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Boundedness of global solutions of one dimensional quasilinear degenerate parabolic equations

By Ryuichi SUZUKI

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1. Introduction.

Let $\Omega = (-L, L)$ be a bounded open interval in $\boldsymbol{R}$. In this paper we shall consider the one-dimensional Dirichlet problem

$$\partial_t \beta(u) = u_{xx} + f(u) \quad \text{in} \quad (x, t) \in \Omega \times (0, T) \tag{1.1}$$

$$u(\pm L, t) = 0 \quad \text{on } t \in (0, T) \tag{1.2}$$

$$u(x, 0) = u_0(x) \quad \text{on } x \in \Omega \tag{1.3}$$

where $\partial_t = \partial/\partial t$ and $\beta(v)$, $f(v)$ with $v \geq 0$ and $u_0(x)$ are nonnegative functions.

Equation (1.1) describes the combustion process in a stationary medium in which the thermal conductivity $\beta'(u)^{-1}$ and the volume heat source $f(u)$ are depending in a nonlinear way on the temperature $\beta(u) = \beta(u(x, t))$ of the medium.

Throughout this paper we assume

(A1) $\beta(v), f(v) \in C^\infty(\boldsymbol{R}_+) \cap C(\bar{\boldsymbol{R}}_+)$ where $\boldsymbol{R}_+ = (0, \infty)$ and $\bar{\boldsymbol{R}}_+ = [0, \infty)$, $\beta(v) > 0$, $\beta'(v) > 0$, $\beta''(v) \leq 0$ and $f(v) > 0$ for $v > 0$, $\lim_{v \to \infty} \beta(v) = \infty$, $f \circ \beta^{-1}(v)$ is locally Lipschitz continuous in $v \geq 0$.

(A2) $u_0(x) \geq 0$, $\in C(\bar{\Omega})$ and $u_0(\pm L) = 0$ (compatibility condition).

With these conditions above Dirichlet problem has a unique local solution $u(x, t) \geq 0$ (in time) which satisfies (1.1)$\sim$(1.3) in a weak sense (e.f.- Aronson-Crandall-Peletier [3], Bertsch-Kersner-Peletier [4], Ladyzenskaja, et al. [11], Oleinik et al. [13]). The definition of "weak" solutions is given in Section 2.

Let $s(x)$ be the principal eigensolution of $-\partial^2/\partial x^2$ in $(-L, L)$ with Dirichlet boundary conditions (s is normalized: $s > 0$ in $\Omega, \int_\Omega s(x)\,dx = 1$) and λ be the first eigenvalue of this problem. We further assume the almost necessary condition to raise the blow-up (see Imai-Mochizuki [9]).

(A3) There exist a continuous function $g(\xi)$ of ξ and a $\xi_1 \geq 0$ such that

$$g(\xi) \leq f(\xi) - \lambda\xi \quad \text{in } \xi \geq 0, \tag{1.4}$$

$$\Gamma \equiv g \circ \beta^{-1} \text{ is convex in } (\beta(0), \infty), \tag{1.5}$$

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$$g(\xi) > 0, \quad \int_\xi^\infty \frac{\beta'(\eta)}{g(\eta)}\, d\eta < \infty \quad \text{if} \quad \xi > \xi_1 \tag{1.6}$$

and $\Gamma(\rho)$ is nondecreasing in $\rho > \beta(\xi_1)$.

Remark 1.1. Equation

$$(u^{1/m})_t = u_{xx} + \mu u^{p/m}\{\log(u+2)\}^b$$
$$(m \geq 1, p > 1, \mu > 0, b \geq 0) \tag{1.7}$$

satisfies (A1)$\sim$(A3), if $p/m > 1$ or $p = m > 1, \mu > \lambda$ or $p = m = 1, b > 1$.

Under these conditions (A1)$\sim$(A3), we have already known that if the initial data $u_0(x)$ is large enough, then the solution of (1.1)$\sim$(1.3) blows up in finite time: Set

$$J(t) = \int_\Omega \beta(u(x,t))s(x)\, dx \quad \text{for } t > 0. \tag{1.8}$$

Proposition 1.2. (Imai-Mochizuki [9]) *Assume* (A1)$\sim$(A3). *Let* $u(x,t)$ *be a weak solution of* (1.1)$\sim$(1.3). *Then, if* $J(0) > \beta(\xi_1)$, $u(x,t)$ *blows up in finite time.*

If the initial data $u_0(x)$ is small enough, then the solution $u(x,t)$ of (1.1)$\sim$(1.3) exists globally in time and stays bounded as $t \to \infty$ (see Lemma 5.3), provided that $f(\xi)$ near $\xi = 0$ satisfies inequality

$$\inf_{\xi \geq 0}\{4LF(\sqrt{2L}\xi) - \xi^2\} < 0 \tag{1.9}$$

where $F(\xi) = \int_0^\xi f(\eta)\, d\eta$.

We are now interested in the problem whether or not the following third case exists:

(P) $u(x,t)$ exists globally but is not uniformly bounded in $\Omega \times (0,\infty)$.

In the semilinear case $\beta(\xi) = \xi$, Fila [7] showed that if we assume

(B) there exist constants $\varepsilon > 0, C_1, C_2, C_3 > 0$ such that

$$\xi f(\xi) + C_1 \geq 2(1+\varepsilon)F(\xi) \quad \text{for } \xi \geq 0 \tag{1.10}$$

with $F(\xi) \geq C_2\xi^{2+\varepsilon} - C_3$ where $F(\xi) = \int_0^\xi f(\eta)\, d\eta$, then the global solution $u(x,t)$ of (1.1)$\sim$(1.3) stays bounded as $t \to \infty$, namely, (P) does not occur. This result can be extended easily to more general $\beta(\xi)$. Then, the condition is changed to (B) with some lower restrictions on the growth order of $f(\xi)$. For example, if $\beta(\xi) = \xi^{1/m}$, the condition becomes (B) with $F(\xi) \geq C_2\xi^{1+1/m+\varepsilon} - C_3$, and under this condition equation (1.7) satisfies $p/m > 1$.

Here, we note that Fila [7] also showed the similar results in the quasilinear case. But, he needed some upper restrictions on the growth order of $f(\xi)$. We also note that in the higher dimension case $N > 1$ he also got the almost complete results for the boundedness of the global solutions (see Fila [7]).

REMARK 1.3. We do not know whether or not the above conditions are stronger than (A3) in general. But, if we restrict ourselves to (1.7), above conditions are stronger than (A3).

On the other hand, if we assume

$$\text{(C)} \qquad \lim_{\xi\to\infty} \sup \frac{F(\xi)}{\xi^2\beta(\xi)} = 0$$

and in the degenerate case $\beta'(0) = \infty$ we further assume that

$$\text{(C}'\text{)} \qquad \begin{aligned} \sigma_{a_1}u_0(x) &\geq u_0(x) \quad \text{for} \quad (x,t) \in [-L, a_1] \\ \sigma_{a_2}u_0(x) &\geq u_0(x) \quad \text{for} \quad (x,t) \in [a_2, L] \end{aligned}$$

for some $-L < a_1 < a_2 < L$ where $\sigma_a u_0(x) \equiv u_0(2a - x)$, we can easily see that (P) does not occur by using the energy estimates and the following corollary which is a direct result from Proposition 1.2.

COROLLARY 1.4. *Assume* (A1)$\sim$(A3). *Let $u(x,t)$ be a global weak solution of* (1.1)$\sim$(1.3) *in time. Then*

$$J(t) \leq \beta(\xi_1) \quad \textit{for all } t \geq 0. \tag{1.11}$$

If $1 \leq p/m < 1 + 1/m$, then (1.7) satisfies condition (C).

Thus, conditions (B) and (C) give some lower and upper restrictions on the growth order of $f(\xi)$ respectively. Our aim in this paper is to remove these restrictions, namely, we show the boundedness of any global weak solution to (1.1)$\sim$(1.3) with the growth order of $f(\xi)$ which satisfies only the blow-up conditions (A3):

THEOREM 1.5. *Assume* (A1)$\sim$(A3). *Let $u(x,t)$ be a global weak solution in time to* (1.1)$\sim$(1.3). *Then, $u(x,t)$ is uniformly bounded in $\Omega \times [0,\infty)$.*

In order to prove this theorem, we need the property of the zero set of $u_x(\cdot,t)$ and show the existence of $\lim_{t\uparrow T} \operatorname{sgn} u_x(x,t)$ which was appeared in Chen-Matano [**6**] and R. Suzuki [**15**]. Using these properties and the energy methods we prove the theorem. Then, it is important to study the property of $u(x,t)$ on the zero set of $u_x(\cdot,t)$.

The higher dimension case $N > 1$ has been discussed by many authors, e.g., Cazenave-Lions [**5**], Giga [**8**] for the semilinear case and Ni-Sacks-Tavantzis [**12**], Fila [**7**] for the semilinear and quasilinear cases.

The rest of the paper is organized as follows. In the next Section 2 we state the definition of a weak solution and give the fundamental tools and lemmas. In Section 3 we show the existence of $\lim_{t\uparrow T} \operatorname{sgn} u_x(x,t) (\neq 0)$, and in Section 4 we study the property of $u(x,t)$ on a local minimum (or maximum) curve which is a set (x,t) satisfying $u_x(x,t) = 0$. In Section 5, we show the boundedness of the solution near the boundary by using the energy methods. Finally, in Section 6 we prove Theorem 1.5.

2. Definitions and preliminaries.

In this section we define weak solutions of (1.1)$\sim$(1.3) and state the fundamental tools and lemmas which are used later.

Throughout this section we assume (A1) (A2).

DEFINITION 2.1. Let G be an open interval in $\boldsymbol{R}$. By a weak solution of equation (1.1) in $G \times (0, T)$, we mean a function $u(x, t)$ such that

1) $u(x, t) \geq 0$ in $\bar{G} \times [0, T)$ and $\in BC(\bar{G} \times [0, \tau])$ (bounded continuous) for each $0 < \tau < T$.

2) For any bounded open interval $D = (\alpha, \beta)$ in G, $0 < \tau < T$ and nonnegative $\varphi(x, t) \in C^2(\bar{D} \times [0, T))$ which vanishes on the boundary ∂D,

$$\begin{aligned} &\int_D \beta(u(x,\tau))\varphi(x,\tau)\,dx - \int_D \beta(u(x,0))\varphi(x,0)\,dx \\ &\quad = \int_0^\tau \int_D \{\beta(u)\partial_t\varphi + u\varphi_{xx} + f(u)\varphi\}\,dx\,dt - \int_0^\tau u\varphi_x\,dt \Big|_{x=\beta}^{x=\alpha}. \end{aligned} \tag{2.1}$$

By a weak solution of the initial boundary value problem for (1.1) in $G \times (0, T)$, we mean a weak solution of (1.1) satisfying the given initial and boundary values on the parabolic boundary of $G \times (0, T)$.

DEFINITION 2.2. A function $u(x, t)$ defined in $G \times (0, T)$ is called a weak super-(or sub-) solution of (1.1) if $u(x, t)$ satisfies 1) 2) of Definition 2.1 with equality in (2.1) replaced by $\geq$ (or $\leq$).

PROPOSITION 2.3 (comparison principle). *Let u (or v) be a super-solution (or sub-solution) of* (1.1) *in $G \times (0, T)$. If $u \geq v$ on the parabolic boundary of $G \times (0, T)$, then we have $u \geq v$ in the whole $\bar{G} \times [0, T)$.*

PROOF. see Aronson-Crandall-Peletier [**3**] and Bertsch-Kersner-Peletier [**4**]. □

PROPOSITION 2.4 (positivity and smoothness principle). *Let u be a solution of* (1.1) *in $G \times (0, T)$. Let $u(\bar{x}, \bar{t}) > 0$ for some $(\bar{x}, \bar{t}) \in G \times (0, T)$, then u is a classical solution in a neighborhood W of $(\bar{x}, \bar{t})$ and*

$$u(\bar{x}, t) > \rho(t) > 0 \text{ in } t \in [\bar{t}, T) \tag{2.2}$$

where $\rho(t), t \geq \bar{t}$ solves the initial value problem

$$\rho' = \frac{-\lambda\rho}{\beta'(\rho)} \text{ in } (\bar{t}, \infty) \text{ with } \rho(\bar{t}) \in (0, u(\bar{x}, \bar{t})). \tag{2.3}$$

PROOF. See Lemma 2.1 in K. Mochizuki-R. Suzuki [**12**] and Lemma 2.3 in R. Suzuki [**15**]. □

Finally we prove the next lemma for a global weak solution of (1.1)$\sim$(1.3).

LEMMA 2.5. *Let $u(x,t)$ be a global weak solution of* (1.1)$\sim$(1.3). *Then, there exists* $T > 0$ *such that*

$$u(x,t) > 0 \quad \textit{for} \quad (x,t) \in \Omega \times (T,\infty). \tag{2.4}$$

PROOF. Using the comparison principle (Proposition 2.3), we may assume that the initial data $u_0(x) \geq 0$ satisfies the following conditions:

$$u_0(x) > 0, x \in (a_1,a_2), \quad = 0, x \notin (a_1,a_2) \text{ for some } -L < a_1 < a_2 < L \tag{2.5}$$

and

$$u_0(x) \text{ has the unique local maximum point } x = (a_1+a_2)/2 \text{ in } (a_1,a_2). \tag{2.6}$$

Set

$$\xi_1(t) = \inf\{x \in \Omega \mid u(x,t) > 0\}$$

and

$$\xi_2(t) = \sup\{x \in \Omega \mid u(x,t) > 0\}$$

for each $t > 0$.
It follows from Proposition 2.3 and Proposition 2.4 that

$$(\xi_1(t),\xi_2(t)) = \{x \in \Omega \mid u(x,t) > 0\} \quad \text{for each } t > 0 \tag{2.7}$$

and

$$-\xi_1(t) \text{ and } \xi_2(t) \text{ are monotone nondecreasing.} \tag{2.8}$$

Assume contrary that

$$\xi_1(t) > -L \quad \text{for all } t \geq 0 \tag{2.9}$$

and put

$$b = \lim_{t \uparrow \infty} \xi_1(t) \geq -L.$$

First, we shall compare u and $\sigma_{b+\delta}u \equiv u(2(b+\delta)-x,t)$ (which is called a reflection of u with respect to $b+\delta$) for small $\delta > 0$. Choosing $0 < \delta < |a_1 - a_2|/4$, we get

$$\sigma_{b+\delta}u_0(x) \geq u_0(x), \quad x \in [b, b+\delta]. \tag{2.10}$$

We also see that $\sigma_{b+\delta}u$ and u satisfy the same equation (1.1) and $\sigma_{b+\delta}u \geq u$ on the parabolic boundary of $(b,b+\delta) \times (0,\infty)$. Hence, applying Proposition 2.3 to u and $\sigma_{b+\delta}u$, we obtain $\sigma_{b+\delta}u \geq u$ in the whole domain of $(b,b+\delta) \times (0,\infty)$. Namely, if we set $w = \sigma_{b+\delta}u - u$, then $w \geq 0$ in $[b,b+\delta] \times [0,\infty)$. It follows from $w(b+\delta,t) = 0$ that

$$u_x(b+\delta,t) = -\frac{1}{2}\, w_x(b+\delta,t) \geq 0 \quad \text{for } t \geq 0. \tag{2.11}$$

On the other hand, by (2.9) we get

$$u_x(b,t) = 0. \tag{2.12}$$

Thus, we see that $u(x,t)$ is a super-solution of the Neumann problem of (1.1) in $(b, b+\delta) \times (0, \infty)$.

It follows from the definition of b that there exists $t_0 > 0$ such that

$$u(b+\delta, t_0) > 0. \tag{2.13}$$

Let $v(x,t)$ be the weak solution to the Neumann problem

$$\begin{cases} \beta(v)_t - v_{xx} = 0 & (x,t) \in (b, b+\delta) \times (t_0, \infty) \\ v_x(b,t) = v_x(b+\delta, t) = 0 & t > t_0 \\ v(x,t_0) = u(x,t_0) & x \in (b, b+\delta). \end{cases} \tag{2.14}$$

Since $v(x,t)$ is a sub-solution of the Neumann problem of (1.1), by the comparison theorem we get

$$u(x,t) \geq v(x,t) \quad \text{in } (x,t) \in (b, b+\delta) \times (t_0, \infty). \tag{2.15}$$

Hence, noting

$$v(x,t) \longrightarrow \frac{1}{\delta} \int_b^{b+\delta} u(x,t_0)\, dx \equiv \tilde{v} > 0 \quad (\text{as } t \to \infty)$$

uniformly with respect to $x \in [b, b+\delta]$ (see Alikakos-Rostamian [**1**]), we obtain for some $t_1 > t_0$

$$u(x,t) \geq \frac{1}{2} \tilde{v} > 0 \quad \text{for} (x,t) \in [b, b+\delta] \times [t_1, \infty).$$

This is a contradiction to $u(b,t) = 0$ for $t \geq 0$ and so we have for some $T' > 0$,

$$\xi_1(T') = -L.$$

By the similar methods, we also get

$$\xi_2(T'') = L$$

for some $T'' > 0$.

Therefore, if we put $T = \max\{T', T''\}$, we obtain the assertions of Lemma 2.5. The proof is complete. □

3. The existence of $\lim_{t \uparrow T} \operatorname{sgn} u_x(x, t)$.

Throughout this section we assume (A1)(A2) and the following condition on the initial data $u_0(x)$:

$$u_0(x) > 0 \quad \text{for } x \in \Omega. \tag{3.1}$$

The purpose of this section is to prove the next proposition.

Proposition 3.1. *Assume* (A1)(A2) *and* (3.1). *Let* $u(x,t)$ *be a weak solution of* (1.1)$\sim$(1.3) *in* $\Omega \times (0,T)$ *(included* $T = \infty$*). Then, for any* $a \in \Omega$ *with* $a \neq 0$, *there* *exists*

$$\lim_{t \uparrow T} \operatorname{sgn} u_x(x, t) \neq 0. \tag{3.2}$$

This proposition plays an important role to prove Theorem 1.5 and was shown by Chen-Matano [**6**] in the semilinear case (also for the Neumann and periodic problems), and was shown by R. Suzuki [**15**] in the degenerate case under the technical condition (C′) in Section 1.

In order to show Proposition 3.1 we need some notations and preliminary lemmas (see Angenent [**2**], Chen-Matano [**6**] and R. Suzuki [**15**]).

NOTATION 3.2. Let $w(x)$ be a continuous real value function on K where K is a bounded closed interval in $\boldsymbol{R}$. We define the *nodal number* of w by

$$\nu_K(w) = \text{the number of points } x \in K \text{ with } w(x) = 0.$$

This defines a functional $\nu_K : C(K) \to \boldsymbol{N} \cup \{0\} \cup \{\infty\}$.

DEFINITION 3.3. We say that $w \in C^1(K)$ poses only simple zeroes if $w'(x) \neq 0$ for any $x \in K$ such that $w(x) = 0$. The set of all such functions is denoted by $\Sigma(K)$.

LEMMA 3.4. (Angenent [2]). *Let $p(x,t), q(x,t), r(x,t)$ be locally bounded continuous functions in $[a,b] \times (t_0, T)$ with $p_{xx}, p_{xt}, p_{tt}, p_x, p_t, q_x, q_t$, all locally bounded continuous. Furthermore, let $p(x,t) > 0$ and let $w(x,t)$ be a classical solution of*

$$w_t = p(x,t) w_{xx} + q(x,t) w_x + r(x,t) w \quad (x,t) \in [a,b] \times (t_0, T). \tag{3.3}$$

Assume that $w(a,t) \neq 0$ and $w(b,t) \neq 0$ for any $t \in (t_0, T)$. Then, putting $K = [a,b]$, we have

(i) *$\nu_K(w(\cdot, t))$ is finite for any $t \in (t_0, T)$ and is monotone nonincreasing in t;*

(ii) *If (x_0, t_1) is a multiple zero of w, then $\nu_K(w(\cdot, t_2)) > \nu_K(w(\cdot, t_3))$ for all $t_0 < t_2 < t_1 < t_3 < T$.*

(iii) *There exists a strictly decreasing sequence of points $\{t_k\} \subset (t_0, T)$ such that $\{t_k\} \downarrow t_0$ and $w(\cdot, t) \in \Sigma(K)$ for any $t \in (t_0, T) \backslash \{t_k\}$.*

Let $\{t_n\}$ be a sequence in (t_0, T) such that $t_n < t_{n+1}$ and $t_n \uparrow T$ as $n \to \infty$. Let $\{K_n\}_{n=0}^{\infty}$ be a sequence of bounded closed intervals in $\Omega = (-L, L)$ such that

$$K_n \subsetneq K_{n+1}. \tag{3.4}$$

Set

$$\tilde{Q} = \cup_{n=1}^{\infty} K_n \times [t_n, t_{n+1}) \bigcup K_0 \times (t_0, t_1) \tag{3.5}$$

and

$$K(t) = K_n \quad \text{if } t \in [t_n, t_{n+1}). \tag{3.6}$$

Then, we can extend the above lemma as follows:

LEMMA 3.5. *Let* $p(x,t), q(x,t), r(x,t)$ *be as in Lemma* 3.4 *with* $[a,b]\times(t_0,T)$ *replaced by* $\tilde{Q}$. *Let* $w(x,t)$ *be a classical solution of* (3.3) *in* $\tilde{Q}$. *Then, if* $w(x,t)\neq 0$ *on the boundary of* $\tilde{Q}$ *in* $\boldsymbol{R}\times(t_0,T)$,

(i) $v_{K(t)}(w(\cdot,t))$ *is finite for any* $t\in(t_0,T)$ *and is monotone nonincreasing in* t;

(ii) *If* (x_0,t_1) *is a multiple zero of* w, *then* $v_{K(t_2)}(w(\cdot,t_2))>v_{K(t_3)}(w(\cdot,t_3))$ *for all* $t_0<t_2<t_1<t_3<T$.

(iii) *There exists a strictly decreasing sequence of points* $\{t_k\}\subset(t_0,T)$ *such that* $\{t_k\}\downarrow t_0$ *and* $w(\cdot,t)\in\Sigma(K(t))$ *for any* $t\in(t_0,T)\backslash\{t_k\}$.

PROOF. Noting $v_{K_n}(w(\cdot,t_{n+1}))=v_{K_{n+1}}(w(\cdot,t_{n+1}))$, by Lemma 3.4 we can show Lemma 3.5 easily. □

PROOF OF PROPOSITION 3.1. We shall show this proposition in case $a>0$.

By (3.1) and Proposition 2.4, we get

$$u(x,t)>0\quad \text{for}(x,t)\in\Omega\times(0,T). \tag{3.7}$$

Set $w=v-u$ where $v=\sigma_a u\equiv u(2a-x,t)$ (which is called a reflection of u with respect to a). Then, since $w(L,t)=\sigma_a u(L,t)-u(L,t)=\sigma_a u(2a-L,t)>0$ in $[0,T)$, for any sequence $\{t_n\}\uparrow T$ there exists $\{\delta_n\}\downarrow 0$ such that

$$w(L-\delta,t)=-w(2a-L+\delta,t)>0\quad\text{for all }(\delta,t)\in(0,\delta_n]\times[0,t_n]. \tag{3.8}$$

Hence, putting $K_n=[2a-L+\delta_{n+1},L-\delta_{n+1}]$ and $\tilde{Q}=\cup_{n=1}^{\infty}K_n\times[t_n,t_{n+1})\bigcup K_0\times(t_0,t_1)$ we have

$$w\neq 0\quad\text{on }\partial\tilde{Q} \tag{3.9}$$

where $\partial\tilde{Q}$ is the boundary of $\tilde{Q}$ in $\boldsymbol{R}\times(t_0,T)$.

On the other hand, since u and $v=\sigma_a u$ satisfy the same equation (1.1) in $\tilde{Q}$, we see that w satisfies equation

$$w_t=\frac{1}{\overline{\beta'(u)}}\{w_{xx}+(\bar{f}-\overline{\beta'}v_t)w\}\quad\text{in }\Omega\times(0,T) \tag{3.10}$$

where $\bar{\varphi}=\bar{\varphi}(u,v)=\int_0^1\varphi'(\theta v+(1-\theta)u)\,d\theta$.

Applying Lemma 3.5 (iii) to w in $\tilde{Q}$, we get that there exists $\tau\in(t_0,T)$ such that

$$w(\cdot,t)\in\Sigma(K(t))\quad\text{for }t\geq\tau, \tag{3.11}$$

where $K(t)=K_n$ if $t\in[t_n,t_{n+1})$. It follows from $w(a,t)=0$ that

$$u_x(a,t)=-\frac{1}{2}w_x(a,t)\neq 0\quad\text{for }t\geq\tau.$$

Thus, by virtue of the continuity of $u_x(a,t)$ we obtain the existence of $\lim_{t\uparrow T}\operatorname{sgn}u_x(x,t)(\neq 0)$. The proof is complete. □

4. Solutions on local minimum (or maximum) curves.

In this section we assume (A1)$\sim$(A3) and (3.1). Let $u(x,t)$ be a global weak solution of (1.1)$\sim$(1.3). Then, by (3.1) we see that $u(x,t) > 0$ in $\Omega \times (0,\infty)$. Our purpose in this section is to study the property of $u(x,t)$ on a local minimum (or maximum) curve which is defined as follows:

DEFINITION 4.1. Let $\ell(t) : [t_0, t_1] \to \Omega$ be a piecewise C^1-function for some $t_1 > t_0 \geq 0$. When for any $t \in [t_0, t_1]$ $\ell(t)$ is a local maximum point of $u(x,t)$ in Ω, we say this continuous curve $\ell(t)$ in $[t_0, t_1]$ a local maximum curve. When for any $t \in [t_0, t_1]$ $\ell(t)$ is a local minimum point of $u(x,t)$ in Ω, we say this continuous curve $\ell(t)$ in $[t_0, t_1]$ a local minimum curve.

Let us begin with the following lemma.

LEMMA 4.2. *For any $t > 0$, the zero set of $u_x(\cdot,t)$ in each compact set of Ω is finite.*

PROOF. For any $t \in (0,T)$, we can choose small $\delta_\pm > 0$ such that

$$u_x(-L+\delta_-,t) \neq 0 \quad \text{and} \quad u_x(L-\delta_+,t) \neq 0.$$

If follows from Lemma 3.4 (i) that $v_{[-L+\delta_-,L-\delta_+]}(u_x(\cdot,t))$ is finite. The proof is complete. $\square$

LEMMA 4.3. *Suppose for the initial data $u_0(x)$ that $u_0(x) \in C^2(\Omega)$, the zero set of $u_{0,x}$ in each compact set of Ω is finite. Then if $x_0 \in \Omega$ is a local maximum (or minimum) point of $u(x,t_1)$ in Ω for some $t_1 \in (0,\infty)$, there exists a local maximum (or minimum) curve $\ell(t)$ in $[0,t_1]$ of $u(x,t)$ satisfying $\ell(t_1) = x_0$.*

PROOF. Noting that $f \circ \beta^{-1}(v)$ is locally Lipschitz continuous in $v \geq 0$, this lemma follows from Lemma 4.2, Angenent's theorem (Lemma 3.4) and the implicit function theorem for $u_x(x,t)$. $\square$

LEMMA 4.4. *Let $\ell(t)$ be a local minimum curve of $u(x,t)$ on $[0,t_1]$. Then*

$$\frac{\partial}{\partial t}\beta(u(\ell(t),t)) \geq f(u(\ell(t),t)) \quad \text{for } t \in [0,t_1]. \tag{4.1}$$

PROOF. Noting $u_x(\ell(t),t) = 0$ and $u_{xx}(\ell(t),t) \geq 0$, we obtain (4.1). $\square$

LEMMA 4.5. *Let $\ell(t)$ in $[0,t_1]$ be as in Lemma 4.4. Then, there exists a decreasing function $C(t)$ in $t \in [0,\infty)$ which is independent of $\ell(t)$ such that*

$$u(\ell(0),0) \leq C(t_1) \tag{4.2}$$

and

$$\lim_{t\to\infty} C(t) = 0. \tag{4.3}$$

PROOF. By (4.1), we obtain

$$\frac{\beta'(u(\ell(t),t))}{f(u(\ell(t),t))}\frac{\partial}{\partial t}u(\ell(t),t) \geq 1. \tag{4.4}$$

Putting $G(v) = \int_v^\infty \frac{\beta'(\xi)}{f(\xi)}\,d\xi < \infty$ and integrating (4.4) over $[0,t_1]$, we get

$$G(u(\ell(0),0)) - G(u(\ell(t_1),t_1)) \geq t_1. \tag{4.5}$$

Since $G(v)$ is a decreasing function, we have

$$u(\ell(0),0) \leq G^{-1}(t_1).$$

If $f(0) > 0$ then $t_1 \leq G(0)$ and if $f(0) = 0$ then $\lim_{t\to\infty} G^{-1}(t) = 0$ by (A1). Hence, if we set $C(t) = G^{-1}(t)$ when $0 \leq t \leq G(0)$ and if we set $C(t) = 0$ when $t > G(0)$, then we get (4.2) and (4.3). The proof is complete. □

LEMMA 4.6. *Let $u_0(x)$ be as in Lemma* 4.3 *and $u(x,t)$ be a global weak solution of* $(1.1)\sim(1.3)$. *Suppose*

$$u_x(\pm a_1, t) \neq 0 \qquad \text{for all } t \geq 0 \tag{4.6}$$

for some $a_1 \in (0,L)$. Then, there exists $T > 0$ such that for any $t \geq T$ $u(x,t)$ does not have a local minimum point in $(-a_1,a_1)$.

PROOF. By virtue of Lemma 4.5 we can choose $T > 0$ such as

$$0 < C(T) < \min_{x\in[-a_1,a_1]} u_0(x). \tag{4.7}$$

We assume contrary that $x_0 \in (-a_1,a_1)$ is a local minimum point of $u(x,t_1)$ in $(-a_1,a_1)$ for some $t_1 \geq T$. Then, it follows from Lemma 4.3 that there exists a local minimum curve $\ell(t)$ in $[0,t_1]$ such that $\ell(t_1) = x_0$. Noting (4.6), we have $\ell(0) \in (-a_1,a_1)$. Therefore, we see from Lemma 4.5 that

$$\min_{x\in[-a_1,a_1]} u_0(x) \leq u(\ell(0),0) \leq C(t_1) \leq C(T).$$

This is a contradiction to (4.7) and so we get the assertions of Lemma 4.6. The proof is complete. □

Finally, we further assume

$$J(t) \equiv \int_\Omega \beta(u(x,t))s(x)\,dx \leq M \quad \text{for all } t \geq 0 \tag{4.8}$$

for some $M > 0$ (see (1.8)), and state the next key lemma for the proof of Theorem 1.5.

LEMMA 4.7. *Assume assumption* (4.8). *Let $a_1 \in (0,L)$ and let $\ell(t)$ be a local maximum curve from $[0,T]$ to $(-L,-a_1)$ (or $(-a_1,a_1)$ or (a_1,L)). Then, there exist $d > 0$ and $R > 0$ which are independent of $\ell(t)$ and T such that if $nd \leq T$ for some $n \in N$*

then

$$\inf_{t\in[(i-1)d,id)} u(\ell(t),t) < R \tag{4.9}$$

for each $i = 1,\cdot,\cdot,\cdot,n$.

PROOF. We shall only prove this lemma in case $\ell(t) \in (a_1, L)$.

Put $b_1 = (L + a_1)/2$. Noting $\lim_{v\to\infty} \beta(v) = \infty$ we can choose large $R > 0$ such as

$$(b_1 - a_1)s(-b_1)\beta\left(\frac{R}{2L}(L - b_1)\right) \geq 3M. \tag{4.10}$$

Let $v(x,t)$ be a nonnegative weak solution of the initial boundary value problem

$$\begin{cases} \beta(v)_t = v_{xx}, & x \in \Omega, t > 0 \\ v(-L,t) = 0, v(L,t) = R, & t > 0 \\ v(x,0) = 0 & x \in \Omega. \end{cases} \tag{4.11}$$

Then, we have already known that $v(x,t) \leq R$ for $(x,t) \in \Omega \times (0,\infty)$ and $v(x,t)$ goes to the stationary solution $v_0(x) \equiv (R/2L)(x + L)$ as $t \to \infty$ uniformly in $x \in \Omega$ (see [9]), and so

$$\int_{-b_1}^{-a_1} \beta(v(x,t))s(x)\,dx \to \int_{-b_1}^{-a_1} \beta(v_0(x))s(x)\,dx \quad \text{as } t \to \infty. \tag{4.12}$$

Therefore, since

$$\int_{-b_1}^{-a_1} \beta(v_0(x))s(x)\,dx \geq (b_1 - a_1)s(-b_1)\beta\left(\frac{R}{2L}(L - b_1)\right) \geq 3M, \qquad (\text{by}(4.10))$$

there exists $t_1 > 0$ such that

$$\int_{-b_1}^{-a_1} \beta(v(x,t))s(x)\,dx \geq 2M \quad \text{for all } t \geq t_1. \tag{4.13}$$

Set $d = t_1$ and assume $nd \leq T$ for some $n \in \boldsymbol{N}$. Suppose contrary that for some $1 \leq i \leq n$

$$\inf_{t\in[(i-1)d,id)} u(\ell(t),t) \geq R. \tag{4.14}$$

Put $v_i(x,t) = v(x,t-(i-1)d)$ and $Q = \bigcup_{(i-1)d\leq t\leq id}(-L,\ell(t)) \times \{t\}$. Then, noting $v_i(x,t)$ is a sub-solution of (1.1) in Q and $u \geq v_i$ on the parabolic boundary of Q, we get $u \geq v_i$ in Q by the comparison theorem. Namely,

$$u(x,id) \geq v_i(x,id) = v(x,d) \quad x \in (-L,\ell(id)). \tag{4.15}$$

Hence, it follows from (4.13) that

$$\begin{aligned} \left.\int_\Omega \beta(u)s(x)\,dx\right|_{t=id} &\geq \int_{-L}^{\ell(id)} \beta(v(x,d))s(x)\,dx \\ &\geq \int_{-b_1}^{-a_1} \beta(v(x,d))s(x)\,dx \geq 2M. \end{aligned}$$

This is a contradiction to (4.8) and so we get (4.9). The proof is complete. □

5. Boundedness of the solution near the boundary.

Throughout this section we assume (A1)(A2) and (3.1), and we further assume the following condition for the initial data $u_0(x)$:

$$\int_{\Omega} \{u_{0,x}(x)\}^2 \, dx < \infty. \tag{5.1}$$

Let $u(x,t)$ be a weak solution of (1.1)$\sim$(1.3) and let $\ell(t)$ be a local minimum curve of $u(x,t)$ from $[0,T]$ to (a_1, L) for some $a_1 \in (0,L)$. Set $\delta = L - a_1 > 0$. We shall show that if δ is small enough then $u(x,t) \le C$ in $\bigcup_{0 \le t \le T} (\ell(t), L) \times \{t\}$ for some positive constant C which is independent of T and $\ell(t)$. For this aim we need the energy estimates for $u(x,t)$ as follows:

LEMMA 5.1. *For $0 \le t' \le t \le T$, we have*

$$\begin{aligned} &\int_{t'}^{t} \int_{\ell(s)}^{L} \{B(u)_t\}^2 \, dx \, ds + \frac{1}{2} \int_{\ell(t)}^{L} u_x{}^2 \, dx \Big|_{t=t'}^{t} \\ &\quad \le \int_{\ell(t)}^{L} F(u) \, dx \Big|_{t=t'}^{t} + F(u(\ell(t),t))(\ell(t) - a_1) \Big|_{t=t'}^{t} \end{aligned} \tag{5.2}$$

where $B(\xi) = \int_0^\xi \sqrt{\beta'(\eta)} \, d\eta$ and $F(\xi) = \int_0^\xi f(\eta) \, d\eta$.

PROOF. We shall prove (5.2) for a classical solution of (1.1)$\sim$(1.3). Then, by the limit procedure we can also prove (5.2) for a weak solution of (1.1)$\sim$(1.3).

Let $u(x,t)$ be a classical solution of (1.1)$\sim$(1.3). Put $Q_{t,t'} = \bigcup_{t' \le s \le t} (\ell(s), L) \times \{s\}$. Multiplying u_t to the both sides of (1.1) and integrating over $Q_{t,t'}$, we get

$$\begin{aligned} &\int_{t'}^{t} \int_{\ell(s)}^{L} \beta(u)_s u_s \, dx \, ds + \int_{t'}^{t} \int_{\ell(s)}^{L} \left\{ \frac{1}{2} (u_x{}^2) \right\}_s dx \, ds \\ &\quad = \int_{t'}^{t} \int_{\ell(s)}^{L} F(u)_s \, dx \, ds. \end{aligned} \tag{5.3}$$

Here, we used the fact that $u_x = 0$ on $x = \ell(t)$.

We estimate the both sides of (5.3). Since $u_x = 0$ on $x = \ell(t)$, we have

$$\begin{aligned} \int_{\ell(s)}^{L} \left\{ \frac{1}{2} (u_x)^2 \right\}_s dx &= \frac{1}{2} u_x{}^2(\ell(s), s) \ell'(s) + \frac{\partial}{\partial s} \int_{\ell(s)}^{L} \frac{1}{2} (u_x)^2 \, dx \\ &= \frac{\partial}{\partial s} \int_{\ell(s)}^{L} \frac{1}{2} (u_x)^2 \, dx. \end{aligned}$$

Using the fact that $u(\ell(s),s)_s \ge 0$ and integrating by parts we get

$$\int_{t'}^{t}\int_{\ell(s)}^{L}\frac{\partial}{\partial s}F(u)\,dx\,ds=\int_{t'}^{t}F(u(\ell(s),s))\ell'(s)\,ds+\int_{t'}^{t}\frac{\partial}{\partial s}\int_{\ell(s)}^{L}F(u)\,dx\,ds$$
$$=F(u(\ell(t),t))(\ell(t)-a_1)\Big|_{t=t'}^{t}-\int_{t'}^{t}F(u(\ell(s),s))_s(\ell(s)-a_1)\,ds$$
$$+\int_{t'}^{t}\frac{\partial}{\partial s}\int_{\ell(s)}^{L}F(u)\,dx\,ds$$
$$\leq F(u(\ell(t),t))(\ell(t)-a_1)\Big|_{t=t'}^{t}+\int_{\ell(t)}^{L}F(u)\,dx\Big|_{t=t'}^{t}.$$

Therefore, combining these estimates and noting the relation $\beta(u)_t u_t=\{B(u)_t\}^2$ we obtain (5.2). The proof is complete. □

We also need the following intermediate value theorem:

Lemma 5.2. *Let $h(t)$ be a real-value function on some closed interval $[0,k]$. Suppose that $h(t)$ satisfies*

$$\lim_{t\downarrow t_0}\sup h(t)\leq h(t_0)\leq \lim_{t\uparrow t_0}\inf h(t) \tag{5.4}$$

for each $t_0\in[0,k]$. Then, if $h(0)<h_1<h(k)$ for some $h_1\in(0,k)$, there exists $t_1\in(0,k)$ such that

$$h_1=h(t_1). \tag{5.5}$$

Proof. Put

$$t_1=\sup\{\tau\in[0,k]\,|\,h(t)<h_1\quad \text{for all } t\in[0,\tau]\}. \tag{5.6}$$

By virtue of (5.4) we have

$$0<t_1<k$$

and

$$h(t_1)\leq\lim_{t\uparrow t_1}\inf h(t)\leq h_1. \tag{5.7}$$

Suppose $h(t_1)<h_1$. Then, from (5.4) we obtain

$$\lim_{t\downarrow t_1}\sup h(t)\leq h(t_1)<h_1,$$

namely, there exists $\delta>0$ such that

$$h(t)<h_1\quad\text{for } t\in[t_1,t_1+\delta). \tag{5.8}$$

This is a contradiction to (5.6) and so we get (5.5). The proof is complete. □

Lemma 5.3 (Ref. Imai-Mochizuki [**9**]). *Assume* (A1)(A2)(3.1) *and* (5.1). *Let $u(x,t)$ be a global weak solution of* $(1.1)\sim(1.3)$ *and $\ell(t)$ be a local minimum curve from $[0,T]$ to (a_1,L). Set $\delta=L-a_1$ and suppose*

$$\inf_{\xi\geq 0}\{2\delta F(\sqrt{\delta}\xi)-\xi^2\}<0 \tag{5.9}$$

where $F(\xi) = \int_0^\xi f(\eta)\, d\eta$. *Then, there exists* $M_1 > 0$ *which is independent of* $\ell(t)$ *and* T *such that if* T *is large enough then*

$$u(x,t) \le M_1 \quad \text{for } 0 \le t \le T, x \in (\ell(t), L). \tag{5.10}$$

PROOF. Set

$$h(t) = \left\{ \int_{\ell(t)}^{L} u(x,t)_x{}^2\, dx \right\}^{1/2}.$$

Then, by (5.2) we get

$$h(t)^2 \le 2 \int_{\ell(t)}^{L} F(u)\, dx \bigg|_{t=t'}^{t} + 2F(u(\ell(t),t))(\ell(t) - a_1) \bigg|_{t=t'}^{t} + h(t')^2$$
$$\text{for } 0 \le t' \le t \le T, \tag{5.11}$$

which leads to (5.4) and

$$h(t)^2 \le h(0)^2 + 2\delta F(\sqrt{\delta} h(t)), \tag{5.12}$$

that is,

$$2\delta F(\sqrt{\delta} h(t)) - h(t)^2 + h(0)^2 \ge 0 \quad \text{for all } t \in [0,T]. \tag{5.13}$$

Here, we used $|L - \ell(t)| + |\ell(t) - a_1| = \delta$ and

$$u(x,t) \le \sqrt{\delta} h(t) \quad \text{for all } t \ge 0, x \in [\ell(t), L]. \tag{5.14}$$

On the other hand, noting assumption (5.9) we can choose a positive constant $K > 0$ such as

$$\inf_{\xi \ge 0} (2\delta F(\sqrt{\delta}\xi) - \xi^2 + K^2) < 0. \tag{5.15}$$

Put

$$\xi_1 = \inf\{\xi \ge 0 \,|\, 2\delta F(\sqrt{\delta}\xi) - \xi^2 + K^2 < 0\} (> 0).$$

Since $u(\ell(0),0) \le C(T) \to 0$ as $T \to \infty$ by Lemma 4.5, we see that $\ell(0) \to 0$ as $T \to \infty$. Hence, if T is large enough, then

$$h(0) = \left\{ \int_{\ell(0)}^{L} u_{0,x}{}^2\, dx \right\}^{1/2} < \min\{K, \xi_1\}. \tag{5.16}$$

It follows from (5.13) that

$$2\delta F(\sqrt{\delta} h(t)) - h(t)^2 + K^2 > 0 \quad \text{for all } t \in [0,T]. \tag{5.17}$$

Therefore, by using Lemma 5.2 we shall show

$$h(t) \le \xi_1 \quad \text{for all } t \in [0,T]. \tag{5.18}$$

In fact, suppose that there is $t_1 \in (0, T]$ such that

$$h(t_1) > \xi_1.$$

Then, by the definition of ξ_1 there exists $\xi_2 \in (h(0), h(t_1))$ such that

$$2\delta F(\sqrt{\delta}\xi_2) - \xi_2^2 + K^2 < 0.$$

It follows from Lemma 5.2 that there exists $t_2 \in (0, t_1)$ such that $\xi_2 = h(t_2)$, and so

$$2\delta F(\sqrt{\delta}h(t_2)) - h(t_2)^2 + K^2 < 0.$$

This is a contradiction to (5.17) and we obtain (5.18).

Thus, using (5.14) we have

$$u(x,t) \leq \sqrt{\delta}\xi_1 \quad \text{for } x \in [\ell(t), L], t \in [0, T].$$

The proof is complete. □

6. Proof of Theorem 1.5.

In this section we assume (A1)$\sim$(A3) and prove Theorem 1.5. By Lemma 2.5, Lemma 4.2 and the energy estimates for $u(x,t)$ we can assume for the initial data $u_0(x)$ that

$$u_0(x) > 0 \quad x \in \Omega, \tag{6.1}$$

$$\text{the zero set of } u_0(x) \text{ in each compact set of } \Omega \text{ is finite} \tag{6.2}$$

and

$$\int_\Omega {u_0(x)_x}^2 \, dx < \infty. \tag{6.3}$$

Let $a_1 \in (0, L)$ satisfy (5.9) with $\delta = L - a_1$ and let $u(x,t)$ be a global weak solution of (1.1)$\sim$(1.3). Furthermore, from Proposition 3.1 and Lemma 4.6 we can assume for $u(x,t)$ that

$$u_x(\pm a_1, t) \neq 0 \quad \text{for all } t \geq 0 \tag{6.4}$$

and

$$u(x,t) \text{ does not have a local minimum point in } (-a_1, a_1) \text{ for } t \geq 0. \tag{6.5}$$

We can now consider the next three cases:

(case I) $u(x,t)$ has the unique local maximum point in $(-a_1, a_1)$ for each $t \geq 0$.

(case II) $u_x(x,t) > 0$ for $(x,t) \in [-a_1, a_1] \times [0, \infty)$.

(case III) $u_x(x,t) < 0$ for $(x,t) \in [-a_1, a_1] \times [0, \infty)$.

Using the Angenent's lemma (Lemma 3.4), we obtain the following lemma soon.

LEMMA 6.1. *There exists the local maximum curve* $\ell(t) : [0, \infty) \to (-L, L)$ *satisfying the following conditions for each one of the above three cases*:

(case I)

$$-a_1 < \ell(t) < a_1 \quad \textit{for } t \geq 0 \tag{6.6}$$

and

$$\{x \in (-a_1, a_1) \mid u_x(x, t) = 0\} = \{x \in (-a_1, a_1) \mid x = \ell(t)\} \quad \textit{for } t \geq 0. \tag{6.7}$$

(case II)

$$\ell(t) > a_1 \quad \textit{for } t \geq 0 \tag{6.8}$$

and

$$u_x(x, t) > 0 \quad \textit{for } t \geq 0, x \in [a_1, \ell(t)). \tag{6.9}$$

(case III)

$$\ell(t) < -a_1 \quad \textit{for } t \geq 0 \tag{6.10}$$

and

$$u_x(x, t) < 0 \quad \textit{for } t \geq 0, x \in (\ell(t), -a_1]. \tag{6.11}$$

PROOF. The proof is obvious. □

With these preliminaries, we can show the next key lemma.

LEMMA 6.2. *Let* $u(x, t)$ *be a global weak solution of* (1.1)$\sim$(1.3). *Then, there exist* $d > 0, R' > 0$ *and* $i_0 \in \boldsymbol{N}$ *such that for any integer* $i \geq i_0$,

$$\max_{x \in [-L, L]} u(x, t_i) < R' \quad \textit{for some } t_i \in [(i-1)d, id). \tag{6.12}$$

PROOF. We shall prove Lemma 6.2 only in case I. In other cases we can also show it similarly.

By Corollary 1.4 we note that inequality (4.8) holds with $M = \beta(\xi_1)$, since $u(x, t)$ is a global weak solution of (1.1)$\sim$(1.3). Let $\ell(t) : [0, \infty) \to (-a_1, a_1)$ be the local maximum curve satisfying (6.6) and (6.7). Then, it follows form Lemma 4.7 that for any $i \in \boldsymbol{N}$ there exists $t_i \in [(i-1)d, id)$ such that

$$u(\ell(t_i), t_i) < R, \tag{6.13}$$

where $R > 0$ and $d > 0$ are appeared in Lemma 4.7.

Put

$$x_i^+ = \min\{\tilde{x} \in (-L, L) \mid \tilde{x} \text{ is a local minimum point of } u(\cdot, t_i) \text{ in } (-L, L) \text{ and } \tilde{x} > \ell(t_i)\}$$

and

$$x_i^- = \max\{\tilde{x} \in (-L, L) \mid \tilde{x} \text{ is a local minimum point of } u(\cdot, t_i) \text{ in } (-L, L) \text{ and } \tilde{x} < \ell(t_i)\}.$$

Then, we see that $x_i^- < -a_1$ and $x_i^+ > a_1$.

Let $\ell_i(t) : [0, t_i] \to [a_1, L)$ be a local minimum curve of $u(x,t)$ with $\ell_i(t_i) = x_i^+$. Noting that $\delta = |a_1 - L|$ satisfies (5.9) and applying Lemma 5.3, we have for large i,

$$u(x, t_i) \le M_1 \quad \text{for } x \in [x_i^+, L],$$

where $M_1 > 0$ is appeared in Lemma 5.3. Similarly we get for large i

$$u(x, t_i) \le M_1 \quad \text{for } x \in [-L, x_i^-].$$

Therefore, putting $R' = \max\{M_1, R\}$ we obtain (6.12). The proof is complete. □

Therefore,

LEMMA 6.3. *Let $u(x,t)$ be a global weak solution of* (1.1) ~ (1.3) *and let t_i and R' be as in Lemma* 6.2. *Then*

$$\begin{aligned}\int_\Omega {u_x}^2\,dx\Big|_{t=t_i} &+ 2\int_0^\infty\int_\Omega \{B(u)_t\}^2\,dx\,dt \\ &\le 2\left\{4LF(R') + \int_\Omega {u_{0,x}}^2\,dx\right\} \equiv M_2.\end{aligned} \tag{6.14}$$

Further, if $u(x,t) \le C$ in $(x,t) \in \Omega \times [t_i, \tau]$ for some $\tau > t_i$, then

$$\int_\Omega {u_x}^2\,dx\Big|_{t=\tau} \le 4LF(C) + M_2. \tag{6.15}$$

PROOF. Combining (5.2) with $\ell(t) = -L$ and Lemma 6.2, we get the assertions of Lemma 6.3. □

If we prove the next lemma, we are ready to show Theorem 1.5.

LEMMA 6.4. *Let $u(x,t)$ be a real-value continuous function in $\Omega \times [t_i, \tau]$ which vanishes on the boundary $\partial\Omega$ and let $u(x,t)$ satisfy inequality*

$$\int_\Omega u_x(x,t)^2\,dx \le C \quad \textit{for } t \in [t_i, \tau] \tag{6.16}$$

for some $C > 0$. Then

$$\int_{t_i}^\tau \int_\Omega {u_t}^2\,dx\,dt < \varepsilon \tag{6.17}$$

implies

$$|u(x,t) - u(x,t')| < \varepsilon^{1/4}(2C^{1/2} + |t - t'|^{1/2}) \quad \text{for } x \in \Omega, t, t' \in [t_i, \tau]. \tag{6.18}$$

PROOF. By (6.16), we have

$$|u(x,t) - u(x',t)| \le C^{1/2}|x - x'|^{1/2} \quad \text{for } x, x' \in \Omega, t \in [t_i, \tau]. \tag{6.19}$$

Choose $\hat{x} \in [x - \varepsilon^{1/2}, x]$ or $\hat{x} \in [x, x + \varepsilon^{1/2}]$ to satisfy

$$\left|\int_{t'}^t u_t(\hat{x}, \mu)^2\,d\mu\right| \le \varepsilon^{-1/2}\left|\int_x^{x \pm \varepsilon^{1/2}}\int_{t'}^t u_t(z,\mu)^2\,d\mu\,dz\right| < \varepsilon^{1/2}.$$

Combining this and (6.19) we get

$$\begin{aligned}|u(x,t)-u(x,t')| &\le |u(x,t)-u(\hat{x},t)|+|u(\hat{x},t)-u(\hat{x},t')|+|u(\hat{x},t')-u(x,t')|\\ &\le 2C^{1/2}|x-\hat{x}|^{1/2}+|t-t'|^{1/2}\left|\int_{t'}^{t} u_t(\hat{x},\mu)^2\,d\mu\right|^{1/2}\\ &\le 2C^{1/2}\varepsilon^{1/4}+|t-t'|^{1/2}\varepsilon^{1/4}=\varepsilon^{1/4}(2C^{1/2}+|t-t'|^{1/2}). \quad \square\end{aligned}$$

PROOF OF THEOREM 1.5. We shall show

$$\lim_{i\to\infty}\sup \max_{\bar{\Omega}\times[t_i,(i+1)d]} u(x,t) < R'+1 \tag{6.20}$$

where R', t_i and d are appeared in Lemma 6.2.

Set

$$\tau_0=\sup\left\{\tau \,\middle|\, \lim_{i\to\infty}\sup \max_{\bar{\Omega}\times[t_i,t_i+\tau]} u(x,t)<R'+1\right\}(>0)$$

and suppose

$$\tau_0<\infty.$$

Then, by the definition of τ_0, we see that for any $\delta\in(0,\tau_0)$ there exists $i_1(\ge i_0)$ such that

$$\max_{\bar{\Omega}\times[t_i,t_i+\tau_0-\delta]} u(x,t)<R'+1 \quad \text{for } i\ge i_1,$$

where i_0 is as in Lemma 6.2.

It follows from Lemma 6.3 that

$$\sup_{t\in[t_i,t_i+\tau_0-\delta]}\int_\Omega {u_x}^2\,dx \le 4LF(R'+1)+M_2\equiv C(R') \quad \text{for } i\ge i_1. \tag{6.21}$$

On the other hand, it follows from (6.14) and $\beta''(v)\le 0$ that

$$\int_{t_i}^{t_i+\tau_0-\delta}\int_\Omega {u_t}^2\,dx\,dt \le \frac{1}{\beta'(R'+1)}\int_{t_i}^{\infty}\int_\Omega \{B(u)_t\}^2\,dx\,dt\equiv\varepsilon_i\to 0 \quad (\text{as } i\to\infty). \tag{6.22}$$

Hence, applying Lemma 6.4 to $u(x,t)$ in $\Omega\times[t_i,t_i+\tau_0-\delta]$, we get

$$|u(x,t_i+\tau_0-\delta)-u(x,t_i)|<\varepsilon_i^{1/4}(2C(R')^{1/2}+(\tau_0-\delta)^{1/2}) \quad \text{for } x\in\Omega.$$

Therefore, by (6.12) we see that if i is large enough then

$$\begin{aligned}u(x,t_i+\tau_0-\delta) &< R'+\varepsilon_i^{1/4}(2C(R')^{1/2}+(\tau_0-\delta)^{1/2})\\ &< R'+\frac{1}{2} \quad \text{for } x\in\Omega.\end{aligned} \tag{6.23}$$

Let $v(t)$ be the solution of the initial value problem

$$\begin{cases} \beta(v)_t = f(v) \\ v(0) = R' + \dfrac{1}{2} \end{cases} \tag{6.24}$$

and choose $\delta > 0$ small enough such as

$$v(2\delta) < R' + \frac{3}{4}.$$

Then, setting $v_i(t) = v(t - t_i - \tau_0 + \delta)$ and applying the comparison theorem to $u(x,t)$ and $v_i(t)$ in $\Omega \times (t_i + \tau_0 - \delta, t_i + \tau_0 + \delta)$, we get for large i,

$$u(x,t) \leq v_i(t) \leq v(2\delta) < R' + \frac{3}{4} \quad \text{for} \quad (x,t) \in \Omega \times [t_i + \tau_0 - \delta, t_i + \tau_0 + \delta].$$

This is a contradiction to the definition of τ_0. Hence we get $\tau_0 = \infty$ and so (6.20). Thus, noting $t_{i+1} < (i+1)d$ we obtain Theorem 1.5. The proof is complete. $\square$

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Ryuichi SUZUKI
Department of Mathematics
Faculty of Engineering
Kokushikan University
4-28-1 Setagaya Setagaya-ku
Tokyo, 154
Japan