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A minimal action of the compact quantum group $SU_q(n)$ on a full factor

By Yoshimichi UEDA

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Abstract. Based on the free product construction we show that a certain full factor of type III_{q^2} admits a minimal coaction of the compact quantum group $SU_q(n)$. Minimal coactions of compact Kac algebras are also investigated by the same technique.

1. Introduction.

The importance of study on quantum groups has been emphasized recently, and Kac algebras ([**ES2**]) and the compact quantum group (or quantum matrix pseudo-group) $SU_q(n)$ ([**Wr1**], [**Wr2**]) are typical examples. In the operator algebra setting, their coactions (i.e., "quantized symmetries") on ambient algebras (i.e., "quantized spaces") are of central importance. Among them minimal coactions (if they exist) are desirable, fixed-point subalgebras and/or crossed products naturally giving rise to infinite-index ([**J**], [**Ks**], [**L1**]) irreducible inclusions of factors of depth 2 (in the sense of A. Ocneanu). In fact, such inclusions have been recently discussed by several authors ([**EN**], [**ILP**], [**Ymg1**] and so on).

However, to the best of author's knowledge no example of a minimal coaction of a compact quantum group on a factor is known so far. In fact, some attempt was made by S. Yamagami to prove its non-existence. The main purpose of the paper is to show that a certain full factor (in the sense of [**C2**]) of type III_{q^2} indeed admits a minimal coaction of the compact quantum group $SU_q(n)$ $(0 < q < 1)$. Minimal coactions of compact Kac algebras are also investigated, and our technical tool here is the free product construction. When dealing with minimal coactions, the main difficulties are the computation of the fixed-point subalgebra and to see its irreducibility. The advantage of the free product construction is its high non-commutativity. We choose a much smaller subalgebra than the fixed-point subalgebra, and we can sometimes easily observe its irreducibility against the original factor. In this way, the current approach enables us to obtain the minimality without determining the fixed-point subalgebra. Note that a similar idea was used in [**P**] for different purposes.

To explain the idea in our construction, we here deal with the compact group case. Let $\mathscr{R}$ be the AFD II_1 factor with the unique normalized trace τ, and G be a compact group with the (probability) Haar measure μ. Note that the tensor product $\mathscr{R} \otimes L^\infty(G)$ is equipped with the natural tensor product trace $\tau \otimes (\int_G \cdot \, d\mu)$, and we then

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perform the free product

$$(\mathscr{R} \otimes L^\infty(G)) * \mathscr{R} \qquad \left(\text{relative to } \tau \otimes \left(\int_G \cdot\, d\mu\right) \text{ and } \tau\right).$$

Using [**D2**, Theorem 4.6.], we can see that this algebra is the free group factor $L(\boldsymbol{F}_2)$. Note that the translation λ_g on the group induces the natural free product action

$$\alpha_g = (\mathrm{id} \otimes \mathrm{Ad}(\lambda_g)) * \mathrm{id}$$

on the free product. The computation of the fixed-point algebra might be difficult, but the obvious subalgebra$(\mathscr{R} \otimes \mathbf{C}1) * \mathscr{R}$ sits in the fixed-point subalgebra, and it is quite standard to see $((\mathscr{R} \otimes L^\infty(G)) * \mathscr{R}) \cap ((\mathscr{R} \otimes \mathbf{C}1) * \mathscr{R})' = \mathbf{C}1$ based on free product machine, i.e., due to its high non-commutativity. Thus, we have obtained a minimal action of a compact group (on the free group factor $L(\boldsymbol{F}_2)$). In general an action of course has to be replaced by a coaction so that the notion of free products of coactions is required.

In §2 we will summarize basic definitions and properties on free products of von Neumann algebras. The result (which appeared in L. Barnett's article [**B**]) guaranteeing the high non-commutativity of the free product construction is important to us. Standard facts on compact Kac algebras, the compact quantum group $SU_q(n)$, and Woronowicz algebras as well as their coactions will be also collected here. From the discussions in the preceding paragraph, it is clear that what is relevant for our purpose is how to justify the notion of free products of two coactions. This will be done in §3 under a natural invariance condition, and in the next §4 we will prove the above-mentioned main result by our free product machine. Minimal coactions of arbitrary compact Kac algebras will also be investigated by the same idea, and we see that the free group factor with n generators admits a minimal coaction of any hyperfinite compact Kac algebra. In §5, based on the justification in §3, we will investigate the free product of two compact Kac algebras, and we would like to point out that the discussions here are closely related to the recent article [**Wn**].

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2. Preliminaries.

In this section, we will summarize basic definitions and properties needed in this paper.

2.1. Free product.

Throughout this subsection, we assume that $\mathcal{N}_i$ is a σ-finite von Neumann algebra with a faithful normal state φ_i $(i = 1, 2)$, and denote by $\mathcal{N}_i^\circ$ the kernel of φ_i $(i = 1, 2)$. We first describe the notion of (reduced) free products of von Neumann algebras with respect to faithful normal states introduced by D. V. Voiculescu [**V1**].

Definition 2.1.1. ([**V1**], [**VDN**]) Let $(\mathcal{M}, \varphi)$ be a von Neumann algebra equipped with a faithful normal state. $(\mathcal{M}, \varphi) = (\mathcal{N}_1, \varphi_1) * (\mathcal{N}_2, \varphi_2)$ is called the free product of $(\mathcal{N}_1, \varphi_1)$ and $(\mathcal{N}_2, \varphi_2)$ if it satisfies the following three conditions:

(1) There exist two injective normal unital $*$homomorphisms $\lambda_{\mathcal{N}_1}$ and $\lambda_{\mathcal{N}_2}$ into $\mathcal{M}$ from $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively whose ranges generate $\mathcal{M}$.

(2) $\varphi \circ \lambda_{\mathcal{N}_1} = \varphi_1$ and $\varphi \circ \lambda_{\mathcal{N}_2} = \varphi_2$.

(3) φ satisfies the freeness in the sense of D. V. Voiculescu ([**V1**]), i.e. for $x_j \in \mathcal{N}_{i_j}^\circ$ with $i_1 \neq \cdots \neq i_n$ and $i_j \in \{1, 2\}$, then

$$\varphi(\lambda_{\mathcal{N}_{i_1}}(x_1) \cdots \lambda_{\mathcal{N}_{i_n}}(x_n)) = 0.$$

The free product was constructed in [**V1**] and [**VDN**], and known to be characterized by the above three conditions. Therefore, we will employ the above as our working definition. We will often identify $\lambda_{\mathcal{N}_i}(x)$ with x itself. When no confusion is possible, we will denote by $\mathcal{N}_1 * \mathcal{N}_2$ the free product von Neumann algebra $\mathcal{M}$ and $\varphi_1 * \varphi_2$ will be referred to as a free state.

The following theorem was proved by L. Barnett in [**B**] based on well-known L. Pukanszky's 14ε-argument.

Theorem 2.1.2. ([**B**, Theorem 11.]) *Let $(\mathcal{M}, \varphi) = (\mathcal{N}_1, \varphi_1) * (\mathcal{N}_2, \varphi_2)$. Suppose that $(\mathcal{N}_i)_{\varphi_i}$ contains a discrete group $\mathcal{G}_i$ of orthogonal unitaries with respect to φ_i $(i = 1, 2)$ with $|\mathcal{G}_1| \geq 2, |\mathcal{G}_2| \geq 3$.*

Fix a in $\mathcal{G}_1 \backslash \{1\}$ and b, c in $\mathcal{G}_2 \backslash \{1\}$, and we have

$$\|x - \varphi(x)1\|_\varphi \leq 14 \max\{\|[x, a]\|_\varphi \|[x, b]\|_\varphi \|[x, c]\|_\varphi\}. \tag{1}$$

for every x in $\mathcal{M}$, where $\| \ \|_\varphi$ is the norm induced by φ.

Hence, $\mathcal{M}$ is a full factor. Furthermore, by (1) *the following holds:*

$$\{a, b, c\}' \cap \mathcal{M} = \mathbf{C}1. \tag{2}$$

The equation (2) indicating the high non-commutativity of free products is our essential tool.

2.2. Compact Kac algebras, the compact quantum group $SU_q(n)$ and their coactions on von Neumann algebras.

We first describe the notion of Woronowicz algebras defined by T. Masuda and Y. Nakagami ([**MN**]). This notion is fitting to our purpose.

Definition 2.2.1. ([**MN**]) Let $\mathcal{A}$ be a von Neumann algebra, $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be an injective normal unital $*$homomorphism (comultiplication), $\kappa : \mathcal{A} \to \mathcal{A}$ be an anti-$*$automorphism (unitary antipode), $\{\tau_t\}_{t \in \mathbf{R}}$ be a 1-parameter s.o.-continuous automorphism group on $\mathcal{A}$ (deformation automorphism), and $h : \mathcal{A}_+ \to \mathbf{R}_+$ be a faithful normal semifinite weight (Haar weight). $\boldsymbol{W} = (\mathscr{A}, \delta, \kappa, \{\tau_t\}, h)$ is called a Woronowicz algebra if it satisfies the following four conditions:

(1) δ satisfies the co-associative law.

(2) $\kappa^2 = \mathrm{id}$ and $(\kappa \otimes \kappa) \circ \delta = \Sigma \circ \delta \circ \kappa$, where Σ is the flip.

(3) $(\tau_t \otimes \tau_t) \circ \delta = \delta \circ \tau_t$ and $\kappa \circ \tau_t = \tau_t \circ \kappa$.

(4) h has the left invariance: $(\mathrm{id} \otimes h)(\delta(a)) = h(a)1$ for a in $\mathscr{A}_+$, strong left invariance:

$$(\mathrm{id} \otimes \delta)((1 \otimes y^*)\delta(x)) = (\tau_{-i/2} \circ \kappa \otimes h)(\delta(y)^*(1 \otimes x))$$

for entire analytic elements x, y in $\{a \in \mathscr{A} : h(a^*a) < \infty\}$ with respect to $\{\tau_t\}$, and the commutativity of σ^h and $\sigma^{h\circ\kappa}$.

When $\tau_t = id$, we call $\boldsymbol{K} = (\mathscr{A}, \delta, \kappa, h)$ a Kac algebra. Also we call $(\mathscr{A}, \delta)$ with the condition (1) a Hopf–von Neumann algebra.

In this paper, we will only deal with compact Woronowicz algebras (i.e. the Haar weight is bounded: $h(1) < +\infty$). For simplicity, we assume $h(1) = 1$ and call h the Haar state. In this case, we can prove the uniqueness of the Haar state, the right invariance of the Haar state ([**MN**, Remark 1.3]), and the Haar state of a compact Kac algebra is tracial ([**ES2**, 6.2.1. Theorem]).

Let (A, u) (A is a unital C^*-algebra and u is a unitary in $A \otimes M_n(\mathrm{C})$) be the compact quantum group $SU_q(n)$ ([**Wr1**], [**Wr2**]). S. L. Woronowicz proved the existence of the unique Haar state ϕ, and the Peter-Weyl type theory. If we denote by $(\mathscr{H}_\phi, \pi_\phi, \xi_\phi)$ the G.N.S. triple associated with (A, ϕ), we can equip $\mathscr{A} = \pi_\phi(A)''$ with a Woronowicz algebra structure ([**MN**, Theorem 5.6.]) and this compact Woronowicz algebra will be denoted by $L^\infty(SU_q(n))$. In fact, the Haar state is given by the vector state ω_{ξ_ϕ} and the comultiplication is induced by the fundamental unitary or Kac-Takesaki operator (the unitarity can be proved):

$$W(\pi_\phi(a)\xi_\phi \otimes \pi_\phi(b)\xi_\phi) = (\pi_\phi \otimes \pi_\phi)(\Phi(b))(\pi_\phi(a)\xi_\phi \otimes \xi_\phi)$$

for every a, b in A, where Φ is the canonical comultiplication of $SU_q(n)$. The adjoint $V = W^*$ is a multiplicative unitary in the sense of [**BS**].

Remark 2.2.2. In [**EV**], M. Enock and L. Vaĭnerman claimed that from every compact quantum group in the sense of S. L. Woronowicz one can construct a Woronowicz algebra as above.

Here, with the above notation, we describe the notion of a coaction of a Hopf–von Neumann algebra on a von Neumann algebra.

Definition 2.2.3. Let $(\mathscr{A}, \delta)$ be a Hopf–von Neumann algebra and $\mathscr{M}$ be a von Neumann algebra. An injective normal unital $*$homomorphism $\alpha : \mathscr{A} \to \mathscr{M} \otimes \mathscr{A}$ is called a coaction if it satisfies

$$(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \delta) \circ \alpha.$$

To deal with the crossed product of a von Neumann algebra by a compact Woronowicz algebra, we consider the dual Woronowicz algebra ([**MN**]) of a given Woronowicz algebra. Let $\boldsymbol{W} = (\mathscr{A}, \delta, \kappa, \{\tau_t\}, h)$ be a compact Woronowicz algebra. We assume that $\mathscr{A}$ acts on the standard Hilbert space $L^2(\mathscr{A})$. Let W be the fundamental unitary on $L^2(\mathscr{A}) \otimes L^2(\mathscr{A})$ defined by

$$W(x\xi_h \otimes y\xi_h) = \delta(y)(x\xi_h \otimes \xi_h)$$

for every x, y in $\mathscr{A}$, where ξ_h is the implementing vector of h in the natural cone.

We define $\hat{\pi}(\phi)$ in $B(L^2(\mathscr{A}))$ for every ϕ in $\mathscr{A}_*$ by

$$\hat{\pi}(\phi) = (\phi \otimes \mathrm{id})(W^*)$$

and the dual Woronowicz algebra $\hat{\boldsymbol{W}} = (\hat{\mathscr{A}}, \hat{\delta}, \hat{\kappa}, \{\hat{\tau}_t\}, \hat{h})$ by

$$\hat{\mathscr{A}} = \{\hat{\pi}(\phi) : \phi \in \mathscr{A}_*\}'',$$

$$\hat{\delta}(x) = \hat{W}(1 \otimes x)\hat{W}$$

for every x in $\hat{\mathscr{A}}$ with $\hat{W} = \Sigma(W^*)$, and

$$\hat{\kappa}(\hat{\pi}(\phi)) = \hat{\pi}(\phi \circ \kappa), \qquad \hat{\tau}_t(\hat{\pi}(\phi)) = \hat{\pi}(\phi \circ \tau_{-t})$$

for every ϕ in $\mathscr{A}_*$.

Remark 2.2.4. Construction of the dual Haar weight requires the left Hilbert algebra technique (see [**MN**]).

Here we recall the notion of crossed products ([**N**]) and minimal coactions.

Definition 2.2.5. Assume that $\boldsymbol{W} = (\mathscr{A}, \delta, \kappa, \{\tau_t\}, h)$ is a compact Woronowicz algebra with the corresponding dual Woronowicz algebra $\hat{\boldsymbol{W}} = (\hat{\mathscr{A}}, \hat{\delta}, \hat{\kappa}, \{\hat{\tau}_t\}, \hat{h})$. Let $\mathscr{M}$ be a von Neumann algebra and $\alpha : \mathscr{M} \to \mathscr{M} \otimes \mathscr{A}$ be a coaction.

(1) The crossed product $\mathscr{M} \rtimes_\alpha \boldsymbol{W}$ of $\mathscr{M}$ by $\boldsymbol{W}$ with respect to α is the von Neumann algebra generated by $\alpha(\mathscr{M})$ and $\mathrm{C}1 \otimes \hat{\mathscr{A}}'$, where $\hat{\mathscr{A}}' = \hat{J}\hat{\mathscr{A}}\hat{J}$ and $\hat{J}$ is the modular conjugation of $\hat{\mathscr{A}}$.

(2) The coaction α is called a minimal coaction if the relative commutant of the fixed-point algebra $\mathscr{M}^\alpha = \{x \in \mathscr{M} : \alpha(x) = x \otimes 1\}$ in $\mathscr{M}$ is trivial and the crossed product algebra $\mathscr{M} \rtimes_\alpha \boldsymbol{W}$ is a factor.

3. Free product of coactions.

Throughout this section, we assume that $(\mathscr{A}, \delta)$ is a σ-finite Hopf–von Neumann algebra (i.e. $\mathscr{A}$ is a σ-finite von Neumann algebra with a comultiplication $\delta : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$), $\mathscr{N}_i$ is a σ-finite von Neumann algebra with a faithful normal state φ_i $(i = 1, 2)$ and $\alpha_i : \mathscr{N}_i \to \mathscr{N}_i \otimes \mathscr{A}$ is a coaction $(i = 1, 2)$.

Assume the following conditions:

(1) There exists a subset $\{u_\gamma\}$ of linearly independent elements in $\mathscr{A}$ whose finite linear combinations form a unital dense $*$subalgebra in $\mathscr{A}$.

(2) φ_i is an invariant state of α_i $(i = 1, 2)$,
i.e.

$$(\varphi_i \otimes \mathrm{id}) \circ \alpha_i(x) = \varphi_i(x)1$$

for every x in $\mathscr{N}_i$ $(i = 1, 2)$.

When α is an ordinary action of a compact group G on a von Neumann algebra $\mathscr{M}$ acting on a Hilbert space $\mathscr{H}$, we have an invariant state φ on $\mathscr{M}$ ($\varphi \circ \alpha_g = \varphi$), and the corresponding coaction $\pi_\alpha : \mathscr{M} \to \mathscr{M} \otimes L^\infty(G)$ is

$$(\pi_\alpha(x)\xi)(g) = \alpha_g(x)\xi(g), \qquad g \in G, x \in \mathscr{M}, \xi \in L^2(G; \mathscr{H}).$$

It is straight-forward to check $(\varphi \otimes \mathrm{id})(\pi_\alpha(x)) = \varphi(x)1$ so that the invariance condition (2) is quite natural in this case.

Let

$$(\mathscr{M}, \varphi) = (\mathscr{N}_1, \varphi_1) * (\mathscr{N}_2, \varphi_2).$$

LEMMA 3.1. *$\lambda_{\mathscr{N}_1}(\mathscr{N}_1) \otimes \mathscr{A}$ and $\lambda_{\mathscr{N}_2}(\mathscr{N}_2) \otimes \mathscr{A}$ are free in $\mathscr{M} \otimes \mathscr{A}$ relative to $\mathscr{A}$ with respect to the conditional expectation $\varphi \otimes id$ onto $\mathbf{C}1 \otimes \mathscr{A} = \mathscr{A}$ in the sense of D. V. Voiculescu* (see [**VDN**, §3.8]).

PROOF. Let $\mathscr{L}_i$ be the set of finite linear combinations of $\lambda_{\mathscr{N}_i}(x) \otimes u_\gamma$ for x in $\mathscr{N}_i$ and γ $(i = 1, 2)$. Then it is clear that $\mathscr{L}_i$ is a unital dense $*$subalgebra in $\lambda_{\mathscr{N}_i}(\mathscr{N}_i) \otimes \mathscr{A}$ by the assumption $(i = 1, 2)$. Note that $(\varphi_i \otimes \mathrm{id})(\mathscr{L}_i) \subseteq \mathscr{L}_i$ $(i = 1, 2)$. Hence, by the same argument as in [**VDN**, Proposition 2.5.7.], it is sufficient to show that $\mathscr{L}_1$ and $\mathscr{L}_2$ are free over $\mathscr{A}$ with respect to $\varphi \otimes \mathrm{id}$.

For each $X = \sum \lambda_{\mathscr{N}_i}(x_\gamma) \otimes u_\gamma$ in $\mathscr{L}_i$ $(i = 1, 2)$, we have

$$(\varphi \otimes \mathrm{id})(X) = \sum \varphi \circ \lambda_{\mathscr{N}_i}(x_\gamma) u_\gamma = \sum \varphi_i(x_\gamma) u_\gamma.$$

Since $\{u_\gamma\}$ is linearly independent, we get

$$(*) \qquad (\varphi \otimes \mathrm{id})(X) = 0 \quad \Rightarrow \quad \varphi_i(x_\gamma) = 0 \qquad \textit{for every } \gamma.$$

Take an arbitrary $X^{(j)}$ in $\mathscr{L}_{i_j}$ $(j = 1, \dots, n)$ with $(\varphi \otimes \mathrm{id})(X^{(j)}) = 0$ for all j, $i_1 \neq \cdots \neq i_n$ and $i_j \in \{1, 2\}$. By $(*)$, each $X^{(j)}$ is of the form

$$X^{(j)} = \sum \lambda_{\mathscr{N}_{i_j}}(x_\gamma^{(j)}) \otimes u_\gamma, \quad \varphi_{i_j}(x_\gamma^{(j)}) = 0.$$

Hence we have

$$(\varphi \otimes \mathrm{id})(X^{(1)} \cdots X^{(n)}) = \sum_{\gamma_1} \cdots \sum_{\gamma_n} \varphi(\lambda_{\mathscr{N}_{i_1}}(x_{\gamma_1}^{(1)}) \cdots \lambda_{\mathscr{M}_{i_n}}(x_{\gamma_n}^{(n)})) u_{\gamma_1} \cdots u_{\gamma_n},$$

and each coefficient is zero since φ is a free state. Therefore $\mathscr{L}_1$ and $\mathscr{L}_2$ are free. $\square$

LEMMA 3.2. *There exists a unique $*$isomorphism*

$$\Gamma : \mathscr{M} \to \{(\lambda_{\mathscr{N}_1} \otimes \mathrm{id}) \circ \alpha_1(\mathscr{N}_1), (\lambda_{\mathscr{N}_2} \otimes \mathrm{id}) \circ \alpha_2(\mathscr{N}_2)\}'' \textit{ in } \mathscr{M} \otimes \mathscr{A}$$

satisfying $\Gamma \circ \lambda_{\mathscr{N}_i} = (\lambda_{\mathscr{N}_i} \otimes \mathrm{id}) \circ \alpha_i$ $(i = 1, 2)$.

PROOF. We consider the subalgebras

$$(\lambda_{\mathscr{N}_1} \otimes \mathrm{id}) \circ \alpha_1(\mathscr{N}_1) \quad \text{and} \quad (\lambda_{\mathscr{N}_2} \otimes \mathrm{id}) \circ \alpha_2(\mathscr{N}_2)$$

isomorphic to $\mathscr{N}_1$, $\mathscr{N}_2$ respectively. For a faithful normal state ψ on $\mathscr{A}$, by the invariance condition we compute

$$(\varphi_i \otimes \psi) \circ (\lambda_{\mathscr{N}_i} \otimes \mathrm{id}) \circ \alpha_i(x) = \varphi_i(x)\psi(1) = \varphi_i(x) \quad (i = 1, 2).$$

Thus, Lemma 3.1 guarantees the freeness of the two subalgebras relative to $\varphi \otimes \psi$, and we can construct the desired isomorphism. $\square$

Lemma 3.3. *The above Γ is a coaction on $\mathscr{M}$ by $(\mathscr{A}, \delta)$.*

Proof. Since Γ and δ are (injective) normal unital $*$homomorphisms, it is sufficient to check $(\Gamma \otimes \mathrm{id}) \circ \Gamma = (\mathrm{id} \otimes \delta) \circ \Gamma$ against generators.

For each x in $\mathscr{N}_i$, we have

$$\begin{aligned}
(\Gamma \otimes \mathrm{id}) \circ \Gamma(\lambda_{\mathscr{N}_i}(x)) &= (\Gamma \otimes \mathrm{id}) \circ (\lambda_{\mathscr{N}_i} \otimes \mathrm{id}) \circ \alpha_i(x) \\
&= (\Gamma \circ \lambda_{\mathscr{N}_i} \otimes \mathrm{id}) \circ \alpha_i(x) \\
&= ((\lambda_{\mathscr{N}_i} \otimes \mathrm{id}) \circ \alpha_i \otimes \mathrm{id}) \circ \alpha_i(x) \\
&= (\lambda_{\mathscr{N}_i} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\alpha_i \otimes \mathrm{id}) \circ \alpha_i(x) \\
&= (\lambda_{\mathscr{N}_i} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta) \circ \alpha_i(x) \\
&= (\mathrm{id} \otimes \delta) \circ (\lambda_{\mathscr{N}_i} \otimes \mathrm{id}) \circ \alpha_i(x) \\
&= (\mathrm{id} \otimes \delta) \circ \Gamma(\lambda_{\mathscr{N}_i}(x)).
\end{aligned}$$

Here, the first, third and seventh equalities come from the definition of Γ while the fifth comes from the fact that α_i is a coaction. Hence we are done. $\square$

Definition 3.4. The above coaction is called the free product of coactions α_1 and α_2 and denoted by $\alpha_1 * \alpha_2$.

Remark 3.5. $L^\infty(SU_q(n))$ and every compact Kac algebra satisfy the condition (1) thanks to the Peter-Weyl type theorem (for example see [**Wr2**]). Indeed every Hopf–von Neumann algebra satisfies the condition (1) by the use of a Hamel basis. This was pointed out by Y. Sekine. Hence the condition (1) is not essential. Also the condition (2) is necessary even in the case of group actions.

Remark 3.6. Lemma 3.1 shows

$$(\mathscr{M} \otimes \mathscr{A}, \varphi \otimes \mathrm{id}) \cong (\mathscr{N}_1 \otimes \mathscr{A}, \varphi_1 \otimes \mathrm{id}) *_{\mathscr{A}} (\mathscr{N}_2 \otimes \mathscr{A}, \varphi_2 \otimes \mathrm{id}),$$

where the right-hand side means the "amalgamated free product" of von Neumann algebras. Indeed, many fundamental operations such as tensor product are compatible with the free products (or amalgamated free products) in a certain sense.

4. Main results.

Let $L^\infty(SU_q(n)) = (\mathscr{A}, \delta, \kappa, \{\tau_t\}, h)$ be the compact Woronowicz algebra associated with the compact quantum group $SU_q(n)$ $(0 < q < 1)$ as in the previous section.

Let $\mathscr{R}$ be the AFD type II_1 factor with the unique normalized trace τ and $\beta : \mathscr{R} \to \mathscr{R} \otimes \mathscr{A}$ be the "trivial" coaction defined by $\beta(x) = x \otimes 1$ for every x in $\mathscr{R}$.

We set

$$(\mathscr{M}, \varphi) = (\mathscr{R} \otimes \mathscr{A}, \tau \otimes h) * (\mathscr{R}, \tau)$$

and define the free product coaction $\Gamma : \mathscr{M} \to \mathscr{M} \otimes \mathscr{A}$ by

$$\Gamma = (\mathrm{id} \otimes \delta) * \beta$$

in the sense of §3.

LEMMA 4.1. *The relative commutant* $(\mathscr{M}^{\Gamma})' \cap \mathscr{M}$ *is trivial.*

PROOF. The AFD type II_1 factor has a sufficiently large group of orthogonal unitaries with respect to the unique normalized trace. Hence we can easily find unitaries a in $\mathscr{R} \otimes \mathrm{C}1$ ($\subseteq \mathscr{R} \otimes \mathscr{A}$) and b, c in $\mathscr{R}$ satisfying the conditions in Theorem 2.1.2. Since $\mathscr{M}^{\Gamma}$ contains $\mathscr{R} \otimes \mathrm{C}1$ and $\mathscr{R}$, we have

$$(\mathscr{M}^{\Gamma})' \cap \mathscr{M} \subseteq ((\mathscr{R} \otimes \mathrm{C}1) * \mathscr{R})' \cap \mathscr{M} \subseteq \{a, b, c\}' \cap \mathscr{M} = \mathrm{C}1$$

by (2) in Theorem 2.1.2. □

LEMMA 4.2. *The crossed product algebra* $\mathscr{M} \rtimes_{\Gamma} L^{\infty}(SU_q(n))$ *is a factor.*

PROOF. Let W be the fundamental unitary associated with $L^{\infty}(SU_q(n))$. We define the unitary u in $\mathscr{M} \otimes B(L^2(\mathscr{A}))$ by $u = 1_{\mathscr{R}} \otimes W^*$ (in $(\mathscr{R} \otimes \mathscr{A}) \otimes \hat{\mathscr{A}} \subseteq \mathscr{M} \otimes B(L^2(\mathscr{A}))$). By the pentagon equation $W_{12}W_{23} = W_{23}W_{13}W_{12}$, $\overline{\Gamma} = (\mathrm{id} \otimes \Sigma) \circ (\Gamma \otimes \mathrm{id})$ satisfies

$$\begin{aligned}
\overline{\Gamma}(u) &= (\mathrm{id} \otimes \Sigma) \circ (\Gamma \otimes \mathrm{id})(u) \\
&= (\mathrm{id} \otimes \Sigma) \circ (\mathrm{id} \otimes \delta \otimes \mathrm{id})(1_{\mathscr{R}} \otimes W^*) \\
&= 1_{\mathscr{R}} \otimes (\mathrm{id} \otimes \Sigma) \circ (\delta \otimes \mathrm{id})(W^*) \\
&= 1_{\mathscr{R}} \otimes (\mathrm{id} \otimes \Sigma)(W_{12}W_{23}^*W_{12}^*) \\
&= 1_{\mathscr{R}} \otimes (\mathrm{id} \otimes \Sigma)(W_{13}^*W_{23}^*) \\
&= 1_{\mathscr{R}} \otimes (\mathrm{id} \otimes \Sigma)(W_{13})^*(\mathrm{id} \otimes \Sigma)(W_{23})^* \\
&= 1_{\mathscr{R}} \otimes (W^* \otimes 1_{B(L^2(\mathscr{A}))})(1_{\mathscr{A}} \otimes \Sigma(W^*)) \\
&= (u \otimes 1_{B(L^2(\mathscr{A}))})(1_{\mathscr{N}} \otimes \Sigma(W^*)).
\end{aligned}$$

Hence we get

$$\mathscr{M} \rtimes_{\Gamma} L^{\infty}(SU_q(n)) \cong (\mathscr{M} \otimes B(L^2(\mathscr{A})))^{\tilde{\Gamma}} \cong (\mathscr{M} \otimes B(L^2(\mathscr{A})))^{\overline{\Gamma}} = \mathscr{M}^{\Gamma} \otimes B(L^2(\mathscr{A}))$$

by the Takesaki duality theorem for $SU_q(n)$ ([**N**]), where $\tilde{\Gamma} = \mathrm{Ad}(1 \otimes \Sigma(W^*)) \circ \overline{\Gamma}$. Therefore, $\mathscr{M} \rtimes_{\Gamma} L^{\infty}(SU_q(n))$ is a factor. □

From now on, we investigate the type of the above $\mathscr{M}$. Note that $(\mathscr{R} \otimes \mathrm{C}1) * \mathscr{R}$ sits in the centralizer $\mathscr{M}_{\varphi}$ (see [**VDN**, Theorem 1.6.5.] also [**B**] and [**V1**]). Hence, by the same reason as in Lemma 4.1, we get

$$(\mathscr{M}_\varphi)' \cap \mathscr{M} = \mathrm{C}1$$

by (2) in Theorem 2.1.2 and hence the centralizer $\mathscr{M}_\varphi$ is a factor. Therefore, $\mathscr{M}$ is not of type III_0 as is well-known. Hence, we can see the type by Connes' T-set

$$T(\mathscr{M}) = \{t \in \mathrm{R} : \sigma_t^\varphi = \mathrm{id}\}.$$

Since

$$\sigma_t^\varphi = (\sigma_t^\tau \otimes \sigma_t^h) * \sigma_t^\tau = (\mathrm{id} \otimes \sigma_t^h) * \mathrm{id}$$

(see [**VDN**, Theorem 1.6.5.] also [**B**] and [**V1**]), we have

$$T(\mathscr{M}) = \{t \in \mathrm{R} : \sigma_t^h = \mathrm{id}\}.$$

Such a type classification result was obtained by L. Barnett in [**B**]. (K. Dykema obtained another type classification results in [**D4**].) Also, by Theorem 2.1.2 $\mathscr{M}$ is full.

Here, using the explicit computation of the modular automorphism of the Haar state we determine the above T-set $T(\mathscr{M})$. For simplicity, we further assume $n = 2$. In this case, $\mathscr{A}$ is generated by two elements α, γ such that

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is the fundamental representation of $SU_q(2)$ ([**Wr1**], [**Wr2**]). By [**Wr2**: equations (5.20), (A1.3)] we can see that

$$\sigma_t^h(\alpha) = f_{it} * \alpha * f_{it} = q^{-2it}\alpha, \quad \sigma_t^h(\gamma) = f_{it} * \gamma * f_{it} = \gamma,$$

where f_z ($z \in \mathrm{C}$) is the Woronowicz character. Hence,

$$\sigma_t^h = \mathrm{id} \quad \Leftrightarrow \quad q^{-2it} = 1$$

because σ^h is an automorphism and α, γ generate $\mathscr{A}$. Therefore, we get

$$T(\mathscr{M}) = \frac{2\pi}{\log q^2}\boldsymbol{Z}$$

so that $\mathscr{M}$ is of type III_{q^2}. For an arbitrary n, combining [**MN**: Proposition 5.3. equation (5.8), Proposition 5.5.] and the above argument, we can see that $\mathscr{M}$ is of type III_{q^2}.

Consequently, we get the following theorem:

Theorem 4.3. *We can construct a full factor of type III_{q^2} admitting a minimal coaction of the compact quantum group $SU_q(n)$.*

As was explained in §1, our method to construct a minimal coaction is to take a suitable "model" coaction and construct a von Neumann algebra admitting a minimal coaction by using free product construction. (In the above, the model coaction is the "regular" representation.) In the original version of this paper, a different coaction was used, but the referee pointed out the resulting crossed product is not a factor. Furthermore, the referee kindly suggested us the use of the coaction on the Cuntz algebra $\mathcal{O}_n$ constructed in [**KNW**] with a certain invariant state on the UHF-part (see [**Kn**] for $n = 2$, [**N**] for an arbitrary n).

Remark 4.4. By the standard technique, we can easily see that there exists a minimal coaction of the compact quantum group $SU_q(n)$ on a type III_1 factor.

The above method to construct a minimal coaction of the compact quantum group $SU_q(n)$ remains valid for arbitrary compact Woronowicz algebras. Constructing a minimal coaction of an arbitrary compact Kac algebra is of independent interest.

Let $\boldsymbol{K} = (\mathscr{A}, \delta, \kappa, h)$ be a given compact Kac algebra and $\mathscr{R}$ be the AFD type II_1 factor with the unique normalized trace τ.

We set

$$(\mathscr{N}, \phi) = (\mathscr{R} \otimes \mathscr{A}, \tau \otimes h) * (\mathscr{R}, \tau)$$

and define the free product coaction $\Psi : \mathscr{N} \to \mathscr{N} \otimes \mathscr{A}$ by

$$\Psi = (\mathrm{id} \otimes \delta) * \beta$$

in the sense of §3, where β is the trivial coaction.

By the same reason as in Lemma 4.1, we have

$$(\mathscr{N}^{\Psi})' \cap \mathscr{N} = \mathrm{C}1$$

because $\mathscr{N}^{\Psi}$ contains $\mathscr{R} \otimes \mathrm{C}1$ $(\subseteq \mathscr{R} \otimes \mathscr{A})$ and $\mathscr{R}$.

Also, since $\tau \otimes h$ and τ are traces, the free state ϕ is a faithful normal normalized trace on $\mathscr{N}$ (see [**VDN**, Proposition 2.5.3.]). Hence $\mathscr{N}$ is of type II_1. And furthermore, if $\mathscr{A}$ has separable predual, $\mathscr{N}$ is full by the same reason as in the previous discussion. Also, we can show the factoriality of the crossed product algebra $\mathscr{N} \rtimes_{\Psi} \boldsymbol{K}$ by repeating the proceeding discussion based on the Takesaki duality theorem for Kac algebras ([**ES1**]).

Consequently, we get the following theorem:

Theorem 4.5. *For each compact Kac algebra $\boldsymbol{K} = (\mathscr{A}, \delta, \kappa, h)$, we can construct a type II_1 factor admitting a minimal coaction of $\boldsymbol{K}$. Furthermore, if $\mathscr{A}$ has separable predual, this type II_1 factor is full.*

If a given compact Kac algebra $\boldsymbol{K}$ is hyperfinite, we can easily see the above algebra $\mathscr{N}$ is the free group factor $L(\boldsymbol{F}_2)$ by the result [**D2**, Theorem 4.6.], and we can construct a minimal coaction of $\boldsymbol{K}$ on all interpolated free group factors (in the sense of K. Dykema [**D3**] and F. Radulescu [**R**]) of free dimension (in the sense of K. Dykema [**D2**]) more than 2.

Consequently, we get

Corollary 4.6. *The free group factor $L(\boldsymbol{F}_n)$ with n generators $(n \geq 2)$ admits a minimal coaction of an arbitrary hyperfinite compact Kac algebra.*

Remark 4.7. In the recent article [**ILP**], M. Izumi, R. Longo and S. Popa gave another definition of the minimality of compact Kac algebra coactions. Their definition consists of irreducibility and faithfulness. They proved that their minimality implies ours. We can easily check the faithfulness of the minimal coactions in this paper.

5. A remark on free products of compact Kac algebras.

In this section, based on the technique in §3 we show that the (reduced!) free product of given compact Kac algebras with respect to their Haar states has a compact Kac algebra structure.

Let $\boldsymbol{K}_i = (\mathscr{A}_i, \delta_i, \kappa_i, h_i)$ be given compact Kac algebras $(i = 1, 2)$. Here we recall the following facts (see [**ES2**]):

(1) The Haar state h_i is a faithful normal trace $(i = 1, 2)$.

(2) The Haar state h_i is left and right invariant $(i = 1, 2)$.

(3) The Haar state h_i is invariant under the unitary antipode κ_i $(i = 1, 2)$.

Let

$$(\mathscr{A}, h) = (\mathscr{A}_1, h_1) * (\mathscr{A}_2, h_2).$$

By (3), we can show the existence of an anti-$*$automorophism κ on $\mathscr{A}$ satisfying $\kappa|_{\mathscr{A}_i} = \kappa_i$ $(i = 1, 2)$. Also h is a faithful normal tracial state on $\mathscr{A}$ by (1) ([**VDN**, Proposition 2.5.3.]). Hence the main difficulty here is how to define the comultiplication on $\mathscr{A}$. However, using the same argument as in Lemma 3.1, 3.2 and 3.3, we can prove the following lemma:

Lemma 5.1. *There exists an injective normal unital $*$homomorphism $\delta : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ satisfying $\delta|_{\mathscr{A}_i} = \delta_i$ $(i = 1, 2)$. In particular, δ is a comultiplication on $\mathscr{A}$.*

Proof. To show this lemma, Lemma 3.1 has to be replaced by the following argument: Let $\{u^i_{\gamma^i}\}$ be the set of linearly independent elements in $\mathscr{A}_i$ whose finite linear combinations form a dense $*$subalgebra $(i = 1, 2)$, (see Remark 3.5), and $\mathscr{L}_i$ be the set in $\mathscr{A}_i \otimes \mathscr{A}_i$ of finite linear combination of $x \otimes u^i_{\gamma^i}$ for x in $\mathscr{A}_i$ and γ^i $(i = 1, 2)$. Note that $\mathscr{L}_i$ is a unital dense $*$subalgebra of $\mathscr{A}_i \otimes \mathscr{A}_i$ $(i = 1, 2)$.

We can easily prove

$$(h \otimes \mathrm{id})(X_1 \cdots X_n) = 0$$

for X_j in $\mathscr{L}_{i_j}$ with $(h \otimes \mathrm{id})(X_j) = 0$ for all j, $i_1 \neq \cdots \neq i_n$, and $i_j \in \{1, 2\}$ by similar argument as in Lemma 3.1. Hence, by the same reason as in Lemma 3.1, we get

$$(h \otimes \mathrm{id})(Y_1 \cdots Y_n) = 0$$

for Y_j in $\mathscr{A}_{i_j} \otimes \mathscr{A}_{i_j}$ $(\subseteq \mathscr{A} \otimes \mathscr{A})$ with $(h \otimes \mathrm{id})(Y_j) = 0$ for all j, $i_1 \neq \cdots \neq i_n$, and $i_j \in \{1, 2\}$.

The rest of the proof is analogous, and details are left to the reader. □

Consequently we get the following theorem:

Theorem 5.2. *The free product of two compact Kac algebras with respect to their Haar states has a natural compact Kac algebra structure.*

Remark 5.3. In this section, we only considered compact Kac algebras for simplicity. However the same argument works for compact Woronowicz algebras without essential changes.

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Yoshimichi Ueda
Graduate School of Mathematics
Kyushu University
Fukuoka 810-8560
Japan