J. Math. Soc. Japan
Vol. 50, No. 4, 1998

Actions of loop groups on simply connected H-surfaces in space forms

By Atsushi FUJIOKA

(Received Nov. 28, 1995)
(Revised Nov. 14, 1996)

Abstract. In this paper we shall define certain loop groups which act on simply connected H-surfaces in space forms preserving conformality, and obtain a criterion for these group actions to be equivariant.

Introduction.

In recent years, there has been much progress in the theory of harmonic maps from Riemann surfaces into Lie groups or symmetric spaces. For example, the discovery of actions of loop groups on harmonic maps has played an important role [**BG**], [**BP**], [**DPW**], [**GO1**], [**GO2**], [**U**]. As a special case of the results in [**BP**], it is shown that a certain loop group of $SO(4)$ acts on harmonic maps from a simply connected Riemann surface into the standard 3-sphere S^3.

Let $\mathfrak{M}^3(c)$ denote the simply connected 3-dimensional space form of curvature c and M a Riemann surface. Since harmonic maps from M to S^3 are H-surfaces* in $\mathfrak{M}^3(c)$ for $H = 0$, $c = 1$, it is natural to expect as an analogue to the results in [**BP**] above mentioned, that a certain loop group of the isometry group of $\mathfrak{M}^3(c)$ acts on simply connected H-surfaces in $\mathfrak{M}^3(c)$ naturally, preserving conformality. In this paper, we shall show that it is true.

Before we state our main theorem precisely, we shall give some notations and definitions.

First we shall describe $\mathfrak{M}^3(c)$ as a Riemannian symmetric space.

We put

$$G_0 = SO(3) \ltimes \boldsymbol{R}^3 = \left\{ \begin{pmatrix} T & s \\ 0 & 1 \end{pmatrix} \in GL(4, \boldsymbol{R}); T \in SO(3), s \in \boldsymbol{R}^3 \right\},$$

$$G_c = SO(4)$$

for $c > 0$, and

$$\begin{aligned} G_c &= SO^+(3,1) \\ &= \{X = (x_{ij}) \in GL(4, \boldsymbol{R}); {}^tXJX = J, \det X = 1, x_{44} > 0\} \end{aligned}$$

This research was partially supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

1991 Mathematics Subject Classification. Primary 53A10 Secondary 58E20

*Here an H-surface means a map which satisfies H-surface equations. Hence H-surfaces in the classical sense are conformal H-surfaces.

for $c < 0$, where $J = \mathrm{diag}(1, 1, 1, -1)$. Set

$$K_c = \left\{ \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} \in GL(4, \boldsymbol{R}); T \in SO(3) \right\}.$$

An involutive automorphism is defined by

$$\sigma_c(X) = JXJ,$$

where $X \in G_c$.

The corresponding Cartan decomposition $\mathfrak{g}_c = \mathfrak{k}_c \oplus \mathfrak{m}_c$ is given by

$$\mathfrak{g}_c = \left\{ \begin{pmatrix} A & b \\ -\mathrm{sign}(c)^t b & 0 \end{pmatrix} \in M(4, \boldsymbol{R}); A \in \mathfrak{so}(3), b \in \boldsymbol{R}^3 \right\},$$

$$\mathfrak{k}_c = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in M(4, \boldsymbol{R}); A \in \mathfrak{so}(3) \right\}$$

and

$$\mathfrak{m}_c = \left\{ \begin{pmatrix} 0 & b \\ -\mathrm{sign}(c)^t b & 0 \end{pmatrix} \in M(4, \boldsymbol{R}); b \in \boldsymbol{R}^3 \right\},$$

where

$$\mathrm{sign}(c) = \begin{cases} 1 & \text{if } c > 0, \\ 0 & \text{if } c = 0, \\ -1 & \text{if } c < 0. \end{cases}$$

An $\mathrm{Ad}_{G_c} K_c$-invariant metric on $\mathfrak{m}_c$ is defined by

$$g_c\left(\begin{pmatrix} 0 & b \\ -\mathrm{sign}(c)^t b & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ -\mathrm{sign}(c)^t b' & 0 \end{pmatrix} \right) = L^2(c)\langle b, b' \rangle,$$

where $b, b' \in \boldsymbol{R}^3$ and

$$L(c) = \begin{cases} 1 & \text{if } c = 0, \\ \dfrac{1}{\sqrt{|c|}} & \text{if } c \neq 0. \end{cases}$$

Then $\mathfrak{M}^3(c)$ is a Riemannian symmetric space corresponding to $(G_c, K_c, \sigma_c, g_c)$. Let $\pi : G_c \to \mathfrak{M}^3(c) = G_c/K_c$ be the natural projection.

Let M be a simply connected Riemann surface. For any map $f : M \to \mathfrak{M}^3(c)$, there always exists a map $F : M \to G_c$ such that $\pi \circ F = f$. Such a map F is called a framing of f. Then we have a decomposition $F^{-1}dF =: \alpha = \alpha_{\mathfrak{k}_c} + \alpha_{\mathfrak{m}_c}$. Since M is a Riemann surface, we have a type decomposition $\alpha_{\mathfrak{m}_c} = \alpha'_{\mathfrak{m}_c} + \alpha''_{\mathfrak{m}_c}$, where $\alpha'_{\mathfrak{m}_c}$ is an $\mathfrak{m}_c^{\boldsymbol{C}}$-valued $(1,0)$-form with complex conjugate $\alpha''_{\mathfrak{m}_c}$. We write the decomposition $\alpha = \alpha_{\mathfrak{k}_c} + \alpha'_{\mathfrak{m}_c} + \alpha''_{\mathfrak{m}_c}$ simply as $\alpha = \alpha_{\mathfrak{k}} + \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$. When we write $\alpha_{\mathfrak{m}}$ as

$$\alpha_{\mathfrak{m}} = \begin{pmatrix} 0 & b \\ -\mathrm{sign}(c)^t b & 0 \end{pmatrix}$$

for $b \in \Omega^1(M, \boldsymbol{R}^3)$, we shall define an $\mathfrak{m}_c$-valued 2-form $\alpha_{\mathfrak{m}} \times \alpha_{\mathfrak{m}}$ by

$$\alpha_{\mathfrak{m}} \times \alpha_{\mathfrak{m}} = \begin{pmatrix} 0 & b \times b \\ -\operatorname{sign}(c)^t(b \times b) & 0 \end{pmatrix},$$

where $\times$ is the exterior product on $\boldsymbol{R}^3$. Then we set

$$n_f = \frac{L(c)}{2} d\pi(\operatorname{Ad} F * (\alpha_{\mathfrak{m}} \times \alpha_{\mathfrak{m}})),$$

where $*$ is the Hodge star operator on M. It is easy to see that n_f is independent of the choice of F.

We now fix $p_0 \in M$ and set $o = \{K_c\}$. For $H, c \in \boldsymbol{R}$, we set

$$\mathscr{H}_{H,c} = \left\{ f : M \to \mathfrak{M}^3(c);\ \frac{1}{2}\operatorname{trace} \nabla\, df = H n_f, f(p_0) = o \right\},$$

$$\mathscr{C}_{H,c} = \{ f \in \mathscr{H}_{H,c};\ f \text{ is weakly conformal}\}.$$

Then $\mathscr{C}_{H,c}$ is the set of based branched conformal immersion with constant mean curvature $= H$.

In the previous paper [F], the author obtained a criterion for the existence of a natural bijective correspondence between simply connected H-surfaces in $\mathfrak{M}^3(c)$ and simply connected H'-surfaces in $\mathfrak{M}^3(c')$:

Proposition ([F]). *Let $H, H', c, c' \in \boldsymbol{R}$. If $\operatorname{sign}(H^2 + c) = \operatorname{sign}(H'^2 + c')$, then there exists a bijective map*

$$\Psi : \mathscr{H}_{H,c} \to \mathscr{H}_{H',c'}$$

such that $\Psi(\mathscr{C}_{H,c}) = \mathscr{C}_{H',c'}$.

In this paper, we shall also obtain a criterion for the above bijective correspondence to be equivariant with respect to the loop group actions:

Main Theorem. *Let $H, H', c, c' \in \boldsymbol{R}$. If $\operatorname{sign}(H^2 + c) = \operatorname{sign}(H'^2 + c')$, then there exists a loop group $\mathscr{G}_{H,c}$ (respectively $\mathscr{G}_{H',c'}$) acting on $\mathscr{H}_{H,c}$ (respectively $\mathscr{H}_{H',c'}$) such that*

(i) *there exists a Lie group isomorphism*

$$\Phi : \mathscr{G}_{H,c} \to \mathscr{G}_{H',c'},$$

(ii) *the bijective map $\Psi : \mathscr{H}_{H,c} \to \mathscr{H}_{H',c'}$ is Φ-equivariant,*

(iii) *$\mathscr{G}_{H,c}$ (respectively $\mathscr{G}_{H',c'}$) acts on $\mathscr{C}_{H,c}$ (respectively $\mathscr{C}_{H',c'}$) and the bijective map $\Psi|_{\mathscr{C}_{H,c}} : \mathscr{C}_{H,c} \to \mathscr{C}_{H',c'}$ is Φ-equivariant.*

Acknowledgement. The author would like to thank Professor Takushiro Ochiai for his useful suggestions and constant encouragement. Thanks are also due to Dr. Sakagawa and Mr. Higaki for valuable discussions and the referee for kind advice.

§1. Definition of loop groups.

We fix $0 < \varepsilon < 1$ and partition the Riemann sphere $\boldsymbol{P}^1 = \boldsymbol{C} \cup \{\infty\}$ as follows. Let C_ε and $C_{1/\varepsilon}$ denote the circles of radius ε and $1/\varepsilon$ about $0 \in \boldsymbol{C}$ and define open sets by

$$I_\varepsilon = \{\lambda \in \boldsymbol{P}^1; |\lambda| < \varepsilon\}, \quad I_{1/\varepsilon} = \{\lambda \in \boldsymbol{P}^1; |\lambda| > 1/\varepsilon\},$$
$$E^{(\varepsilon)} = \{\lambda \in \boldsymbol{P}^1; \varepsilon < |\lambda| < 1/\varepsilon\}.$$

Now put $I^{(\varepsilon)} = I_\varepsilon \cup I_{1/\varepsilon}$ and $C^{(\varepsilon)} = C_\varepsilon \cup C_{1/\varepsilon}$ so that $\boldsymbol{P}^1 = I^{(\varepsilon)} \cup C^{(\varepsilon)} \cup E^{(\varepsilon)}$. For a map $\xi : C^{(\varepsilon)} \to \mathfrak{g}_c^{\boldsymbol{C}}$, define $A_\xi : C^{(\varepsilon)} \to \mathfrak{so}(3)^{\boldsymbol{C}}$ and $b_\xi : C^{(\varepsilon)} \to \boldsymbol{C}^3$ by

$$\xi(\lambda) = \begin{pmatrix} A_\xi(\lambda) & b_\xi(\lambda) \\ -\operatorname{sign}(c)^t b_\xi(\lambda) & 0 \end{pmatrix},$$

where $\lambda \in C^{(\varepsilon)}$. We define a bijective map $\iota : \boldsymbol{C}^3 \to \mathfrak{so}(3)^{\boldsymbol{C}}$ by

$$\iota(p) = \begin{pmatrix} 0 & -p^3 & p^2 \\ p^3 & 0 & -p^1 \\ -p^2 & p^1 & 0 \end{pmatrix} \qquad \text{for } p = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} \in \boldsymbol{C}^3.$$

First we define a loop group $\Lambda^\varepsilon G_c$ by

$$\Lambda^\varepsilon G_c = \{g \in C^\infty(C^{(\varepsilon)}, G_c^{\boldsymbol{C}}); \overline{g(\lambda)} = g(1/\bar{\lambda}) \text{ for } \lambda \in C^{(\varepsilon)}\},$$

where conjugation is the Cartan involution of $G_c^{\boldsymbol{C}}$ fixing G_c. Then the Lie algebra $\Lambda^\varepsilon \mathfrak{g}_c$ of the Lie group $\Lambda^\varepsilon G_c$ is defined by

$$\Lambda^\varepsilon \mathfrak{g}_c = \{\xi \in C^\infty(C^{(\varepsilon)}, \mathfrak{g}_c^{\boldsymbol{C}}); \overline{\xi(\lambda)} = \xi(1/\bar{\lambda}) \text{ for } \lambda \in C^{(\varepsilon)}\}.$$

(For the definition of the manifold structure, we refer to [**M**], [**OMYK**], [**PS**].)

Then the following loop algebras are important for us to define loop groups.

For $c = 0, \rho \in \sqrt{-1}\boldsymbol{R}\backslash\{0\}$, we set

$$\Lambda^{\varepsilon,\rho}\mathfrak{g}_c = \left\{\xi \in \Lambda^\varepsilon \mathfrak{g}_c; \lambda \frac{dA_\xi}{d\lambda} = \rho\iota(b_\xi) \text{ for } \lambda \in C^{(\varepsilon)}\right\}.$$

For $c = 0, \rho = 0$, we set

$$\Lambda^{\varepsilon,\rho}\mathfrak{g}_c = \{\xi \in \Lambda^\varepsilon \mathfrak{g}_c; b_\xi(\lambda) = 0 \text{ for } \lambda \in C^{(\varepsilon)}\}.$$

For $c \neq 0$, $\rho \in \sqrt{-1}\boldsymbol{R}$, we set

$$\alpha_n = \begin{cases} \dfrac{1}{2}\{(\rho+1)^n + (\rho-1)^n\} & \text{if } c > 0, \\ \dfrac{1}{2}\{(\rho-\sqrt{-1})^n + (\rho+\sqrt{-1})^n\} & \text{if } c < 0, \end{cases}$$

$$\beta_n = \begin{cases} \dfrac{1}{2}\{(\rho+1)^n - (\rho-1)^n\} & \text{if } c > 0, \\ \dfrac{\sqrt{-1}}{2}\{(\rho-\sqrt{-1})^n - (\rho+\sqrt{-1})^n\} & \text{if } c < 0 \end{cases}$$

for $n \in \boldsymbol{Z}$. Here we put $0^0 = 1$. Then we set

$$\Lambda^{\varepsilon,\rho}\mathfrak{g}_c = \{\xi \in \Lambda^\varepsilon \mathfrak{g}_c; A_\xi \text{ and } b_\xi \text{ have Fourier series } (*)\},$$

where

$$(*) \qquad A_\xi(\lambda) = \sum_{n\in\mathbf{Z}} \alpha_n \iota(a_n)\lambda^n, b_\xi(\lambda) = \sum_{n\in\mathbf{Z}} \beta_n a_n \lambda^n \quad \text{for } a_n \in \mathbf{C}^3, \lambda \in C_\varepsilon.$$

It is easy to see that $\Lambda^{\varepsilon,\rho}\mathfrak{g}_c$ is a Lie subalgebra of $\Lambda^\varepsilon \mathfrak{g}_c$. Then we have the following:

Lemma 1.1. *Suppose one of the following conditions are satisfied:*
(i) $c = 0$ (ii) $c > 0$, $(1-\rho^2)\varepsilon^2 < 1$ (iii) $c < 0$, $(1+|\rho|)\varepsilon < 1$.
Then we have a Lie subgroup $\Lambda^{\varepsilon,\rho}G_c \subset \Lambda^\varepsilon G_c$ *such that its Lie algebra is* $\Lambda^{\varepsilon,\rho}\mathfrak{g}_c$.

Proof.
Case I: $c = 0$, $\rho = 0$.
For $g \in \Lambda^\varepsilon G_c$, set $T_g \in C^\infty(C^{(\varepsilon)}, SO(3)^{\mathbf{C}})$ and $s_g \in C^\infty(C^{(\varepsilon)}, \mathbf{C}^3)$ by

$$g(\lambda) = \begin{pmatrix} T_g(\lambda) & s_g(\lambda) \\ 0 & 1 \end{pmatrix},$$

where $\lambda \in C^{(\varepsilon)}$. Then it is obvious to see that

$$\Lambda^{\varepsilon,\rho}G_c = \{g \in \Lambda^\varepsilon G_c; s_g(\lambda) = 0 \text{ for } \lambda \in C^{(\varepsilon)}\}.$$

Case II: $c = 0$, $\rho \in \sqrt{-1}\mathbf{R}\backslash\{0\}$.
By direct computation, it is easy to see that

$$\Lambda^{\varepsilon,\rho}G_c = \left\{g \in \Lambda^\varepsilon G_c; \lambda\frac{dT_g}{d\lambda} = \rho\iota(s_g)T_g \text{ for } \lambda \in C^{(\varepsilon)}\right\}.$$

Case III: $c > 0$, $\rho = 0$.
In this case, we have

$$\Lambda^{\varepsilon,\rho}\mathfrak{g}_c = \{\xi \in \Lambda^\varepsilon\mathfrak{g}_c; \xi(-\lambda) = \sigma_c\xi(\lambda), \overline{\xi(\lambda)} = \xi(1/\bar{\lambda}) \text{ for } \lambda \in C^{(\varepsilon)}\}.$$

Hence we have

$$\Lambda^{\varepsilon,\rho}G_c = \{g \in \Lambda^\varepsilon G_c; g(-\lambda) = \sigma_c g(\lambda), \overline{g(\lambda)} = g(1/\bar{\lambda}) \text{ for } \lambda \in C^{(\varepsilon)}\}.$$

Case IV: $c > 0$, $\rho \in \sqrt{-1}\mathbf{R}\backslash\{0\}$, $(1-\rho^2)\varepsilon^2 < 1$.
It is well-known that the condition $\xi \in C^\infty(C^{(\varepsilon)}, \mathfrak{g}_c^{\mathbf{C}})$ is equivalent to

$$(1.1) \qquad \sum_{n\in\mathbf{Z}} (1+n^2)^l(|\alpha_n|^2 + |\beta_n|^2)\varepsilon^{2n}|a_n|^2 < \infty$$

for any $l > 0$. Since $|\alpha_n|^2 + |\beta_n|^2 = (1-\alpha^2)^n$, (1.1) is equivalent to

$$(1.2) \qquad \sum_{n\in\mathbf{Z}} (1+n^2)^l(\varepsilon\sqrt{1-\rho^2})^{2n}|a_n|^2 < \infty$$

for any $l > 0$. We set

$$(1.3) \qquad \psi(\xi)(\lambda) = \begin{pmatrix} \sum_{n\in\mathbf{Z}} \iota(a_{2n})\lambda^{2n} & \sum_{n\in\mathbf{Z}} a_{2n+1}\lambda^{2n+1} \\ -{}^t(\sum_{n\in\mathbf{Z}} a_{2n+1}\lambda^{2n+1}) & 0 \end{pmatrix}$$

for $\lambda \in C_\varepsilon$. Note that from (1.2), ψ defines a well-defined Lie algebra isomorphism:

$$\psi : \Lambda^{\varepsilon,\rho}\mathfrak{g}_c \to \Lambda^{\varepsilon\sqrt{1-\rho^2},0}\mathfrak{g}_1.$$

Then by [**OMYK**, Theorem 3.2], we have a homomorphism:

$$\varphi : \widetilde{\Lambda^{\varepsilon\sqrt{1-\rho^2},0}G_1} \to \Lambda^\varepsilon G_c$$

such that $d\varphi = \psi^{-1}$, where we denote the universal covering group of a Lie group G as $\tilde{G}$. Set $\Lambda^{\varepsilon,\rho}G_c = \operatorname{Im}\varphi$. Then $\Lambda^{\varepsilon,\rho}G_c$ is the desired Lie subgroup of $\Lambda^\varepsilon G_c$.

Case V: $c < 0$, $(1 + |\rho|)\varepsilon < 1$.

Since $|\alpha_n|^2 + |\beta_n|^2 = 1/2\{(1 + |\rho|)^{2n} + (1 - |\rho|)^{2n}\}$, $\xi \in C^\infty(C^{(\varepsilon)}, \mathfrak{g}_c^{\boldsymbol{C}})$ is equivalent to

$$\sum_{n\in\boldsymbol{Z}}(1 + n^2)^l\{(1 \pm |\rho|)\varepsilon\}^{2n}|a_n|^2 < \infty \tag{1.4}$$

for any $l > 0$. If we define $\psi(\xi)(\lambda)$ by (1.3), from (1.4), ψ defines a well-defined Lie algebra isomorphism:

$$\psi : \Lambda^{\varepsilon,\rho}\mathfrak{g}_c \to \begin{cases} \Lambda^{(1+|\rho|)\varepsilon,0}\mathfrak{g}_1 \cap \Lambda^{|1-|\rho||\varepsilon,0}\mathfrak{g}_1 & \text{if } |\rho| \neq 1, \\ \Lambda^{(1+|\rho|)\varepsilon,0}\mathfrak{g}_1 & \text{if } |\rho| = 1. \end{cases}$$

Similar to the case IV, we have the desired Lie subgroup $\Lambda^{\varepsilon,\rho}G_c$. □

We identify K_c with $SO(3)$ and fix an Iwasawa decomposition of $SO(3)^{\boldsymbol{C}}$: $SO(3)^{\boldsymbol{C}} = SO(3)B$, where B is a Borel subgroup of $SO(3)^{\boldsymbol{C}}$. We define subgroups of $\Lambda^{\varepsilon,\rho}G_c$ as follows.

$$\Lambda_E^{\varepsilon,\rho}G_c = \{g \in \Lambda^{\varepsilon,\rho}G_c; g \text{ extends holomorphically to } g : E^{(\varepsilon)} \to G_c^{\boldsymbol{C}}\},$$

$$\Lambda_I^{\varepsilon,\rho}G_c = \{g \in \Lambda^{\varepsilon,\rho}G_c; g \text{ extends holomorphically to } g : I^{(\varepsilon)} \to G_c^{\boldsymbol{C}}\}.$$

It is easy to see that $g \in \Lambda_I^{\varepsilon,\rho}G_c$ satisfies $g(0) \in K_c^{\boldsymbol{C}}$. We define a subgroup of $\Lambda_I^{\varepsilon,\rho}G_c$ by

$$\Lambda_{I,B}^{\varepsilon,\rho}G_c = \{g \in \Lambda_I^{\varepsilon,\rho}G_c; g(0) \in B\}.$$

Then we obtain the following Iwasawa type decomposition for $\Lambda^{\varepsilon,\rho}G_c$.

LEMMA 1.2. *Suppose one of the following conditions are satisfied:*
(i) $c = 0$ (ii) $c > 0$, $(1 - \rho^2)\varepsilon^2 < 1$ (iii) $c < 0$, $(1 + |\rho|)\varepsilon < 1$.
Then a map defined by multiplication

$$\Lambda_E^{\varepsilon,\rho}G_c \times \Lambda_{I,B}^{\varepsilon,\rho}G_c \to \Lambda^{\varepsilon,\rho}G_c$$

is bijective.

PROOF.
Case I: $c = 0$, $\rho = 0$ or $c > 0$, $\rho = 0$.
This is a special case of the result due to McIntosh [**M**, Proposition 6.2].
Case II: $c = 0$, $\rho \in \sqrt{-1}\boldsymbol{R}\backslash\{0\}$.
Similar to the case I, for $g \in \Lambda^{\varepsilon,\rho}G_c$, since T_g satisfies $\overline{T_g(\lambda)} = T_g(1/\bar{\lambda})$ for $\lambda \in C^{(\varepsilon)}$, there exist $T_1, T_2 \in C^\infty(C^{(\varepsilon)}, SO(3)^{\boldsymbol{C}})$ such that (i) $T_g = T_1T_2$, (ii) $\overline{T_i(\lambda)} = T_i(1/\bar{\lambda})$ for

$\lambda \in C^{(\varepsilon)}$, $i = 1,2$, (iii) T_1 extends holomorphically to $T_1 : E^{(\varepsilon)} \to SO(3)^{\boldsymbol{C}}$, (iv) T_2 extends holomorphically to $T_2 : I^{(\varepsilon)} \to SO(3)^{\boldsymbol{C}}$, (v) $T_2(0) \in B$. We define $s_1, s_2 \in C^\infty(C^{(\varepsilon)}, \boldsymbol{C}^3)$ by

$$\lambda \frac{dT_i}{d\lambda} = \rho\iota(s_i)T_i \qquad \text{for} \qquad i = 1,2.$$

By definition, we have (i) $\overline{s_i(\lambda)} = s_i(1/\bar{\lambda})$ for $\lambda \in C^{(\varepsilon)}$, $i = 1,2$, (ii) s_1 extends holomorphically to $s_1 : E^{(\varepsilon)} \to \boldsymbol{C}^3$, (iii) s_2 extends holomorphically to $s_2 : I^{(\varepsilon)} \to \boldsymbol{C}^3$, (iv) $s_2(0) = 0$. On $C^{(\varepsilon)}$ we have

$$\begin{pmatrix} T_1 & s_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} T_2 & s_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} T_1T_2 & T_1s_2 + s_1 \\ 0 & 1 \end{pmatrix}.$$

Direct computation shows that

$$\lambda \frac{d(T_1T_2)}{d\lambda} = \rho\iota(T_1s_2 + s_1)T_1T_2.$$

Hence we have the desired decomposition.

Case III: $c > 0$, $\rho \in \sqrt{-1}\boldsymbol{R}\backslash\{0\}$, $(1 - \rho^2)\varepsilon^2 < 1$.

Using the decomposition of $\Lambda^{\varepsilon\sqrt{1-\rho^2},0}G_1$ and the covering map

$$\varphi : \widetilde{\Lambda^{\varepsilon\sqrt{1-\rho^2},0}G_1} \to \Lambda^\varepsilon G_c,$$

we have the following decomposition:

$$\widetilde{\Lambda^{\varepsilon,\rho}G_c} = \widetilde{\Lambda_E^{\varepsilon,\rho}G_c}\widetilde{\Lambda_{I,B}^{\varepsilon,\rho}G_c}.$$

Since any $g \in \Lambda_{I,B}^{\varepsilon,\rho}G_c$ is homotopic to a constant loop by definition of $\Lambda_{I,B}^{\varepsilon,\rho}G_c$, $\Lambda_{I,B}^{\varepsilon,\rho}G$ is simply connected. Hence we have the desired decomposition.

Case IV: $c < 0$, $(1 + |\rho|)\varepsilon < 1$.

Similar to the case III, we have the desired decomposition. □

§2. Extended framings.

Let $f \in \mathscr{H}_{H,c}$ and F be a framing of f. Then direct computation shows that $1/2\,\mathrm{trace}\,\nabla df = Hn_f$ is equivalent to

$$(d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}]) - (d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}]) = 2\sqrt{-1}HL(c)\alpha'_{\mathfrak{m}} \times \alpha''_{\mathfrak{m}}. \tag{2.1}$$

Taking the $\mathfrak{m}_c$- and $\mathfrak{k}_c$-parts of the Maurer-Cartan equations for α, we have

$$(d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}]) + (d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}]) = 0 \tag{2.2}$$

and

$$d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}] = 0. \tag{2.3}$$

Equations (2.1) and (2.2) are equivalent to

$$d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}] = \sqrt{-1}HL(c)\alpha'_{\mathfrak{m}} \times \alpha''_{\mathfrak{m}}. \tag{2.4}$$

If we write α'_{m} as

$$\alpha'_{\mathrm{m}} = \begin{pmatrix} 0 & b' \\ -\mathrm{sign}(c){}^t b' & 0 \end{pmatrix} \tag{2.5}$$

for $b' \in \Omega^{1,0}(M, \boldsymbol{C}^3)$, we shall define $\iota(\alpha'_{\mathrm{m}}) \in \Omega^{1,0}(M, \mathfrak{k}_c^{\boldsymbol{C}})$ by

$$\iota(\alpha'_{\mathrm{m}}) = \begin{pmatrix} \iota(b') & 0 \\ 0 & 0 \end{pmatrix}.$$

We define $\iota(\alpha''_{\mathrm{m}}) \in \Omega^{0,1}(M, \mathfrak{k}_c^{\boldsymbol{C}})$ similarly. For $\lambda \in \boldsymbol{C}\backslash\{0\}$, define a $\mathfrak{g}_c^{\boldsymbol{C}}$-valued 1-form by

$$\alpha_\lambda = \lambda^{-1}\alpha'_{\mathrm{m}} + \lambda\alpha''_{\mathrm{m}} + \alpha_{\mathfrak{k}} - \sqrt{-1}HL(c)(\lambda^{-1} - 1)\iota(\alpha'_{\mathrm{m}}) + \sqrt{-1}HL(c)(\lambda - 1)\iota(\alpha''_{\mathrm{m}}).$$

It is computed in [F] that α satisfies equations (2.3) and (2.4) if and only if

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$$

for all $\lambda \in \boldsymbol{C}\backslash\{0\}$. Then from the standard fact, there exists a map $F_\lambda : M \to G_c^{\boldsymbol{C}}$, unique up to left translation by a constant element of $G_c^{\boldsymbol{C}}$, satisfying $F_\lambda^{-1}\, dF_\lambda = \alpha_\lambda$. Furthermore, we set

$$\Lambda^{\rho}_{\mathrm{hol}} G_c = \bigcap_{0<\varepsilon<\varepsilon_0} \Lambda_E^{\varepsilon,\rho} G_c,$$

where ε_0 satisfies one of the following conditions:

(i) $c = 0$, $\varepsilon_0 = 1$, (ii) $c > 0$, $(1 - \rho^2)\varepsilon_0{}^2 = 1$, (iii) $c < 0$, $(1 + |\rho|)\varepsilon_0 = 1$.

These observations lead us to the following definition.

DEFINITION 2.1. *A map $F_\lambda : M \to \Lambda^{\rho}_{\mathrm{hol}} G_c$ is an extended framing if $\lambda F_\lambda^{-1}\partial F_\lambda$ is holomorphic at $\lambda = 0$, where $F_\lambda^{-1}\partial F_\lambda$ is the $(1,0)$-part of $F_\lambda^{-1} dF_\lambda$.*

Then it is easy to see that $f \in \mathscr{H}_{H,c}$ admits an extended framing $F_\lambda : M \to \Lambda^{\sqrt{-1}HL(c)}_{\mathrm{hol}} G_c$ such that F_1 is a framing of f. Conversely, if $F_\lambda : M \to \Lambda^{\sqrt{-1}HL(c)}_{\mathrm{hol}} G_c$ is an extended framing with $F_\lambda(p_0) \in K_c$, then $\pi \circ F_1 \in \mathscr{H}_{H,c}$. We set

$$\mathscr{E}_{H,c} = \{F_\lambda : M \to \Lambda^{\sqrt{-1}HL(c)}_{\mathrm{hol}} G_c; F_\lambda \text{ is an extended framing, } F_\lambda(p_0) \in K_c\},$$

$$\mathscr{K}_c = C^\infty(M, K_c).$$

Then we have a bijective correspondence

$$\mathscr{H}_{H,c} \cong \mathscr{E}_{H,c}/\mathscr{K}_c$$

which maps $f \in \mathscr{H}_{H,c}$ to $\{F_\lambda\} \in \mathscr{E}_{H,c}/\mathscr{K}_c$, where $\mathscr{K}_c$ acts by point-wise multiplication on the right.

§3. Action of $\Lambda_I^{\varepsilon,\rho} G_c$ on extended framings.

For any $g \in \Lambda^{\varepsilon,\rho} G_c$ such that (i) $c = 0$, or (ii) $c > 0$, $(1 - \rho^2)\varepsilon^2 < 1$ or (iii) $c < 0$, $(1 + |\rho|)\varepsilon < 1$, we have a unique factorization by Lemma 1.2

$$g = g_E g_I,$$

where $g_E \in \Lambda_E^{\varepsilon,\rho} G_c$, $g_I \in \Lambda_{I,B}^{\varepsilon,\rho} G_c$. Then we define an action of $\Lambda_I^{\varepsilon,\rho} G_c$ on $\Lambda_E^{\varepsilon,\rho} G_c$ by

$$g \# h = (gh)_E,$$

where $g \in \Lambda_I^{\varepsilon,\rho} G_c$, $h \in \Lambda_E^{\varepsilon,\rho} G_c$.

PROPOSITION 3.1 ([**BP**, Proposition 2.9]). *Let $\rho = \sqrt{-1} HL(c)$, $g \in \Lambda_I^{\varepsilon,\rho} G_c$ and $F_\lambda : M \to \Lambda_{\mathrm{hol}}^{\rho} G_c$ be an extended framing. Define $g \# F_\lambda : M \to \Lambda_{\mathrm{hol}}^{\rho} G_c$ by*

$$(g \# F_\lambda)(p) = g \# (F_\lambda(p)),$$

for $p \in M$. Then

(i) *$g \# F_\lambda$ is also an extended framing.*

(ii) *If F_λ is based (that is, $F_\lambda \in \mathscr{E}_{H,c}$) then so is $g \# F_\lambda$.*

(iii) *If $k \in \mathscr{K}_c$ then*

$$g \# (F_\lambda k) = (g \# F_\lambda) \tilde{k}$$

with $\tilde{k} \in \mathscr{K}_c$.

Thus $\Lambda_I^{\varepsilon,\rho} G_c$ acts on $\mathscr{H}_{H,c} = \mathscr{E}_{H,c} / \mathscr{K}_c$.

Furthermore, we have

LEMMA 3.2. *Let $g \in \Lambda_I^{\varepsilon,\sqrt{-1}HL(c)} G_c$ and $F_\lambda \in \mathscr{E}_{H,c}$. If $\pi \circ F_\lambda$ is weakly conformal, so is $\pi \circ (g \# F_\lambda)$.*

PROOF. Write

$$g F_\lambda = pq,$$

where $p = g \# F_\lambda$, $q : M \to \Lambda_{I,B}^{\varepsilon,\sqrt{-1}HL(c)} G_c$. Then

$$p^{-1} dp = \mathrm{Ad}\, q (F_\lambda^{-1} dF_\lambda - q^{-1} dq).$$

Hence if we write $\alpha'_{\mathfrak{m}}$ as (2.5), the $(1,0)$-part of the $\mathfrak{m}_c$-part of $p^{-1} dp$ is

$$\begin{pmatrix} 0 & Qb' \\ -\mathrm{sign}(c)^t (Qb') & 0 \end{pmatrix},$$

where

$$q(0) = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}$$

with $Q \in B$. This completes the proof. □

LEMMA 3.3. *There exist Lie group isomorphisms*

(i) *$\Lambda_I^{\varepsilon,\rho} G_c \cong \Lambda_I^{\varepsilon,0} G_1$, if $c = 0$,*

(ii) *$\Lambda_I^{\varepsilon,\rho} G_c \cong \Lambda_I^{\varepsilon\sqrt{1-\rho^2},0} G_1$ if $c > 0$, $\varepsilon\sqrt{1-\rho^2} < 1$,*

(iii) *$\Lambda_I^{\varepsilon,\rho} G_c \cong \Lambda_I^{\varepsilon(1+|\rho|),0} G_1$ if $c < 0$, $\varepsilon(1+|\rho|) < 1$.*

PROOF. By the proof of Lemma 1.1 and Lemma 1.2, we have (ii) and (iii). We have only to show (i).

Case I: $\rho = 0$.

For $\xi \in \Lambda^{\varepsilon,0}\mathfrak{g}_c$, we have a Fourier series of A_ξ:

$$A_\xi(\lambda) = \sum_{n \in \boldsymbol{Z}} \iota(a_n)\lambda^n,$$

where $a_n \in \boldsymbol{C}^3$, $\lambda \in C_\varepsilon$. We set $\psi(\xi)(\lambda)$ by (1.3). Then ψ defines a well-defined Lie algebra isomorphism:

$$\psi : \Lambda^{\varepsilon,0}\mathfrak{g}_c \to \Lambda^{\varepsilon,0}\mathfrak{g}_1.$$

Hence we have the desired isomorphism.

Case II: $\rho \in \sqrt{-1}\boldsymbol{R}\backslash\{0\}$.

For $\xi \in \Lambda^{\varepsilon,\rho}\mathfrak{g}_c$, it is straightforward to see that we have Fourier series:

$$A_\xi(\lambda) = \iota(a_0) + \sum_{n \neq 0} \frac{\rho}{n}\iota(a_n)\lambda^n, \quad b_\xi(\lambda) = \sum_{n \in \boldsymbol{Z}} a_n\lambda^n,$$

where $a_n \in \boldsymbol{C}^3$, $\lambda \in C_\varepsilon$. We set $\psi(\xi)(\lambda)$ by

$$\psi(\xi)(\lambda) = \begin{pmatrix} \iota(a_0) + \sum_{n \neq 0} \frac{\rho}{2n}\iota(a_{2n})\lambda^{2n} & \sum_{n \in \boldsymbol{Z}} \frac{a_{2n+1}}{2n+1}\lambda^{2n+1} \\ -{}^t\sum_{n \in \boldsymbol{Z}} \frac{a_{2n+1}}{2n+1}\lambda^{2n+1} & 0 \end{pmatrix},$$

for $\lambda \in C_\varepsilon$. Then ψ defines a well-defined Lie algebra isomorphism:

$$\psi : \Lambda^{\varepsilon,\rho}\mathfrak{g}_c \to \Lambda^{\varepsilon,0}\mathfrak{g}_1.$$

Hence we have the desired isomorphism. □

If we define $\mathscr{G}_{H,c} = \Lambda_I^{\varepsilon,\sqrt{-1}HL(c)}G_c$, $\mathscr{G}_{H',c'} = \Lambda_I^{\varepsilon',\sqrt{-1}H'L(c')}G_{c'}$ for suitable ε, ε' with $0 < \varepsilon$, $\varepsilon' < 1$, then using Lemma 3.2, Lemma 3.3 and Proposition in the introduction, we obtain our main theorem.

References

[BG] M. J. Bergvelt and M. A. Guest, Actions of loop groups on harmonic maps, Trans. Amer. Math. Soc. **326** (1991), 861–886.

[BP] F. E. Burstall and F. Pedit, Dressing orbits of harmonic maps, Duke Math. J. **80** (1995), 353–382.

[DPW] J. Dorfmeister, F. Pedit and H. Wu, Weierstrass type representation of harmonic maps into symmetric spaces, preprint (1994).

[F] A. Fujioka, Harmonic maps and associated maps from simply connected Riemann surfaces into the 3-dimensional space forms, Tôhoku Math. J. **47** (1995), 431–439.

[GO1] M. A. Guest and Y. Ohnita, Group actions and deformations for harmonic maps, J. Math. Soc. Japan **45** (1993), 671–704.

[GO2] M. A. Guest and Y. Ohnita, Loop group actions on harmonic maps and their applications, Harmonic Maps and Integrable Systems, A. Fordy and J. C. Wood, eds., Aspects of Mathematics E23 (1994), 273–292.

[Mc] I. McIntosh, Global solutions of the elliptic 2D periodic Toda lattice, Nonlinearity **7** (1994), 85–108.

[M] J. W. Milnor, Remarks on infinite dimensional Lie groups, in: Relativity, Groups and Topology II, B.S. de Witt, R. Stora (editors) (1984), North-Holland, Amsterdam.

[OMYK] H. Omori, Y. Maeda, A. Yoshioka and O. Kobayashi, On Regular Fréchet-Lie Groups V, Tokyo J. Math. **6** (1983), 39–64.
[PS] A. N. Pressley and G. B. Segal, Loop groups, Oxford University Press, 1986.
[U] K. Uhlenbeck, Harmonic maps into Lie groups (Classical solutions of the chiral model), J. Differential Geom. **30** (1989), 1–50.

Atsushi Fujioka
Graduate School of Natural Science
and Technology, Kanazawa University
Kakuma-machi, Kanazawa, 920-1192
Japan,
E-mail: fujioka@kappa.s.kanazawa-u.ac.jp