L)$-module V by characters of irreducible K- and L-modules. In Section 3, we find a gap G-module for a certain direct product group of symmetric groups. The groups S_4 and S_5 are not gap groups but $S_4 \times S_5$ is a gap group. In Section 4, we introduce a condition NGC and show that G holds NGC if and only if G is not a gap group. We define a dimension matrix and give the condition equivalent to one being a gap group by using a dimension submatrix. In Section 5, by using the results in Section 4, we show that $K \times C_2$ is an almost gap group if so is K. In Section 6, we show that there are many finite groups G such that $\mathscr{P}(G) \cap \mathscr{L}(G) = \varnothing$ holds but G are not gap groups. As an application we completely decide when a direct product group of symmetric groups is a gap group. Since a gap group which is a direct product of symmetric groups is an Oliver group, it can act smoothly on a standard sphere with one fixed point.

2. Direct product groups.

Let $G = K \times L$ be a finite group. We denote by χ_V the character for a G-module V. Let P and H be subgroups of G such that $[H:P] = 2$. Then

$$(2.1)\quad d_{V\otimes W}(P,H) = \frac{1}{|P|}\sum_{x\in P}\chi_V(\pi_1(x))\chi_W(\pi_2(x)) - \frac{2}{|H|}\sum_{y\in H}\chi_V(\pi_1(y))\chi_W(\pi_2(y))$$
$$= -\frac{1}{|P|}\sum_{h\in H\setminus P}\chi_V(\pi_1(h))\chi_W(\pi_2(h)),$$

where V (resp. W) is a K- (resp. L-) module and $\pi_1 : G \to K$ and $\pi_2 : G \to L$ are the canonical projections.

We set

$$\mathscr{D}^2(G) = \{(P,H) \in \mathscr{D}(G) \mid [H:P] = [HG^{\{2\}} : PG^{\{2\}}] = 2 \text{ and }$$
$$PG^{\{q\}} = G \text{ for all odd primes } q\}.$$

and

$$\underline{\mathscr{D}}^2(G) = \underline{\mathscr{D}}(G) \cap \mathscr{D}^2(G).$$

Then $d_{V(G)}$ is positive on $\underline{\mathscr{D}}(G) \setminus \underline{\mathscr{D}}^2(G)$.

We have shown a restriction formula that reads as follows:

Proposition 2.2 (cf. [**5**, Proposition 3.1]). *Let K be a subgroup of an almost gap group G such that $G^{\{2\}} < K \le G$. Then K is an almost gap group. Furthermore, if the order of $G^{\{2\}}$ is not a power of a prime, then $G^{\{2\}}$ is an almost gap group.*

Let $RO(G)_{\mathscr{L}(G)}$ be an additive subgroup of $RO(G)$ generated by $\mathscr{L}(G)$-free irreducible real G-modules. There is a group epimorphism $\varphi : RO(G) \to RO(G)_{\mathscr{L}(G)}$ which is a left inverse of the inclusion $RO(G)_{\mathscr{L}(G)} \hookrightarrow RO(G)$. For a G-module V, we set $V_{\mathscr{L}(G)} = \varphi(V)$. Then $V_{\mathscr{L}(G)}$ is an $\mathscr{L}(G)$-free G-module and

$$V_{\mathscr{L}(G)} = (V - V^G) - \bigoplus_{p \mid |G|} (V - V^G)^{G^{\{p\}}}. \tag{2.3}$$

holds. In particular, $V(G) = \boldsymbol{R}[G]_{\mathscr{L}(G)}$ holds. Here the minus sign is interpreted as follows. For some integer $\ell > 0$, we regard V as a G-submodule of $\ell \boldsymbol{R}[G]$ with some G-invariant inner product. For a G-submodule W of V, we denote by $V - W$ the G-module which is orthogonal complement of W in V. For distinct primes p and q, $V^{G^{\{p\}}} \cap V^{G^{\{q\}}} = V^G$ holds, since $G^{\{p\}}G^{\{q\}} = G$. Then the direct sum of $(V - V^G)^{G^{\{p\}}}$ is a G-submodule of $V - V^G$.

The following is a restriction formula for an odd prime p.

Proposition 2.4. *Let K be a subgroup of G such that $G^{\{p\}} < K \le G$ for a prime p. Suppose there is a normal p-subgroup L of G such that $LK = G$. If G is a gap group then so is K.*

Proof. Let W be a gap G-module. Since $W^K = 0$, we set $V = (\mathrm{Res}^G_K W^L)_{\mathscr{L}(K)}$. We show that V is a gap K-module. It suffices to show that V is positive on $\mathscr{D}^2(K)$. Let $(P, H) \in \mathscr{D}^2(K)$. Then P is a p-group and thus LP is also a p-group. Therefore it follows that $(LP, HP) \in \mathscr{D}^2(G)$ and $d_V(P, H) = d_W(LP, HP) - \sum_q d_W(LPK^{\{q\}}, HPK^{\{q\}}) = d_W(LP, HP) - d_W(LPK^{\{2\}}, HPK^{\{2\}})$. We claim that $LK^{\{2\}} = G^{\{2\}}$ and thus $d_V(P, H) = d_W(LP, LH) > 0$. For $g = \ell k \in G = LK$, we obtain $g^{-1}LK^{\{2\}}g = (k^{-1}Lk)(k^{-1}K^{\{2\}}k) = LK^{\{2\}}$. Hence $LK^{\{2\}}$ is a normal subgroup of G. Clearly $LK^{\{2\}} \le G^{\{2\}}$.

$$\begin{tikzcd}
K \arrow[d] & K \cap G^{\{2\}} \arrow[l, "\triangleleft"'] \arrow[d] & K^{\{2\}} \arrow[l, "\triangleleft"'] \arrow[d] \\
G = LK & G^{\{2\}} = L(K \cap G^{\{2\}}) \arrow[l, "\triangleleft"'] & LK^{\{2\}} \arrow[l, "\triangleleft"']
\end{tikzcd}$$

Since $L \cap (K \cap G^{\{2\}}) = (L \cap G^{\{2\}}) \cap K = L \cap K = L \cap K^{\{2\}}$, it follows that $[G^{\{2\}} : K \cap G^{\{2\}}] = [LK^{\{2\}} : K^{\{2\}}]$. Therefore we obtain that $[G^{\{2\}} : LK^{\{2\}}]$ is a power of 2 and thus $G^{\{2\}} = LK^{\{2\}}$. $\square$

Corollary 2.5. *Let p be an odd prime, L a nontrivial p-group and K a finite group such that $K \times L$ is a gap group. Then the following holds.*

(1) *$K \times N$ is a gap group for any nontrivial subgroup N of L.*

(2) *If $K^{\{p\}} < K$, then K is a gap group.*

Proof. In (2) we let N be a trivial group. Let V be a gap $(K \times L)$-module. Regarding V^L as a $(K \times L)$-module, set $W = \mathrm{Res}^{K\times L}_{K\times N} V^L$. Then $W^{K^{\{q\}}} = V^{K^{\{q\}}\times L} \subseteq V^{(K\times L)^{\{q\}}} = 0$ for any prime q, namely W is $\mathscr{L}(K \times N)$-free. For $(P, H) \in \mathscr{D}^2(K \times N)$, it follows that P is a p-group, $(PL, HL) \in \mathscr{D}^2(K \times L)$ and then $d_W(P, H) = d_V(PL, HL) > 0$. Therefore W is positive on $\mathscr{D}^2(K \times N)$ and hence $W \oplus (\dim W + 1)V(K \times N)$ is a gap $(K \times N)$-module. $\square$

3. Product with a symmetric group.

Let C_n be a cyclic group of order n. In this section, by constructing appropriate gap modules, we show that $S_5 \times S_4$, $S_5 \times S_5$ and $S_5 \times S_4 \times C_2$ are all gap groups. The proof depends on [**5**, Theorem 3.5] and the fact that $A_4 \times C_2$ is an almost gap group.

Let $\mathscr{C}(G)$ be a complete set of cyclic groups C of G generated by elements in $H \backslash P$ of 2-power order, for all $(P, H) \in \mathscr{D}^2(G)$. Let $\mathfrak{C}(G)$ be a complete set of representatives of conjugacy classes of elements $C \in \mathscr{C}(G)$. We denote by $G_{\{p\}}$ a p-Sylow subgroup of G for a prime p.

Proposition 3.1. *$G = A_4 \times C_2$ is an almost gap group but not a gap group.*

Proof. $\mathscr{P}(G) \cap \mathscr{L}(G) = \{G^{\{3\}}\}$ causes that G is not a gap group. Since $G^{\{3\}} = G_{\{2\}}$, the set $\underline{\mathscr{D}}^2(G)$ consists of four elements of type $(G_{\{3\}}, G_{\{3\}} \times C_2)$. Thus

$$\left(\mathrm{Ind}^G_{C_2} \boldsymbol{R}_{\pm} - (\mathrm{Ind}^G_{C_2} \boldsymbol{R}_{\pm})^{G^{\{2\}}}\right) \oplus 2V(G)$$

is a required almost gap G-module, where $\boldsymbol{R}_{\pm}$ is the nontrivial irreducible C_2-module. $\square$

Proposition 3.2. *The G-module $V(G)$ is an almost gap G-module for any nilpotent group G not of prime power order.*

Proof. Note that G is isomorphic to $\prod_p G_{\{p\}}$. Thus if the order of G is divisible by three district primes, $V(G)$ is a gap group by [**5**, Theorem 0.2]. We may assume $|G| = p^a q^b$ for primes p and q $(p > q)$. Let $(P, H) \in \mathscr{D}^2(G)$. Since $PG^{\{p\}} = G$ implies $P = G_{\{p\}} = G^{\{q\}} \in \mathscr{L}(G)$, there are no elements $(P, H) \in \mathscr{D}^2(G)$ such that $P \notin \mathscr{L}(G)$. Thus $V(G)$ is an almost gap G-module by [**5**, Lemma 0.1]. $\square$

Proposition 3.3. *$G = A_5 \times C_2$ is a gap group.*

Proof. Let $K = A_4 \times C_2$ and W_0 be an almost gap K-module. Set $W = \mathrm{Ind}_K^G W_0$ and $V = W \oplus (\dim W + 1)V(G)$. We show that V is a gap G-module. It suffices to show that W is positive at all $(P,H) \in \mathscr{D}^2(G)$. Note that

$$d_W(P,H) = \sum_{PgK \in (P\backslash G/K)^{H/P}} d_{W_0}(K \cap g^{-1}Pg, K \cap g^{-1}Hg) \geq 0.$$

Since $K_{\{2\}}$ is a Sylow 2-subgroup of G, we have $(P\backslash G/K)^{H/P} \neq \emptyset$. It suffices to show that $K \cap g^{-1}Pg \notin \mathscr{L}(K)$. Suppose $K \cap g^{-1}Pg \in \mathscr{L}(K)$. Then $K \cap g^{-1}Pg = K_{\{2\}}$. Thus P is a Sylow 2-subgroup of G but this contracts the existence of H. Hence $K \cap g^{-1}Pg \notin \mathscr{L}(K)$ and W is positive at all $(P,H) \in \mathscr{D}^2(G)$. □

Recalling (2.3), given a subgroup L of G, we define a G-module $V(L;G) = (\mathrm{Ind}_L^G(\boldsymbol{R}[L] - \boldsymbol{R}))_{\mathscr{L}(G)}$, namely an $\mathscr{L}(G)$-free G-module removing non-$\mathscr{L}(G)$-free part $\bigoplus_p(\mathrm{Ind}_L^G(\boldsymbol{R}[L] - \boldsymbol{R}))^{G^{\{p\}}}$ from $\mathrm{Ind}_L^G(\boldsymbol{R}[L] - \boldsymbol{R})$.

Proposition 3.4. *$G = S_5 \times S_4$ and $S_5 \times S_5$ are gap groups.*

Proof. We regard G as a subgroup of S_9. Set $K_1 = S_5 \times A_4$, $K_2 = A_5 \times S_4$ and $K_3 = C_6 \times S_4$, which are all gap groups. (Also see [**5**, Lemma 5.6].) We define $V_m = \mathrm{Ind}_{K_m}^G W_m$ for $m = 1,2,3$, where W_m is a gap K_m-module. It follows that

$$\mathfrak{C}(G) = \{C_{2,1}, C_{4,1}, C_{1,2}, C_{1,4}, C_{2,2}, C_{2,4}, C_{4,2}, C_{4,4}, S_2, S_4, T_2, T_4\}.$$

Here $C_{i,1}$, $C_{1,i}$, $C_{i,j}$, S_i and T_i are cyclic subgroups generated by a_i, b_i, a_ib_j, s_i and t_i respectively $(i,j = 2,4)$, where $a_2 = (1,3)$, $a_4 = (1,2,3,4)$, $b_2 = (6,8)$, $b_4 = (6,7,8,9)$, $s_i = a_ib_4^2$ and $t_i = a_4^2b_i$.

Let $(P,H) \in \mathscr{D}^2(G)$. If $H\backslash P$ has an element which is conjugate to an element in

$$\{a_i, s_i \mid i = 2,4\}, \quad (\text{resp. } \{b_i, t_i \mid i = 2,4\}, \text{ resp. } \{a_2, b_i, a_2b_i \mid i = 2,4\})$$

then V_1 (resp. V_2, resp. V_3) is positive at (P,H).

Let L be a subgroup of G of order 16 generated by a_4b_2, b_4^2, and $(6,7)(8,9)$. Now assume $H\backslash P$ consists of elements which are conjugate to elements in $\{a_4b_i \mid i = 2,4\}$. For such a pair (P,H), there is an element a of G such that $a^{-1}Ha$ is a subgroup of L. (Note that $G_{\{2\}} = D_8 \times D_8$ has just 4 elements conjugate to g for each $g = a_4b_4$, a_4b_2.) Since $N_G(L)$ is a Sylow subgroup $G_{\{2\}}$, it follows that

$$\begin{aligned} d_{V(L;G)}(P,H) &\geq \frac{|N_G(L)|}{|N_G(L) \cap a^{-1}PaL|} - |(G^{\{2\}}P\backslash G/L)^{H/P}| \\ &\geq |N_G(L)/L| - 2 = 2 > 0. \end{aligned}$$

Putting all together, $V(L;G) \oplus 3(V(G) \oplus \bigoplus_{i=1}^3 V_i)$ is a gap G-module.

Since $[S_5 \times S_5 : G] = 5$ is odd, $S_5 \times S_5$ is a gap group by [**5**, Lemma 0.3]. $\square$

Remark 3.5. Consider the following subgroups of $G = S_5 \times S_4$: $P = \langle a_4^2, b_4^2 \rangle$, $H_4 = \langle a_4 b_4, a_4 b_4^3 \rangle$ and $H_2 = \langle a_4 b_2, a_4 b_2 b_4^2 \rangle$. Then (P, H_4) and (P, H_2) are elements of $\mathscr{D}^2(G)$. $N_4 = N_G(C_{4,4}) = \langle a_4, b_4, a_2 b_2 \rangle$ of order 32 has just 4 elements which are conjugate to $a_4 b_4$ and no elements conjugate to $a_4 b_2$. Thus $|(H_4 \backslash P) \cap N_4| = 4$ and so $H_4 \cap N_4 = 8$. Therefore if $H_4 \geq C_{4,4}$, then $|N_4/PC_{4,4} \cap N_4| = |N_4/H_4 \cap N_4| = 4$ and $d_{V(C_{4,4})}(P, H_4) \geq 4 - 2 = 2$. Similarly since $N_2 = N_G(C_{4,2}) = \langle a_4, b_2, b_4^2, a_2 \rangle \cong D_8 \times C_2 \times C_2$ of order 32 has only 4 elements conjugate to $a_4 b_2$ and no elements conjugate to $a_4 b_4$, it follows that $d_{V(C_{4,2})}(P, H_2) \geq 2$. Then in this estimation we only obtain that $V(C_{4,2}) \oplus V(C_{4,4})$ is nonnegative at (P, H_4) and (P, H_2). However $|(P \backslash G/C_{4,j})^{H_j/P}| = 8$ in fact and thus $d_{V(C_{4,j})}(P, H_j) = 6$ for $j = 2, 4$. Thus $W = V(C_{4,2}) \oplus V(C_{4,4})$ is positive at (P, H_j) for $j = 2, 4$, and hence $5(V(G) \oplus V_1 \oplus V_2 \oplus V_3) \oplus W$ is a gap G-module.

For $G = S_5 \times S_4 \times C_2$, the set $\mathfrak{C}(G)$ consists of 28 elements. Let $K_1 = S_5 \times A_4 \times C_2$, $K_2 = A_4 \times S_4 \times C_2$ and $K_3 = C_6 \times S_4 \times C_2$ be subgroups of G. They are gap groups by Proposition 3.1 and [**5**, Theorem 3.5]. Similarly by using their gap groups, we can prove the next proposition.

Proposition 3.6. *Let V_i $(i = 1, 2, 3)$ be G-modules induced from gap K_i-modules. Let K_5 be a subgroup of G generated by* $(1, 2, 3, 4)(6, 7)$, $(6, 7)(8, 9)$, $(6, 8)(7, 9)$, *and* $(10, 11)$, *viewing naturally $G = S_5 \times S_4 \times C_2$ as a subgroup of S_{11}. Then*

$$3(V_1 \oplus V_2 \oplus V_3 \oplus V(G)) \oplus V(K_5; G)$$

is a gap G-module. Furthermore, $S_5 \times S_5 \times C_2$ is a gap group.

Considering the similar argument of the proof of Propositions 3.4 and 3.6, we obtain the following proposition.

Proposition 3.7. *Let G be a finite group such that $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$ and $\mathfrak{F}$ a subset of $\mathfrak{C}(G)$. We assume that*:

(1) *For any element C of $\mathfrak{F}$, there is a gap group K such that $C \leq K \leq G$.*

(2) *There is an $\mathscr{L}(G)$-free G-module W which is positive at any $(P, H) \in \mathscr{D}^2(G)$ such that H contains an element of $\mathfrak{C}(G) \backslash \mathfrak{F}$ as a subgroup.*

Then G is a gap group.

Proof. For each element C of $\mathfrak{F}$, pick up a gap subgroup K_C of G which includes C and a gap K_C-module W_C. Then $\mathrm{Ind}_{K_C}^G W_C$ is nonnegative on $\mathscr{D}(G)$, and is positive at $(P,H)\in\mathscr{D}^2(G)$ if $C\cap g^{-1}(H\backslash P)g\neq\emptyset$ for some $g\in G$. Therefore $W\oplus(\dim W+1)(V(G)\oplus\bigoplus_{C\in\mathfrak{F}}\mathrm{Ind}_{K_C}^G W_C)$ is a gap G-module. $\square$

Let $n\geq 9$. Note that $K=S_{n-5}\times A_5$ $(\leq S_n)$ is a gap group, since $A_5\times C_2$ is a gap group. Let $\mathfrak{F}\subset\mathfrak{C}(G)$ be a set of all elements of order $<k_2$, where k_2 is a power of 2 such that $k_2\leq n<2k_2$. Then K contains any element of $\mathfrak{F}$ up to conjugate in G. $W=V(\langle(1,2,\dots,k_2)\rangle;S_n)$ fulfills (2) in Proposition 3.7. Hence S_n is a gap group which has been already shown in [**2**].

4. Farkas lemma and the condition NGC.

Throughout this section, we assume that G is a finite group not of prime power order. We consider the following condition NGC: There are a nonempty subset $S\subset\mathscr{D}(G)$ and positive integers $m(P,H)$ for $(P,H)\in S$ such that

$$\sum_{(P,H)\in S} m(P,H)d_V(P,H)=0 \tag{4.1}$$

for any $\mathscr{L}(G)$-free irreducible G-module V.

We denote by NGC(G) the condition NGC for a group G. If $G_{\{p\}}=G^{\{q\}}$, then setting $S=\{(G^{\{q\}},G)\}$ and $m(G^{\{q\}},G)=1$, we obtain (4.1). If $\mathscr{P}(G)\cap\mathscr{L}(G)=\emptyset$, then S must be a subset of $\mathscr{D}^2(G)$ by existence of $V(G)$.

We give two examples. For a dihedral group D_{2n} of order $2n$, any $\mathscr{L}(D_{2n})$-free irreducible module is zero at $(\{1\},C_2)$. Let $P_1=\langle(1,3)(2,4)\rangle$, $H_1=\langle(1,2,3,4)\rangle$, $P_2=\langle(1,2,3)\rangle$, and $H_2=\langle(1,2,3),(1,2)\rangle$ be subgroups of S_5, and set $S=\{(P_1,H_1),(P_2,H_2)\}$. Then $d_W(P_1,H_1)+d_W(P_2,H_2)=0$ for any $\mathscr{L}(S_5)$-free irreducible module W. (See [**6**].) Therefore D_{2n} and S_5 satisfy the condition NGC. Hereafter we show that if G is not a gap group, G satisfies the condition NGC.

We write $\boldsymbol{x}\geq\boldsymbol{y}$ (resp. $\boldsymbol{x}>\boldsymbol{y}$), if $x_i\geq y_i$ (resp. $x_i>y_i$) for any i, where $\boldsymbol{x}={}^t[x_1,\dots,x_n]$ and $\boldsymbol{y}={}^t[y_1,\dots,y_n]$.

Theorem 4.2 (The duality theorem cf. [**1**, p. 248]). *For an $n\times m$ matrix A with entries in $\boldsymbol{Q}$, let*

$$\begin{array}{ll} \textit{minimize} & {}^t\boldsymbol{c}\boldsymbol{x} \\ \textit{subject to} & A\boldsymbol{x}\geq\boldsymbol{b},\quad \boldsymbol{x}\geq\boldsymbol{0} \end{array}$$

be a primal problem and let

$$\begin{array}{ll} \textit{maximize} & {}^t\boldsymbol{b}\boldsymbol{y} \\ \textit{subject to} & {}^tA\boldsymbol{y}\leq\boldsymbol{c},\quad \boldsymbol{y}\geq\boldsymbol{0} \end{array}$$

be a problem which is called the dual problem. Then the following relationship between the primal and dual problems holds.

		Dual		
		Optimal	*Infeasible*	*Unbounded*
Primal	*Optimal*	*Possible*	*Impossible*	*Impossible*
	Infeasible	*Impossible*	*Possible*	*Possible*
	Unbounded	*Impossible*	*Possible*	*Impossible*

The duality theorem is proved by applying a linear programming over $\boldsymbol{Q}$. A key point of the proof is that the (revised) simplex method is closed over $\boldsymbol{Q}$. We omit the detail.

Lemma 4.3 (Farkas Lemma). *Let A be an $n \times m$ matrix with entries in $\boldsymbol{Q}$. For $\boldsymbol{b} \in \boldsymbol{Q}^n$, set*

$$X(A, \boldsymbol{b}) = \{\boldsymbol{x} \in \boldsymbol{Q}^m \mid A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} \quad \text{and} \quad Y(A, \boldsymbol{b}) = \{\boldsymbol{y} \in \boldsymbol{Q}^n \mid {}^tA\boldsymbol{y} \leq \boldsymbol{0}, {}^t\boldsymbol{b}\boldsymbol{y} > \boldsymbol{0}\}.$$

Then either $X(A, \boldsymbol{b})$ or $Y(A, \boldsymbol{b})$ is empty but not both.

Proof. First suppose $X(A, \boldsymbol{b}) \neq \emptyset$. If it might holds $Y(A, \boldsymbol{b}) \neq \emptyset$, then ${}^t\boldsymbol{y}A\boldsymbol{x} = {}^t\boldsymbol{y}\boldsymbol{b} = {}^t\boldsymbol{b}\boldsymbol{y} > \boldsymbol{0}$ but the inequalities ${}^t\boldsymbol{y}A \leq {}^t\boldsymbol{0}$ and $\boldsymbol{x} \geq \boldsymbol{0}$ implies ${}^t\boldsymbol{y}A\boldsymbol{x} \leq \boldsymbol{0}$ which is contradiction. Thus $Y(A, \boldsymbol{b})$ is empty. Next suppose $X(A, \boldsymbol{b}) = \emptyset$. Consider a primal problem

$$\begin{aligned} &\text{minimize} \quad {}^t\boldsymbol{0}\boldsymbol{x} \\ &\text{subject to} \quad \begin{bmatrix} A \\ -A \end{bmatrix} \boldsymbol{x} \geq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix}, \quad \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

which has an infeasible solution. Then the dual problem is

$$\begin{aligned} &\text{maximize} \quad {}^t\boldsymbol{b}\boldsymbol{z} \\ &\text{subject to} \quad {}^tA\boldsymbol{z} \leq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{z} = \boldsymbol{y}_1 - \boldsymbol{y}_2$ and $\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}$. This problem has a solution $\boldsymbol{z} = \boldsymbol{0}$ and thus it has an unbounded solution. Therefore there exists a solution $\boldsymbol{z}$ such that ${}^t\boldsymbol{b}\boldsymbol{z} > 0$ and then $Y(A, \boldsymbol{b}) \neq \emptyset$. □

Let n be a number of $\mathscr{L}(G)$-free irreducible G-modules and $m = |\mathscr{D}(G)|$. We denote by $M(m, n; \boldsymbol{Z})$ the set of $m \times n$ matrices with entries in $\boldsymbol{Z}$. We say that D is a dimension matrix of G, if $D \in M(m, n; \boldsymbol{Z})$ is a matrix whose (i, j)-entry is $d_{V_j}(P_i, H_i)$, where V_j runs over $\mathscr{L}(G)$-free irreducible G-modules and (P_i, H_i) runs over elements of $\mathscr{D}(G)$. For a subset S of $\mathscr{D}(G)$, a submatrix

$D' \in M(|S|, n; \boldsymbol{Z})$ of a dimension matrix D of G is called a dimension submatrix of G over S. Set

$$\boldsymbol{Z}^k_{\geq \mathbf{0}} = \{\boldsymbol{x} \in \boldsymbol{Z}^k \mid \boldsymbol{x} \geq \mathbf{0}\}$$

and

$$Z_S(G) = \left\{ \boldsymbol{y} = {}^t[y_1, \dots, y_k] \in \boldsymbol{Z}^k_{\geq \mathbf{0}} \;\middle|\; {}^tD'\boldsymbol{y} \leq \mathbf{0}, \sum_i y_i > 0 \right\}.$$

If G is a gap group, then there is $\boldsymbol{x} \in \boldsymbol{Z}^n_{\geq \mathbf{0}}$ such that $D\boldsymbol{x} > \mathbf{0}$. The converse is also true, since $W = \sum_i x_i V_i$ is a gap G-module, where x_i is the i-th entry of $\boldsymbol{x}$.

Proposition 4.4. *The followings are equivalent.*

(1) *G is not a gap group.*

(2) *$Z_{\mathscr{D}(G)}(G) \neq \emptyset$.*

(3) *There are a nonempty subset $S \subseteq \mathscr{D}(G)$ and positive integers $m(P, H)$ for $(P, H) \in S$ such that $\sum_{(P,H) \in S} m(P, H) d_V(P, H) \leq 0$ for any $\mathscr{L}(G)$-free irreducible G-module V.*

Proof. Let D be a dimension matrix of G. Set $A = [D, -E]$, where E is the identity matrix, and $\boldsymbol{b} = {}^t[1, \dots, 1]$. If there is $\boldsymbol{x} = {}^t[\boldsymbol{x}_1, \boldsymbol{x}_2] \in X(A, \boldsymbol{b})$, then $D\boldsymbol{x}_1 - \boldsymbol{x}_2 = \boldsymbol{b}$ and thus $D\boldsymbol{x}_1 \geq \boldsymbol{b}$. Take a positive integer k such that $k\boldsymbol{x}_1 \in \boldsymbol{Z}^m$. Then $D(k\boldsymbol{x}_1) \geq k\boldsymbol{b} \geq \boldsymbol{b}$. Therefore $X(A, \boldsymbol{b}) \neq \emptyset$ implies that G is a gap group. Clearly if G is a gap group, then $X(A, \boldsymbol{b}) \neq \emptyset$ holds. Then by Lemma 4.3, G is a gap group if and only if $Y(A, \boldsymbol{b}) = \emptyset$, equivalently $Z_{\mathscr{D}(G)}(G) = \emptyset$ holds. Therefore (1) and (2) are equivalent.

It is clear that (3) implies (2). To finish the proof we show that (2) implies (3). Take $\boldsymbol{z} \in Z_{\mathscr{D}(G)}(G)$. Set S as a set of (P_{i_t}, H_{i_t})'s such that the i_t-th entry of $\boldsymbol{z}$ is nonzero, and let $m(P_{i_t}, H_{i_t})$ be the i_t-th entry of $\boldsymbol{z}$. Then $\sum_{(P_{i_t}, H_{i_t}) \in S} m(P_{i_t}, H_{i_t}) \cdot d_{V_j}(P_{i_t}, H_{i_t}) \leq 0$ clearly holds. □

Thus if NGC(G) holds, then G is not a gap group.

Proposition 4.5. *Suppose that there are an $\mathscr{L}(G)$-free G-module W and a subset $T \subseteq \mathscr{D}(G)$ such that $d_W(P, H) \geq 0$ for any $(P, H) \in \mathscr{D}(G)$ and $d_W(P, H) > 0$ for any $(P, H) \in T$. Then the followings are equivalent.*

(1) *G is not a gap group.*

(2) *$Z_{\mathscr{D}(G) \setminus T}(G) \neq \emptyset$.*

(3) *There is a nonempty subset $S \subseteq \mathscr{D}(G) \setminus T$ and integers $m(P, H) > 0$ for $(P, H) \in S$ such that $\sum_{(P,H) \in S} m(P, H) d_V(P, H) \leq 0$ for any $\mathscr{L}(G)$-free irreducible G-module V.*

Proof. Clearly (2) implies (1) by Proposition 4.4. Suppose that G is not a gap group. Let D_1 be a dimension submatrix of G over S and D_2 a dimension submatrix of G over $\mathscr{D}(G) \setminus S$. Then $D = {}^t[D_1, D_2]$ is a dimension matrix of G.

Let V_j $(1 \le j \le k)$ be a complete set of $\mathscr{L}(G)$-free irreducible G-modules. Set $\boldsymbol{y} = {}^t[y_1, \dots, y_k]$, where $W = \sum_{j=1}^{k} y_j V_j$. Since there is a nonzero vector $\boldsymbol{x} = {}^t[\boldsymbol{x}_1, \boldsymbol{x}_2] \ge \boldsymbol{0}$ such that ${}^t\boldsymbol{x}D = {}^t\boldsymbol{x}_1 D_1 + {}^t\boldsymbol{x}_2 D_2 \le {}^t\boldsymbol{0}$, we have ${}^t\boldsymbol{x}_1 D_1 \boldsymbol{y} + {}^t\boldsymbol{x}_2 D_2 \boldsymbol{y} \le 0$. Since $D_1\boldsymbol{y} > \boldsymbol{0}$ and $D_2\boldsymbol{y} \ge \boldsymbol{0}$, we obtain that ${}^t\boldsymbol{x}_1 D_1 \boldsymbol{y} = {}^t\boldsymbol{x}_2 D_2 \boldsymbol{y} = 0$ and thus $\boldsymbol{x}_1 = \boldsymbol{0}$. Therefore ${}^t\boldsymbol{x}_2 D_2 \le {}^t\boldsymbol{0}$ for the nonzero vector $\boldsymbol{x}_2 \ge \boldsymbol{0}$ and hence (2) holds. $\square$

Corollary 4.6. *Let G be a finite group such that $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$. The group G is a gap group if and only if $Z_{\mathscr{D}^2(G)}(G) = \emptyset$ holds.*

This holds from the existence of $V(G)$.

The following proposition can be proven by the same manner of the proof of Proposition 4.5. Recall that $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$ implies $\underline{\mathscr{D}}(G) = \mathscr{D}(G)$.

Proposition 4.7. *Let G be a finite group not of prime power order. Suppose that there are an $\mathscr{L}(G)$-free G-module W and a subset $T \subseteq \underline{\mathscr{D}}^2(G)$ such that $d_W(P,H) \ge 0$ for any $(P,H) \in \mathscr{D}(G)$ and $d_W(P,H) > 0$ for any $(P,H) \in T$. Then the followings are equivalent.*

(1) *G is not an almost gap group.*

(2) *$Z_{\underline{\mathscr{D}}^2(G)\setminus T}(G) \neq \emptyset$.*

(3) *There is a nonempty subset $S \subseteq \underline{\mathscr{D}}^2(G)\setminus T$ and integers $m(P,H) > 0$ for $(P,H) \in S$ such that $\sum_{(P,H)\in S} m(P,H) d_V(P,H) \le 0$ for any $\mathscr{L}(G)$-free irreducible G-module V.*

Theorem 4.8. *Let G be a finite group not of prime power order. Then*

$$Z_S(G) = \{\boldsymbol{y} \in \boldsymbol{Z}^k_{\ge \boldsymbol{0}} \mid {}^tD\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{y} \neq \boldsymbol{0}\},$$

where D is a dimension submatrix over S. In particular G is not a gap group if and only if NGC(G) *holds.*

Proof. Since (1) and (3) of Proposition 4.4 are equivalent, NGC(G) implies G is not a gap group. Suppose that G is not a gap group. We show that NGC(G) holds. Let D be a dimension submatrix of G over $\mathscr{D}^2(G)$ and set $\boldsymbol{c} = {}^t[1, \dots, 1] \in \boldsymbol{Q}^n$. Note that $\boldsymbol{R}[G]$ includes all irreducible G-modules. Since $V(G)$ is a module removing non-$\mathscr{L}(G)$-free, (irreducible) G-modules from $\boldsymbol{R}[G]$, the G-module $V(G)$ includes any $\mathscr{L}(G)$-free irreducible G-modules. Let $\boldsymbol{a} \in \boldsymbol{Z}^n_{\ge \boldsymbol{0}}$ be a vector corresponding with $V(G)$. Thus we obtain that both $D\boldsymbol{a} = \boldsymbol{0}$ and ${}^t\boldsymbol{b}\boldsymbol{a} > 0$ for any $\boldsymbol{b} \in \boldsymbol{Q}^n$ such that $\boldsymbol{b} \ge 0$ and ${}^t\boldsymbol{c}\boldsymbol{b} > 0$. Then

$$Y({}^tD, \boldsymbol{b}) \cap Y(-{}^tD, \boldsymbol{b}) \neq \emptyset.$$

By Lemma 4.3 we get

$$X({}^tD, \boldsymbol{b}) \cup X(-{}^tD, \boldsymbol{b}) = \emptyset,$$

namely,

$$\{\boldsymbol{x} \in \boldsymbol{Q}^m \mid {}^tD\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0}\} \cup \{\boldsymbol{x} \in \boldsymbol{Q}^m \mid {}^tD\boldsymbol{x} = -\boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0}\} = \emptyset.$$

Then defining a map $f : \boldsymbol{Z}^m_{\geq \boldsymbol{0}} \to \boldsymbol{Z}^n$ by $f(\boldsymbol{x}) = {}^t D\boldsymbol{x}$, the image of f is a subset of

$$\boldsymbol{Z}^n \backslash \{\pm \boldsymbol{b} \mid \boldsymbol{b} \geq \boldsymbol{0}, {}^t\boldsymbol{c}\boldsymbol{b} > 0\} = (\boldsymbol{Z}^n \backslash \{\pm \boldsymbol{b} \mid \boldsymbol{b} \geq \boldsymbol{0}\}) \cup \{\boldsymbol{0}\}.$$

Taking $\boldsymbol{z} \in Z_{\mathscr{D}^2(G)}(G)$ by Proposition 4.5, $f(\boldsymbol{z}) \leq \boldsymbol{0}$ holds. On the other hand, the vector $f(\boldsymbol{z})$ belongs to $(\boldsymbol{Z}^n \backslash \{\pm \boldsymbol{b} \mid \boldsymbol{b} \geq \boldsymbol{0}\}) \cup \{\boldsymbol{0}\}$. Hence we obtain $f(\boldsymbol{z}) = \boldsymbol{0}$. We complete the proof. □

5. Product with the cyclic group of order 2.

The purpose of this section is to prove the following lemma.

LEMMA 5.1. *If K is an almost gap group, then so is $G = K \times C_2$.*

Combining [**5**, Theorem 0.4] and Lemma 5.1, we obtain Theorem 1.2.

Now we show Lemma 5.1. To apply Proposition 4.5, we define a subset T of $\underline{\mathscr{D}}(G)$. Let W be an almost gap K-module. Let $\pi_1 : G \to K$ and $\pi_2 : G \to C_2$ be canonical projections. First, set $T_1 = \underline{\mathscr{D}}(G) \backslash \underline{\mathscr{D}}^2(G)$. The module $V_1 = V(G)$ is nonnegative on $\mathscr{D}(G)$ and positive on T_1. Second, set $T_2 = \{(P, H) \in \underline{\mathscr{D}}^2(G) \mid \pi_2(P) = \pi_2(H)\}$. Then $V_2 = \mathrm{Ind}_K^G W$ is nonnegative on $\mathscr{D}(G)$ and positive on T_2. It is clear that V_1 and V_2 are $\mathscr{L}(G)$-free. Note that $V(P \times C_2)$ is an almost gap group and particularly, nonnegative on $\mathscr{D}(G)$ for any p-group P $(p \neq 2)$. Third, set $T_3 = \{(P, P \times C_2) \in \mathscr{D}^2(G) \mid P \in \mathscr{P}(K) \backslash \mathscr{L}(K)\}$. We show that there is an $\mathscr{L}(G)$-free G-module V_3 such that V_3 is nonnegative on $\mathscr{D}(G)$ and positive on T_3, by dividing two cases. Let $(P, H) \in T_3$.

The first case is one where $|K|$ is divisible by at least two odd primes. Take an odd prime q such that q divides $|K|$ and addly if $P \neq \{1\}$ then P is not a q-group. Then $\mathrm{Ind}_{C_q \times C_2}^G V(C_q \times C_2)$ is positive at (P, H). Set $V_3 = \bigoplus_p \mathrm{Ind}_{C_p \times C_2}^G V(C_p \times C_2)$, where p ranges over all odd primes which divide $|K|$. Then V_3 is positive on T_3.

The second case is one where $|K| = 2^a p^b$ for some odd prime p and some integer $a, b \geq 1$. Set $L = K_{\{p\}} \times C_2$ and $V_3 = \mathrm{Ind}_L^G V(L)$. If $\mathscr{P}(K) \cap \mathscr{L}(K) = \emptyset$, then $K_{\{p\}}$ is not a normal subgroup of K and thus there is an element $g \in G$ for any $P \in \mathscr{P}(K)$ such that $L \cap g^{-1}Pg < K_{\{p\}}$. If $K_{\{p\}}$ is a normal subgroup of K, then $L \cap P < K_{\{p\}}$ for any $P \in \mathscr{P}(K) \backslash \mathscr{L}(K)$. Therefore we obtain that

$$\begin{aligned} d_{V_3}(P, H) &= \sum_{PgL \in (P \backslash G/L)^{H/P}} d_{V(L)}(L \cap g^{-1}Pg, L \cap g^{-1}Hg) \\ &= \sum_{\pi_1(P)gK_{\{p\}} \in \pi_1(P) \backslash K/K_{\{p\}}} \dim V(L)^{L \cap g^{-1}Pg} > 0. \end{aligned}$$

Then V_3 is positive on T_3.

Putting all together, $V = V_1 \oplus V_2 \oplus V_3$ is nonnegative on $\mathscr{D}(G)$ and positive on $T = T_1 \cup T_2 \cup T_3$.

Let V_j $(1 \leq j \leq \gamma)$ be all irreducible K-modules such that V_j is $\mathscr{L}(K)$-free whenever $1 \leq j \leq \alpha$, ${V_j}^{K\{2\}} = 0$ but ${V_j}^{K\{p\}} \neq 0$ for some odd prime p whenever $\alpha < j \leq \beta$, and ${V_j}^{K\{2\}} \neq 0$ whenever $\beta < j \leq \gamma$. Then any $\mathscr{L}(G)$-free irreducible G-module is one of $U_j = V_j \otimes \boldsymbol{R}$ $(1 \leq j \leq \alpha)$ and $W_k = V_k \otimes \boldsymbol{R}_{\pm}$ $(1 \leq k \leq \beta)$. Here $\boldsymbol{R}$ (resp. $\boldsymbol{R}_{\pm}$) is the irreducible trivial (resp. nontrivial) C_2-module.

Suppose that G is not an almost gap group. By Proposition 4.7, there are a nonempty subset $S \subseteq \mathscr{D}(G) \backslash T$ and a nonzero vector $\boldsymbol{x} \in \boldsymbol{Z}_{\geq \boldsymbol{0}}^{\gamma}$ such that ${}^t\boldsymbol{x}D \leq {}^t\boldsymbol{0}$. Here $D = [d_{U_j}(P,H), d_{W_k}(P,H)]$ is a dimension submatrix of G over S, where $1 \leq j \leq \alpha$ and $1 \leq k \leq \beta$. For $(P,H) \in S$, we obtain that $P = H \cap K$, $\pi_1(H) > P$,

$$d_{U_j}(P,H) = -\frac{1}{|P|}\sum_{h \in H \backslash P} \chi_{V_j}(\pi_1(h))\chi_{\boldsymbol{R}}(\pi_2(h)) = d_{V_j}(P, \pi_1(H)),$$

and

$$d_{W_k}(P,H) = -\frac{1}{|P|}\sum_{h \in H \backslash P} \chi_{W_k}(\pi_1(h))\chi_{\boldsymbol{R}_{\pm}}(\pi_2(h)) = -d_{V_k}(P, \pi_1(H)).$$

Let $F = [d_{U_j}(P,H)]$ be a submatrix of D such that $D = [F, -F, -F']$ for some matrix F'. Then ${}^t\boldsymbol{x}D \leq {}^t\boldsymbol{0}$ implies that ${}^t\boldsymbol{x}F \leq {}^t\boldsymbol{0}$ and $-{}^t\boldsymbol{x}F \leq {}^t\boldsymbol{0}$. Hence ${}^t\boldsymbol{x}F = {}^t\boldsymbol{0}$ holds. On the other hand, a map $\mathscr{D}(G) \backslash T \to \mathscr{D}^2(K)$ assigning (P,H) to $(P, \pi_1(H))$ is a bijection. (If $|K|$ is odd, then $\mathscr{D}(G) \backslash T$ and $\mathscr{D}^2(K)$ are both empty.) Then $F = [d_{V_j}(P, \pi_1(H))]$ is a dimension submatrix of K. By Proposition 4.7, K is not an almost gap group, which is contradiction. Therefore $K \times C_2$ is also an almost gap group.

Corollary 5.2. *The wreath product $K \int L$ is a gap group for any finite group K, if L is a gap group.*

Proof. It is clear from the existence of epimorphisms $K \int L \to L$. □

6. Product with a dihedral group.

Let $D_{2n} = \langle a, b \mid a^2 = b^n = (ab)^2 = 1 \rangle$ be a dihedral group of order $2n$. In this section we study which $K \times D_{2n}$ is a gap group. If K is a gap group, then so is $K \times D_{2n}$. We are also interesting in the converse problem.

We set

$$\mathscr{D}_p^2(G) = \{(P,H) \in \mathscr{D}^2(G) \mid P \text{ is a } p\text{-group}\}$$

for a prime p and

$$\mathscr{D}_1^2(G) = \{(\{1\}, C_2) \in \mathscr{D}^2(G)\}.$$

Proposition 6.1. *Let G be a finite group not of prime power order, p an odd prime and Q a nontrivial p-group. The natural projection $\pi : G \times Q \to G$ induces a surjection $Z_{\mathscr{D}_p^2(G \times Q)}(G \times Q) \to Z_{\mathscr{D}_p^2(G)}(G)$. Furthermore, it is a bijection if $|G|$ and p are coprime.*

Proof. Let $(P,H) \in \mathscr{D}_p^2(G \times Q)$. Note that $H \cap Q = P \cap Q$, $(G \times Q)^{\{2\}} \cap Q = Q$ and $\pi(P(G \times Q)^{\{r\}}) = \pi(P)\pi(G \times Q)^{\{r\}} = \pi(P)G^{\{r\}}$ for any prime r. Thus $(\pi(P), \pi(H)) \in \mathscr{D}_p^2(G)$. For a G-module V, it follows that

$$d_{V \otimes \boldsymbol{R}}(P,H) = -\frac{1}{|P|}\sum_{h \in H \setminus P} \chi_V(\pi(h)) = -\frac{|P \cap Q|}{|P|}\sum_{x \in \pi(H) \setminus \pi(P)} \chi_V(x) = d_V(\pi(P), \pi(H)),$$

where $\boldsymbol{R}$ regards as the trivial W-module and χ_V is the character for V. Thus the projection π induces a map $Z_{\mathscr{D}_p^2(G \times Q)}(G \times Q) \to Z_{\mathscr{D}_p^2(G)}(G)$. We show that the map is surjective. Set $S = \{(A \times Q, B \times Q) \mid (A,B) \in \mathscr{D}_p^2(G)\}$ which is a subset of $\mathscr{D}_p^2(G \times Q)$. Let $(P,H) \in S$. Then $d_{V \otimes W}(P,H) = d_V(\pi(P), \pi(H)) \dim W^Q$ for a G-module V and a Q-module W. If $V \times W$ is $\mathscr{L}(G \times Q)$-free and W is the trivial irreducible Q-module, then V is $\mathscr{L}(G)$-free. Thus a dimension submatrix $D = [d_{V \otimes W}(P,H)]$ over S coincides with $[d_V(\pi(P), \pi(H)), \mathbf{0}, \dots, \mathbf{0}]$. Note that $[d_V(\pi(P), \pi(H))]$ is a dimension submatrix over $\mathscr{D}_p^2(G)$. For $\boldsymbol{x} \in Z_{\mathscr{D}_p^2(G)}(G)$, take $\boldsymbol{y} \in Z_{\mathscr{D}_p^2(G \times Q)}(G \times Q)$ whose entry corresponding to $(P,H) \in \mathscr{D}_p^2(G \times Q)$ is the entry of $\boldsymbol{x}$ corresponding to $(\pi(P), \pi(H))$ if $(P,H) \in S$ and zero otherwise. Then the map sends $\boldsymbol{y}$ to $\boldsymbol{x}$. Therefore the map is surjective. If $|G|$ is a coprime to p, then $S = \mathscr{D}_p^2(G \times Q)$ which implies that the map is bijective. We complete the proof. $\square$

This proposition implies as follows. If $\mathscr{P}(G) \cap \mathscr{L}(G) = \emptyset$, then $Z_{\mathscr{D}_p^2(G)}(G) \neq \emptyset$ is equivalent to that there is a nontrivial p-group Q such that $G \times Q$ is not a gap group. Furthermore, if $G \times Q$ is a gap group for some nontrivial p-group Q, so is $G \times R$ for any nontrivial p-group R. It also holds in the case where $p = 2$, by Theorem 1.2 and Proposition 2.2.

Corollary 6.2. *Let K be a p-group. The group $G = K \times D_{2n}$ is not a gap group.*

Proof. Since $(\{1\}, \langle a \rangle) \in \mathscr{D}_p(D_{2n})$, Proposition 6.1 yields the assertion. $\square$

Proposition 6.3. *Let p be a prime and let K_1 and K_2 be finite groups not of prime power order. If $Z_{\mathscr{D}_p^2(K_1)}(K_1)$ and $Z_{\mathscr{D}_p^2(K_2)}(K_2)$ are both nonempty, then $Z_{\mathscr{D}_p^2(K_1 \times K_2)}(K_1 \times K_2) \neq \emptyset$.*

Proof. We define $(P,H) \in \mathscr{D}_p^2(K_1 \times K_2)$ for $(P_1, H_1) \in \mathscr{D}_p^2(K_1)$ and $(P_2, H_2) \in \mathscr{D}_p^2(K_2)$ as follows. Set $P = P_1 \times P_2$, which is a p-group. Take $h_j \in H_j$ such that $h_j \notin P_j$ and h_j is an element of 2-power order for $j = 1, 2$, and denote by H a subgroup of $K_1 \times K_2$ generated by P and $h = h_1 h_2$. It is clear that $(P,H) \in \mathscr{D}_p^2(K_1 \times K_2)$. Let S be a subset of $\mathscr{D}_p^2(K_1 \times K_2)$ which is the image of the above assignment and $D = [d_{V \otimes W}(P,H)]$ a dimension submatrix over S. Since

$$\begin{aligned} d_{V\otimes W}(P,H) &= -\frac{1}{|P|}\sum_{x\in P}\chi_V(\pi_1(hx))\chi_W(\pi_2(hx)) \\ &= -\frac{1}{|P|}\sum_{(p_1,p_2)\in P}\chi_V(h_1p_1)\chi_W(h_2p_2) \\ &= -\frac{1}{|P|}\sum_{p_1\in P_1}\chi_V(h_1p_1)\sum_{p_2\in P_2}\chi_W(h_2p_2) \\ &= -d_V(P_1,H_1)d_W(P_2,H_2), \end{aligned}$$

we have $[d_{V\otimes W}(P,H)] = -[d_V(P_1,H_1)]\otimes[d_W(P_2,H_2)]$. Recall that $[d_V(P_1,H_1)]$ (resp. $[d_W(P_2,H_2)]$) is a dimension submatrix over $\mathscr{D}_p^2(K_1)$ (resp. $\mathscr{D}_p^2(K_2)$). Thus $\boldsymbol{x}_j \in Z_{\mathscr{D}_p^2(K_j)}(K_j)$ $(j=1,2)$ implies $\boldsymbol{x}_1\otimes\boldsymbol{x}_2 \in Z_{\mathscr{D}_p^2(K_1\times K_2)}(K_1\times K_2)$. $\square$

Remarking $\bigcap_p \mathscr{D}_p^2(G) = \mathscr{D}_1^2(G)$, similarly as in the proof of Proposition 6.3, we obtain the following proposition.

Proposition 6.4. *Let K_1 and K_2 be finite groups not of prime power order such that $Z_{\mathscr{D}^2(K_1)}(K_1)\neq\varnothing$ and $Z_{\mathscr{D}_1^2(K_2)}(K_2)\neq\varnothing$. Then $Z_{\mathscr{D}^2(K_1\times K_2)}(K_1\times K_2)\neq\varnothing$ holds.*

On the other hand, the G-module $V(G)$ gives some restriction:

Proposition 6.5. *Let G be a finite group such that $\{1\} < G^{\{p\}} < G$ for some odd prime p. Then $d_{V(G)}$ is positive on $\mathscr{D}_1^2(G)$. In particular, $Z_{\mathscr{D}_1^2(G)}(G)=\varnothing$ holds.*

Example 6.6. Let $D_4 = \langle(1,2)(3,4),(1,3)(2,4)\rangle$ and $D_8 = \langle(1,2)(3,4),(1,2,3,4)\rangle$ be subgroups of S_4. Then $(D_4,D_8)\in Z_{\mathscr{D}_2^2(S_4)}(S_4)$. Thus $Z_{\mathscr{D}_2^2(S_4\times S_4)}(S_4\times S_4)\neq\varnothing$ which implies that $S_4\times S_4$ is not a gap group. Repeating, $\prod_{i=1}^n S_4$ is also not a gap group.

Now we prove the main theorem.

Proof of Theorem 1.1. By Theorem 1.2, G is a gap group if so is K. If K is of prime order, Corollary 6.2 yields the assertion. Let K be a finite group not of prime power order which is not a gap group. Then there is a vector $\boldsymbol{x}\in Z_{\mathscr{D}^2(K)}(K)$. By Proposition 2.2, it suffices to show $\mathrm{NGC}(G)$ under the assumption that n is odd, say $n=2\gamma+1$. Let $D=[d_{V_j}(P,H)]\in M(s,t;\boldsymbol{Z})$ be a dimension submatrix of K over $\mathscr{D}^2(K)$. We define $(P,H')\in\mathscr{D}^2(G)$ for $(P,H)\in\mathscr{D}^2(K)$ as follows. Take an element $h\in H\backslash P$ of 2-power order. Let H' be a subgroup of G which are generated by P and ha. Note that H' does not depend on the choice of h. Set $F=[d_{V_j'}(P,H')]\in M(s,t';\boldsymbol{Z})$, where t' is a number of $\mathscr{L}(G)$-free irreducible G-modules. We claim that ${}^tF\boldsymbol{x}=\boldsymbol{0}$, which implies $Z_{\mathscr{D}^2(G)}(G)\neq\varnothing$ and thus G is not a gap group. Let W_1 (resp. W_2) be trivial (resp. nontrivial) 1-dimensional D_{2n}-module and W_k $(3\le k\le\gamma+2)$ be all irreducible 2-dimensional D_{2n}-modules. Let V_j $(1\le j\le\beta)$ be all irreducible K-

modules such that V_j is $\mathscr{L}(K)$-free whenever $1 \leq j \leq \alpha$ but V_j is not whenever $\alpha < j \leq \beta$. Then an $\mathscr{L}(G)$-free irreducible G-module is one of $V_j \otimes W_1$ $(1 \leq j \leq \alpha)$, $V_j \otimes W_2$ $(1 \leq j \leq \alpha)$ and $V_j \otimes W_k$ $(1 \leq j \leq \beta, 3 \leq k \leq \gamma + 2)$. Thus $t' = 2\alpha + \beta\gamma$. We obtain that

$$d_{V_j \otimes W_1}(P, H') = -\frac{1}{|P|} \sum_{h \in H' \setminus P} \chi_{V_j}(\pi_1(h)) \chi_{W_1}(a) = d_{V_j}(P, H)$$

by (2.1), where $\pi_1 : G \to K$ is a canonical projection. Similarly, we get $d_{V_j \otimes W_2}(P, H') = -d_{V_j}(P, H)$ and $d_{V_j \otimes W_k}(P, H') = 0$. Thus $F = [D, -D, \mathbf{0}]$ and then ${}^t F\boldsymbol{x} = \mathbf{0}$. We complete the proof. □

COROLLARY 6.7. *Let K be a p-group, $\prod_{k=1}^{\alpha} S_4$, or S_5. Then $G = K \times \prod_{j=1}^{\beta} D_{2n_j}$ is not a gap group for any $\beta \geq 0$ and any $n_j \geq 1$.*

PROOF. Since K is not a gap group, Corollary 6.2 and Theorem 1.1 imply $\mathrm{NGC}(K \times D_{2n_1})$. Thus the proof is completed applying Theorem 1.1 each step by induction on β. □

THEOREM 6.8. *Let n_k $(1 \leq k \leq \alpha)$ be an integer such that $n_1 \geq n_2 \geq \cdots \geq n_\alpha > 1$ and let $G = \prod_{k=1}^{\alpha} S_{n_k}$ be a direct product group of symmetric groups. Then G is a gap group if and only if either $\alpha \geq 1$ and $n_1 \geq 6$ or $\alpha \geq 2$ and $n_1 = 5$, $n_2 \geq 4$.*

This holds from Propositions 2.2, 3.4, Corollary 6.7 and a result of Dovermann and Herzog [**2**]: A symmetric group S_n is a gap group for $n \geq 6$.

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