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Compact Riemann surfaces with large automorphism groups

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Abstract. In this paper we study the relation between automorphism groups of branched coverings over the complex projective line and automorphism groups of compact Riemann surfaces. We give a criterion for the coincidence of them. We also give examples when the criterion does not hold.

1. Introduction.

Let X be a compact Riemann surface of genus $g \geq 2$. Let $\pi : X \to \boldsymbol{P}^1$ be a finite Galois branched covering of the complex projective line $\boldsymbol{P}^1$. A covering transformation of π is by definition an automorphism φ of X such that $\pi \circ \varphi = \pi$. We denote by $\mathrm{Aut}(X)$ the automorpism group of X and by $\mathrm{Aut}(\pi)$ the covering transformation group of π. $\mathrm{Aut}(\pi)$ is a subgroup of $\mathrm{Aut}(X)$. We say X has a large automorpism group $\mathrm{Aut}(X)$ if its order $\#\mathrm{Aut}(X)$ is strictly greater than $4(g-1)$. Let G be a subgroup of $\mathrm{Aut}(X)$ with its order $\# G > 4(g-1)$. It is a well-known fact that the quotient space X/G is biholomorphic to $\boldsymbol{P}^1$ and that the canonical quotient map $\pi : X \to X/G \cong \boldsymbol{P}^1$ can be considered as a finite Galois covering of $\boldsymbol{P}^1$. We can naturally identify the covering transformation group $\mathrm{Aut}(\pi)$ of π with G (See, e.g., [**3**] [**6**]). Let $B_\pi = m_1 Q_1 + \cdots + m_d Q_d$ $(2 \leq m_1 \leq m_2 \leq \cdots \leq m_d)$ be the branch locus of π. Here m_j is called the ramification index of π at Q_j. That is, if R is a point of $\pi^{-1}(Q_j)$, then there are local coordinate systems t and x around R and Q_j respectively with $t(R) = 0$ and $x(Q_j) = 0$ such that π is locally given as: $t \mapsto x = t^{m_j}$. We say π has a branching type $(m_1, m_2, \ldots, m_d)$ if $B_\pi = m_1 Q_1 + \cdots + m_d Q_d$. If the covering degree of π is strictly greater than $4(g-1)$, then only the following possibilities for branching indices can occur:

(A) 4 branch points (infinite family): $(2,2,2,n)$ for $3 \leq n$. **(B) 4 branch points (other cases):** $(2,2,3,n)$ for $3 \leq n \leq 5$. **(C) 3 branch points (infinite family):** $(2,3,n)$ for $7 \leq n$, $(2,4,n)$ for $5 \leq n$, $(2,m,n)$ for $5 \leq m \leq n$, $(3,3,n)$ for $4 \leq n$, $(3,4,n)$ for $4 \leq n$, $(3,5,n)$ for $5 \leq n$, $(3,6,n)$ for $6 \leq n$, $(4,4,n)$ for $4 \leq n$. **(D) 3 branch points (other cases):** $(3,7,n)$ for $7 \leq n \leq 41$, $(3,8,n)$ for $8 \leq n \leq 23$, $(3,9,n)$ for $9 \leq n \leq 17$, $(3,10,n)$ for $10 \leq n \leq 14$, $(3,11,n)$ for $11 \leq n \leq 13$, $(4,5,n)$ for $5 \leq n \leq 19$, $(4,6,n)$ for $6 \leq n \leq 11$, $(4,7,n)$ for $7 \leq n \leq 9$, $(5,5,n)$ for $5 \leq n \leq 9$, $(5,6,n)$ for $6 \leq n \leq 7$.

These are easy consequences of the Riemann-Hurwitz formula (For the proof refer, for example [**1**]). In this paper we investigate the relation between $\mathrm{Aut}(X)$ and $\mathrm{Aut}(\pi)$. To mention our results, we divide the above list of branching types of Galois coverings of $\boldsymbol{P}^1$ into the following two lists (List 1 and List 2):

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List 1 **(a) 3 branch points infinite family:** $(2,3,n)$ for $7 \leq n$, $(2,4,n)$ for $5 \leq n$, $n \neq 8$, $(2,m,n)$ for $5 \leq m \leq n$, $n \neq m, 2m$, $(3,4,n)$ for $4 \leq n$, $n \neq 4, 12$, $(3,5,n)$ for $5 \leq n$, $n \neq 5, 15$, $(3,6,n)$ for $6 \leq n$, $n \neq 6, 18$. **(b) 3 branch points (other cases):** $(3,7,n)$ for $7 \leq n \leq 41$, $n \neq 7, 21$, $(3,8,n)$ for $8 \leq n \leq 23$, $n \neq 8$, $(3,9,n)$ for $9 \leq n \leq 17$, $n \neq 9$, $(3,10,n)$ for $10 \leq n \leq 14$, $n \neq 10$, $(3,11,n)$ for $11 \leq n \leq 13$, $n \neq 11$, $(4,5,n)$ for $6 \leq n \leq 19$, $(4,6,n)$ for $7 \leq n \leq 11$, $(4,7,n)$ for $8 \leq n \leq 9$, $(5,6,n)$ for $6 \leq n \leq 7$, $n \neq 6$.

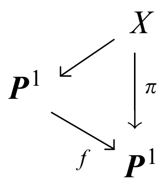
List 2 **(c) 4 branch points (infinite family):** $(2,2,2,n)$ for $3 \leq n$. **(d) 4 branch points (other cases):** $(2,2,3,n)$ for $3 \leq n \leq 5$. **(e) 3 branch points infinite family:** $(3,3,n)$ for $4 \leq n$, $(4,4,n)$ for $4 \leq n$. **(f) 3 branch points (other cases):** $(2,4,n)$ for $n = 8$, $(2,m,n)$ for $5 \leq m \leq n$, $n = m, 2m$, $(3,4,n)$ for $n = 4, 12$, $(3,5,n)$ for $n = 5, 15$, $(3,6,n)$ for $n = 6, 18$, $(3,7,n)$ for $n = 7, 21$, $(3,8,n)$ for $n = 8$, $(3,9,n)$ for $n = 9$, $(3,10,n)$ for $n = 10$, $(3,11,n)$ for $n = 11$, $(4,5,n)$ for $n = 5$, $(4,6,n)$ for $n = 6$, $(4,7,n)$ for $n = 7$, $(5,5,n)$ for $5 \leq n \leq 9$, $(5,6,n)$ for $n = 6$.

We have the following theorem:

Theorem 1. *Let* $\pi : X \to \boldsymbol{P}^1$ *be a finite Galois covering of* $\boldsymbol{P}^1$ *with* $\deg(\pi) > 4(g-1)$. **(i)** *If the branching type of* π *is one of the* List 1, *then* $\mathrm{Aut}(X) = \mathrm{Aut}(\pi)$. **(ii)** *For the branching types of* List 2, *there are compact Riemann surfaces* X *and Galois coverings* $\pi : X \to \boldsymbol{P}^1$ *such that* $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$.

In §2 we will give the outline of the proof of the part (i) of Theorem 1 and in §3 we will give the concrete examples of Galois coverings such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ for all branching types in List 2. We note that if there is a Galois covering $\pi : X \to \boldsymbol{P}^1$ with branch divisor B such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$, then, by the consideration of characteristic normal subgroup of the fundamental group $\pi_1(X)$ (See [**5**]), there are infinitely many Galois coverings $\pi' : X' \to \boldsymbol{P}^1$ with branch divisor B such that $\mathrm{Aut}(X') \neq \mathrm{Aut}(\pi')$.

Next, let $f : \boldsymbol{P}^1 \to \boldsymbol{P}^1$ be a finite surjective holomorphic mapping, i.e., a rational function. The Galois closure $\pi : X \to \boldsymbol{P}^1$ of f is the minimal finite Galois covering which makes the following diagram commutative.



That is, the extention $\pi^* : \boldsymbol{C}(\boldsymbol{P}^1) \hookrightarrow \boldsymbol{C}(X)$ is the Galois closure of $f^* : \boldsymbol{C}(\boldsymbol{P}^1) \hookrightarrow \boldsymbol{C}(\boldsymbol{P}^1)$, where $\boldsymbol{C}(\boldsymbol{P}^1)$ and $\boldsymbol{C}(X)$ are the fields of rational (i.e., meromorphic) functions of $\boldsymbol{P}^1$ and X, respectively.

Theorem 2. *Let* p, q, r $(p > q > r)$ *be three prime numbers. Let* $\pi : X \to \boldsymbol{P}^1$ *be a finite Galois covering of* $\boldsymbol{P}^1$ *branched at* $0, 1, \infty$ *with the ramification indices* p, q, r, *respectively. If there is a rational function* $f : \boldsymbol{P}^1 \to \boldsymbol{P}^1$ *of degree* p *whose Galois closure is* π, *then* $\mathrm{Aut}(\pi)$ *is a simple group.*

We will give the proof of Theorem 2 and a few examples of Theorem 2 in §4.

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2. The proof of the part (i) of Theorem 1.

The idea is similar to that of classifying commensurability classes of Fuchsian groups (See [**8**], [**9**]). There is the following commutative diagram:

$$\begin{array}{ccc} & & X \\ & \overset{\lambda}{\swarrow} & \downarrow \mu \\ X/\mathrm{Aut}(\pi) & \underset{f}{\searrow} & \\ & & X/\mathrm{Aut}(X) \end{array}$$

Here μ and f are natural projections. It is clear that $\mu : X \to X/\mathrm{Aut}(X) \cong \boldsymbol{P}^1$ is a Galois covering of $\boldsymbol{P}^1$ and that f is a rational function. Assume $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$. Let $B_\mu = l_1P_1 + \cdots + l_sP_s$ be the branch divisor of μ. Then the branch divisor of π must be written as;

$$B_\pi = m_{11}Q_{11} + \cdots + m_{1t_1}Q_{1t_1} + \cdots + m_{s1}Q_{s1} + \cdots + m_{st_s}Q_{st_s},$$

where Q_{ij} are points of $f^{-1}(P_i)$. m_{ij} must be integers such that $m_{ij}|l_i$, since the following lemma holds.

Lemma 1. *Let $f : X \to Y$ and $g : Y \to Z$ be surjective holomorphic mapping between compact Riemann surfaces X, Y and Z. Let P be a point of X. Let e, e' and e'' be the ramification indices of f at P, of g at $f(P)$ and of $g \circ f$ at P, respectively. Then $e'' = ee'$.* (*For an unramified point, the ramification index is defined to be* 1.)

The proof of Lemma 1 is easy and is ommited.

From the Riemann-Hurwitz formula,

$$2g - 2 = -2\deg(\mu) + \sum \frac{\deg(\mu)}{l_j}(l_j - 1),$$

$$2g - 2 = -2\deg(\pi) + \sum \frac{\deg(\pi)}{m_{jk}}(m_{jk} - 1).$$

Taking ratios of the two equations, m_{jk} must satisfy the following condition.

Condition 1.

$$\deg(f) = \frac{t - 2 - \sum(1/m_{jk})}{s - 2 - \sum(1/l_j)} \text{ is an integer strictly greater than } 1,$$

where $m_{jk}|l_j$ and $t = t_1 + \cdots t_s$.

Furthermore, let $\pi_1(\boldsymbol{P}^1 - \{P_1, \ldots, P_s\}) = \langle x_1, \ldots, x_s \,|\, x_1 \ldots x_s = 1\rangle$ be the fundamental group of $\boldsymbol{P}^1 - \{P_1, \ldots, P_s\}$. Here x_j is a loop rounding once counterclockwise around P_j. Since f is a rational function which satisfies the above commutative diagram, the following condition also holds, which is obtained by the local behavior of f around the ramification points.

Condition 2. There exist a finite permutation group G transitive on $\deg(f)$ points and a surjective group homomorphism $\theta : \pi_1(\boldsymbol{P}^1 - \{P_1, \ldots, P_s\}) \to G$ satisfying the following condition: The permutation $\theta(x_j)$ has precisely t_j cycles of lengths, $l_j/m_{j1}, \ldots, l_j/m_{jt_j}$. ($\theta$ is in fact the monodromy representation of the mapping f.)

Thus, for $B_\pi = m_1Q_1 + \cdots + m_dQ_d$, if there is no $B_\mu = l_1P_1 + \cdots + l_sP_s$ satisfying above two conditions, then $\mathrm{Aut}(X) = \mathrm{Aut}(\pi)$. Using this assertion and by direct case by case calculations, we have (i) of Theorem 1 as follows:

The case $nQ_1 + 3Q_2 + 2Q_3$ $(7 \le n)$: By the Riemann-Hurwitz formula, the covering degree $\deg(\pi)$ of π is equal to $(12n/(n-6))(g-1)$.

Suppose that $\mathrm{Aut}(\pi) \ne \mathrm{Aut}(X)$. Then $\deg(f) > 2$. Hence

$$\deg(\mu) > 2\frac{12n}{n-6}(g-1) = \frac{24n}{n-6}(g-1)$$
$$> 24(g-1).$$

So the Galois covering μ must have just three branch points (P_1, P_2, P_3). Let l_1, l_2 and l_3 be the ramification indices of μ at P_1, P_2, P_3, respectively. Moreover the condition $\deg(\mu) > 24(g-1)$ implies that the triple (l_1, l_2, l_3) is very restricted by the Riemann-Hurwitz formula. That is, (l_1, l_2, l_3) must be equal to either

$$(n', 3, 2) \quad (7 \le n' \le 11) \quad \text{or} \quad (5, 4, 2).$$

But these cases cannot occur. In fact, if $(l_1, l_2, l_3) = (n', 3, 2)$ $(7 \le n' \le 11)$, then n' must be a multiple of n (≥ 7). Hence $n' = n$ and so $\deg(\mu) = \deg(\pi)$, a contradiction. In a similar way, we can show that the case $(l_1, l_2, l_3) = (5, 4, 2)$ cannot occur. Hence $\deg(\mu) = \deg(\pi)$ and so $\mathrm{Aut}(\pi) = \mathrm{Aut}(X)$ in this case.

The case $nQ_1 + 4Q_2 + 2Q_3$ $(n \ge 5)$**:** A similar argument to the above case shows that $\mathrm{Aut}(\pi) = \mathrm{Aut}(X)$, except the case $8Q_1 + 4Q_2 + 2Q_3$. This exceptional case cannot be eliminated. For there is a divisor $B_\mu = 2P_1 + 3P_2 + 8P_3$ which satisfies the condition 1 for this branch divisor B_π of π (renumbering indicies of points $\{Q_1, Q_2, Q_3\}$ as $\{Q_{11}, Q_{31}, Q_{32}\}$), and also there is a monodromy representation which satisfies the condition 2 for this B_μ, defined by:

$$\theta(x_1) = (1\ 2), \quad \theta(x_2) = (1\ 2\ 3), \quad \theta(x_3) = (1\ 3).$$

For the rest cases of (i) of Theorem 1, the argument is similar. But here we remark that there are a few cases that satisfy the condition 1 but do not satisfy the condition 2. For example, take $3Q_{21} + 7Q_{31} + 7Q_{32}$ as B_π and $2P_1 + 3P_2 + 14P_3$ as B_μ. In this case $\deg(f)$ is 4 and the condition 1 holds. But there are no monodromy representation of a rational function of degree 4 such that $\theta(x_1) = (\mathit{length}\,2)(\mathit{length}\,2)$, $\theta(x_2) = (\mathit{length}\,3) \cdot (\mathit{length}\,1)$, $\theta(x_3) = (\mathit{length}\,2)(\mathit{length}\,2)$.

3. Examples of Galois coverings with branching types in List 2.

$(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{n})$ **for** $3 \le n$; $\deg(f) = 2$**:** Let $\pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\}) = \langle x_1, x_2, x_3 \mid x_1x_2x_3 = 1\rangle$ be the fundamental group of $\boldsymbol{P}^1 - \{P_1, P_2, P_3\}$. Let $G = \langle A, B\rangle \subset S_{2n}$ be the group generated by A, B in the symmetric group S_{2n} of $2n$ letters. Suppose $n = 4k$ $(k \in Z_{>0})$. Put

$$A = (1\ 2 \cdots 2n)$$
$$B = (1\ 5)(8\ 2n)(11\ 2n-1)\cdots(8+3t\ 2n-t)\cdots(6k-1\ 6k+3)(6k\ 6k+2).$$

Then $AB = (1\ 2\ 3\ 4)(5\ 6\ 7\ 2n)\cdots(6k-4\ 6k-3\ 6k-2\ 6k+3)$
$(6k-1\ 6k+2)(6k\ 6k+1).$

Suppose $n = 4k + 1$ ($k \in \boldsymbol{Z}_{>0}$). Put

$$A = (1\ 2 \cdots 2n)$$

$$B = (1\ 5)(8\ 2n)(11\ 2n-1) \cdots (8+3t\ 2n-t) \cdots (6k-1\ 6k+5)(6k+1\ 6k+3).$$

Then $AB = (1\ 2\ 3\ 4)(5\ 6\ 7\ 2n) \cdots (6k-4\ 6k-3\ 6k-2\ 6k+5)$

$$(6k-1\ 6k\ 6k+3\ 6k+4)(6k+1\ 6k+2).$$

Suppose $n = 4k + 2$ ($k \in \boldsymbol{Z}_{>0}$). Put

$$A = (1\ 2 \cdots 2n)$$

$$B = (1\ 5)(8\ 2n)(11\ 2n-1) \cdots (8+3t\ 2n-t) \cdots (6k+2\ 6k+6).$$

Then $AB = (1\ 2\ 3\ 4)(5\ 6\ 7\ 2n) \cdots (6k-1\ 6k\ 6k+1\ 6k+6)$

$$(6k+2\ 6k+3\ 6k+4\ 6k+5).$$

Suppose $n = 4k + 3$ ($k \in \boldsymbol{Z}_{\geq 0}$). Put

$$A = (1\ 2 \cdots 2n)$$

$$B = (1\ 5)(8\ 2n)(11\ 2n-1) \cdots (8+3t\ 2n-t) \cdots (6k+2\ 6k+8)$$

$$(6k+3\ 6k+7)(6k+4\ 6k+6).$$

Then $AB = (1\ 2\ 3\ 4)(5\ 6\ 7\ 2n) \cdots (6k-1\ 6k\ 6k+1\ 6k+8)$

$$(6k+2\ 6k+7)(6k+3\ 6k+6)(6k+4\ 6k+5).$$

Let $\Phi : \pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\}) \to G$ be the surjective homomorphism defined by:

$$\Phi(x_1) = B^{-1}, \quad \Phi(x_2) = A^{-1}, \quad \Phi(x_3) = AB.$$

Let $\mu : X \to \boldsymbol{P}^1$ be the Galois covering of $\boldsymbol{P}^1$ associated with $\mathrm{Ker}(\Phi)$. Then the branch divisor of μ is $B_\mu = 2P_1 + 4P_2 + 2nP_3$. By the Riemann-Hurwitz formula $\deg(\mu) = (8n/(n-2))(g(X)-1)$. Put $H = G \cap A_{2n}$. Here A_{2n} is the alternating group of $2n$ letters. Then the index $[G : H]$ is 2. Since $\#H = (4n/(n-2))(g(X)-1)$, the quotient space X/H is biholomorphic to $\boldsymbol{P}^1$. Let $\pi : X \to \boldsymbol{P}^1$ be the Galois covering corresponding to H. Since $A \notin H$, $A^2 \in H$, $B \in H$, $AB \notin H$ and $(AB)^2 \in H$, the branch divisor of π must be

$$B_\pi = 2Q_{11} + 2Q_{12} + 2Q_{21} + nQ_{31}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(2, 2, 2, n)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ and $\deg(f) = 2$.

$(2, 2, 2, 4)$; $\deg(f) = 5$: Put

$$A = (4\ 5)$$

$$B = (5\ 3\ 2\ 1).$$

Then $AB = (5\ 4\ 3\ 2\ 1)$.

Let $G = \langle A, B\rangle$ be a group generated by A and B in S_5. Let $\theta : \pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\}) \to G$ be a group homomorphism defined by: $\theta(x_1) = A$, $\theta(x_2) = B$ and $\theta(x_3) = (AB)^{-1}$. Since G is transitive in S_5, there is a covering $f : Y \to \boldsymbol{P}^1$ of degree 5 with monodromy representation θ. By the Riemann-Hurwitz formula Y is biholomorphic to $\boldsymbol{P}^1$. So f is a rational function. Let $\mu : X \to \boldsymbol{P}^1$ be the Galois closure of f. μ is a Galois covering of $\boldsymbol{P}^1$ with branch divisor $B_\mu = 2P_1 + 4P_2 + 5P_3$. Let $\pi : X \to Y \cong \boldsymbol{P}^1$ be the morphism such that $f \circ \pi = \mu$. π is a Galois covering of $\boldsymbol{P}^1$. The branching divisor of π must be

$$B_\pi = 2Q_{11} + 2Q_{12} + 2Q_{13} + 4Q_{21}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(2,2,2,4)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ and $\deg(f) = 5$.

$(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{5})$; $\deg(f) = 6$: Put

$$A = (1\ 6)(3\ 5)$$

$$B = (6\ 5\ 2\ 1)(3\ 4).$$

$$\text{Then } AB = (5\ 4\ 3\ 2\ 1).$$

Let $G = \langle A, B\rangle$ be a group generated by A and B in S_6. G is transitive in S_6. A similar argument to the above case shows that there is a Galois covering $\pi : X \to \boldsymbol{P}^1$ with the branching type $(2,2,2,5)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$, $\deg(f) = 6$ and the branch divisor of μ is $B_\mu = 2P_1 + 4P_2 + 5P_3$.

$(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{3})$; $\deg(f) = 7$: Put

$$A = (1\ 4)(5\ 6)$$

$$B = (3\ 2\ 1)(7\ 6\ 4).$$

$$\text{Then } AB = (7\ 6\ 5\ 4\ 3\ 2\ 1).$$

Let $G = \langle A, B\rangle$ be a group generated by A and B in S_7. G is transitive in S_7. A similar argument to the above case shows that there is a Galois covering $\pi : X \to \boldsymbol{P}^1$ with the branching type $(2,2,2,3)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$, $\deg(f) = 7$ and the branch divisor of μ is $B_\mu = 2P_1 + 3P_2 + 7P_3$.

$(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{8})$; $\deg(f) = 9$: Put

$$A = (1\ 9)(2\ 8)(4\ 7)$$

$$B = (9\ 8\ 1)(7\ 3\ 2)(6\ 5\ 4).$$

$$\text{Then } AB = (8\ 7\ 6\ 5\ 4\ 3\ 2\ 1).$$

Let $G = \langle A, B\rangle$ be a group generated by A and B in S_9. G is transitive in S_9. A similar argument to the above case shows that there is a Galois covering $\pi : X \to \boldsymbol{P}^1$ with the branching type $(2,2,2,8)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$, $\deg(f) = 9$ and the branch divisor of μ is $B_\mu = 2P_1 + 3P_2 + 8P_3$.

(2, 2, 2, 7); deg(*f*) = 15: Put

$$A = (1\ 8)(2\ 4)(3\ 15)(5\ 14)(6\ 12)(7\ 10)$$

$$B = (14\ 4\ 1)(3\ 15\ 2)(13\ 12\ 5)(11\ 10\ 6)(9\ 8\ 7).$$

$$\text{Then } AB = (7\ 6\ 5\ 4\ 3\ 2\ 1)(14\ 13\ 12\ 11\ 10\ 9\ 8).$$

Let $G = \langle A, B \rangle$ be a group generated by A and B in S_{15}. G is transitive in S_{15}. A similar argument to the above case shows that there is a Galois covering $\pi : X \to \boldsymbol{P}^1$ with the branching type $(2, 2, 2, 7)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$, $\deg(f) = 15$ and the branch divisor of μ is $B_\mu = 2P_1 + 3P_2 + 7P_3$.

(2, 2, 3, n) for $3 \le n \le 5$**; deg(*f*) = 2**
If $n = 3$, put

$$A = (1\ 7)(5\ 6)$$

$$B = (7\ 6\ 4\ 3\ 2\ 1)$$

$$AB = (6\ 5\ 4\ 3\ 2\ 1).$$

If $n = 4$, put

$$A = (1\ 2)(3\ 4)$$

$$B = (8\ 7\ 6\ 5\ 4\ 2)$$

$$AB = (8\ 7\ 6\ 5\ 4\ 3\ 2\ 1).$$

If $n = 5$, put

$$A = (1\ 3)(4\ 6)$$

$$B = (10\ 9\ 8\ 7\ 6\ 3)(1\ 2)(4\ 5)$$

$$AB = (10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1).$$

Let $\mu : X \to \boldsymbol{P}^1$ be the Galois covering of $\boldsymbol{P}^1$ associated with $\mathrm{Ker}(\Phi)$. Then the branch divisor of μ is $B_\mu = 2P_1 + 6P_2 + 2nP_3$. By the Riemann-Hurwitz formula $\deg(\mu) = (12n/(2n-3))(g(X) - 1)$. Put $H = G \cap A_{2n}$. Since $\# H = (6n/(2n-3)) \cdot (g(X) - 1) > 4(g(X) - 1)$ for $n = 3, 4, 5$, the quotient space X/H is biholomorphic to $\boldsymbol{P}^1$. Let $\pi : X \to \boldsymbol{P}^1$ be the Galois covering corresponding to H. Since $A \in H$, $B \notin H$, $B^2 \in H$, $AB \notin H$ and $(AB)^2 \in H$, the branch divisor of π must be

$$B_\pi = 2Q_{11} + 2Q_{12} + 3Q_{21} + nQ_{31}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(2, 2, 3, n)$ $(n = 3, 4, 5)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ and $\deg(f) = 2$.

(2, 2, 3, 3); deg(*f*) = 8: Put

$$A = (1\ 4)(5\ 6)(7\ 8)$$

$$B = (3\ 2\ 1)(8\ 6\ 4)$$

$$AB = (8\ 7\ 6\ 5\ 4\ 3\ 2\ 1).$$

Let $G = \langle A, B \rangle$ be a group generated by A and B in S_8. G is transitive in S_8. A similar argument shows that there is a Galois covering $\pi : X \to \boldsymbol{P}^1$ with the branching type $(2, 2, 3, 3)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$, $\deg(f) = 8$ and the branch divisor of μ is $B_\mu = 2P_1 + 3P_2 + 8P_3$.

(2, 2, 3, 3); deg(f) = 14: Put

$$A = (1\ 8)(2\ 13)(4\ 12)(5\ 6)(7\ 11)(9\ 10)$$

$$B = (14\ 13\ 1)(12\ 3\ 2)(13\ 12\ 5)(11\ 6\ 4)(10\ 8\ 7)$$

$$AB = (7\ 6\ 5\ 4\ 3\ 2\ 1)(14\ 13\ 12\ 11\ 10\ 9\ 8).$$

Let $G = \langle A, B \rangle$ be a group generated by A and B in S_{14}. G is transitive in S_{14}. A similar argument shows that there is a Galois covering $\pi : X \to \boldsymbol{P}^1$ with the branching type $(2, 2, 3, 3)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$, $\deg(f) = 14$ and the branch divisor of μ is $B_\mu = 2P_1 + 3P_2 + 7P_3$.

(3, 3, n) for $4 \leq n$ deg(f) = 2:
If $n = 3k$, put

$$A = (1\ 4)(6\ 2n) \cdots (6 + 2t\ 2n - t) \cdots (4k\ 4k + 3)$$

$$B = (3\ 2\ 1)(2n\ 5\ 4) \cdots (2n - t\ 2t + 5\ 2t + 4) \cdots (4k + 3\ 4k - 1\ 4k - 2)$$
$$(4k + 2\ 4k + 1\ 4k)$$

$$AB = (2n\ 2n - 1 \cdots 2\ 1).$$

If $n = 3k + 1$, put

$$A = (1\ 4)(6\ 2n) \cdots (6 + 2t\ 2n - t) \cdots (4k\ 4k + 5)(4k + 3\ 4k + 4)$$

$$B = (3\ 2\ 1)(2n\ 5\ 4) \cdots (2n - t\ 2t + 5\ 2t + 4) \cdots (4k + 5\ 4k - 1\ 4k - 2)$$
$$(4k + 4\ 4k + 2\ 4k)$$

$$AB = (2n\ 2n - 1 \cdots 2\ 1).$$

If $n = 3k + 2$, put

$$A = (1\ 4)(6\ 2n) \cdots (6 + 2t\ 2n - t) \cdots (4k + 2\ 4k + 6)$$
$$(4k + 1\ 4k + 2)(4k + 3\ 4k + 4)$$

$$B = (3\ 2\ 1)(2n\ 5\ 4) \cdots (2n - t\ 2t + 5\ 2t + 4) \cdots (4k + 6\ 4k + 1\ 4k)$$
$$(4k + 5\ 4k + 4\ 4k + 2)$$

$$AB = (2n\ 2n - 1 \cdots 2\ 1).$$

Let $G = \langle A, B \rangle$ be a group generated by A and B in S_{2n}. Let $\Phi : \pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\}) \to G$ be the surjective homomorphism defined by:

$$\Phi(x_1) = A, \quad \Phi(x_2) = B, \quad \Phi(x_3) = (AB)^{-1}.$$

Let $\mu : X \to \boldsymbol{P}^1$ be the Galois covering of $\boldsymbol{P}^1$ associated with $\mathrm{Ker}(\Phi)$. Then the branch divisor of μ is $B_\mu = 2P_1 + 3P_2 + 2nP_3$. By the Riemann-Hurwitz formula $\deg(\mu) = (12n/(n-3))(g(X)-1)$. Put $H = G \cap A_{2n}$. Since $\# H = (6n/(n-3))(g(X)-1) > 4(g(X)-1)$, the quotient space X/H is biholomorphic to $\boldsymbol{P}^1$. Let $\pi : X \to \boldsymbol{P}^1$ be the Galois covering corresponding to H. Since $B \in H$, $AB \notin H$ and $(AB)^2 \in H$, the branch divisor of π must be

$$B_\pi = 3Q_{21} + 3Q_{22} + nQ_{31}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(3,3,n)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ and $\deg(f) = 2$.

$(4,4,n)$ for $4 \le n$ $\deg(f) = 2$**:** If $n = 4k$, put

$$A = (1\ 5)(8\ 2n)\cdots(8+3t\ 2n-t)\cdots(6k-1\ 6k+3)$$
$$B = (4\ 3\ 2\ 1)(2n\ 7\ 6\ 5)\cdots\cdots(6k+2\ 6k+1\ 6k\ 6k-1)$$
$$AB = (2n\ 2n-1\cdots 2\ 1).$$

If $n = 4k+1$, put

$$A = (1\ 5)(8\ 2n)\cdots(8+3t\ 2n-t)\cdots(6k-1\ 6k+5)$$
$$(6k\ 6k+4)(6k+1\ 6k+3)$$
$$B = (4\ 3\ 2\ 1)(2n\ 7\ 6\ 5)\cdots\cdots(6k+5\ 6k-2\ 6k-3\ 6k-4)$$
$$(6k-1\ 6k+4)(6k\ 6k+3)(6k+1\ 6k+2)$$
$$AB = (2n\ 2n-1\cdots 2\ 1).$$

If $n = 4k+2$, put

$$A = (1\ 5)(8\ 2n)\cdots(8+3t\ 2n-t)\cdots(6k+2\ 6k+6)(6k+3\ 6k+5)$$
$$B = (4\ 3\ 2\ 1)(2n\ 7\ 6\ 5)\cdots\cdots(6k+6\ 6k+1\ 6k\ 6k-1)$$
$$(6k+2\ 6k+5)(6k+3\ 6k+4)$$
$$AB = (2n\ 2n-1\cdots 2\ 1).$$

If $n = 4k+3$, put

$$A = (1\ 5)(8\ 2n)\cdots(8+3t\ 2n-t)\cdots(6k+2\ 6k+8)(6k+5\ 6k+7)$$
$$B = (4\ 3\ 2\ 1)(2n\ 7\ 6\ 5)\cdots\cdots(6k+7\ 6k+4\ 6k+3\ 6k+2)(6k+5\ 6k+6)$$
$$AB = (2n\ 2n-1\cdots 2\ 1).$$

Let $G = \langle A, B\rangle$ be a group generated by A and B in S_{2n}. Let $\Phi : \pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\}) \to G$ be the surjective homomorphism defined by:

$$\Phi(x_1) = A, \quad \Phi(x_2) = B, \quad \Phi(x_3) = (AB)^{-1}.$$

Let $\mu : X \to \boldsymbol{P}^1$ be the Galois covering of $\boldsymbol{P}^1$ associated with $\mathrm{Ker}(\Phi)$. Then the branch divisor of μ is $B_\mu = 2P_1 + 4P_2 + 2nP_3$. By the Riemann-Hurwitz formula $\deg(\mu) = (8n/(n-2))(g(X)-1)$. Put $H = G \cap A_{2n}$. Since $\# H = (4n/(n-2))(g(X)-1) > 4(g(X)-1)$, the quotient space X/H is biholomorphic to $\boldsymbol{P}^1$. Let $\pi : X \to \boldsymbol{P}^1$ be the Galois covering corresponding to H. Since $B \in H$, $AB \notin H$ and $(AB)^2 \in H$, the branch divisor of π must be

$$B_\pi = 4Q_{21} + 4Q_{22} + nQ_{31}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(4,4,n)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ and $\deg(f) = 2$.

$(2, m, 2m)$ for $4 \leq m$ $\deg(f) = 3$**:** Let $X = \{[Z_0; Z_1; Z_2] \in \boldsymbol{P}^2 \mid Z_0^m + Z_1^m + Z_2^m = 0\}$ be the Fermat curve of degree m in the complex projective plane $\boldsymbol{P}^2$. It is known that the automorphism group of X is generated by 4 projective transformations

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta \end{pmatrix},$$

in $PGL(3, C)$, where $\zeta = \exp(2\pi i/m)$. The order $\# \mathrm{Aut}(X)$ of $\mathrm{Aut}(X)$ is $6m^2$ (For the proof see, for example, [**7**]). The genus of X is $g(X) = (1/2)(m-1)(m-2)$ by genus formula. Thus

$$\# \mathrm{Aut}(X) = \frac{12m}{m-3}(g(X)-1).$$

$X/\mathrm{Aut}(X)$ is biholomorphic to $\boldsymbol{P}^1$. Let $P = [1; \exp(2\pi i/m); 0]$ be a point in X. Then the isotropy group I_P of P is generated by the following two projective transformations:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \exp\left(\frac{2(m-1)}{m}\pi i\right) \\ 0 & \exp(2\pi i/m) & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(2\pi i/m) & 0 \\ 0 & 0 & \exp(2\pi i/m) \end{pmatrix}.$$

A direct calculation shows that the order $\# I_P$ of I_p is $2m$. Let $\mu : X \to X/\mathrm{Aut}(X)$ be the natural projection. The Riemann-Hurwitz formula implies that μ is a branched covering of $\boldsymbol{P}^1$ with the branching type $(2, 3, 2m)$. Let H be the subgroup of $\mathrm{Aut}(X)$ which is generated by 3 projective transformations:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta \end{pmatrix}.$$

Note that I_P is contained in H. A direct calculation shows that the order $\# H$ of H is $2m^2$. $[\mathrm{Aut}(X) : H] = 3$. It is easy to see that X/H is biholomorphic to $\boldsymbol{P}^1$. Let $\pi : X \to X/G$ be the natural projection. Then the Riemann-Hurwitz formula implies that π is a branched covering of $\boldsymbol{P}^1$ with branching type $(2, m, 2m)$. Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(2, m, 2m)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ and $\deg(f) = 3$.

$(2, m, m)$ for $5 \le m$ $\deg(f) = 2$**:** Suppose that $m = 4k+1$. If k is even, put

$$A = (1\ m+1)(3\ m)(6\ m-1)\cdots(3t\ m-t+1)\cdots(3k-3\ 3k+3)(3k-2\ 3k)$$

$$B = (m+1\ m\ 2\ 1)(m-1\ 5\ 4\ 3)\cdots(3k-6\ 3k-5\ 3k-4\ 3k+3)$$

$$AB = (m\ m-1\cdots 2\ 1).$$

If k is odd, put

$$A = (4\ m)(7\ m-1)(6\ m-1)\cdots(3t+1\ m-t+1)\cdots(3k-2\ 3k+3)(3k-1\ 3k)$$

$$B = (m\ 3\ 2\ 1)(m-1\ 6\ 5\ 4)\cdots(3k-5\ 3k-4\ 3k-3\ 3k+3)(3k-2\ 3k\ 3k+1\ 3k+2)$$

$$AB = (m\ m-1\cdots 2\ 1).$$

Suppose that $m = 4k+3$. If k is even, put

$$A = (4\ m)(7\ m-1)(6\ m-1)\cdots(3t+1\ m-t+1)\cdots(3k+1\ 3k+4)$$
$$(3k+2\ 3k+3)$$

$$B = (m\ 3\ 2\ 1)(m-1\ 6\ 5\ 4)\cdots(3k-2\ 3k-1\ 3k\ 3k+1)(3k+1\ 3k+3)$$

$$AB = (m\ m-1\cdots 2\ 1).$$

If k is odd, put

$$A = (1\ m+1)(3\ m)(6\ m-1)\cdots(3t\ m-t+1)\cdots(3k\ 3k+4)(3k+1\ 3k+3)$$

$$B = (m+1\ m\ 2\ 1)(m-1\ 5\ 4\ 3)\cdots(3k-3\ 3k-2\ 3k-1\ 3k+4)$$
$$(3k\ 3k+3)(3k+1\ 3k+2)$$

$$AB = (m\ m-1\cdots 2\ 1).$$

Suppose that m is even. If $m = 3k-1$ and $k = 4k'+1$, put

$$A = (1\ m+1)(m\ m+4)(m-1\ m+7)\cdots(m-t\ m+3t+4)\cdots(2k+1\ 6k-3)$$
$$(2k\ 6k-2)(3\ 2k-1)\cdots(3s\ 2k-s)\cdots(3(2k'-1)\ 6k'+3)$$
$$(6k'-2\ 6k'+2)(6k'-1\ 6k'+1)$$

$$B = (m+3\ m+2\ m+1\ m)(m+6\ m+5\ m+4\ m-1)\cdots$$
$$(2m-2\ 2m-3\ 2m-4\ 2k+1)(2m-1\ 2k)(2m\ 2k-1\ 2\ 1)$$
$$(2k-2\ 5\ 4\ 3)\cdots(6k'+3\ 6k'-4\ 6k'-5\ 6k'-6)(6k'-3\ 6k'+2)$$
$$(6k'-2\ 6k'+1)(6k'-1\ 6k')$$

$$AB = (2m\ 2m-1\cdots 1)(m\ m-1\cdots 1).$$

If $m = 3k - 1$ and $k = 4k' + 3$, put

$$A = (1\ m+1)(m\ m+4)(m-1\ m+7)\cdots(m-t\ m+3t+4)\cdots(2k+1\ 6k-3)$$
$$(2k\ 6k-2)(3\ 2k-1)\cdots(3s\ 2k-s)\cdots(6k'\ 6k'+6)(6k'+2\ 6k'+4)$$
$$B = (m+3\ m+2\ m+1\ m)(m+6\ m+5\ m+4\ m-1)\cdots$$
$$(2m-2\ 2m-3\ 2m-4\ 2k+1)(2m-1\ 2k)$$
$$(2m\ 2k-1\ 2\ 1)(2k-2\ 5\ 4\ 3)\cdots(6k'+5\ 6k'+2\ 6k'+1\ 6k')(6k'+3\ 6k'+5)$$
$$AB = (2m\ 2m-1\cdots1)(m\ m-1\cdots1).$$

Suppose that m is even. If $m = 3k$ and $k = 4k'$, put

$$A = (1\ m+1)(m\ m+4)(m-1\ m+7)\cdots(m-t\ m+3t+4)\cdots(2k+2\ 6k-2)$$
$$(2k\ 6k)(3\ 2k-1)\cdots(3s\ 2k-s)\cdots(6k'-3\ 6k'+1)(6k'-2\ 6k')$$
$$B = (m+3\ m+2\ m+1\ m)(m+6\ m+5\ m+4\ m-1)\cdots(2m-2\ 2m-3\ 2m-4\ 2k+1)$$
$$(2m-1\ 2k)(2m\ 2k-1\ 2\ 1)(2k-2\ 5\ 4\ 3)\cdots(6k'+1\ 6k'-6\ 6k'-5\ 6k'-4)$$
$$(6k'-3\ 6k')(6k'-2\ 6k'-1)$$
$$AB = (2m\ 2m-1\cdots1)(m\ m-1\cdots1).$$

If $m = 3k$ and $k = 4k' + 2$, put

$$A = (1\ m+1)(m\ m+4)(m-1\ m+7)\cdots(m-t\ m+3t+4)\cdots(2k+2\ 6k-2)$$
$$(2k\ 6k)(3\ 2k-1)\cdots(3s\ 2k-s)\cdots(6k'\ 6k'+4)$$
$$B = (m+3\ m+2\ m+1\ m)(m+6\ m+5\ m+4\ m-1)\cdots(2m-2\ 2m-3\ 2m-4\ 2k+1)$$
$$(2m-1\ 2k)(2m\ 2k-1\ 2\ 1)(2k-2\ 5\ 4\ 3)\cdots(6k'+4\ 6k'-3\ 6k'-2\ 6k'-1)$$
$$(6k'+3\ 6k'+2\ 6k'+1\ 6k')$$
$$AB = (2m\ 2m-1\cdots1)(m\ m-1\cdots1).$$

Suppose that m is even. If $m = 3k + 1$ and $k = 4k' + 1$, put

$$A = (1\ m+1)(m\ m+4)(m-1\ m+7)\cdots(m-t\ m+3t+4)\cdots(2k+1\ 6k+2)$$
$$(3\ 2k+1)\cdots(3s\ 2k-s+1)\cdots(6k'-3\ 6k'+2)(6k'-2\ 6k'+1)$$
$$(6k'-1\ 6k')(6k'-2\ 6k')$$
$$B = (m+3\ m+2\ m+1\ m)(m+6\ m+5\ m+4\ m-1)\cdots(2m-2\ 2m-3\ 2m-4\ 2k+1)$$
$$(2m-1\ 2k)(2m\ 2k-1\ 2\ 1)(2k-2\ 5\ 4\ 3)\cdots(6k'+2\ 6k'-6\ 6k'-5\ 6k'-4)$$
$$(6k'-3\ 6k'+1)(6k'-2\ 6k')$$
$$AB = (2m\ 2m-1\cdots1)(m\ m-1\cdots1).$$

If $m = 3k + 1$ and $k = 4k' + 3$, put

$$A = (1\ m+1)(m\ m+4)(m-1\ m+7)\cdots(m-t\ m+3t+4)\cdots(2k+1\ 6k+2)$$
$$(3\ 2k+1)\cdots(3s\ 2k-s+1)\cdots(6k'\ 6k'+5)(6k'+3\ 6k'+4)$$
$$B = (m+3\ m+2\ m+1\ m)(m+6\ m+5\ m+4\ m-1)\cdots(2m-2\ 2m-3\ 2m-4\ 2k+1)$$
$$(2m-1\ 2k)(2m\ 2k-1\ 2\ 1)(2k-2\ 5\ 4\ 3)\cdots(6k'+5\ 6k'-3\ 6k'-2\ 6k'-1)$$
$$(6k'+4\ 6k'+2\ 6k'+1\ 6k')$$
$$AB = (2m\ 2m-1\cdots 1)(m\ m-1\cdots 1).$$

Let $G = \langle A, B\rangle$ be a group generated by A and B in S_{2m}. Let $\Phi : \pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\}) \to G$ be the surjective homomorphism defined by:

$$\Phi(x_1) = A, \quad \Phi(x_2) = B, \quad \Phi(x_3) = (AB)^{-1}.$$

Let $\mu : X \to \boldsymbol{P}^1$ be the Galois covering of $\boldsymbol{P}^1$ associated with $\mathrm{Ker}(\Phi)$. Then the branch divisor of μ is $B_\mu = 2P_1 + 4P_2 + mP_3$. By the Riemann-Hurwitz formula $\deg(\mu) = (8m/(m-4))(g(X)-1)$. Put $H = G \cap A_{2n}$. Since $\# H = (4m/(m-4))(g(X)-1) > 4(g(X)-1)$, the quotient space X/H is biholomorphic to $\boldsymbol{P}^1$. Let $\pi : X \to \boldsymbol{P}^1$ be the Galois covering corresponding to H. Since $A \in H$, $B \notin H$, $B^2 \in H$, the branch divisor of π must be

$$B_\pi = 2Q_{21} + mQ_{31} + mQ_{32}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(2, m, m)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ and $\deg(f) = 2$.

$(3, m, m)$ for $4 \le m \le 11$ $\deg(f) = 2$:
If $m = 4$, put

$$A = (1\ 5)$$
$$B = (4\ 3\ 2\ 1)(5\ 6)$$
$$AB = (6\ 5\ 4\ 3\ 2\ 1).$$

If $m = 5$, put

$$A = (1\ 6)$$
$$B = (5\ 4\ 3\ 2\ 1)$$
$$AB = (6\ 5\ 4\ 3\ 2\ 1).$$

Let $G = \langle A, B\rangle$ be a group generated by A and B in S_m. Let $\Phi : \pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\}) \to G$ be the surjective homomorphism defined by:

$$\Phi(x_1) = A, \quad \Phi(x_2) = B, \quad \Phi(x_3) = (AB)^{-1}.$$

Let $\mu : X \to \boldsymbol{P}^1$ be the Galois covering of $\boldsymbol{P}^1$ associated with $\mathrm{Ker}(\Phi)$. Then the branch divisor of μ is $B_\mu = 2P_1 + mP_2 + 6P_3$. Put $H = G \cap A_{2n}$. Let $\pi : X \to \boldsymbol{P}^1$ be the

Galois covering corresponding to H. Since $B \in H$, $AB \notin H$ and $(AB)^2 \in H$, the branch divisor of π must be

$$B_\pi = mQ_{21} + mQ_{22} + 3Q_{31}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(3, m, m)$ (for $m = 4, 5$) such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ and $\deg(f) = 2$.

If $m = 6$, put

$$A = (1\ 2)(3\ 4)(5\ 7)$$
$$B = (1\ 2\ 3)(4\ 5\ 6\ 7\ 8\ 9)$$
$$AB = (1\ 3\ 5\ 8\ 9\ 4)(6\ 7).$$

If $m = 7$, put

$$A = (1\ 7)$$
$$B = (6\ 5\ 4\ 3\ 2\ 1)$$
$$AB = (7\ 6\ 5\ 4\ 3\ 2\ 1).$$

If $m = 8$, put

$$A = (1\ 9)(5\ 6)(7\ 16)(8\ 11)(12\ 15)$$
$$B = (16\ 6\ 4\ 3\ 2\ 1)(15\ 11\ 7)(10\ 9\ 8)(14\ 13\ 12)$$
$$AB = (8\ 7 \cdots 2\ 1)(16\ 15 \cdots 9).$$

If $m = 9$, put

$$A = (1\ 6)$$
$$B = (6\ 5\ 4\ 3\ 2\ 1)(9\ 8\ 7)$$
$$AB = (9\ 8 \cdots 2\ 1).$$

If $m = 10$, put

$$A = (1\ 11)(3\ 8)(4\ 6)(5\ 21)(10\ 16)(12\ 15)(17\ 20)$$
$$B = (20\ 16\ 9\ 8\ 2\ 1)(15\ 11\ 10)(19\ 18\ 17)(7\ 6\ 3)(5\ 21\ 4)$$
$$AB = (10\ 9 \cdots 2\ 1)(20\ 19 \cdots 11).$$

If $m = 11$, put

$$A = (1\ 7)(9\ 11)(10\ 12)$$
$$B = (6\ 5\ 4\ 3\ 2\ 1)(11\ 8\ 7)(10\ 12\ 9)$$
$$AB = (11\ 10\ 9 \cdots 2\ 1).$$

Let $G = \langle A, B \rangle$ be a group generated by A and B in S_m. Let $\Phi : \pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\})$

$\to G$ be the surjective homomorphism defined by:

$$\Phi(x_1) = A, \quad \Phi(x_2) = B, \quad \Phi(x_3) = (AB)^{-1}.$$

Let $G = \langle A, B\rangle$ be a group generated by A and B. Let $\mu : X \to \boldsymbol{P}^1$ be the Galois covering of $\boldsymbol{P}^1$ associated with $\mathrm{Ker}(\Phi)$. Then the branch divisor of μ is $B_\mu = 2P_1 + 6P_2 + mP_3$. Put $H = G \cap A_{2n}$. Let $\pi : X \to \boldsymbol{P}^1$ be the Galois covering corresponding to H. Since $B \notin H$, $B^2 \in H$ and $AB \in H$, the branch divisor of π must be

$$B_\pi = 3Q_{21} + mQ_{31} + mQ_{32}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(3, m, m)$ (for $6 \le m \le 11$) such that $\mathrm{Aut}(X) \ne \mathrm{Aut}(\pi)$ and $\deg(f) = 2$.

$(\mathbf{3}, \mathbf{m}, \mathbf{3m})$ for $4 \le m \le 7$ $\deg(f) = 4$:
If $m = 4$, put

$$A = (1\ 13)(3\ 15)(5\ 14)(6\ 12)(8\ 11)$$

$$B = (15\ 2\ 1)(14\ 4\ 3)(13\ 12\ 5)(11\ 7\ 6)(10\ 9\ 8)$$

$$AB = (12\ 11 \cdots 2\ 1)(15\ 14\ 13).$$

If $m = 5$, put

$$A = (1\ 4)(6\ 15)(8\ 14)(10\ 13)$$

$$B = (3\ 2\ 1)(15\ 5\ 4)(14\ 7\ 6)(13\ 9\ 8)(12\ 11\ 10)$$

$$AB = (15\ 14 \cdots 2\ 1).$$

If $m = 6$, put

$$A = (1\ 19)(3\ 21)(5\ 20)(6\ 18)(8\ 17)(10\ 16)(12\ 15)$$

$$B = (21\ 2\ 1)(20\ 4\ 3)(19\ 18\ 5)(17\ 7\ 6)(16\ 9\ 8)(15\ 11\ 10)(14\ 13\ 12)$$

$$AB = (18\ 17 \cdots 2\ 1)(21\ 20\ 19).$$

If $m = 7$, put

$$A = (1\ 22)(3\ 24)(5\ 23)(6\ 21)(8\ 20)(10\ 19)(12\ 18)(14\ 17)$$

$$B = (24\ 2\ 1)(23\ 4\ 3)(22\ 21\ 5)(20\ 7\ 6)(19\ 9\ 8)(18\ 11\ 10)(17\ 13\ 12)(16\ 15\ 14)$$

$$AB = (21\ 20 \cdots 2\ 1)(24\ 23\ 22).$$

Let $G = \langle A, B\rangle$ be a group generated by A and B. Let $\Phi : \pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\}) \to G$ be the surjective homomorphism defined by:

$$\Phi(x_1) = A, \quad \Phi(x_2) = B, \quad \Phi(x_3) = (AB)^{-1}.$$

Let $\mu : X \to \boldsymbol{P}^1$ be the Galois covering of $\boldsymbol{P}^1$ associated with $\mathrm{Ker}(\Phi)$. Then the branch divisor of μ is $B_\mu = 2P_1 + 3P_2 + 3mP_3$. Put $H = \langle A^3, AB\rangle$. By the calculations using computer soft 'GAP', we have $\# G = 5184000$ $\# H = 1296000$ if $m = 4$, $\# G = 2592000$

$\# H = 648000$ if $m = 5$, $\# G = 384072192000$ $\# H = 96018048000$ if $m = 6$, $\# G = 98322481152000$ $\# H = 24580620288000$ if $m = 7$. Thus $[G : H]$ is equal to 4. Let $\pi : X \to \boldsymbol{P}^1$ be the Galois covering corresponding to H. Since A, $A^2 \notin H$, $A^3 \in H$ and $AB \in H$, the branch divisor of π must be

$$B_\pi = 3Q_{21} + mQ_{31} + mQ_{32}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(3, m, 3m)$ (for $m = 4, 5, 6, 7$) such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ and $\deg(f) = 4$.

$(\mathbf{4}, \mathbf{m}, \mathbf{m})$ for $5 \leq m \leq 7$ $\deg(f) = 2$:
If $m = 5$, put

$$A = (1\ 9)(4\ 8)(6\ 10)$$

$$B = (9\ 8\ 3\ 2\ 1)(7\ 6\ 10\ 5\ 4)$$

$$AB = (8\ 7\ 6\ 5\ 4\ 3\ 2\ 1).$$

If $m = 6$, put

$$A = (6\ 8)$$

$$B = (8\ 5\ 4\ 3\ 2\ 1)(6\ 7)$$

$$AB = (8\ 7\ 6\ 5\ 4\ 3\ 2\ 1).$$

If $m = 7$, put

$$A = (7\ 8)$$

$$B = (8\ 6\ 5\ 4\ 3\ 2\ 1)$$

$$AB = (8\ 7\ 6\ 5\ 4\ 3\ 2\ 1).$$

Let $G = \langle A, B \rangle$ be a group generated by A and B. Let $\Phi : \pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\}) \to G$ be the surjective homomorphism defined by:

$$\Phi(x_1) = A, \quad \Phi(x_2) = B, \quad \Phi(x_3) = (AB)^{-1}.$$

Let $\mu : X \to \boldsymbol{P}^1$ be the Galois covering of $\boldsymbol{P}^1$ associated with $\mathrm{Ker}(\Phi)$. Then the branch divisor of μ is $B_\mu = 2P_1 + mP_2 + 8P_3$. Put $H = G \cap A_{2n}$. Let $\pi : X \to \boldsymbol{P}^1$ be the Galois covering corresponding to H. Since $A \in H$, $B \in H$, $AB \notin H$ and $(AB)^2 \in H$, the branch divisor of π must be

$$B_\pi = mQ_{21} + mQ_{22} + 4Q_{31}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(4, m, m)$ (for $5 \leq m \leq 7$) such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ and $\deg(f) = 2$.

$(\mathbf{5}, \mathbf{5}, \mathbf{n})$ for $5 \leq n \leq 9$ $\deg(f) = 2$:
If $n = 5$, put

$$A = (1\ 6)$$

$$B = (5\ 4\ 3\ 2\ 1)(10\ 9\ 8\ 7\ 6)$$

$$AB = (10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1).$$

If $n = 6$, put

$$A = (1\ 6)(7\ 8)(9\ 10)$$
$$B = (5\ 4\ 3\ 2\ 1)(12\ 11\ 10\ 8\ 6)$$
$$AB = (12\ 11 \cdots 2\ 1).$$

If $n = 7$, put

$$A = (1\ 7)(2\ 3)(8\ 9)$$
$$B = (6\ 5\ 4\ 3\ 1)(14\ 13\ 11\ 9\ 7)$$
$$AB = (14\ 13 \cdots 2\ 1).$$

If $n = 8$, put

$$A = (1\ 6)(10\ 16)(11\ 12)$$
$$B = (5\ 4\ 3\ 2\ 1)(16\ 9\ 8\ 7\ 6)(15\ 14\ 13\ 12\ 10)$$
$$AB = (18\ 17 \cdots 2\ 1).$$

If $n = 9$, put

$$A = (1\ 6)(10\ 18)(11\ 12)(13\ 14)(15\ 16)$$
$$B = (5\ 4\ 3\ 2\ 1)(18\ 9\ 8\ 7\ 6)(17\ 16\ 14\ 12\ 10)$$
$$AB = (18\ 17 \cdots 2\ 1).$$

Let $G = \langle A, B \rangle$ be a group generated by A and B. Let $\Phi : \pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\}) \to G$ be the surjective homomorphism defined by:

$$\Phi(x_1) = A, \quad \Phi(x_2) = B, \quad \Phi(x_3) = (AB)^{-1}.$$

Let $\mu : X \to \boldsymbol{P}^1$ be the Galois covering of $\boldsymbol{P}^1$ associated with $\mathrm{Ker}(\Phi)$. Then the branch divisor of μ is $B_\mu = 2P_1 + 5P_2 + 2nP_3$. Put $H = G \cap A_{2n}$. Let $\pi : X \to \boldsymbol{P}^1$ be the Galois covering corresponding to H. Since $A \in H$, $B \in H$, $AB \notin H$ and $(AB)^2 \in H$, the branch divisor of π must be

$$B_\pi = 5Q_{21} + 5Q_{22} + nQ_{31}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(5, 5, n)$ (for $5 \le n \le 9$) such that $\mathrm{Aut}(X) \ne \mathrm{Aut}(\pi)$ and $\deg(f) = 2$.

$(\mathbf{5}, \mathbf{m}, \mathbf{m})$ **for** $m = 6$ $\deg(f) = 2$**:**
Put

$$A = (2\ 4)(5\ 6)(7\ 8)$$
$$B = (10\ 9\ 8\ 6\ 4\ 1)(2\ 3)$$
$$AB = (10\ 9 \cdots 2\ 1).$$

Let $G = \langle A, B \rangle$ be a group generated by A and B. Let $\Phi : \pi_1(\boldsymbol{P}^1 - \{P_1, P_2, P_3\}) \to G$ be the surjective homomorphism defined by:

$$\Phi(x_1) = A, \quad \Phi(x_2) = B, \quad \Phi(x_3) = (AB)^{-1}.$$

Let $\mu : X \to \boldsymbol{P}^1$ be the Galois covering of $\boldsymbol{P}^1$ associated with $\mathrm{Ker}(\Phi)$. Then the branch divisor of μ is $B_\mu = 2P_1 + 6P_2 + 10P_3$. Put $H = G \cap A_{2n}$. Let $\pi : X \to \boldsymbol{P}^1$ be the Galois covering corresponding to H. Since $B \in H$, $AB \notin H$ and $(AB)^2 \in H$, the branch divisor of π must be

$$B_\pi = 6Q_{21} + 6Q_{22} + 5Q_{31}.$$

Thus π is a Galois covering of $\boldsymbol{P}^1$ with the branching type $(5, 6, 6)$ such that $\mathrm{Aut}(X) \neq \mathrm{Aut}(\pi)$ and $\deg(f) = 2$.

4. Proof of Theorem 2.

We first consider the case $p = 7$. Let

$$\Phi : \pi_1(\boldsymbol{P}^1 - \{0, 1, \infty\}, *) \to S_p$$

be the monodromy representation of the covering

$$f : \boldsymbol{P}^1 = S \to \boldsymbol{P}^1 = M,$$

where S_p is the p-th symmetric group. Let G be the image of Φ. Then G is a transitive subgroup of S_p generated by two permutations

$$A = \Phi(\gamma_0) \quad \text{and} \quad B = \Phi(\gamma_1) \quad ((AB)^{-1} = \Phi(\gamma_\infty)),$$

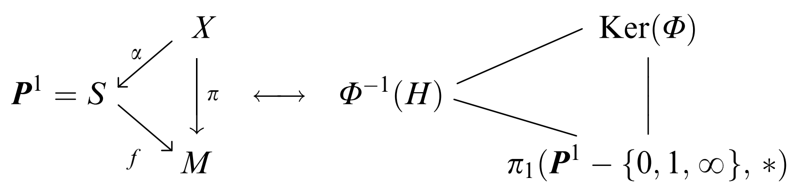
where γ_0, γ_1 and γ_∞ are lassos in $\pi_1(\boldsymbol{P}^1 - \{0, 1, \infty\}, *)$. Let H be the isotropy subgroup of G fixing a letter. Then

$$[G : H] = p. \tag{1}$$

Moreover, we have

$$\bigcap_{a \in G} aHa^{-1} = \{1\}.$$

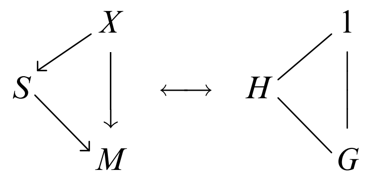
Since π is the Galois closure of f, there is the following Galois correspondence:



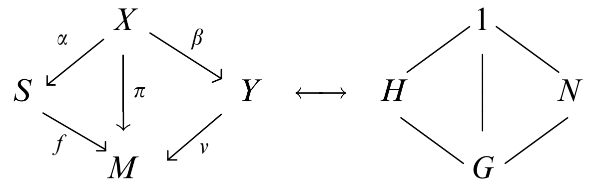
Note that

$$\mathrm{Aut}(\pi) \cong \pi_1(\boldsymbol{P}^1 - \{0, 1, \infty\}, *)/\mathrm{Ker}(\Phi) \cong G.$$

Hence we also have the following Galois correspondence:



Now, we show that $G(\cong \mathrm{Aut}(\pi))$ is a simple group. Assume the converse. Let N be a normal subgroup of G such that $N \neq \{1\}$ and $N \neq G$. By (2), N is not contained in H. Consider the Galois correspondence:



Lemma 2. *(1) β is unbranched. (2) ν is a Galois covering branched at $0, 1, \infty$ with ramification indices p, q, r, respectively.*

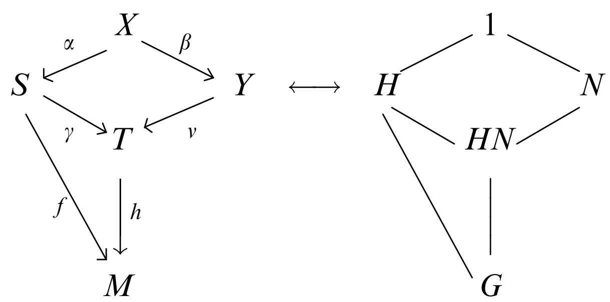
Proof. ν is a Galois covering, since N is normal. The relation $\pi = \nu \circ \beta$ implies that ν branches at most at $0, 1, \infty$. There exists no Galois covering of $M = \boldsymbol{P}^1$ which branches (i) at one point nor (ii) at two points with different ramification indices. Hence, by Lemma 1 and by the assumption that p, q, r are different prime numbers, (iii) ν is unbranched or (iv) ν is branched at $0, 1, \infty$ with the ramification indices p, q, r, respectively. But (iii) does not occur. For, if ν is unbranched, then ν must be homeomorphism since $M = \boldsymbol{P}^1$ is simply connected. Hence $N = G$, a contradiction. Hence (iv) occurs. Finally, the relation $\pi = \nu \circ \beta$ implies that β is unbranched. □

Lemma 3. *H is not contained in N.*

Proof. In fact, if H is contained in N, then there is a covering $S = \boldsymbol{P}^1 \to Y$. Hence Y is biholomorphic to $\boldsymbol{P}^1$. But the genus of Y is greater than 1 by Lemma 2, a contradiction. □

Lemma 4. *$HN = G$.*

Proof. Consider the following Galois correspondence:



By the relation $f = h \circ \gamma$, we have

$$p = \deg(f) = \deg(h)\deg(\gamma).$$

Hence either $\deg(\gamma) = p$ or $\deg(\gamma) = 1$. If $\deg(\gamma) = 1$, then $H = HN \supset N$, a contradiction. Hence $\deg(\gamma) = p$ and so $\deg(h) = 1$. Hence $HN = G$. □

Now, let P be a point in $v^{-1}(0)$. Since the ramification index of v at P is p, there are local coordinate systems t and x around P and 0 with $t(P) = 0$, $x(0) = 0$ such that v is locally given by

$$v : t \mapsto x = t^p.$$

Put $\zeta = \exp(2\pi i/p)$. Then the holomorphic mapping

$$\varphi : t \mapsto \zeta t$$

defined around P satisfies $v \circ \varphi = v$. Since v is a Galois covering, φ can be uniquely extended to an automorphism φ of v. Note that

$$\varphi^p = 1.$$

Note also that

$$\mathrm{Aut}(v) \cong G/N = HN/N \cong H/H \cap N.$$

Hence the order of H can be divided by p. On the other hand, since H is the isotropy subgroup of $G \subset S_p$ fixing a letter, say 1, H is regarded as a subgroup of S_{p-1}, a contradiction.

In the above proof of Theorem 2, we assumed $p = 7$. In the case $p = 5$, we necessary have $q = 3$ and $r = 2$. In this case, the Galois closure $\pi : X \to \boldsymbol{P}^1$ of f satisfies that X is biholomorphic to P^1 and $\mathrm{Aut}(\pi)$ is isomorphic to the alternating group A_5 of 5 letters. (See Hochstadt [**2**].) Hence Theorem 2 holds in this case.

This completes the proof of Theorem 2.

EXAMPLE 1. Consider the permutations

$$A = (7\ 6\ 5\ 4\ 3\ 2\ 1),$$

$$B = (1\ 2\ 3)(4\ 6\ 7),$$

$$(AB)^{-1} = (1\ 4)(5\ 6).$$

They generate the simple group G of order 168. (For the computation, we used the computer soft 'GAP'.) G is a transitive subgroup of S_7. Hence there is a covering $f : S \to \boldsymbol{P}^1$ branched at $0, 1, \infty$ whose monodromy representation Φ satisfies

$$\Phi(\gamma_0) = A, \quad \Phi(\gamma_1) = B \quad \text{and} \quad \Phi(\gamma_\infty) = (AB)^{-1}.$$

By the Riemann-Hurwitz formula, S is biholomorphic to $\boldsymbol{P}^1$. The Galois closure $\pi : X \to \boldsymbol{P}^1$ of f branches at $0, 1, \infty$, with the ramification indices

$$\mathrm{ord}(A) = 7, \quad \mathrm{ord}(B) = 3, \quad \mathrm{ord}((AB)^{-1}) = 2,$$

respectively. As was noted above, $\mathrm{Aut}(\pi)$ ($\cong \mathrm{Aut}(X)$ by Theorem 1) is isomorphic to G. In this case the genus of X is 3. (See Klein [**4**].)

EXAMPLE 2. Consider the permutations

$$A = (11\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1),$$

$$B = (1\ 2\ 3\ 4\ 5)(6\ 8\ 9\ 10\ 11),$$

$$(AB)^{-1} = (1\ 6)(7\ 8).$$

They generate the alternating group A_{11} of 11 letters. (For the computation, we again used the computer soft 'GAP'.) Let $f : \boldsymbol{P}^1 \to \boldsymbol{P}^1$ be a rational function defined as in Example 1. Let $\pi : X \to \boldsymbol{P}^1$ be the Galois closure of f. Then π branches at $0, 1, \infty$ with the ramification indices

$$\mathrm{ord}(A) = 11, \quad \mathrm{ord}(B) = 5, \quad \mathrm{ord}((AB)^{-1}) = 2,$$

respectively. $\mathrm{Aut}(\pi)$ ($\cong \mathrm{Aut}(X)$ by Theorem 1) is isomorphic to A_{11}. In this case the genus of X is 1512001.

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