J. Math. Soc. Japan
Vol. 52, No. 1, 2000

Asymptotic behavior of the transition probability of a simple random walk on a line graph

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(Received Jun. 8, 1998)

Abstract. For simple random walks $\{P_G^n\}$ on a homogeneous graph G and $\{P_{L(G)}^n\}$ on its line graph $L(G)$, we obtain the relationship between the asymptotic behavior of the n-step transition probability $P_G^n(x,x)$ and that of $P_{L(G)}^n(x,x)$ as $n \to \infty$.

1. Introduction.

Let G be an infinite connected graph and $P_G^n(x,x)$ the probability that a simple random walk (the definition will be given in Section 2) on G starting at x returns to x at time n. It is well-known that for even n,

$$P_{\boldsymbol{Z}^d}^n(x,x) \sim \frac{2d^{d/2}}{(2\pi n)^{d/2}} \quad (n \to \infty), \tag{1.1}$$

where $\boldsymbol{Z}^d$ is the d-dimensional lattice [**8**]. Similarly, for the hexagonal lattice and the Kagome lattice, one can show

$$P_{Hexagonal}^n(x,x) \sim 3\sqrt{3}\frac{1}{(2\pi n)^{d/2}} \quad (\text{even } n \to \infty), \quad P_{Kagome}^n(x,x) \sim \frac{4\sqrt{3}}{3}\frac{1}{(2\pi n)^{d/2}} \quad (n \to \infty) \tag{1.2}$$

by the calculation of Fourier series. Here the power d equals 2, which depends on the fact that the vertices of both infinite lattices can be embedded in $\boldsymbol{Z}^2$ periodically.

Now when the transition probability of a random walk on a graph G which has periodic structure in some sense behaves asymptotically as

$$P_G^n(x,x) \sim \frac{C_G}{(2\pi n)^{d/2}} \quad (n \to \infty), \tag{1.3}$$

what is the meaning of the constant C_G ([**6**])? One geometrical interpretation of C_G is given in [**4**]. In this paper, in connection with the problem above, we investigate how the constant C_G changes under the graph theoretical operation of G which is called line graph.

First we prepare some definitions. Let $G = (V(G), E(G))$ be a connected infinite graph, where the sets $V(G)$ and $E(G)$ are the vertex set and the unordered edge set of

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1991 *Mathematics Subject Classification.* Primary 58G25; Secondary 05C50.

Key words and phrases. Line graph, simple random walk, transition probability, discrete Laplacian, spectrum.

Supported by J.S.P.S. Research Fellowships for Young Scientists.

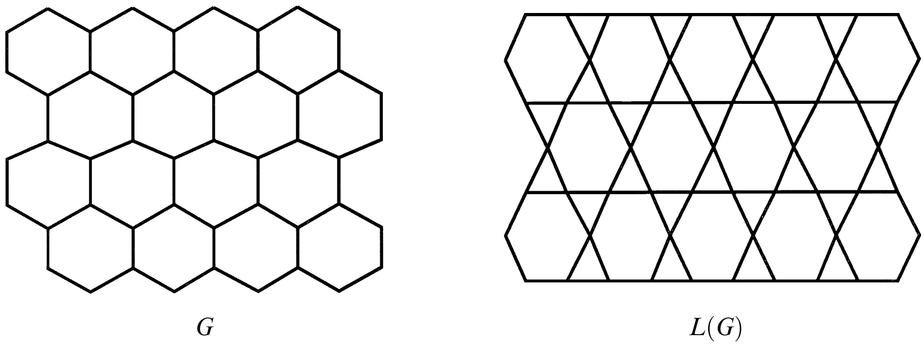

Figure 1: Hexagonal-lattice and Kagome-lattice.

G, respectively. We assume a graph G is simple, that is, G has no self loops and no multiple edges. A set $N_x = \{y \in V(G); xy \in E(G)\}$ is the neighborhood of a vertex x. A graph G is called d-regular if $|N_x| \equiv d$ for all $x \in V(G)$, where $|A|$ is the cardinality of a set A. Throughout this paper, we deal with only d-regular graphs.

Now we define a line graph $L(G)$ of G as follows:

- $V(L(G)) = E(G)$
- $E(L(G)) = \{(x, y)(y, z);\ \ xy \in E(G)$ and $yz \in E(G),\ \ x \neq z\}$

The vertex set of $L(G)$ is the edge set of G and vertices α and β in $L(G)$ are adjacent if α and β as edges in G have a common vertex in G.

REMARK 1.1. One can check in Figure 1 that the line graph of the hexagonal-lattice is the Kagome-lattice, that is, L(hexagonal-lattice) = Kagome-lattice

Next we define a notion of homogeneity of graphs. A graph G is said to be homogeneous if for any pair of vertices x and y, there exists a graph automorphism which maps x to y. (We remark that the homogeneity in the sense above is usually called vertex transitivity in graph theory.) When G is homogeneous, G is necessarily a regular graph and for all $n \in \boldsymbol{N}$ there exists a constant $0 \leq C_n \leq 1$ such that

$$P_G^n(x, x) = C_n \quad (\forall x \in V(G)).$$

For example, $\boldsymbol{Z}^d$(d-dimensional lattice), triangular-lattice, hexagonal-lattice, Kagome-lattice, T_d(d-regular tree) and etc. are homogeneous in the sense above. Before we

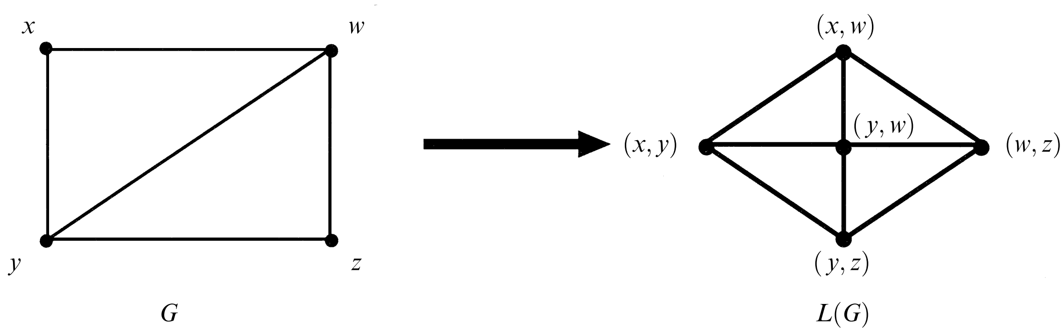

Figure 2: Line graph.

mention our main theorem, we recall the definition of a bipartite graph. A graph G is called a bipartite graph if G has no cycles of odd length, in other words, the vertex set $V(G)$ can be partitioned into two disjoint subsets V_1 and V_2 in such a way that $V(G) = V_1 \amalg V_2$ and every edge in $E(G)$ connects a vertex in V_1 with a vertex in V_2. If G is bipartite, the simple random walk on G has period 2 and the spectrum $\sigma(P_G)$ is symmetric with respect to the origin (see Lemma 2.3).

Our main theorem is the following:

Theorem. *Let G be a homogeneous d-regular graph with $d \geq 3$, P_G the transition operator associated with a simple random walk on G, and $\lambda_0(G) = \sup \sigma(P_G)$ and $\lambda_1(G) = \inf \sigma(P_G)$. Assume that there exists a positive constant $C_G > 0$ and $p \geq 0$ such that*

$$P_G^n(x,x) \sim \frac{C_G \lambda_0(G)^n}{n^p} \tag{1.4}$$

as $n \to \infty$ (as even $n \to \infty$ for (2)).

(1) *When $\lambda_0(G) > |\lambda_1(G)|$,*

$$P_{L(G)}^n(\alpha,\alpha) \sim \frac{2C_G}{d}\left(\frac{(2d-2)\lambda_0(L(G))}{d\lambda_0(G)}\right)^p \frac{\lambda_0(L(G))^n}{n^p} \tag{1.5}$$

for any $\alpha \in V(L(G))$ as $n \to \infty$. Especially, for $\lambda_0(G) = 1$, as $n \to \infty$,

$$P_{L(G)}^n(\alpha,\alpha) \sim \frac{2C_G}{d}\left(\frac{2d-2}{d}\right)^p \frac{1}{n^p}. \tag{1.6}$$

(2) *When G is bipartite (automatically $\lambda_0(G) = |\lambda_1(G)|$), the asymptotic formulas* (1.5) *and* (1.6) *with the coefficient $2C_G$ replaced by C_G hold, that is,*

$$P_{L(G)}^n(\alpha,\alpha) \sim \frac{C_G}{d}\left(\frac{(2d-2)\lambda_0(L(G))}{d\lambda_0(G)}\right)^p \frac{\lambda_0(L(G))^n}{n^p} \tag{1.7}$$

for any $\alpha \in V(L(G))$ as $n \to \infty$. Especially, for $\lambda_0(G) = 1$,

$$P_{L(G)}^n(\alpha,\alpha) \sim \frac{C_G}{d}\left(\frac{2d-2}{d}\right)^p \frac{1}{n^p} \tag{1.8}$$

as $n \to \infty$.

Remark 1.2. The upper bound of the spectrum of $P_{L(G)}$, $\lambda_0(L(G))$ in equations (1.5) and (1.7), can be expressed by $\lambda_0(G)$ by Lemma 2.1(4)$'$, namely,

$$\lambda_0(L(G)) = \frac{1}{2d-2}(d\lambda_0(G) + (d-2)). \tag{1.9}$$

In particular, $\lambda_0(G) = 1$ and $\lambda_0(L(G)) = 1$ are equivalent.

Remark 1.3. If G is the hexagonal lattice, then it is a bipartite 3-regular graph, and it is easy to check that $\lambda_0(G) = 1 = |\lambda_1(G)|$, $d = 3$, $p = 1$. Noting Remark 1.1 we obtain from (1.8)

$$C_{L(G)} = \frac{4}{9} C_G. \tag{1.10}$$

This is the relationship between the coefficients $\frac{4\sqrt{3}}{3}$ and $3\sqrt{3}$ in (1.2).

Remark 1.4. In the case where $\lambda_0(G) = 1$, there are many examples for which the assumption (1.4) holds, for example, abelian covering graphs [**4**]. In the case where $\lambda_0(G) < 1$, there are only a few examples such as d-regular trees. However, we conjecture that the assumption (1.4) holds for all homogeneous graphs.

2. Lemmas.

Let G be a homogeneous d-regular graph and $L(G)$ its line graph which is automatically $(2d-2)$-regular graph. We note that $L(G)$ is not in general a homogeneous graph even if G is homogeneous. We consider a simple random walk on G, that is, $(P_G(x,y))_{x,y \in V(G)}$ is the transition probability matrix which is defined as follows:

$$P_G(x,y) = \begin{cases} 1/d, & \text{if } y \in N_x, \\ 0, & \text{otherwise,} \end{cases}$$

where N_x is the neighborhood of x. Then P_G is a bounded self-adjoint operator on $\ell^2(G)$ which is the set of real-valued functions on $V(G)$ which satisfy $\sum_{x \in V(G)} d \cdot f(x)^2 < \infty$ with the inner product $\langle f, g \rangle = \sum_{x \in V(G)} d \cdot f(x) g(x)$. Since P_G is a contraction operator, its spectrum is contained in $[-1, 1]$. We denote the transition probability of a simple random walk on $L(G)$ by $P_{L(G)}$. We have obtained the relationship between the spectrum of P_G and that of $P_{L(G)}$ in [**7**].

Lemma 2.1. *Let $\phi : \ell^2(G) \to \ell^2(L(G))$ and $\phi^* : \ell^2(L(G)) \to \ell^2(G)$ be defined by*

$$\phi f(x,y) = C_d(f(x) + f(y)), \quad \phi^* F(x) = C_d^{-1} \sum_{r \in N_x} F(x,r),$$

where $C_d = (d/(2d-2))^{1/2}$ and $\ell^2(L(G))$ is identified with the space of symmetric ℓ^2-functions $\{F(x,y); xy \in E(G), \|F\|^2 = \sum_{xy \in E(G)} (2d-2)|F(x,y)|^2 < \infty\}$. Then

(1) *ϕ and ϕ^* are linear bounded operators and ϕ^* is the adjoint operator of ϕ,*

(2) *$\phi^*\phi = d(P_G + 1)$, $\phi\phi^* = (2d-2)(P_{L(G)} + 1/(d-1))$,*

(3) *$\phi^* P_{L(G)} = h(P_G)\phi^*$, where $h(x) = (1/(2d-2))\{dx + (d-2)\}$,*

(4) *$\sigma(P_{L(G)}) = \{-1/(d-1)\} \cup h(\sigma(P_G))$, where $\{-1/(d-1)\}$ are eigenvalues of infinite multiplicity. In particular,*

(4)′ *$\lambda_0(L(G)) = h(\lambda_0(G))$, where $\lambda_0(G)$ (resp. $\lambda_0(L(G))$) is the upper bound of the spectrum $\sigma(P_G)$ (resp. $\sigma(P_{L(G)})$).*

Proof. The proof can be found in [**7**]. □

We remark that $\overline{\ell^2(L(G))}$ is decomposed into two closed subspaces, that is, $\ell^2(L(G)) = \overline{\phi(\ell^2(G))} \oplus \overline{\phi(\ell^2(G))}^{\perp}$. The spectrum of $P_{L(G)}$ restricted to the subspace $\overline{\phi(\ell^2(G))}$ is $h(\sigma(P_G))$ and that of $P_{L(G)}$ restricted to $\overline{\phi(\ell^2(G))}^{\perp}$ is $\{-1/(d-1)\}$.

Let $e_x \in \ell^2(G)$ and $e_\alpha \in \ell^2(L(G))$ be defined by $e_x = d^{-1/2}\delta_x \in \ell^2(G)$, $e_\alpha = (2d-2)^{-1/2}\delta_\alpha \in \ell^2(L(G))$. Then $\{e_x\}_{x\in V(G)}$ (resp. $\{e_\alpha\}_{\alpha\in V(L(G))}$) is an orthonormal basis of $\ell^2(G)$ (resp. $\ell^2(L(G))$). We can show the following lemma.

Lemma 2.2. *Let G be a homogeneous d-regular graph. Then for each $\alpha = xy \in V(L(G)) = E(G)$,*

$$(d-1)P^{n+1}_{L(G)}(\alpha,\alpha) + P^n_{L(G)}(\alpha,\alpha) = \langle (1+P_G)h(P_G)^n e_x, e_x\rangle. \tag{2.1}$$

Proof. We calculate $I_n = \langle \phi^* P^n_{L(G)} e_\alpha, \phi^* e_\alpha\rangle$ in two ways. Firstly by Lemma 2.1 (1) and (2), we obtain

$$\begin{aligned} I_n = \langle \phi\phi^* P^n_{L(G)} e_\alpha, e_\alpha\rangle &= \left\langle (2d-2)\left(P_{L(G)} + \frac{1}{d-1}\right) P^n_{L(G)} e_\alpha, e_\alpha \right\rangle \\ &= (2d-2)P^{n+1}_{L(G)}(\alpha,\alpha) + 2P^n_{L(G)}(\alpha,\alpha). \end{aligned}$$

On the other hand, using Lemma 2.1 (3) and the definition of ϕ^*, we have

$$\begin{aligned} I_n &= \langle h(P_G)^n \phi^* e_\alpha, \phi^* e_\alpha\rangle = \langle h(P_G)^n (e_x + e_y), (e_x + e_y)\rangle \\ &= 2(\langle h(P_G)^n e_x, e_x\rangle + \langle h(P_G)^n e_x, e_y\rangle), \end{aligned}$$

where $\alpha = xy$ and we used the homogeneity of G for the last equality. Then we obtain

$$(d-1)P^{n+1}_{L(G)}(\alpha,\alpha) + P^n_{L(G)}(\alpha,\alpha) = \langle h(P_G)^n e_x, e_x\rangle + \langle h(P_G)^n e_x, e_y\rangle, \tag{2.2}$$

where the function h is the same one as in Lemma 2.1 (3).

For any homogeneous graph G, it is easy to see that for $\lambda \in \boldsymbol{C}\backslash\sigma(P_G)$

$$\lambda g_\lambda(x,x) = 1 + g_\lambda(r,x) \quad (\forall r \in N_x),$$

where $g_\lambda(x,y)$ is a green function (or a resolvent kernel), that is, $g_\lambda(x,y) = (\lambda - P_G)^{-1}(x,y)$. Then using the functional calculus, for any function k analytic on a neighborhood of the spectrum $\sigma(P_G)$, especially for any polynomial k, we have

$$P_G k(P_G)(x,x) = k(P_G)(r,x) \quad (\forall r \in N_x). \tag{2.3}$$

Hence by using the equality (2.3) in (2.2) we obtain the lemma. $\square$

Next lemma can be considered as part of an extension of the Perron–Frobenius theorem for positive matrices, which is essentially obtained in [**2**].

Lemma 2.3. *Let T be a bounded self-adjoint operator on a Hilbert space H having the positivity preserving property, that is, $Tf \geq 0$ if $f \geq 0$. Put $\lambda_0(T) = \sup(\sigma(T))$ and $\lambda_1(T) = \inf(\sigma(T))$. Then,*

$$\lambda_0(T) + \lambda_1(T) \geq 0. \tag{2.4}$$

In particular, for the transition operator P_G associated with a simple random walk on G,

$$\lambda_0(G) + \lambda_1(G) \geq 0, \tag{2.5}$$

where $\lambda_0(G) = \sup(\sigma(P_G))$ *and* $\lambda_1(G) = \inf(\sigma(P_G))$. *The equality* $\lambda_0(G) + \lambda_1(G) = 0$ *holds if* G *is bipartite.*

PROOF. By the positivity preserving property, for any $f \in H$, we obtain

$$\begin{aligned}\lambda_0(T)\|f\|^2 + \langle Tf, f\rangle &\geq \langle T|f|, |f|\rangle + \langle Tf, f\rangle \\ &= \frac{1}{2}(\langle T(|f| + f), |f| + f\rangle + \langle T(|f| - f), |f| - f\rangle) \\ &= 2(\langle Tf_+, f_+\rangle + \langle Tf_-, f_-\rangle) \\ &\geq 0, \end{aligned} \tag{2.6}$$

where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. We can choose a sequence of f_n such that $\|(T - \lambda_1(T))f_n\| \to 0$ and $\|f_n\| = 1$ by Weyl's criterion [**5**]. Consequently, putting $f = f_n$ in (2.6) and letting $n \to \infty$, we obtain

$$\lambda_0(T) + \lambda_1(T) \geq 0. \tag{2.7}$$

For a bipartite graph G with bipartition $V(G) = V_1 \amalg V_2$, we define a unitary operator $U : \ell^2(G) \to \ell^2(G)$ as $Uf(x) = f(x)$ if $x \in V_1$; $Uf(x) = -f(x)$ if $x \in V_2$. It is easy to check that $-P_G = UP_GU^{-1}$ and so P_G and $-P_G$ are unitarily equivalent. Consequently, $\lambda_0(G) = -\lambda_1(G)$. □

The asymptotic behaviors of $P_{L(G)}^{n+1}(\alpha, \alpha)$ and $P_{L(G)}^{n}(\alpha, \alpha)$ are a little different in strongly transient case. Indeed, we have the following lemma.

LEMMA 2.4. *Let* G *be a homogeneous d-regular graph,* P_G *the transition operator associated with a simple random walk on* G *and* $\lambda_0(G) = \sup \sigma(P_G)$.
1) *When* $\lambda_0(G) > |\lambda_1(G)|$,

$$\lim_{n\to\infty} \frac{P_G^{n+1}(x, x)}{P_G^n(x, x)} = \lambda_0(G) \quad \textit{for any } x \in V(G). \tag{2.8}$$

In particular, for the line graph $L(G)$ *of a homogeneous d-regular graph* G, $\lambda_0(L(G)) > |\lambda_1(L(G))|$ *holds if* $d \geq 3$, *and then*

$$\lim_{n\to\infty} \frac{P_{L(G)}^{n+1}(\alpha, \alpha)}{P_{L(G)}^n(\alpha, \alpha)} = \lambda_0(L(G)) \quad \textit{for any } \alpha \in V(L(G)). \tag{2.9}$$

2) *When* G *is bipartite (and necessarily* $\lambda_0(G) = |\lambda_1(G)|$),

$$\lim_{n\to\infty} \frac{P_G^{2n+2}(x, x)}{P_G^{2n}(x, x)} = \lambda_0(G)^2 \quad \textit{for any } x \in V(G). \tag{2.10}$$

PROOF. 1) By the assumption and Lemma 2.3, G is non-bipartite. Then a simple random walk on G is aperiodic; $P_G^n(x, x) > 0$ for any sufficiently large integer n. Let $E_G(\xi)$ is the resolution of the identity of P_G and put $d\nu(\xi) = d\|E_G(\xi)e_x\|^2$ (independent of $x \in V(G)$ due to the homogeneity). Since $\lambda_0(G)$ is in the spectrum $\sigma(P_G)$ and G is homogeneous, it can be easily checked that $\nu([\lambda_0(G) - \varepsilon, \lambda_0(G)]) > 0$ for any $\varepsilon > 0$.

Then we obtain

$$P_G^n(x,x) = \int_{\lambda_1(G)}^{\lambda_0(G)} \xi^n \, d\nu(\xi) \sim \int_{\lambda_0(G)-\varepsilon}^{\lambda_0(G)} \xi^n \, d\nu(\xi). \tag{2.11}$$

Hence we have

$$\lambda_0(G) - \varepsilon \le \liminf_{n\to\infty} \frac{P_G^{n+1}(x,x)}{P_G^n(x,x)} \le \limsup_{n\to\infty} \frac{P_G^{n+1}(x,x)}{P_G^n(x,x)} \le \lambda_0(G)$$

and since $\varepsilon > 0$ is arbitrary, (2.8) holds.

For the second assertion, we remark on the structure of the spectrum of $P_{L(G)}$. Since $\sigma(P_G)$ is contained in $[-1,1]$ for any G, the image $h(\sigma(P_G))$ is contained in $[-1/(d-1), 1]$, where h is the same one as in Lemma 2.1 (3). So because of Lemma 2.1 (4) we have

$$\lambda_1(L(G)) = \inf \sigma(P_{L(G)}) = \frac{-1}{d-1}.$$

We also note that the upper bound of the spectra of d-regular graphs is greater than that of the d-regular tree T_d, that is, $\lambda_0(G) \ge \lambda_0(T_d) = 2\sqrt{d-1}/d$ for any d-regular graph G [**1**]. When $d \ge 3$, we obtain

$$|\lambda_1(L(G))| = \left|\frac{-1}{d-1}\right| < \frac{1}{2d-2}(2\sqrt{d-1} + (d-2)) = h(\lambda_0(T_d)) \le \lambda_0(L(G)). \tag{2.12}$$

2) When G is bipartite, it is sufficient to note that $P_G^{2n}(x,x) > 0$ and $P_G^{2n+1}(x,x) = 0$ for any n. $\square$

Remark 2.5. This lemma holds for more general symmetric random walks on infinite graphs under appropriate modification.

Next we consider the asymptotic behavior of moments.

Lemma 2.6. *Let $|a| < \lambda_0$ and μ be a probability measure supported on $[a, \lambda_0]$ such that for $p \ge 0$*

$$\int_a^{\lambda_0} \xi^n \, d\mu(\xi) \sim \frac{A\lambda_0^n}{n^p} \tag{2.13}$$

as $n \to \infty$. Let $v \in C^2([a, \lambda_0])$ be a function which has the unique maximum at λ_0 in $[a, \lambda_0]$ and $v'(\lambda_0) > 0$, and u be a function continuous at λ_0. Then we have

$$\int_a^{\lambda_0} u(\xi) v(\xi)^n \, d\mu(\xi) \sim \frac{Au(\lambda_0)v(\lambda_0)^p}{(\lambda_0 v'(\lambda_0))^p} \frac{v(\lambda_0)^n}{n^p} \tag{2.14}$$

as $n \to \infty$.

Proof. The asymptotic behavior depends only on ξ near λ_0 and since u is continuous at λ_0

$$\int_a^{\lambda_0} u(\xi)v(\xi)^n \, d\mu(\xi) \sim u(\lambda_0) \int_a^{\lambda_0} v(\xi)^n \, d\mu(\xi). \tag{2.15}$$

We first assume that $\lambda_0 = 1$ and $v(1) = 1$. Since $v \in C^2$, one can check that

$$\log v(\xi) = (v'(1) + O(|1-\xi|)) \log \xi \tag{2.16}$$

as $\xi \to 1$. So for any $\varepsilon > 0$ there exists a positive constant $0 < \xi_0 < 1$ such that

$$\int_\eta^1 v(\xi)^n \, d\mu(\xi) \le \int_\eta^1 \xi^{n(v'(1)-\varepsilon)} \, d\mu(\xi) \tag{2.17}$$

for $\xi_0 < \forall \eta < 1$. Since $v'(1) > 0$, for $|a| < \eta < 1$, we have

$$\begin{aligned} \limsup_{n\to\infty} n^p \int_a^1 v(\xi)^n \, d\mu(\xi) &= \limsup_{n\to\infty} n^p \int_\eta^1 v(\xi)^n \, d\mu(\xi) \\ &\le \limsup_{n\to\infty} n^p \int_\eta^1 \xi^{n(v'(1)-\varepsilon)} \, d\mu(\xi) \\ &= \frac{A}{(v'(1)-\varepsilon)^p}. \end{aligned} \tag{2.18}$$

Similarly, we obtain

$$\liminf_{n\to\infty} n^p \int_a^1 v(\xi)^n \, d\mu(\xi) \ge \frac{A}{(v'(1)+\varepsilon)^p}. \tag{2.19}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\int_a^1 v(\xi)^n \, d\mu(\xi) \sim \frac{A}{v'(1)^p} \frac{1}{n^p} \tag{2.20}$$

as $n \to \infty$. In general case, it is sufficient to consider $v(\lambda_0\xi)/v(\lambda_0)$ as a function $v(\xi)$ and $d\mu(\lambda_0\xi)$ as a measure $d\mu(\xi)$. $\square$

3. Proof of the theorem.

Using the lemmas which are obtained in the previous section, we can compute the asymptotic behavior of $P^n_{L(G)}(\alpha,\alpha)$ as $n \to \infty$.

Assume that the following asymptotic behavior holds:

$$P^n_G(x,x) \sim \frac{C_G \lambda_0(G)^n}{n^p} \quad (n \to \infty) \tag{3.1}$$

and first assume that

$$\lambda_0(G) > |\lambda_1(G)|, \tag{3.2}$$

where $\lambda_0(G) = \sup \sigma(P_G)$ and $\lambda_1(G) = \inf \sigma(P_G)$. (Note that in general $\lambda_0(G) \ge |\lambda_1(G)|$ by Lemma 2.3.)

THEOREM 3.1. *Let G be a homogeneous d-regular graph. The assumptions* (3.1), (3.2) *hold. Then, for any $\alpha \in V(L(G))$, as $n \to \infty$,*

$$P^n_{L(G)}(\alpha, \alpha) \sim \frac{2C_G}{d}\left(\frac{(2d-2)\lambda_0(L(G))}{d\lambda_0(G)}\right)^p \frac{\lambda_0(L(G))^n}{n^p}. \tag{3.3}$$

Especially, for $\lambda_0(G) = 1$, as $n \to \infty$,

$$P^n_{L(G)}(\alpha, \alpha) \sim \frac{2C_G}{d}\left(\frac{2d-2}{d}\right)^p \frac{1}{n^p}. \tag{3.4}$$

PROOF. The assumption (3.1) says that

$$\int_{\lambda_1(G)}^{\lambda_0(G)} \xi^n d\|E_G(\xi)e_x\|^2 \sim \frac{C_G \lambda_0(G)^n}{n^p}, \tag{3.5}$$

where $E_G(\xi)$ is the resolution of the identity of P_G. Therefore by (3.2) and Lemma 2.6, for the function h in Lemma 2.1 (3), we obtain

$$\begin{aligned}\langle (1+P_G)h(P_G)^n e_x, e_x\rangle &= \int_{\lambda_1(G)}^{\lambda_0(G)} (1+\xi)h(\xi)^n d\|E_G(\xi)e_x\|^2 \\ &\sim \frac{C_G(1+\lambda_0(G))h(\lambda_0(G))^p}{(\lambda_0(G)h'(\lambda_0(G)))^p}\frac{h(\lambda_0(G))^n}{n^p} \\ &= C_G(1+\lambda_0(G))\left(\frac{(2d-2)\lambda_0(L(G))}{d\lambda_0(G)}\right)^p \frac{\lambda_0(L(G))^n}{n^p} \end{aligned} \tag{3.6}$$

as $n \to \infty$. By Lemma 2.4 and Lemma 2.1 (4)$'$, we have

$$\begin{aligned}(d-1)P^{n+1}_{L(G)}(\alpha, \alpha) + P^n_{L(G)}(\alpha, \alpha) &\sim ((d-1)\lambda_0(L(G)) + 1)\cdot P^n_{L(G)}(\alpha, \alpha) \\ &= \frac{d}{2}(\lambda_0(G)+1)P^n_{L(G)}(\alpha, \alpha)\end{aligned} \tag{3.7}$$

as $n \to \infty$. Therefore using Lemma 2.2 we obtain the theorem. □

COROLLARY 3.2. *If G is a bipartite homogeneous d-regular graph* (*and so $\lambda_0(G) = |\lambda_1(G)|$*), *and the assumption* (3.1) *holds for even $n \to \infty$, then*

$$P^n_{L(G)}(\alpha, \alpha) \sim \frac{C_G}{d}\left(\frac{(2d-2)\lambda_0(L(G))}{d\lambda_0(G)}\right)^p \frac{\lambda_0(L(G))^n}{n^p}. \tag{3.8}$$

Especially, for $\lambda_0(G) = 1$, as $n \to \infty$,

$$P^n_{L(G)}(\alpha, \alpha) \sim \frac{C_G}{d}\left(\frac{2d-2}{d}\right)^p \frac{1}{n^p}. \tag{3.9}$$

PROOF. It is sufficient to note that if a graph G is bipartite then P_G and $-P_G$ are unitarily equivalent, which implies that (3.1) is equivalent to

$$\int_0^{\lambda_0(G)} \xi^n d\|E_G(\xi)e_x\|^2 \sim \frac{C_G \lambda_0(G)^n}{2n^p} \tag{3.10}$$

as $n \to \infty$. □

REMARK 3.3. The assumption (3.2) should be replaced with G being non-bipartite. We conjecture that if G is homogeneous and the spectrum is symmetric (in the sense that $\lambda_0(G) = |\lambda_1(G)|$) then G is bipartite. In general, if G is not homogeneous, the conjecture above is not true. For example, let $\boldsymbol{Z}^1 = (V(\boldsymbol{Z}^1), E(\boldsymbol{Z}^1))$ be the ordinary one-dimensional lattice and $G = (V(G), E(G))$ the graph such that $V(G) = V(\boldsymbol{Z}^1) \cup \{a\}$ and $E(G) = E(\boldsymbol{Z}^1) \cup \{(0,a),(1,a)\}$. Since the compact perturbation does not change the essential spectrum we obtain $\sigma(P_G) = \sigma(P_{\boldsymbol{Z}^1}) = [-1,1]$ and so the spectrum of G is symmetric. However, G has a cycle of length 3 and so G is not bipartite. So far we have shown that if G is homogeneous and non-bipartite, and $\lambda_0(G) = 1$ then $|\lambda_1(G)|$ is strictly less than 1, in other words, the spectrum is not symmetric [**3**].

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